CHARACTER SUMS, RECIPROCITY AND FUNCTIONAL EQUATIONS

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ABSTRACT. In this article, we examine the role of the recent banner of *Spectral Reciprocity* within the Langlands *Beyond Endoscopy* and Braverman–Kazhdan programmes. We present a novel four-variable character sum identity, which serves as a pivotal arithmetic input for the functional equations of automorphic L-functions. This identity can also be interpreted as a twisted, non-archimedean counterpart to Weber-type integrals from the theory of special functions.

1. INTRODUCTION

In his seminal work *Beyond Endoscopy* [26], Langlands proposed a new approach to the general *Functoriality Conjecture* via trace formulae and poles of automorphic L-functions at $s = 1$. This method was first implemented in the thesis of Venkatesh [33, 34], specifically for the symmetric square L-function of the group GL(2). He showed, by very impressive techniques from analytic number theory, that the pole of this L-function detects the functorial transfer from a one-dimensional torus to GL(2), or in more classical terms, the automorphic induction of Hecke Grössencharakters to dihedral forms of GL(2).

More recently, Ngô [28] and Sakellaridis [31, 30] have advocated for a synthesis of Beyond Endoscopy with Braverman–Kazhdan's programme [11], proposing an alternative to the functoriality conjecture through *functional equations* and *converse theorems*. A central theme of [11] is to systematically deduce functional equations of automorphic L-functions from generalized non-abelian Fourier transforms and Poisson summation, guided by the foundational work of Godement–Jacquet [14]. The application of trace formulae in the spirit of Beyond Endoscopy illuminates the constructions of the conjectural ingredients in [11], see [28, Section 7.2].

Venkatesh's thesis [33] is remarkable in its depth, offering a number of elegant yet simple ideas that broaden the scope of Beyond Endoscopy. For instance, [33, Chapter 3] provides a new analytic proof of the classical converse theorem for modular forms of level 1 using a limiting trace formula. His approach has gained considerable recent interest and been further generalized, e.g., Booker–Farmer–Lee [10] and Blomer–Leung [9].

In this article, we build on the ideas from [33] to prove the functional equation of an L-function by embedding it into a *spectral identity* for moments of L-functions. This approach aligns with current developments in *Spectral Reciprocity* after Blomer–Khan [6, 7], but with several important variations. A key novelty of our work is a surprising reciprocity identity for a four-variable character sum, whose proof is somewhat non-trivial.

Theorem 1.1. *Let $q \geq 1$, a, b, u, v be integers such that $ab | q^\infty$ and $(uv, q) = 1$. Let $\chi \pmod q$ be a Dirichlet character. Define*

$$\mathscr{C}_\chi^\ell(a,u,b,v) := \chi(u) \sideset{}{^*}\sum_{\substack{\alpha \,(\bmod\, a) \\ bv \equiv -\alpha q \,(\bmod\, a)}} e\left(\frac{\ell \overline{\alpha u}}{a}\right) \overline{\chi}\left(\frac{\alpha q + bv}{a}\right). \tag{1.1}$$

Then we have

$$\mathscr{C}_\chi^\ell(a,u,b,v) = e\left(-\frac{\ell q \overline{uv}}{ab}\right) \overline{\mathscr{C}_\chi^\ell(b,v,a,u)}. \tag{1.2}$$

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The identity (1.2) features the switching of parameters $a \leftrightarrow b$ and $u \leftrightarrow v$, despite their distinct roles in the definition (1.1) for the character sum. From an arithmetic perspective, it is important for determining the *root numbers* of functional equations, as we will explain in Section 1.2.

1.1. **Global applications.** Let $q \geq 1$ be an integer, $e(z) := e^{2\pi i z}$, $e_q(z) := e^{2\pi i z/q}$, and χ be a primitive Dirichlet character $\pmod q$. The Gauss sum associated with χ is defined by

$$\epsilon_\chi := \frac{1}{\sqrt{q}} \sum_{\alpha \pmod q} \chi(\alpha) e_q(\alpha). \tag{1.3}$$

We focus on a simple setting to illustrate the role of Theorem 1.1 in Beyond Endoscopy. Pick any orthogonal basis $\mathscr{B}_k(1)$ of holomorphic cusp forms of level 1 and weight k, for which the Fourier expansion of $f \in \mathscr{B}_k(1)$ is given by

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \tag{1.4}$$

where $\lambda_f(1) = 1$, and $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$. The twisted automorphic L-function $L(s, f \times \chi)$ associated with f and χ is defined by the Dirichlet series

$$L(s, f \times \chi) := \sum_{n \geq 1} \frac{\lambda_f(n)\chi(n)}{n^s}, \tag{1.5}$$

which converges absolutely for $\operatorname{Re} s \gg 1$. In this article, k is an even integer with $k \geq 6$. [1]

We obtain a novel proof of the functional equation for $L(s, f \times \chi)$ from the following *spectral identity* of L-functions, rather than using an Eulerian integral representation.

Theorem 1.2. *Let $g \in C_c^\infty(0,\infty)$ and $\mathscr{G}(s) := \int_0^\infty g(x) x^{s-1}\, dx$ be its Mellin transform. Then we have*

$$\int_{(\sigma)} \mathscr{G}(s) \left\{ \sideset{}{^h}\sum_{f \in \mathscr{B}_k(1)} \lambda_f(\ell) \left(L(s, f\times\chi) - i^k \epsilon_\chi^2 q^{1-2s} \frac{\gamma_k(1-s)}{\gamma_k(s)} L(1-s, f \times \overline{\chi}) \right) \right\} \frac{ds}{2\pi i} = 0 \tag{1.6}$$

for any $\sigma \in (0,1)$ and $\ell \geq 1$, where

$$\sideset{}{^h}\sum_{f \in \mathscr{B}_k(1)} \alpha_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathscr{B}_k(1)} \frac{\alpha_f}{||f||^2}, \tag{1.7}$$

$$\gamma_k(s) := c_k (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) = \pi^{-s} \Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right), \quad c_k := 2^{(3-k)/2} \sqrt{\pi}. \tag{1.8}$$

Implicit in (1.6) are the *analytic continuation* and *polynomial growth* of $L(s, f \times \chi)$. A number of subtle points require careful discussion, which have not yet been fully addressed in the literature for our applications, see Section 1.3. To isolate a single form from the spectral average (1.6), we follow an elegant argument from [33, Section 2.6 and Section 3.2 (Proposition 8)], which can be viewed more broadly in [29, Section 1.6 and p. 353] related to *'fundamental lemma'*. In Section 9, we deduce that:

Corollary 1.3. *Let $f \in \mathscr{B}_k(1)$ and $\chi \pmod q$ be a primitive Dirichlet character. Then for any $s \in \mathbb{C}$, we have*

$$L(s, f \times \chi) = i^k \epsilon_\chi^2 q^{1-2s} \frac{\gamma_k(1-s)}{\gamma_k(s)} L(1-s, f \times \overline{\chi}). \tag{1.9}$$

The argument used in Theorem 1.2 applies equally well to additive twists, thus providing a new proof of the *Voronoi formula*, which is stated as follows.

[1] It is well-known, by the Riemann–Roch theorem, that there are no non-zero holomorphic cusp form of level 1 and weight $k < 12$.

Corollary 1.4. *For any $q \geq 1$, $(a,q)=1$ and $g \in C_c^\infty(0,\infty)$, we have*

$$\sum_{n=1}^{\infty} \lambda_f(n) e_q(an) g(n) \;=\; \frac{2\pi i^k}{q} \sum_{n=1}^{\infty} \lambda_f(n)\, e_q\left(-\overline{a}n\right) \int_0^\infty g(x) J_{k-1}\left(\frac{4\pi\sqrt{nx}}{q}\right) dx, \tag{1.10}$$

where $a\overline{a} \equiv 1 (\bmod q)$ and $J_{k-1}(\cdot)$ denotes the J-Bessel function of order $k-1$.

Alternatively, Corollary 1.4 follows from Corollary 1.3 using [24, Theorem 1.3]. Recently, there has been interest in interpreting the Voronoi formulae within various frameworks, see [22], [23], [32], [4], [27].

1.2. **A road map for Theorem 1.2.** Although our argument begins *globally*, using Dirichlet series and the Petersson Trace Formula, [2] the proof of our main arithmetic input (Theorem 1.1) is *local.*

The first summand of (1.6) is given by

$$I_k(\ell;\chi) := \int_{(\sigma)} \mathscr{G}(s) \sum_{f\in\mathscr{B}_k(1)}^{h} \lambda_f(\ell) L(s, f\times\chi)\, \frac{ds}{2\pi i} \;=\; \sum_{n=1}^{\infty} g(n)\chi(n) \sum_{f\in\mathscr{B}_k(1)}^{h} \lambda_f(\ell)\lambda_f(n), \tag{1.11}$$

where its geometric expansion consists of $\mathscr{D}_\chi^\ell(m,c) := \chi(m) S(m,\ell;c)$, where $S(m,\ell;c)$ denotes the Kloosterman sum, and the J-Bessel function (J_{k-1}). With an application of Poisson summation, we obtain

$$I_k(\ell;\chi) \;=\; g(\ell)\chi(\ell) \;+\; \frac{2\pi i^{-k}}{q} \sum_{c\geq 1} \frac{1}{c^2} \sum_{m\in\mathbb{Z}} \widehat{\mathscr{D}}_\chi^\ell(m,c) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c}\right) e_{cq}(-my)\, dy, \tag{1.12}$$

where

$$\widehat{\mathscr{D}}_\chi^\ell(m,c) \;:=\; \sum_{\gamma\,(cq)} \mathscr{D}_\chi^\ell(\gamma,c) e_{cq}(m\gamma) \;=\; \sum_{\gamma\,(cq)} \chi(\alpha) S(\gamma,\ell;c) e_{cq}(m\gamma). \tag{1.13}$$

This is the content of Section 3.1–3.2.

1.2.1. *Two dualities.* The adelic viewpoint suggests that J_{k-1} serves as the archimedean counterpart of $S(m,\ell;n)$. Correspondingly, our argument hinges on two intriguing *geometric dualities*, one for the *Fourier–Hankel transform* of J_{k-1} and the other for the *finite Fourier transform* $\widehat{\mathscr{D}}_\chi^\ell(\,\cdot\,,c)$.

The first duality is known in the classical literature of special functions, commonly known as the *Weber second exponential integral* (see Lemma 2.4):

$$\int_0^\infty e(\alpha y) J_{k-1}(4\pi\beta\sqrt{y}) J_{k-1}(4\pi\gamma\sqrt{y})\, dy \;=\; \frac{i}{2\pi\alpha} e\left(\operatorname{sgn}(\alpha)\frac{k-1}{4}\right) J_{k-1}\left(\frac{4\pi\beta\gamma}{|\alpha|}\right) e\left(-\frac{\beta^2+\gamma^2}{\alpha}\right). \tag{1.14}$$

In particular, the *Hankel inversion formula* (Lemma 2.3) can be interpreted as a limiting form of (1.14) and it plays a crucial role in our argument. The second duality is a twisted, non-archimedean analogue of (1.14):

Proposition 1.5. *Let $\ell, m, c \in \mathbb{Z}$ with $m, c \geq 1$ and $\chi\,(\bmod q)$ be a primitive character. Then*

$$\widehat{\mathscr{D}}_\chi^\ell(m,c) \;=\; \frac{c}{m} e\left(-\frac{1}{mc}\right) \cdot (\epsilon_\chi)^2 \cdot \overline{\widehat{\mathscr{D}}_\chi^\ell(c,m)}, \tag{1.15}$$

where ϵ_χ is the Gauss sum defined in (1.3).

The proof of Proposition 1.5 is non-trivial and a sketch will be provided in Section 1.2.3.

1.2.2. *An analytic-arithmetic cancellation.* We aim to follow the general plan of Spectral Reciprocity. The natural next step is to apply Poisson summation to the c-sum in (1.12), though an observant reader might question its applicability due to the apparent singularity at $c=0$. However, the factor $1/c^2$ of (1.12) is canceled out upon inserting (1.15) and (1.14) (with appropriately chosen α,β,γ)! Additionally, the factor $e(-1/mc)$ from (1.15) perfectly cancels the final exponential factor in (1.14), paving the way for the two upcoming steps.

[2] In terms of adelic settings, this corresponds to the use of *'nonstandard test functions'* of Sakellaridis [29, p. 357, footnote 3].

This technical feature is also crucial in previous works on Beyond Endoscopy, where the Arthur–Selberg trace formula was analyzed and the singularities from the archimedean orbital integrals are more subtle (see [3, Section 4], [15], [13, Section 6]). Once again, smoothing of these singularities is necessary for Poisson summation, but this requires delicate use of an *approximate functional equation* and the *class number formula*.

1.2.3. *A three-fold reciprocity and local analysis.* In Section 3.3, we apply a *first* additive reciprocity

$$\frac{\overline{m}}{c} + \frac{\overline{c}}{m} \equiv \frac{1}{mc} \pmod 1 \tag{1.16}$$

to (1.13), where $m\overline{m} \equiv 1 \pmod c$ and $c\overline{c} \equiv 1 \pmod m$. In Section 4.2, we split the c-sum and m-sum in (1.12) appropriately. One roughly arrives at

$$\frac{\epsilon_\chi}{q^{3/2}} \sum_{c_0 \mid q^\infty} \sum_{m_0 \mid q^\infty} \sum_{\substack{m' \geq 1 \\ (m',q)=1}} \frac{1}{m_0 m'} \sum_{\substack{c' \neq 0 \\ (c', m'q)=1}} e_{c_0 m_0 m'}\left(\ell q \overline{c'}\right) \mathscr{C}_\chi^\ell(c_0, c', m_0, m') \times (\text{archimedean part}), \tag{1.17}$$

where $\mathscr{C}_\chi^\ell(\cdots)$ is the character sum defined in Theorem 1.1. See (4.11) for the precise expressions.

The character sum (1.1) exhibits a twisted multiplicativity property (Lemma 4.10), enabling a *local* analysis. Theorem 1.1 follows from a p-adic stationary phase argument ([18, Chapter 12.3]) at both ramified and unramified places (Section 5), along with a *second* use of (1.16) (see (5.2)). Our proof of (1.2), and hence Theorem 1.2, is non-trivial even when the conductor of χ is a prime. This crucial ingredient behind Theorem 1.2 is previously unknown, despite the extensive literature on moments of L-functions similar to (1.11) over the past few decades.

Now, a *third* use of the additive reciprocity (1.16) allows us to combine the exponential phase of (1.2) with that of (1.17), then a second copy of ϵ_χ arises upon opening up $\overline{\mathscr{C}_\chi^\ell}(\cdots)$ by its definition (1.1) and gluing variables back together, see (6.5). Theorem 1.1 will be proved in Section 5, and Proposition 1.5 follows from it. Note that (1.16) also resolves the analytic issues in Section 1.2.2.

1.2.4. *Back to the global aspect.* We are now in a position to swap the roles of the c-sum and m-sum in (1.12). This is less straightforward compared to other implementations of *Spectral Reciprocity*, particularly given the discussions in Section 1.2.3. Setting aside a couple of analytic subtleties to be addressed in Sections 1.3 and 7–8, Theorem 1.2 follows from *backward* applications of the Poisson and Petersson formulae (Section 6), with the factor $(\epsilon_\chi)^2$ from (1.15) becoming the root number for the functional equation (1.9).

1.3. **Analytic continuation.** In the context of Beyond Endoscopy, the challenge of obtaining analytic continuation should not be underestimated. Indeed, *limiting forms* of trace formulae were sufficient for showing the automorphic induction of characters ([34]) and converse theorems ([33, 10, 9]). In other words, it suffices to study the poles of L-functions at $s = 1$, or to obtain a continuation slightly past the line $\operatorname{Re} s = 1$.

Furthermore, using his refined analysis of the trace formula for GL(2) in [3, 1], Altuğ proved that the L-function of the Ramanujan Δ-function admits an analytic continuation to the region $\operatorname{Re} s > 31/32$ in [2]. See also Duke–Iwaniec [12] and White [38] for other approaches. They were unable to obtain analytic continuation past $\operatorname{Re} s = 1/2$. However, an analytic continuation to $\operatorname{Re} s > 1/2 - \delta$ (for some small absolute $\delta > 0$) is essential to even begin discussing the functional equation.

In his thesis [33, Chapter 2.6], Venkatesh came up with the idea of embedding the holomorphic cusp forms of weight k and level 1 into the *full spectrum* of L^2-automorphic forms of level 1, consisting of Maass cusp forms of weight 0, holomorphic cusp forms of *all weights* and the Eisenstein series. This allows an arbitrary smooth compactly test function on $(0, \infty)$ can be put on the *geometric* side of the trace formula, i.e., using the *arithmetic Kuznetsov (or Petersson–Kuznetsov)* trace formula (see [18, Theorem 16.5]) instead. This simplifies the analysis, reducing it to handling a basic Fourier integral, and one obtains

$$\sum_{\substack{f:\text{ full spectrum}\\ \text{of level 1}}}^{h} H(t_f)\lambda_f(\ell)\sum_n g(n/X)\lambda_f(n)\chi(n) \;=\; O_A(X^{-A}) \qquad (X\to\infty). \tag{1.18}$$

However, the space of admissible *spectral* test functions H is somewhat limited and the real technical challenge lies in constructing a test function in this space that 'effectively' isolates a specific part of the spectrum. This problem is addressed in [33, Chapter 6.3], which involves an elaborate analysis of the Sears–Titchmarsh transform. In the case of [33, p. 50], an extra technical assumption concerning the growth in spectral parameters (Laplace eigenvalues/weights) is needed (cf. [33, eq. (2.35)]).

This is why we chose to work with the Petersson formula and its integral transform. By the stationary calculus of [8], we find that $L(s, f\times\chi)$ admits an analytic continuation and has polynomial growth in the region $\operatorname{Re} s > -(k-6)/2$, see Section 7. To our knowledge, this argument has not yet appeared in the literature.

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2. Preliminaries

Let $\phi\in C_c^\infty(\mathbb{R})$ and $\psi\in C_c^\infty(0,\infty)$. The *Fourier transform* of ϕ and the *Mellin transform* of ψ are given by

$$\widehat{\phi}(y) \;:=\; \int_{\mathbb{R}} \phi(x)e(-xy)\,dx \qquad (y\in\mathbb{R}) \tag{2.1}$$

and

$$\widetilde{\psi}(s) \;:=\; \int_0^\infty \psi(x)x^{s-1}\,dx \qquad (s\in\mathbb{C}) \tag{2.2}$$

respectively, where $e(z) := e^{2\pi i z}$. Their respective inversion formula, i.e.,

$$\phi(x) \;=\; \int_{\mathbb{R}} \widehat{\phi}(y)e(xy)\,dy \tag{2.3}$$

and

$$\psi(x) \;=\; \int_{(\sigma)} \widetilde{\psi}(s)x^{-s}\,\frac{ds}{2\pi i}, \tag{2.4}$$

holds provided that the integral converges absolutely.

In this article, the Hankel transform also plays an important role. To define such a transform, we first introduce the *J-Bessel functions* via the following generating series:

$$e^{\frac12 z(\xi-1/\xi)} \;=\; \sum_{n\in\mathbb{Z}} J_n(z)\xi^n \tag{2.5}$$

for $z,\xi\in\mathbb{C}$. By standard complex analysis, the following power series expansion holds

$$J_n(z) \;=\; \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r+n)!}\left(\frac{z}{2}\right)^{n+2r}. \tag{2.6}$$

It converges pointwise absolutely and uniformly on every compact subset of $\mathbb{C}$. See Watson [37, Section 2.1] for the details. The J-Bessel function also admits the following *Mellin-Barnes representation:*

Lemma 2.1. *Let $k\ge 2$, $c_k := 2^{(3-k)/2}\sqrt{\pi}$, and*

$$\gamma_k(s) \;:=\; c_k(2\pi)^{-s}\Gamma\left(s+\frac{k-1}{2}\right) \;=\; \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right). \tag{2.7}$$

Then we have

$$J_{k-1}(4\pi x) \;=\; \frac{1}{2\pi}\int_{(\sigma)} \frac{\gamma_k(1-s)}{\gamma_k(s)} x^{2(s-1)} \frac{ds}{2\pi i} \tag{2.8}$$

for $x>0$, *where the integral converges absolutely whenever* $1<\sigma<(k+1)/2$.

Proof. See [16, eq. 8.412.4] with the substitutions $s \to -s - \frac{k-1}{2}$ and $x \to 4\pi x$. In fact, (2.8) can be verified by comparing the residual series obtained from contour-shifting to $-\infty$ in (2.8) with (2.6). The absolute convergence can be verified easily by Stirling's formula. □

Let $F \in C_c^\infty(0,\infty)$ and $k \geq 2$. Then the *Hankel transform* of F of order $k-1$ is defined by

$$(\mathscr{H}_k F)(a) \;:=\; 2\pi \int_0^\infty F(x) J_{k-1}(4\pi\sqrt{ax})\, dx \qquad (a \,>\, 0). \tag{2.9}$$

The rapid decay of $\mathscr{H}_k F$ can be deduced by integrating-by-parts many times in (2.9) using

$$J_{k-1}(2\pi x) \;=\; W_k(x) e(x) \,+\, \overline{W_k}(x) e(-x) \tag{2.10}$$

for $x>0$, where W_k is a smooth function satisfying $x^j(\partial^j W_k)(x) \ll_{j,k} 1/\sqrt{x}$ for any $j \geq 0$, see [37, p. 206].

Lemma 2.2. *For* $k>2$ *and* $j \in \{0,1,2,3\}$ *we have*

$$(\mathscr{H}_k F)^{(j)}(a) \;\ll_k\; a^{(k-2j-1)/2} \qquad \textit{for} \quad 0<a<1. \tag{2.11}$$

Proof. Follows directly from the recurrence $2J_k'(z) = J_{k-1}(z) - J_{k+1}(z)$, and the estimate

$$J_{k-1}(y) \;\ll_k\; y^{k-1} \quad \text{for} \quad y \,>\, 0. \tag{2.12}$$

□

We also have the following well-known inversion formula.

Lemma 2.3. *For any* $F \in C_c^\infty(0,\infty)$, *we have*

$$(\mathscr{H}_k \circ \mathscr{H}_k F)(b) \;=\; F(b) \qquad (b \,>\, 0). \tag{2.13}$$

Proof. See [37, Section 14.3–4] with minor adjustments. □

The Hankel inversion formula can be obtained as a limiting case of the following result, which is also essential to our argument. While Bessel functions with *positive arguments* are more commonly encountered in the analytic theory of automorphic forms (e.g., (1.10) or (2.16) below), it turns out to be technically convenient to invoke those with *complex arguments* as intermediates in our case:

Lemma 2.4. *Let* $k \geq 2$, $\operatorname{Re}\alpha > 0$, *and* $\beta, \gamma > 0$. *Then*

$$\int_0^\infty e^{-2\pi\alpha y} J_{k-1}(4\pi\beta\sqrt{y}) J_{k-1}(4\pi\gamma\sqrt{y})\, dy \;=\; \frac{i^{1-k}}{2\pi\alpha} J_{k-1}\left(\frac{4\pi i\beta\gamma}{\alpha}\right) \exp\left(-2\pi\frac{\beta^2+\gamma^2}{\alpha}\right). \tag{2.14}$$

The integral converges absolutely.

Proof. Follows directly from [37, Chapter 13.31] or [16, eq. 6.676.1-2] with minor adjustments. □

Remark 1. *The integral identity (2.14) is a variant of (1.14), where the latter does not converge absolutely. The integral of (1.14) can be understood as follows. It is a sum of two integrals, one over* $(0,1)$ *and the other over* $(1,\infty)$. *Using (2.12), the integral over* $(0,1)$ *converges absolutely. The integral over* $(1,\infty)$ *can be shown to converge by an integration-by-parts argument with (2.10).*

Lemma 2.5. *Let $h \in C^\infty[\alpha,\beta]$ be a real-valued function and $w \in C_c^\infty[\alpha,\beta]$. Suppose there exist $W,V,H,G,R>0$ such that the following bounds hold for any $t \in [\alpha,\beta]$:*

- $w^{(j)}(t) \ll_j W/V^j$ *for any* $j \geq 0$,
- $h^{(j)}(t) \ll_j H/G^j$ *for any* $j \geq 2$, *and*
- $|h'(t)| \geq R$.

Then for any $A \geq 0$, we have

$$\int_{\mathbb{R}} w(t)e(h(t))\, dt \ll_A (\beta-\alpha)W\left(\frac{1}{RV}+\frac{1}{RG}+\frac{H}{(RG)^2}\right)^A. \tag{2.15}$$

Proof. [8, Lemma 8.1]. □

Lemma 2.6 (Petersson trace formula)**.** *Let $\mathscr{B}_k(1)$ be an orthogonal basis of holomorphic cuspidal Hecke eigenforms of level 1 and weight k. For any $\ell, n \geq 1$, we have*[3]

$$\sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell)\lambda_f(n) = \delta(n=\ell) + 2\pi i^{-k}\sum_{c=1}^\infty \frac{S(n,\ell;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{n\ell}}{c}\right), \tag{2.16}$$

where $\lambda_f(1)=1$, the notation $\sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)}$ was defined in (1.7), and the Kloosterman sum is given by

$$S(n,\ell;c) := \sideset{}{^*}\sum_{x(c)} e_c\left(nx+\ell\overline{x}\right) \qquad \text{with} \qquad e_c(x) := e^{2\pi i x/c}.$$

Proof. See [18, Proposition 14.5 and Lemma 14.10]. □

Lemma 2.7 (Poisson summation $(\bmod\, c)$)**.** *Let $c, X>0$ and $c \in \mathbb{Z}$. Let $V \in C_c^\infty(\mathbb{R})$ and $K:\mathbb{Z}\to\mathbb{C}$ be a c-periodic function. Then*

$$\sum_{n\in\mathbb{Z}} V(n/X)K(n) = \frac{X}{c}\sum_{m\in\mathbb{Z}}\left(\sum_{\gamma(c)} K(\gamma)\, e_c\left(m\gamma\right)\right)\int_0^\infty V(y)\, e\left(-\frac{mXy}{c}\right) dy. \tag{2.17}$$

Proof. Follows from the standard Poisson summation formula. □

3. Setting the stage for Theorem 1.2: Petersson–Poisson–Reciprocity

Let $g \in C_c^\infty(0,\infty)$ be a given test function. Define

$$I_k(\ell;\chi) := \sum_{n=1}^\infty g(n)\chi(n) \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell)\lambda_f(n). \tag{3.1}$$

Section 3 through 6 are devoted to proving the following identity:

$$I_k(\ell;\chi) = 2\pi i^k \frac{\epsilon_\chi^2}{q}\sum_{c=1}^\infty \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell)\lambda_f(c)\overline{\chi}(c)\int_0^\infty g(y)J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy. \tag{3.2}$$

3.1. **Step 1: Petersson trace formula.** Applying (2.16) to (3.1), we have

$$I_k(\ell;\chi) = g(\ell)\chi(\ell) + 2\pi i^{-k}\sum_{c=1}^\infty c^{-1}\sum_n g(n)\chi(n)\, S(n,\ell;c)\, J_{k-1}\left(\frac{4\pi\sqrt{n\ell}}{c}\right).$$

[3] In this article, we use $\delta(\cdots)$ to denote the indicator function with respect to the condition $(\cdots)$.

This is followed by opening up the Kloosterman sum by its definition, i.e.,

$$I_k(\ell;\chi) \;=\; g(\ell)\chi(\ell) \;+\; 2\pi i^{-k}\sum_{c=1}^{\infty} c^{-1}\sideset{}{^*}\sum_{x\,(c)} e_c\left(\ell\overline{x}\right)\sum_n g(n)\chi(n)\,J_{k-1}\left(\frac{4\pi\sqrt{n\ell}}{c}\right)e_c(nx). \tag{3.3}$$

3.2. **Step 2: Poisson summation.** We apply Lemma 2.7 to the n-sum of (3.3), which gives

$$I_k(\ell;\chi) \;=\; g(\ell)\chi(\ell) \;+\; 2\pi i^{-k}\sum_{c=1}^{\infty} c^{-1}\sideset{}{^*}\sum_{x\,(c)} e_c\left(\ell\overline{x}\right)\cdot\frac{1}{cq}\sum_m\left(\sum_{\gamma\,(cq)}\chi(\gamma)e_c\left(\gamma x\right)e_{cq}\left(m\gamma\right)\right)$$
$$\times\int_0^{\infty} g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c}\right)e\left(-\frac{my}{cq}\right)\,dy.$$

The γ-sum can be decomposed via $\gamma=\alpha+\beta q$ with $\alpha\,(\mathrm{mod}\; q)$ and $\beta\,(\mathrm{mod}\; c)$, i.e.,

$$\begin{aligned}\sum_{\gamma\,(cq)}\chi(\gamma)e_c\left(\gamma x\right)e_{cq}\left(m\gamma\right) \;&=\; \sum_{\alpha\,(q)}\sum_{\beta\,(c)}\chi(\alpha)e_{cq}\left((\alpha+\beta q)(qx+m)\right)\\ &= c\,\delta(xq\equiv -m\,(c))\sum_{\alpha\,(q)}\chi(\alpha)e_q\left(\alpha(qx+m)/c\right)\\ &= c\sqrt{q}\epsilon_\chi\delta(xq\equiv -m\,(c))\overline{\chi}\left(\frac{qx+m}{c}\right),\end{aligned}\tag{3.4}$$

where the last line follows from the primitivity of $\chi\,(\mathrm{mod}\; q)$, see [18, eq. (3.12)]. Hence, we get

$$I_k(\ell;\chi) \;=g(\ell)\chi(\ell)$$
$$+\;2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c=1}^{\infty} c^{-1}\sideset{}{^*}\sum_{x\,(c)} e_c\left(\ell\overline{x}\right)\sum_{m\equiv -xq\,(c)}\overline{\chi}\left(\frac{qx+m}{c}\right)\int_0^{\infty} g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c}\right)e\left(-\frac{my}{cq}\right)\,dy. \tag{3.5}$$

To address coprimality issues, we split the c-sum of (3.5) by $c=c_0c'$, where $c_0:=(c,q^\infty)$ and $(c',q)=1$. In particular, we have

$$\sideset{}{^*}\sum_{\substack{x\,(c)\\ m\equiv -xq\,(c)}} e_c\left(\ell\overline{x}\right)\overline{\chi}\left(\frac{qx+m}{c}\right) \;=\; \chi(c')e_{c'}\left(-\ell q\overline{c_0 m}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv -\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right). \tag{3.6}$$

Notice that the dual zeroth frequency (i.e., the term $m=0$) of (3.5) is non-vanishing only when $c_0|q$ and $c'=1$. In this case, the α-sum is given by

$$\sideset{}{^*}\sum_{\alpha\,(c_0)} e_{c_0}\left(\ell\overline{\alpha}\right)\overline{\chi}\left(\alpha\frac{q}{c_0}\right) \;=\; \delta(c_0=q)\sum_{\alpha\,(q)}\overline{\chi}(\alpha)e_q\left(\ell\overline{\alpha}\right) \;=\; \sqrt{q}\epsilon_\chi\overline{\chi}(\ell).$$

We extract the zeroth frequency from the rest of the frequencies, resulting in the expression:

$$I_k(\ell;\chi) \;=\; g(\ell)\chi(\ell) \;+\; 2\pi i^{-k}\frac{\epsilon_\chi^2}{q}\overline{\chi}(\ell)\int_0^{\infty} g(y)J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{q}\right)dy \;+\; S_k(\ell;\chi), \tag{3.7}$$

where

$$S_k(\ell;\chi) \;=\; 2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{\substack{c'\geq 1\\ (c',q)=1}}\sum_{\substack{m\neq 0\\ (m,c')=1}}\frac{\chi(c')}{c_0c'}e_{c'}\left(-\ell q\overline{c_0 m}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv -\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right)$$
$$\times\int_0^{\infty} g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(-\frac{my}{c_0c'q}\right)\,dy. \tag{3.8}$$

3.3. Step 3: Reciprocity. Applying the additive reciprocity (1.16) to the factor $e_{c'}\left(-\ell q \overline{c_0 m}\right)$ in (3.8), it follows that

$$S_k(\ell;\chi) = 2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{\substack{c'\geq 1\\(c',q)=1}}\sum_{\substack{m\neq 0\\(m,c')=1}}\frac{\chi(c')}{c_0c'}\,e\left(\frac{\ell q\overline{c'}}{c_0 m}-\frac{\ell q}{c_0c'm}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right)$$
$$\times\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(-\frac{my}{c_0c'q}\right)dy. \tag{3.9}$$

4. Step 4: Preparation on the c_0, c'-sums

To prepare for Poisson summation in the c_0, c'-sums and for swapping the roles of m and c_0c' at a later stage, two crucial components are needed:

- (Analytic) Transform the c'-sum with $c'\geq 1$ in (3.9) to a sum over all integers, and remove the singularity at $c'=0$ by suitable analytic manipulations;
- (Arithmetic) Transform the character sum in (3.9) into a suitable form that facilitates the eventual combination of c_0- and c'-sum.

The second point requires a fairly intricate analysis of character sums, see Section 4.2 – 6.1.

4.1. Step 4.1: Analytic preparation via Hankel inversion. The c'-sum of (3.9) can be rewritten to sum over all non-zero integers. Indeed, observe that the contribution from $m<0$ in (3.9) is given by

$$2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{\substack{c'\geq 1\\(c',q)=1}}\sum_{\substack{m\geq 1\\(m,c')=1}}\frac{\chi(c')}{c_0c'}\,e\left(-\frac{\ell q\overline{c'}}{c_0 m}+\frac{\ell q}{c_0c'm}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(-\ell\overline{\alpha c'}\right)\overline{\chi}\left(-\frac{\alpha q+m}{c_0}\right)$$
$$\times\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(\frac{my}{c_0c'q}\right)dy, \tag{4.1}$$

upon making the changes of variables $m\mapsto -m$ and $\alpha\mapsto-\alpha$. The change of variables $c'\mapsto -c'$ and the fact that $J_{k-1}(-x)=-J_{k-1}(x)$ (follows from (2.6) and $k\in 2\mathbb{N}$) allow us to rewrite (4.1) as:

$$2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{\substack{c'\leq -1\\(c',q)=1}}\sum_{\substack{m\geq 1\\(m,c')=1}}\frac{\chi(c')}{c_0c'}\,e\left(\frac{\ell q\overline{c'}}{c_0 m}-\frac{\ell q}{c_0c'm}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right)$$
$$\times\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(-\frac{my}{c_0c'q}\right)dy.$$

Hence, the expression (3.9) becomes:

$$S_k(\ell;\chi) = 2\pi i^{-k}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{\substack{c'\neq 0\\(c',q)=1}}\sum_{\substack{m\geq 1\\(m,c')=1}}\frac{\chi(c')}{c_0c'}\,e\left(\frac{\ell q\overline{c'}}{c_0 m}-\frac{\ell q}{c_0c'm}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right)$$
$$\times\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(-\frac{my}{c_0c'q}\right)dy. \tag{4.2}$$

The last integral converges absolutely since $g\in C_c^\infty(0,\infty)$. By the dominated convergence theorem, we have

$$\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)e\left(-\frac{my}{c_0c'q}\right)dy$$
$$=\lim_{\epsilon\to 0+}\int_0^\infty g(y)\,J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right)\exp\left(-2\pi\left(\epsilon+\frac{im}{c_0c'q}\right)y\right)dy. \tag{4.3}$$

The Hankel inversion formula (Lemma 2.3) with the change of variables $x \to x/q^2$ give

$$\int_0^\infty g(y)\, J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right) e\left(-\frac{my}{c_0c'q}\right)\, dy$$
$$= \frac{2\pi}{q^2} \lim_{\epsilon\to 0+} \int_0^\infty (\mathscr{H}_k g)\left(x/q^2\right) \int_0^\infty J_{k-1}\left(\frac{4\pi\sqrt{xy}}{q}\right) J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right) \exp\left(-2\pi\left(\epsilon+\frac{im}{c_0c'q}\right)y\right)\, dy\, dx, \tag{4.4}$$

where the interchange of the order of integration is permitted by absolute convergence and the decay of the Hankel transform $\mathscr{H}_k g$ of g. Now, we are in a position to apply Lemma 2.4. In other words,

$$\int_0^\infty g(y)\, J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right) e\left(-\frac{my}{c_0c'q}\right)\, dy$$
$$= \frac{2\pi}{q^2} \lim_{\epsilon\to 0+} \int_0^\infty (\mathscr{H}_k g)\left(x/q^2\right) \frac{i^{1-k}}{2\pi\left(\epsilon+\frac{im}{c_0c'q}\right)} J_{k-1}\left(\frac{4\pi i\sqrt{\ell x}}{c_0c'q\epsilon+im}\right) \exp\left(-2\pi\,\frac{x/q^2+\ell/c^2}{\epsilon+\frac{im}{c_0c'q}}\right)\, dx. \tag{4.5}$$

By the decay of $\mathscr{H}_k g$ and dominated convergence again, it follows from continuity that

$$\int_0^\infty g(y)\, J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{c_0c'}\right) e\left(-\frac{my}{c_0c'q}\right)\, dy$$
$$= \frac{i^{-k}}{q}\frac{c_0c'}{m}\, e\left(\frac{\ell q}{c_0c'm}\right) \int_0^\infty (\mathscr{H}_k g)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m}\right) e\left(\frac{c_0c'x}{qm}\right)\, dx. \tag{4.6}$$

Substitute the last expression back into (4.2) and observe the cancellations in (1) a pair of exponential phases, and (2) a pair of factors c_0c'. We thus obtain

$$S_k(\ell;\chi) = \frac{2\pi}{q}\frac{\epsilon_\chi}{\sqrt{q}} \sum_{c_0|q^\infty} \sum_{\substack{c'\neq 0\\ (c',q)=1}} \sum_{\substack{m\geq 1\\ (m,c')=1}} \frac{1}{m}\chi(c')\, e_{c_0m}\left(\ell q\overline{c'}\right) \sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right)$$
$$\times \int_0^\infty (\mathscr{H}_k g)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m}\right) e\left(\frac{c_0c'x}{qm}\right)\, dx. \tag{4.7}$$

We must show that the order of summation can be interchanged to set the stage for the Poisson summation in c_0, c'-sums in Section 6.2. This requires $k \geq 6$ and follows from trivially bounding the sums (the α-sum by c_0) and integration by parts thrice in (4.7). Indeed, by Lemma 2.2 and (2.10), observe that

$$\left(\frac{c_0c'}{m}\right)^3 \int_0^\infty (\mathscr{H}_k g)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m}\right) e\left(\frac{c_0c'x}{qm}\right)\, dx \ll_{\ell,q,k} \frac{1}{m^{k-1}}. \tag{4.8}$$

4.2. **Step 4.2: Arithmetic preparation on the c_0, c'-sums.** Now, we move on to analyze the arithmetic component of (4.7), i.e.,

$$\chi(c')e_{c_0m}\left(\ell q\overline{c'}\right) \sideset{}{^*}\sum_{\substack{\alpha\,(c_0)\\ m\equiv-\alpha q\,(c_0)}} e_{c_0}\left(\ell\overline{\alpha c'}\right)\overline{\chi}\left(\frac{\alpha q+m}{c_0}\right). \tag{4.9}$$

For technical convenience, we consider a slightly more general character sum which we define as follows. Let a, b, h, r, u, v be integers such that $ab|r^\infty$ and $(uv, r) = 1$. Let ψ be a primitive character mod r. Define

$$\mathscr{C}_\psi^h(a,u,b,v) := \psi(u) \sideset{}{^*}\sum_{\substack{\alpha\,(a)\\ bv\equiv-\alpha r\,(a)}} e_a\left(h\overline{\alpha u}\right)\overline{\psi}\left(\frac{\alpha r+bv}{a}\right). \tag{4.10}$$

We decouple the m-sum of (4.7) by $m = m_0m'$ (as before), where $m_0 := (m, q^\infty)$ and $(m', q) = 1$. Then (4.9) is equal to

$$e_{c_0m_0m'}\left(\ell q\overline{c'}\right)\mathscr{C}^\ell_\chi(c_0, c', m_0, m'),$$

and

$$S_k(\ell;\chi) = \frac{2\pi}{q}\frac{\epsilon_\chi}{\sqrt{q}}\sum_{c_0|q^\infty}\sum_{m_0|q^\infty}\sum_{\substack{m'\geq 1\\(m',q)=1}}\frac{1}{m_0m'}\sum_{\substack{c'\neq 0\\(c',m'q)=1}} e_{c_0m_0m'}\left(\ell q\overline{c'}\right)\mathscr{C}^\ell_\chi(c_0,c',m_0,m')$$
$$\times\int_0^\infty (\mathscr{H}_kg)\left(x/q^2\right)J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0m'}\right)e\left(\frac{c_0c'x}{qm_0m'}\right)dx. \tag{4.11}$$

We have the following twisted multiplicativity for the character sum (4.10).

Lemma 4.1. *Let $a_1, a_2, b_1, b_2, h, r_1, r_2, u, v$ be integers such that $a_1b_1|r_1^\infty$, $a_2b_2|r_2^\infty$, and $(r_1, r_2) = (uv, r_1r_2) = 1$. Let ψ_1 and ψ_2 be primitive characters mod r_1 and r_2 respectively. Then*

$$\mathscr{C}^h_{\psi_1\psi_2}(a_1a_2,u,b_1b_2,v) = \overline{\psi_1(b_2^2)\psi_2(b_1^2)}\mathscr{C}^{hr_2}_{\psi_1}(a_1,a_2b_2u,b_1,v)\mathscr{C}^{hr_1}_{\psi_2}(a_2,a_1b_1u,b_2,v)$$
$$= \overline{\psi_1(a_2^2)\psi_2(a_1^2)}\mathscr{C}^{hr_2}_{\psi_1}(a_1,u,b_1,a_2b_2v)\mathscr{C}^{hr_1}_{\psi_2}(a_2,u,b_2,a_1b_1v).$$

Proof. By the Chinese remainder theorem, we have

$$\mathscr{C}^h_{\psi_1\psi_2}(a_1a_2,u,b_1b_2,v) = \psi_1(a_2u)\sideset{}{^*}\sum_{\substack{\alpha_1 \ (\mathrm{mod}\ a_1)\\ b_1b_2v\equiv-\alpha_1r_1r_2\,(a_1)}} e_{a_1}(h\overline{\alpha_1a_2u})\overline{\psi_1}\left(\frac{\alpha_1r_1r_2+b_1b_2v}{a_1}\right)$$
$$\times\psi_2(a_1u)\sideset{}{^*}\sum_{\substack{\alpha_2 \ (\mathrm{mod}\ a_2)\\ b_1b_2v\equiv-\alpha_2r_1r_2\,(a_2)}} e_{a_2}(h\overline{\alpha_2a_1u})\overline{\psi_2}\left(\frac{\alpha_2r_1r_2+b_1b_2v}{a_2}\right).$$

Applying the change of variables $\alpha_1\mapsto\alpha_1b_2\overline{r_2}$ and $\alpha_2\mapsto\alpha_2b_1\overline{r_1}$ yields the first equality. To get the second equality, apply the change of variables $\alpha_1\mapsto\alpha_1\overline{a_2r_2}$ and $\alpha_2\mapsto\alpha_2\overline{a_1r_1}$ instead. □

Remark 2. *This lemma constitutes the main reason why the factor $e_{abv}\left(\ell q\overline{u}\right)$ is purposefully left out in the definition of $\mathscr{C}^\ell_\psi(\cdots)$. Otherwise, no multiplicative relation would hold as such an exponential factor is not multiplicative in a and b.*

5. Local analysis of character sums: proof of Theorem 1.1

With Lemma 4.1, we prove Theorem 1.1, which is restated as follows.

Restatement of Theorem 1.1. *Let a, b, h, r, u, v be integers such that $ab|r^\infty$ and $(uv, r) = 1$. Let ψ be a character mod r, then we have*

$$\mathscr{C}^h_\psi(a,u,b,v) = e_{ab}\left(-hr\overline{uv}\right)\overline{\mathscr{C}^h_\psi(b,v,a,u)}.$$

Proof. We first consider the case when $r = p^k$, where p is a prime and $k\geq 1$. Since $ab|r^\infty = p^\infty$, we may write $a = p^s$ and $b = p^t$ for some $s, t\geq 0$. In this, we have

$$\mathscr{C}^h_\psi(p^s,u,p^t,v) = \psi(u)\sideset{}{^*}\sum_{\substack{\alpha\,(p^s)\\ p^tv\equiv-\alpha p^k\,(p^s)}} e_{p^s}\left(h\overline{\alpha u}\right)\overline{\psi}\left(\frac{\alpha p^k+p^tv}{p^s}\right)$$

and

$$e_{p^{s+t}}\left(-hp^k\overline{uv}\right)\overline{\mathscr{C}_\psi^h(p^t,v,p^s,u)} \;=\; \overline{\psi}(v)e_{p^{s+t}}\left(-hp^k\overline{uv}\right)\sideset{}{^*}\sum_{\substack{\alpha\,(p^t)\\ p^s u\equiv -\alpha p^k\,(p^t)}} e_{p^t}\left(-h\overline{\alpha v}\right)\psi\left(\frac{\alpha p^k+p^s u}{p^t}\right).$$

Claim: We have

$$\mathscr{C}_\psi^h(p^s,u,p^t,v) \;=\; 0 \;=\; \overline{\mathscr{C}_\psi^h(p^t,v,p^s,u)}$$

unless one of the following holds:

(1) $s=t\le k$,

(2) $t=k<s$, or

(3) $s=k<t$.

Indeed, the congruence condition for $\mathscr{C}_\psi^h(p^s,u,p^t,v)$ implies that it is equal to 0 unless one of the following holds: $s\le t\le k$, (2) or (3). If $s<t\le k$, then

$$\overline{\psi}\left(\frac{\alpha p^k+p^t v}{p^s}\right) \;=\; \overline{\psi}\left(\alpha p^{k-s}+p^{t-s}v\right) \;=\; 0,$$

and hence $\mathscr{C}_\psi^h(p^s,u,p^t,v)=0$ unless (1) or (2) or (3) holds. Similarly, the congruence condition implies that $\overline{\mathscr{C}_\psi^h(p^t,v,p^s,u)}=0$ unless $t\le s\le k$ or (2) or (3), and the presence of $\psi\left(\frac{\alpha p^k+p^s u}{p^t}\right)$ implies that it is also 0 if $t<s\le k$. This proves our claim.

Before we proceed to analyse the character sum in the remaining three cases above, we apply a change of variable $\alpha\mapsto\alpha v$ to obtain

$$\mathscr{C}_\psi^h(p^s,u,p^t,v) \;=\; \psi(u\overline{v})\sideset{}{^*}\sum_{\substack{\alpha\,(p^s)\\ p^t\equiv -\alpha p^k\,(p^s)}} e_{p^s}\left(h\overline{\alpha uv}\right)\overline{\psi}\left(\frac{\alpha p^k+p^t}{p^s}\right).$$

<u>Case 1:</u> $s=t\le k$. In this case, we have

$$\begin{aligned}\mathscr{C}_\psi^h(p^s,u,p^s,v) &= \psi(u\overline{v})\sideset{}{^*}\sum_{\alpha\,(p^s)} e_{p^s}\left(h\overline{\alpha uv}\right)\overline{\psi}\left(\alpha p^{k-s}+1\right)\\ &= \psi(u\overline{v})e_{p^{2s}}\left(-hp^k\overline{uv}\right)\sideset{}{^*}\sum_{\alpha\,(p^s)} e_{p^s}\left(h\overline{uv}(\overline{\alpha}+p^{k-s})\right)\overline{\psi}\left(\alpha p^{k-s}+1\right).\end{aligned}$$

Notice that since $p\nmid\alpha$, we have $p^{k-s}||(\overline{\alpha p^{k-s}+1}-1)$. Hence, the change of variables

$$\beta=(\overline{\alpha p^{k-s}+1}-1)/p^{k-s}$$

is admissible. When $k=s$, the quantity $\overline{\alpha+1}$ is well-defined due to the presence of ψ. With the above change of variables (mod p^s), it follows that

$$\overline{\alpha p^{k-s}+1} \;\equiv\; \beta p^{k-s}+1 \pmod{p^s}$$

and

$$\begin{aligned}\overline{\alpha} &\equiv \overline{(\overline{\beta p^{k-s}+1}-1)/p^{k-s}}\\ &\equiv \overline{(1-\beta p^{k-s}-1)\overline{\beta p^{k-s}+1}/p^{k-s}}\\ &\equiv -\overline{\beta}(\beta p^{k-s}+1) \;\equiv\; -p^{k-s}-\overline{\beta} \pmod{p^s}.\end{aligned}$$

Hence,

$$\mathscr{C}_\psi^h(p^s,u,p^s,v)=\psi(u\overline{v})e_{p^{2s}}\left(-hp^k\overline{uv}\right)\sum_{\beta\,(p^s)}{}^{*}e_{p^s}\left(-h\overline{\beta uv}\right)\psi\left(\beta p^{k-s}+1\right).$$

With another change of variables $\beta=\gamma\overline{u}$, we get

$$\mathscr{C}_\psi^h(p^s,u,p^s,v)\ =\ e_{p^{2s}}\left(-hp^k\overline{uv}\right)\overline{\mathscr{C}_\psi^h(p^s,v,p^s,u)}.$$

<u>Case 2</u>: $t=k<s$. In this case, we have

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)\ =\ \psi(u\overline{v})\sum_{\substack{\alpha\,(p^s)\\ \alpha\equiv-1\,(p^{s-t})}}{}^{*}e_{p^s}\left(h\overline{\alpha}uv\right)\overline{\psi}\left(\frac{\alpha+1}{p^{s-t}}\right).$$

Apply the change of variables $p^{s-t}\beta=\alpha+1$ and observe the fact that $s>t$, we have

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)\ =\ \psi(u\overline{v})\sum_{\beta\,(p^t)}{}^{*}e_{p^s}\left(h\overline{uv(p^{s-t}\beta-1)}\right)\overline{\psi}\left(\beta\right).$$

Another change of variables $\beta=\overline{\gamma+p^{s-t}}$, which is admissible as $s>t$, yields

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)\ =\ \psi(\overline{v})\sum_{\gamma\,(p^t)}{}^{*}e_{p^s}\left(h\overline{uv(p^{s-t}\overline{\gamma+p^{s-t}}-1)}\right)\psi\left(\gamma u+p^{s-t}u\right).$$

Using

$$\begin{aligned}p^{s-t}\overline{\gamma+p^{s-t}}-1&\equiv p^{s-t}\overline{\gamma}\overline{1+p^{s-t}\overline{\gamma}}\ -\ 1\\&\equiv(1+p^{s-t}\overline{\gamma}-1)\overline{1+p^{s-t}\overline{\gamma}}\ -\ 1\\&\equiv\ -\overline{1+p^{s-t}\overline{\gamma}}\ (\mathrm{mod}\ p^s),\end{aligned}$$

we arrive at

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)\ =\ \psi(\overline{v})\sum_{\gamma\,(p^t)}{}^{*}e_{p^s}\left(-h\overline{uv}(1+p^{s-t}\overline{\gamma})\right)\psi\left(\gamma u+p^{s-t}u\right).$$

A final change of variables $\gamma\mapsto\gamma\overline{u}$ implies

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)\ =\ e_{p^s}\left(-h\overline{uv}\right)\overline{\mathscr{C}_\psi^h(p^t,v,p^s,u)}.$$

<u>Case 3</u>: $s=k<t$. In this final case, we make use of Case 2 to deduce the answer. For $s=k<t$, we have

$$\mathscr{C}_\psi^h(p^t,v,p^s,u)=e_{p^t}\left(-h\overline{uv}\right)\overline{\mathscr{C}_\psi^h(p^s,u,p^t,v)}.$$

This yields

$$\mathscr{C}_\psi^h(p^s,u,p^t,v)=e_{p^t}\left(-h\overline{uv}\right)\overline{\mathscr{C}_\psi^h(p^t,v,p^s,u)}$$

as desired.

Combining all three cases together, we have proved that for $r=p^k$ for some prime p and $k\geq1$,

$$\mathscr{C}_\psi^h(a,u,b,v)\ =\ e_{ab}\left(-hr\overline{uv}\right)\overline{\mathscr{C}_\psi^h(b,v,a,u)}\tag{5.1}$$

for any a,b,u,v with $ab|r^\infty$, $(uv,r)=1$ and any character mod r. Finally, applying the first and second equality of Lemma 4.1 to the left-hand and right-hand side of (5.1) respectively, with the observation that

$$e\left(-\frac{hr\overline{uv}}{a_1a_2b_1b_2}\right)\ =\ e\left(-\frac{hr\overline{a_2b_2uv}}{a_1b_1}-\frac{hr\overline{a_1b_1uv}}{a_2b_2}\right),\tag{5.2}$$

we see that both sides satisfy the same multiplicative relations. This yields the result for general r and concludes the proof. □

6. Backward maneuver

6.1. **Step 5: Reciprocity and recombining sums.** From Theorem 1.1 and the additive reciprocity, observe that

$$\begin{aligned} e_{c_0 m_0 m'}\left(\ell q \overline{c'}\right) \mathscr{C}_\chi^\ell(c_0, c', m_0, m') &= e\left(\frac{\ell q \overline{c'}}{c_0 m_0 m'} - \frac{\ell q \overline{c' m'}}{c_0 m_0}\right) \overline{\mathscr{C}_\chi^\ell(m_0, m', c_0, c')} \\ &= e_{m'}\left(\ell q \overline{c_0 c' m_0}\right) \overline{\mathscr{C}_\chi^\ell(m_0, m', c_0, c')}. \end{aligned} \tag{6.1}$$

Inserting this back into (4.11) and open up $\overline{\mathscr{C}_\chi^\ell}(\cdots)$ by its definition, we obtain

$$\begin{aligned} S_k(\ell;\chi) \;=\; \frac{2\pi}{q}\frac{\epsilon_\chi}{\sqrt{q}} \sum_{c_0, m_0 | q^\infty} \sum_{\substack{m' \geq 1 \\ (m', q) = 1}} \sum_{\substack{c' \neq 0 \\ (c', m' q) = 1}} \frac{\overline{\chi}(m')}{m_0 m'} e_{m'}\left(\ell q \overline{c_0 c' m_0}\right) \sideset{}{^*}\sum_{\substack{\alpha\, (m_0) \\ c_0 c' \equiv -\alpha q\, (m_0)}} e_{m_0}\left(-\ell \overline{\alpha m'}\right) \chi\left(\frac{\alpha q + c_0 c'}{m_0}\right) \\ \times \int_0^\infty \left(\mathscr{H}_k g\right)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0 m'}\right) e\left(\frac{c_0 c' x}{q m_0 m'}\right) dx. \end{aligned}$$

Upon recombining the c_0-sum and c'-sum via $c = c_0 c'$, it follows that

$$\begin{aligned} S_k(\ell;\chi) \;=\; \frac{2\pi}{q}\frac{\epsilon_\chi}{\sqrt{q}} \sum_{m_0 | q^\infty} \sum_{\substack{m' \geq 1 \\ (m', q) = 1}} \sum_{\substack{c \neq 0 \\ (c, m') = 1}} \frac{\overline{\chi}(m')}{m_0 m'} e_{m'}\left(\ell q \overline{c m_0}\right) \sideset{}{^*}\sum_{\substack{\alpha\, (m_0) \\ c \equiv -\alpha q\, (m_0)}} e_{m_0}\left(-\ell \overline{\alpha m'}\right) \chi\left(\frac{\alpha q + c}{m_0}\right) \\ \times \int_0^\infty \left(\mathscr{H}_k g\right)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0 m'}\right) e\left(\frac{c x}{q m_0 m'}\right) dx. \end{aligned} \tag{6.2}$$

This completes the arithmetic preparation as described in Section 4.

6.2. **Step 6: A second Poisson summation.** With the key preparations carried out above, we first rewrite the c-sum of (6.2) as

$$\sum_{\substack{c \neq 0 \\ (c, m') = 1}} \mathscr{D}_\chi(c; m_0, m') \int_0^\infty \left(\mathscr{H}_k g\right)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0 m'}\right) e\left(\frac{c x}{m_0 m' q}\right) dx \;=\; T_1 \;-\; T_2, \tag{6.3}$$

where

$$T_1 \;:=\; \sum_{c \in \mathbb{Z}} \mathscr{D}_\chi(c; m_0, m') \int_0^\infty \left(\mathscr{H}_k g\right)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0 m'}\right) e\left(\frac{c x}{m_0 m' q}\right) dx,$$

$$T_2 \;:=\; \delta(m' = 1) \mathscr{D}_\chi(0; m_0, 1) \int_0^\infty \left(\mathscr{H}_k g\right)\left(x/q^2\right) J_{k-1}\left(\frac{4\pi\sqrt{\ell x}}{m_0}\right) dx,$$

and

$$\mathscr{D}_\chi(c; m_0, m') \;:=\; \delta((c, m') = 1) \overline{\chi}(m') e_{m'}\left(\ell q \overline{c m_0}\right) \sideset{}{^*}\sum_{\substack{\alpha\, (m_0) \\ c \equiv -\alpha q\, (m_0)}} e_{m_0}\left(-\ell \overline{\alpha m'}\right) \chi\left(\frac{\alpha q + c}{m_0}\right).$$

The treatment of T_2 is simpler. Observe that

$$\begin{aligned} \mathscr{D}_\chi(0; m_0, 1) \;&=\; \sideset{}{^*}\sum_{\substack{\alpha\, (m_0) \\ 0 \equiv \alpha q\, (m_0)}} e_{m_0}\left(-\ell \overline{\alpha}\right) \chi\left(\frac{\alpha q}{m_0}\right) \\ &= \delta(m_0 = q) \sideset{}{^*}\sum_{\alpha\, (q)} \overline{\chi}(\alpha) e\left(-\frac{\ell \alpha}{q}\right) \\ &= \delta(m_0 = q) \sqrt{q}\, \overline{\epsilon_\chi} \chi(\ell). \end{aligned}$$

A change of variables $x \to q^2 x$ and the Hankel inversion formula (Lemma 2.3) imply that

$$T_2 \;=\; \delta(m_0 = q, m' = 1)\,\overline{\epsilon_\chi}\chi(\ell)\,\frac{q^{5/2} g(\ell)}{2\pi}. \tag{6.4}$$

For T_1, the character sum $\mathscr{D}_\chi$ can be expressed in terms of Kloosterman sums via (3.6) and (3.4):

$$\begin{aligned}\mathscr{D}_\chi(c; m_0, m') &= \sideset{}{^*}\sum_{\substack{x\,(m)\\ c \equiv -xq\,(m)}} e_m(-\ell\overline{x})\chi\left(\frac{qx+c}{m}\right)\\ &= \frac{\epsilon_\chi}{m\sqrt{q}} \sum_{\gamma\,(mq)} \overline{\chi}(\gamma) \sideset{}{^*}\sum_{x\,(m)} e_m(-\gamma x - \ell\overline{x}) e_{mq}(-c\gamma)\\ &= \frac{\epsilon_\chi}{m\sqrt{q}} \sum_{\gamma\,(mq)} \overline{\chi}(\gamma) S(\gamma, \ell; m) e_{mq}(-c\gamma).\end{aligned} \tag{6.5}$$

Note: $m = m_0 m'$. Plugging (6.5) into T_1 and making a change of variables $x \to mqx$, we arrive at

$$T_1 = \epsilon_\chi\sqrt{q} \sum_{c\in\mathbb{Z}} \sum_{\gamma\,(mq)} \overline{\chi}(\gamma) S(\gamma, \ell; m) e_{mq}(-c\gamma) \int_0^\infty (\mathscr{H}_k g)\left(mx/q\right) J_{k-1}\left(4\pi\sqrt{\frac{\ell q x}{m}}\right) e(cx)\,dx.$$

We may now readily observe the role reversal of the c-sum and m-sum as discussed in Section 1.2.1!

The bounds for the Hankel transform and Bessel functions recorded in Section 2 allow us to apply the dominated convergence theorem and obtain

$$T_1 = \epsilon_\chi\sqrt{q} \sum_{\gamma\,(mq)} \overline{\chi}(\gamma) S(\gamma, \ell; m) \lim_{\epsilon\to 0+} \sum_{c\in\mathbb{Z}} e_{mq}(-c\gamma) \int_{\mathbb{R}} (\mathscr{H}_k g)\left(mx/q\right) J_{k-1}\left(4\pi\sqrt{\frac{\ell q x}{m}}\right) h_\epsilon(x) e(cx)\,dx,$$

where h_ϵ is a smooth function on $\mathbb{R}$ such that $h_\epsilon \equiv 1$ on $[\epsilon, \infty)$, $h_\epsilon \equiv 0$ on $(-\infty, 0]$, and $0 \le h_\epsilon \le 1$ on $(0, \epsilon)$. Inside the limit, we apply Poisson summation $(\bmod\ mq)$ to the c-sum. Using dominated convergence again, we have

$$T_1 \;=\; 2\pi\epsilon_\chi\sqrt{q} \sum_{c=1}^\infty \overline{\chi}(c) S(c, \ell; m) J_{k-1}\left(\frac{4\pi\sqrt{\ell c}}{m}\right) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy \tag{6.6}$$

upon taking the limit $\epsilon \to 0+$. Inserting (6.4)–(6.6) into (6.3) and (6.2), we readily observe that

$$S_k(\ell;\chi) \;=\; 4\pi^2 \frac{\epsilon_\chi^2}{q} \sum_{m=1}^\infty \frac{1}{m} \sum_{c=1}^\infty \overline{\chi}(c) S(c, \ell; m) J_{k-1}\left(\frac{4\pi\sqrt{\ell c}}{m}\right) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy \;-\; g(\ell)\chi(\ell) \tag{6.7}$$

by combining m_0-sum and m'-sum.

6.3. **Step 7: Petersson in reverse.** Inserting (6.7) back into (3.7) yields

$$\begin{aligned} I_k(\ell;\chi) \;&= 2\pi i^{-k} \frac{\epsilon_\chi^2}{q} \overline{\chi}(\ell) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{y\ell}}{q}\right) dy\\ &\quad + 4\pi^2 \frac{\epsilon_\chi^2}{q} \sum_{m=1}^\infty \frac{1}{m} \sum_{c=1}^\infty \overline{\chi}(c) S(c, \ell; m) J_{k-1}\left(\frac{4\pi\sqrt{\ell c}}{m}\right) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy. \end{aligned} \tag{6.8}$$

Applying the Petersson trace formula (2.16) in reverse to the m-sum, we conclude that

$$I_k(\ell;\chi) \;= 2\pi i^k \frac{\epsilon_\chi^2}{q} \sum_{c=1}^\infty \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell)\lambda_f(c)\overline{\chi}(c) \int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy, \tag{6.9}$$

where we used $i^k = i^{-k}$ as k is even.

Remark 3. *Readers should note that the diagonal term $g(\ell)\chi(\ell)$ of (3.3) from the initial application of the Petersson trace formula conveniently canceled with the dual zeroth frequency from the Poisson summation. Only after this does the crucial 'role-reversal' of sums takes place.*

7. Analytic continuation and polynomial growth

In this section, we show that the L-series (1.5) for $L(s, f\times\chi)$ can be analytically continued beyond the line $\operatorname{Re} s = 1/2$, by proving *cancellations* in the *smooth* sums of $\lambda_f(n)\chi(n)$ when averaged over the *family* $\mathcal{B}_k(1)$. To deduce the functional equation from (6.9), it is essential to establish also the *polynomial growth* for

$$\mathcal{A}_\ell(s,\chi) := \sum_{f\in\mathcal{B}_k(1)}^{h} \lambda_f(\ell)L(s,f\times\chi) \tag{7.1}$$

within the critical strip and as $|\operatorname{Im} s|\to\infty$. This section is dedicated to proving the following:

Proposition 7.1. *Let $\ell\ge 1$ and $k\ge 6$ be integers. Then the function $\mathcal{A}_\ell(s,\chi)$ defined in (7.1) admits a holomorphic continuation to the half-plane $\operatorname{Re} s > -(k-6)/2$ and satisfies the estimate*

$$\mathcal{A}_\ell(s,\chi) \ll (1+|t|)^{k-3} \tag{7.2}$$

for any $s=\sigma+it$ with $\sigma > -(k-6)/2$ and $t\in\mathbb{R}$. The implicit constant depends only on k,ℓ,q,σ.

We begin on the region of absolute convergence. More precisely, the Dirichlet series (1.5) converges absolutely on $\operatorname{Re} s > 5/4$. This follows from the bound $|\lambda_f(n)| = O(n^{\frac{1}{4}})$, which can be deduced from the Petersson trace formula (Lemma 2.6), and bounds on Kloosterman sums and Bessel functions, see [18, Chapter 14]. Take $g\in C_c^\infty[1,2]$ such that $1=\sum_{u\in\mathbb{Z}} g(x/2^u)$ for any $x>0$. On the region $\operatorname{Re} s>5/4$, we may rewrite (1.5) as

$$L(s,f\times\chi) = \sum_{u=-1}^{\infty}\frac{1}{2^{us}}\sum_n \lambda_f(n)\chi(n)G_s(n/2^u), \tag{7.3}$$

where $G_s(y) := y^{-s}g(y)$. Then

$$\mathcal{A}_\ell(s,\chi) = \sum_{u=-1}^{\infty}\frac{\mathcal{I}_s(2^u;\ell,\chi)}{2^{us}}, \tag{7.4}$$

where

$$\mathcal{I}_s(X;\ell,\chi) := \sum_{f\in\mathcal{B}_k(1)}^{h} \lambda_f(\ell)\sum_n \lambda_f(n)\chi(n)G_s(n/X). \tag{7.5}$$

For $X>2q^2\ell$, we have $G_s(\ell/X)=0$, and from (3.7)–(3.8), it follows that

$$\frac{1}{X}\mathcal{I}_s(X;\ell,\chi) \ll \left|\int_0^\infty G_s(y)J_{k-1}\left(\frac{4\pi\sqrt{yX\ell}}{q}\right)dy\right| + \sum_{c\ge1}\sum_{m\neq0}\left|\int_0^\infty G_s(y)J_{k-1}\left(\frac{4\pi\sqrt{yX\ell}}{c}\right)e\left(-\frac{myX}{cq}\right)dy\right|. \tag{7.6}$$

Bounding the second summand on the right-hand side is harder, and this will be our focus. We split the c-sum on the right-hand side of (7.6) into two parts, according to the conditions $c>\sqrt{\ell X}/20$ and $c\le\sqrt{\ell X}/20$. These two parts are donated by $\mathcal{I}^{(2)}_{s,>}(X;\ell,\chi)$ and $\mathcal{I}^{(2)}_{s,\le}(X;\ell,\chi)$ respectively.

Let $1<a<(k+1)/2$. By Lemma 2.1, we have

$$\int_{\mathbb{R}} G_s(y)J_{k-1}\left(\frac{4\pi\sqrt{yX\ell}}{c}\right)e\left(-\frac{myX}{cq}\right)dy = \frac{1}{2\pi}\int_{(a)}\frac{\gamma_k(1-v)}{\gamma_k(v)}\left(\frac{\sqrt{X\ell}}{c}\right)^{2(v-1)}\int_{\mathbb{R}} G_s(y)y^{v-1}e\left(-\frac{myX}{cq}\right)dy\,\frac{dv}{2\pi i}.$$

Then integrating by parts $r\ge 2$ times, it follows that the above expression is

$$\ll \left(\frac{\sqrt{\ell X}}{c}\right)^{2(a-1)}\left(\frac{|m|X}{cq}\right)^{-r}\int_{(a)}\left|\frac{\gamma_k(1-v)}{\gamma_k(v)}\right|\int_{\mathbb{R}}\left|\frac{d^r}{dy^r}\left[G_s(y)y^{v-1}\right]\right|dy\,|dv|.$$

Let $v = a + i\tau$ and $s = \sigma + it$ with $\sigma, t, \tau \in \mathbb{R}$. Then Stirling's formula gives

$$\left|\frac{\gamma_k(1-v)}{\gamma_k(v)}\right| \asymp_{k,a} (1+|\tau|)^{1-2a}. \tag{7.7}$$

For any $r \geq 2$, we have

$$\max_{y\in\mathbb{R}} \left|G_s^{(r)}(y)\right| \ll_r \max_{\substack{i+j=r \\ i,j\geq 0 \\ y\in[1,2]}} \left|g^{(i)}(y)\right| \cdot \left|s(s+1)\cdots(s+j-1)\right| y^{-\sigma-j} \ll_{r,\sigma} (1+|t|)^r, \tag{7.8}$$

and

$$\left|\frac{d^r}{dy^r}\left[G_s(y)y^{v-1}\right]\right| \ll_r \max_{\substack{i+j=r \\ i,j\geq 0 \\ y\in[1,2]}} \left|G_s^{(i)}(y)\right|\left|(v-1)\cdots(v-j)y^{v-j-1}\right| \ll_{a,r,\sigma} ((1+|t|)(1+|\tau|))^r. \tag{7.9}$$

Hence, if $1 < a < (k+1)/2$ and $r \in \mathbb{Z}$ such that $2 \leq r < 2a-3$, then

$$\left|\int_{\mathbb{R}} G_s(y) J_{k-1}\left(\frac{4\pi\sqrt{yX\ell}}{c}\right) e\left(-\frac{myX}{cq}\right) dy\right| \ll (1+|t|)^r \frac{X^{a-r-1}}{|m|^r c^{2a-2-r}} \tag{7.10}$$

and

$$\mathscr{I}_{s,>}^{(2)}(X;\ell,\chi) \ll (1+|t|)^r \frac{X^{a-r-1}}{(\sqrt{X})^{2a-3-r}} = (1+|t|)^r X^{(1-r)/2} \tag{7.11}$$

where the implicit constants may depend on ℓ, q, k, σ, r, a.

Next, we estimate $\mathscr{I}_{s,\leq}^{(2)}(X;\ell,\chi)$ with $X > 2q^2\ell$. By (2.10), it suffices to estimate the oscillatory integral:

$$\int_{\mathbb{R}} G_s(y) W_k\left(\frac{2\sqrt{y\ell X}}{c}\right) e\left(\frac{2q\sqrt{y\ell X} - mXy}{cq}\right) dy.$$

To this end, we make use of Lemma 2.5 with the functions

$$w_s(y) := G_s(y) W_k\left(\frac{2\sqrt{y\ell X}}{c}\right) \qquad \text{and} \qquad h(y) := \frac{2q\sqrt{y\ell X} - mXy}{cq}.$$

Suppose $y \in [1,2]$ and $r \geq 2$. We observe the following bounds:

$$h'(y) = \frac{qy^{-1/2}\sqrt{\ell X} - mX}{cq}, \qquad |h'(y)| \gg \frac{|m|X}{cq}\left(1 - \frac{1}{|m|}\sqrt{\frac{q^2\ell}{X}}\right) \gg \frac{|m|X}{cq}, \qquad |h^{(r)}(y)| \asymp_r \frac{\sqrt{\ell X}}{c} \geq 20, \tag{7.12}$$

as well as

$$\begin{aligned}
\left|w_s^{(r)}(y)\right| &\ll_r \max_{i+j=r} \left|G_s^{(i)}(y)\right| \cdot \left|\partial_y^j\left[W_k\left(\frac{2\sqrt{y\ell X}}{c}\right)\right]\right| \\
&\ll_{r,\sigma} (1+|t|)^r \max_{0\leq j\leq r} \max_{\substack{m_1+2m_2+\cdots+j\cdot m_j=j \\ m_1,\ldots,m_j\geq 0}} \left|W_k^{(m_1+\cdots+m_j)}\right|\left(\frac{2\sqrt{y\ell X}}{c}\right) \cdot \prod_{i=1}^{j}\left|\partial_y^i\left(\frac{2\sqrt{y\ell X}}{c}\right)\right|^{m_i} \\
&\ll_r (1+|t|)^r \max_{0\leq j\leq r} \max_{\substack{m_1+2m_2+\cdots+j\cdot m_j=j \\ m_1,\ldots,m_j\geq 0}} \left|W_k^{(m_1+\cdots+m_j)}\right|\left(\frac{2\sqrt{y\ell X}}{c}\right) \cdot \left(\frac{2\sqrt{\ell X}}{c}\right)^{m_1+\cdots+m_j} \\
&\ll_{r,k} (1+|t|)^r \left(\frac{2\sqrt{\ell X}}{c}\right)^{-1/2},
\end{aligned} \tag{7.13}$$

where Leibniz's rule and Faà di Bruno's formula were used. Apply Lemma 2.5 with the choice of parameters:

$$W = \left(\frac{2\sqrt{\ell X}}{c}\right)^{-1/2}, \quad V = (1+|t|)^{-1}, \quad H = \frac{\sqrt{\ell X}}{c}, \quad G = 1, \quad R = \frac{|m|X}{cq}, \tag{7.14}$$

we conclude that:

$$\left|\int_{\mathbb{R}} w_s(y)e\big(h(y)\big)\,dy\right| \ll \left(\frac{2\sqrt{\ell X}}{c}\right)^{-1/2}(1+|t|)^A\left(\frac{cq}{|m|X}+\frac{\sqrt{\ell X}}{c}\left(\frac{cq}{|m|X}\right)^2\right)^A$$
$$\ll \left(\frac{\sqrt{\ell X}}{c}\right)^{-1/2}\left(\frac{cq\,(1+|t|)}{|m|X}\right)^A\left(1+\frac{1}{|m|}\sqrt{\frac{q^2\ell}{X}}\right)^A$$
$$\ll \frac{\sqrt{c}}{X^{1/4}}\left(\frac{c\,(1+|t|)}{|m|X}\right)^A \tag{7.15}$$

for $c>0$, $m\neq 0$, $X>2q^2\ell$, where the implicit constants depend on k,ℓ,q,A,σ. When $A>1$, we have

$$\mathscr{I}^{(2)}_{s,\leq}(X;\ell,\chi) \ll X^{-1/4}\left(\frac{(1+|t|)}{X}\right)^A\sum_{c\ll\sqrt{X}}c^{A+1/2} \ll \sqrt{X}\left(\frac{(1+|t|)}{\sqrt{X}}\right)^A. \tag{7.16}$$

As a result, we obtain the estimate:

$$\frac{1}{X}\mathscr{I}_s(X;\ell,\chi) \ll (1+|t|)^r X^{(1-r)/2}+\sqrt{X}\left(\frac{(1+|t|)}{\sqrt{X}}\right)^A,$$

provided that $1<a<(k+1)/2$, $r\in\mathbb{Z}$ with $2\leq r<2a-3$, $X>2q^2\ell$, and $A>1$. By taking $r=k-3$, $a=(k+1)/2-\epsilon$, $A=k-3$ with $k\geq 6$, we have

$$\mathscr{I}_s(X;\ell,\chi) \ll (1+|t|)^{k-3}X^{-(k-6)/2}, \tag{7.17}$$

where the implicit constants depend on k,ℓ,q,σ.

Remark 4. *If s lies in the vertical strip $\sigma_1\leq\sigma\leq\sigma_2$, then the same estimates hold and the implicit constants only depend on $k,\ell,q,\sigma_1,\sigma_2$.*

We now turn to $\mathscr{A}_\ell(s,\chi)$. On the region $\operatorname{Re} s>5/4$, recall that

$$\mathscr{A}_\ell(s,\chi) = \left(\sum_{\substack{2^u>2\ell\\ u\geq -1}}+\sum_{\substack{2^u\leq 2\ell\\ u\geq -1}}\right)\frac{\mathscr{I}_s(2^u;\ell,\chi)}{2^{us}} \tag{7.18}$$

Apply (7.17) to the first sum of (7.18), we have

$$\sum_{\substack{2^u>2\ell\\ u\geq -1}}\left|\frac{\mathscr{I}_s(2^u;\ell,\chi)}{2^{us}}\right| \ll (1+|t|)^{k-3}\sum_{u\geq 0}\frac{(2^u)^{-\frac{k-6}{2}}}{2^{u\sigma}}. \tag{7.19}$$

The last infinite series converges absolutely provided that $\sigma>-(k-6)/2$. Whenever $\sigma>-(k-6)/2$ and $t\in\mathbb{R}$, the following holds:

$$\sum_{\substack{2^u>2\ell\\ u\geq -1}}\left|\frac{\mathscr{I}_s(2^u;\ell,\chi)}{2^{us}}\right| \ll (1+|t|)^{k-3}, \tag{7.20}$$

where implicit constants depend only on k,ℓ,q,σ. On the other hand, we have

$$\sum_{\substack{2^u\leq 2\ell\\ u\geq -1}}\left|\frac{\mathscr{I}_s(2^u;\ell,\chi)}{2^{us}}\right| \ll \sum_{n\leq 4\ell}\sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)}\left|\lambda_f(\ell)\lambda_f(n)\right|n^{-\sigma} \ll \sum_{n\leq 4\ell}\sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)}\left(|\lambda_f(\ell)|^2+|\lambda_f(n)|^2\right) \ll 1, \tag{7.21}$$

where the implicit constant depends on k,ℓ,q. The last estimate follows from bounding the geometric side of the Petersson formula.

From (7.20) and (7.21), both series on the right side of (7.18) converge absolutely pointwise and uniformly on every compact subset of the region $\operatorname{Re} s>-(k-6)/2$. As a result, $\mathscr{A}_\ell(s,\chi)$ admits an analytic continuation to $\operatorname{Re} s>-(k-6)/2$, and the bound (7.2) holds in the same region. This completes the proof of Proposition 7.1.

8. Proof of Theorem 1.2

Let $g \in C_c^\infty(0,\infty)$ and $\mathscr{G}(s) = \int_0^\infty g(x)x^{s-1}\,dx$ be the Mellin transform of g. Then $\mathscr{G}(s)$ is entire and it follows from repeated integration by parts that

$$|\mathscr{G}(s)| \ll (1+|\mathrm{Im}\, s|)^{-A} \tag{8.1}$$

for any $A>0$, where the implicit constant depends only on $\mathrm{Re}\, s$ and A.

Let $\ell \geq 1$ and $k \geq 6$ is an even integer. In (3.1), apply the Mellin inversion formula (2.4) and rearrange sums and integrals, we have

$$I_k(\ell;\chi) = \int_{(3/2)} \mathscr{G}(s) \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell) L(s, f\times\chi)\, \frac{ds}{2\pi i}.$$

By Proposition 7.1 and (8.1), we may shift the line of integration to $\mathrm{Re}\, s = \sigma \in (0,1)$, i.e.,

$$I_k(\ell;\chi) = \int_{(\sigma)} \mathscr{G}(s) \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell) L(s, f\times\chi)\, \frac{ds}{2\pi i}. \tag{8.2}$$

By (2.8) and (2.4), we have

$$\int_0^\infty g(y) J_{k-1}\left(\frac{4\pi\sqrt{cy}}{q}\right) dy = \frac{1}{2\pi}\int_{(3/2)} \mathscr{G}(s)\frac{\gamma_k(1-s)}{\gamma_k(s)}\left(\frac{c}{q^2}\right)^{s-1}\frac{ds}{2\pi i}. \tag{8.3}$$

Using (8.1), (7.7) and the holomorphy of $\mathscr{G}(s)$, we shift the line of integration to $\mathrm{Re}\, s = -1/2$ in (8.3). Inserting the resultant into (6.9), we deduce that

$$I_k(\ell;\chi) = i^k\epsilon_\chi^2 \sum_{c=1}^\infty \lambda_f(c)\overline{\chi}(c) \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell)\int_{(-1/2)} \mathscr{G}(s)\frac{\gamma_k(1-s)}{\gamma_k(s)}\left(\frac{c}{q^2}\right)^{s-1}\frac{ds}{2\pi i}.$$

Upon exchanging the order of sums and integrals, we find that the c-sum converges absolutely and is precisely the Dirichlet series $L(1-s, f\times\overline{\chi})$. In other words, we have

$$I_k(\ell;\chi) = i^k\epsilon_\chi^2 \int_{(-1/2)} \mathscr{G}(s) q^{1-2s}\frac{\gamma_k(1-s)}{\gamma_k(s)} \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell) L(1-s, f\times\overline{\chi})\,\frac{ds}{2\pi i}. \tag{8.4}$$

By Proposition 7.1, (8.1) and (7.7) again, we may shift the line of integration for (8.4) to $\mathrm{Re}\, s = \sigma \in (0,1)$. Upon comparing with (8.2), Theorem 1.2 follows!

9. Proof of Corollary 1.3

Let $\mathscr{B}_k(1) := \{f_1, \ldots, f_d\}$.[4] The vectors $(\lambda_{f_i}(1), \lambda_{f_i}(2), \ldots)$ for $i = 1, \ldots, d$ are clearly linearly independent over $\mathbb{C}$. Thus, there exists $\ell_1 < \cdots < \ell_d$ such that the submatrix $A := (\lambda_{f_i}(\ell_j))_{1\leq i,j\leq d}$ is invertible.

By Proposition 7.1, the function $s \mapsto \sideset{}{^h}\sum_{f\in\mathscr{B}_k(1)} \lambda_f(\ell_j) L(s, f\times\chi)$ admits a holomorphic continuation (say $G_{\ell_j}(s,\chi)$) to the region $\mathrm{Re}\, s > -(k-6)/2$ for each $j = 1, \ldots, d$. By definition, we have

$$\left(\frac{L(s,f_1\times\chi)}{||f_1||^2}, \ldots, \frac{L(s,f_d\times\chi)}{||f_d||^2}\right) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}\left(G_{\ell_1}(s,\chi), \ldots, G_{\ell_d}(s,\chi)\right) A^{-1}. \tag{9.1}$$

Thus, each of $L(s, f_i\times\chi)$ admits a holomorphic continuation to the same region.

Since the spectral identity (1.6) holds for any $\sigma \in (0,1)$ and for any choice of $g \in C_c^\infty(0,\infty)$, a standard result in elementary analysis allows us to infer that, on the vertical strip $0 < \mathrm{Re}\, s < 1$,

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \left(\frac{L(s,f_i\times\chi)}{||f_i||^2} - i^k\epsilon_\chi^2 q^{1-2s}\frac{\gamma_k(1-s)}{\gamma_k(s)}\frac{L(1-s,f_i\times\overline{\chi})}{||f_i||^2}\right)_{1\leq i\leq d} \cdot A = (0,\ldots,0).$$

[4] Here, we use the finite dimensionality of the linear space of holomorphic cusp forms of a given weight and level.

Since A is invertible, the functional equation (1.9) for $f = f_i$ on $0 < \mathrm{Re}\, s < 1$ follows immediately, where $i = 1, \ldots, d$.

From the holomorphic continuation of $L(s, f_i \times \chi)$ to $\mathrm{Re}\, s > -(k-6)/2$ and the functional equation just proved, $L(s, f_i \times \chi)$ also extends holomorphically to $\mathrm{Re}\, s < k/2 - 2$. As a result, $L(s, f_i \times \chi)$ admits an entire continuation and now the functional equation holds for all $s \in \mathbb{C}$. This completes the proof.

APPENDIX A. BROADER CONTEXTS

The theory of automorphic representations encompasses a wealth of surprising formulae involving automorphic L-functions, periods, or other interesting arithmetic invariants. A notable example is *Waldspurger's formula* [35, 36], which relates the central L-values of integral weight automorphic forms twisted by quadratic characters to the Fourier coefficients of half-integral weight forms via the Shimura correspondence.

In his pioneering works [19, 20, 21], Jacquet gave a new proof of Waldspurger's formula using his Relative Trace Formula (RTF). The development of the RTF programme has since been fruitful. This theme was recently revisited by Sakellaridis [29], who offered another proof of Waldspurger's formula using Langlands' Beyond Endoscopy. His method features a novel, non-standard comparison of the Jacquet RTF (J-RTF) and the Kuznetsov Trace Formula (KTF), uncovering non-trivial spectral identities involving global automorphic invariants.

Sakellaridis' work [29, Section 5] also sheds new light on the analytic theory of automorphic L-functions. The philosophy of Beyond Endoscopy asserts that the functional equations of L-functions should be viewed as *parallel to* Waldspurger's formula at a high level. This was further elaborated by Sakellaridis ([31, Section 6], [30, Section 1.1, 6–9]) and Ngô [28, Section 7.2].

According to [31], both Waldspurger's formula and the functional equation for the standard L-function $L(s,\pi,\mathrm{std})$ of a cuspidal automorphic representation π of $G := \mathrm{PGL}(2)$ can be interpreted as forms of '*Poisson summation*' between two distinct spaces of Schwartz measures. Indeed, according to [31, Diagram (6.2)-(6.3)] and [30, Section 8], Waldspurger's formula arises from the comparison between the J-RTF and the KTF as functionals on the spaces of Schwartz measures for the quotients $T \setminus G/T$ and $N \setminus G/N$ respectively, where E/F be a quadratic extension, $T := \mathrm{Res}_{E/F}\mathbb{G}_m/\mathbb{G}_m$ be a torus of G, and $N := \left\{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\right\} \subset G$ be the standard unipotent subgroup equipped with a non-trivial character of N. On the other hand, the functional equation for $L(s,\pi,\mathrm{std})$ arises from the comparison between two functional of KTFs defined on the Schwartz spaces for $N \setminus G/N$, distinguished by the parameters s and $1-s$. In both cases, there are significant subtleties in formalizing the aforementioned schemes. For instance, the correct notions for the 'Fourier/Hankel transforms' behind the 'Poisson summations' or the 'functionals' behind the RTFs are far from obvious (see [29, Section 1.5 and 4]).

In light of the above, it is worth comparing our work to that of Iwaniec [17], who also relied on a Spectral Reciprocity strategy in his study of Waldspurger's formula. However, there are subtle distinctions in the implementations. In [17], the use of (1.16) stems from the *exact evaluation* of a different type of character sum, known as the Salié sum (see [17, Lemma 1]), which also plays a crucial role in 'detecting' the correspondence between the levels and weights. However, this particular ingredient is *absent* in our context, resulting in different arithmetic and comparisons of RTFs at a fine scale. The Hankel transform (2.9) and inversion (Lemma 2.3) do not play a role in [17], and the essential Bessel integrals in Iwaniec [17] (and Baruch–Mao [5]) are not (1.14).

An early exploration of the relationships between character sum identities and Hecke algebras was undertaken by Kuznetsov [25] in the 1980s, as a byproduct while developing his celebrated trace formula (KTF). He provided the first proof of a famous identity originally stated by Selberg in the 1930s without proof:

$$S(m,\ell;c) = \sum_{d|(m,\ell,c)} S\left(1, m\ell/d^2; c/d\right) d, \tag{A.1}$$

using his trace formula and the Hecke relation for aHecke-Maass cusp form F of level 1:

$$\lambda_F(m)\lambda_F(\ell) = \sum_{d|(m,\ell)} \lambda_F(m\ell/d^2). \tag{A.2}$$

However, a direct proof of (A.1), i.e., one that does not rely on automorphic inputs, has resisted straightforward attempts (as commented by [7, p. 2318]). Our work introduces a new instance of such comparisons in the context of automorphic L-functions. We wish to continue this line of study.

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