

# SIGN CHANGES OF FOURIER COEFFICIENTS FOR HOLOMORPHIC ETA-QUOTIENTS

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ABSTRACT. In this paper we study sign changes of an infinite class of  $\eta$ -quotients which are holomorphic modular forms. There is also a relation to Hurwitz class numbers.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

There is wide interest in sign changes in  $q$ -series  $f(q) = \sum_{n \in \mathbb{Z}} c(n)q^n$  with  $c(n) \in \mathbb{R}$ . For example, if  $f(q)$  is the Fourier expansion of positive real weight cusp form on some congruence subgroup, then Knopp, Kohnen, and Pribitkin [15, Theorem 1] showed that the  $c(n)$  change signs infinitely often. Sign changes in special subsets of  $n \in \mathbb{N}$  in case of half-integral weight cusp forms were considered in [7, 19]. Moreover, Kowalski, Lau, Soundararajan, and Wu [17, Corollary 2] proved that if  $f$  is an integral weight normalized newform, then the set of signs  $\text{sgn}(c(p))$  with  $p$  prime uniquely determines  $f$ . These results do not extend to general holomorphic modular forms; for example, the Eisenstein series for  $\text{SL}_2(\mathbb{Z})$  have at most one sign change, and half of them have no sign change, so the signs of their Fourier coefficients do not uniquely determine them. Define the *Dedekind  $\eta$ -function*

$$\eta(z) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \quad (q := e^{2\pi iz}).$$

In this paper, we investigate sign changes of Fourier coefficients of  $\eta$ -quotients  $\prod_{j=1}^m \eta(jz)^{\delta_j}$  with  $m \in \mathbb{N}$  and  $\delta_j \in \mathbb{Z}$  for  $1 \leq j \leq m$ . Many such quotients are connected to combinatorial counting problems via the corresponding products

$$\prod_{j=1}^m (q^j; q^j)_{\infty}^{\delta_j} =: \sum_{n \geq 0} C_{1^{\delta_1} 2^{\delta_2} \dots m^{\delta_m}}(n) q^n,$$

where  $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$  for  $n \in \mathbb{N}_0 \cup \{\infty\}$  is the  *$q$ -Pochhammer symbol*. For example, Euler showed that  $\frac{1}{\eta(z)}$  is basically equals the partition generating function.

Many interesting examples of sign changes of the  $C_{1^{\delta_1} 2^{\delta_2} \dots m^{\delta_m}}(n)$  appear in the literature. For example, Andrews and Lewis [2, Conjecture 2] made a conjecture pertaining to the so-called crank of partitions which is related to  $\text{sgn}(C_{1^{33-1}}(n))$ ; this conjecture was proven by Kane [14, Corollary 2]. Motivated by work of Borwein [4], Andrews [1, Theorem 2.1] showed that  $\text{sgn}(C_{1^{1p-1}}(n))$  is periodic in  $n$  with period  $p$ . Schlosser and Zhou [24] further investigated  $\text{sgn}(C_{1^{\delta p - \delta}}(n))$  for  $\delta$  real. In all of these examples, the signs  $\text{sgn}(C_{1^{\delta p - \delta}}(n))$  exhibit a regular pattern, contrary to the behaviour of the signs of the Fourier coefficients of newforms that uniquely determine the form. In this paper, we are interested in other cases

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for which  $\text{sgn}(C_{1^{\delta_1} 2^{\delta_2} \dots m^{\delta_m}}(n))$  satisfies some kind of regularity. We focus on  $\eta$ -quotients which are holomorphic modular forms, investigating similar questions for weakly holomorphic modular forms in a forthcoming paper. We study a few different types of regular sign patterns in this paper. We first record examples where  $\text{sgn}(C_{1^{\delta_1} 2^{\delta_2} \dots m^{\delta_m}}(n))$  has some period  $M \in \mathbb{N}$ .

We start with  $M = 1$ . A number of such examples with fixed sign appear via connections with combinatorial counting problems. For instance, Chen and Garvan [8, (1.9) and Theorem 2.1] investigated  $\frac{\eta(2z)^2 \eta(3z)^3}{\eta(z)^2}$ , showing that for  $n \in \mathbb{N}_0$  we have

$$C_{1-22233}(n) = \frac{1}{24} r_3(24n + 11) > 0,$$

where  $r_3(n)$  is the number of representations of  $n$  as a sum of 3 squares. We obtain a similar result for  $\frac{\eta(2z)^3 \eta(4z)^2}{\eta(z)^2}$ .

**Theorem 1.1.** *For  $n \in \mathbb{N}_0$ , we have*

$$C_{1-22342}(n) > 0.$$

For  $M = 2$ , we have the example  $\frac{\eta(z)^3 \eta(3z)^3}{\eta(2z)^2}$ .

**Theorem 1.2.** *The sequence  $\{\text{sgn}(C_{1^3 2-233}(n))\}_{n \geq 1}$  has period 2. In particular, we have*

$$(-1)^n C_{1^3 2-233}(n) > 0.$$

For  $M = 3$ , we find the example  $\frac{\eta(z)^4 \eta(2z)^4}{\eta(3z)^2}$ .

**Theorem 1.3.** *The sequence  $\{\text{sgn}(C_{1^4 2^4 3-2}(n))\}_{n \geq 1}$  has period 3. In particular, we have*

$$\text{sgn}(C_{1^4 2^4 3-2}(n)) = \begin{cases} 1 & \text{if } 3 \mid n, \\ -1 & \text{otherwise.} \end{cases}$$

For  $M = 8$ , we find two examples,  $\frac{\eta(z)^4 \eta(2z)^2}{\eta(4z)^2}$  and  $\frac{\eta(z)^4 \eta(2z)^4}{\eta(4z)^3}$ .

**Theorem 1.4.**

(1) *The sequence  $\{\text{sgn}(C_{1^4 2^2 4-2}(n))\}_{n \geq 1}$  has period 8. In particular, we have*

$$\text{sgn}(C_{1^4 2^2 4-2}(n)) = \begin{cases} 1 & \text{if } n \equiv 0, 3, 7 \pmod{8}, \\ -1 & \text{if } n \equiv 1, 4, 5 \pmod{8}, \\ 0 & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

(2) *The sequence  $\{\text{sgn}(C_{1^4 2^4 4-3}(n))\}_{n \geq 1}$  has period 8. In particular, we have*

$$\text{sgn}(C_{1^4 2^4 4-3}(n)) = \begin{cases} 1 & \text{if } n \equiv 0, 3, 6, 7 \pmod{8}, \\ -1 & \text{if } n \equiv 1, 2, 4, 5 \pmod{8}. \end{cases}$$

For a prime  $p$ , we next consider sign changes for the pair of infinite families of  $\eta$ -quotients

$$Q_p(z) := \frac{\eta(z)^{p^2}}{\eta(pz)^p}, \quad P_p(z) := \frac{\eta(z)^p}{\eta(pz)}.$$

These families appeared throughout the literature, and their Fourier expansions were computed for certain small  $p$ . The function  $Q_2$  is an Eisenstein series (see [16, Example 10.6

(10.14)] and  $P_2$  is a theta function (see [22, Theorem 1.60])

$$Q_2(z) = 1 - 4 \sum_{n \geq 1} (-1)^{n+1} \sum_{d|n} \left( \frac{-1}{d} \right) q^n, \quad P_2(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}.$$

with  $(\cdot)$  the extended Legendre symbol. The Fourier expansion (see [16, Example 11.4])

$$P_3(z) = 1 - 3 \sum_{n \geq 1} \sum_{d|n} \left( \frac{d}{3} \right) q^n + 9 \sum_{n \geq 1} \sum_{d|n} \left( \frac{d}{3} \right) q^{3n}$$

closely resembles the Fourier expansion of  $Q_2$ . These functions are *lacunary* (see [10, Theorem 1.2] for the statement for  $Q_2$ ), meaning that the set of  $n$  for which the  $n$ -th Fourier coefficient is non-zero has density zero, and hence their Fourier coefficients cannot exhibit regular sign changes. Although the Fourier expansion (see [16, Example 12.16 (12.34)])

$$P_5(z) = 1 - 5 \sum_{n \geq 1} \sum_{d|n} \left( \frac{d}{5} \right) dq^n$$

also seems to resemble the Fourier expansions of  $Q_2$  and  $P_3$ , we next see that the signs of the coefficients of  $P_5$  satisfy a somewhat regular pattern related to the prime factorization of  $n$  which is part of a more general phenomenon for larger  $p$ .

**Theorem 1.5.**

(1) Let  $p \geq 3$  be prime and  $a \in \mathbb{N}_0$ . Then for  $m$  is sufficiently large coprime to  $p$  we have

$$\text{sgn} \left( C_{1^{p^2} p^{-p}}(p^a m) \right) = \left( \frac{2}{p} \right) \left( \frac{m}{p} \right).$$

(2) Let  $p \geq 5$  be prime and  $a \in \mathbb{N}_0$ . Then for  $m$  is sufficiently large coprime to  $p$  we have

$$\text{sgn} \left( C_{1^p p^{-1}}(p^a m) \right) = \left( \frac{-2}{p} \right) \left( \frac{m}{p} \right).$$

Although the sign changes in Theorem 1.5 only hold for  $m$  sufficiently large, in special cases this holds for all  $n$  if the underlying space of cusp forms is trivial.

**Corollary 1.6.**

(1) For  $a \in \mathbb{N}_0$  and  $m \in \mathbb{N}$  with  $\gcd(3, m) = 1$ , we have

$$\text{sgn} \left( C_{1^9 3^{-3}}(3^a m) \right) = - \left( \frac{m}{3} \right).$$

(2) For  $a \in \mathbb{N}_0$  and  $m \in \mathbb{N}$  with  $\gcd(5, m) = 1$ , we have

$$\text{sgn} \left( C_{1^{25} 5^{-1}}(5^a m) \right) = - \left( \frac{m}{5} \right).$$

Finally, we consider a half-integral weight case for which the sign of the  $n$ -th Fourier coefficient resembles Corollary 1.6.

**Theorem 1.7.** For  $n \in \mathbb{N}_0$ , write  $8n + 1 = 3^a m$  with  $3 \nmid m$ . Then we have

$$\text{sgn} \left( C_{1^{2^2} 2^3 - 1}(n) \right) = \begin{cases} \left( \frac{m}{3} \right) & \text{if } a = 0, \\ -\frac{\left( \frac{m}{3} \right) + 1}{2} & \text{if } a = 1, \\ -1 & \text{if } a = 2, \\ \text{sgn} \left( C_{1^{2^2} 2^3 - 1} \left( \frac{n-1}{3} \right) \right) & \text{if } a \geq 3. \end{cases}$$

The paper is organized as follows. In Section 2, we recall some basic facts on modular forms and Hurwitz class numbers. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we show Theorem 1.3, in Section 6 we prove Theorem 1.4, and in Section 7 we prove Theorem 1.5, Corollary 1.6, and Theorem 1.7.

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## 2. PRELIMINARIES

**2.1. Modular forms.** We briefly introduce modular forms, but refer the reader to [22] for more details. As usual, for  $d$  odd, we let

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

For  $k \in \frac{1}{2}\mathbb{Z}$ ,  $N \in \mathbb{N}$  ( $4 \mid N$  if  $k \in \mathbb{Z} + \frac{1}{2}$ ), and a character  $\chi \pmod{N}$ , a function  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfies *modularity of weight  $k$  on  $\Gamma_0(N)$  with character  $\chi$*  if for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$f|_k \gamma = \chi(d)f.$$

Here the weight  $k$  slash operator is defined by

$$f|_k \gamma(z) := \begin{cases} \left(\frac{c}{d}\right) \varepsilon_d^{2k} (cz+d)^{-k} f(\gamma z) & \text{if } k \in \mathbb{Z} + \frac{1}{2}, \\ (cz+d)^{-k} f(\gamma z) & \text{if } k \in \mathbb{Z}. \end{cases}$$

We call  $f : \mathbb{H} \rightarrow \mathbb{C}$  a (*holomorphic*) *modular form of weight  $k$  on  $\Gamma_0(N)$  with character  $\chi$*  if  $f$  is holomorphic on  $\mathbb{H}$ , satisfies modularity of weight  $k$  on  $\Gamma_0(N)$  with character  $\chi$ , and for every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ ,  $(cz+d)^{-k} f(\gamma z)$  is bounded as  $z \rightarrow i\infty$ . We denote the space of such forms by  $M_k(\Gamma_0(N), \chi)$ . Modular forms for which  $(cz+d)^{-k} f(\gamma z)$  vanishes as  $z \rightarrow i\infty$  for all  $\gamma \in \text{SL}_2(\mathbb{Z})$  are called *cusp forms*. The corresponding subspace is denoted by  $S_k(\Gamma_0(N), \chi)$ . We drop  $\chi$  from the notation if it is trivial. The *Petersson inner product* between  $f, g \in S_k(\Gamma_0(N), \chi)$  is defined by ( $z = x + iy$ )

$$\langle f, g \rangle := \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2}.$$

**2.2. Operators on (non-holomorphic) modular forms.** We recall the action of certain operators on non-holomorphic functions which satisfy weight  $\kappa \in \frac{1}{2}\mathbb{Z}$  modularity on some group  $\Gamma_0(N)$  with some character. For  $f(z) = \sum_{n \in \mathbb{Z}} c_{f,y}(n) q^n$  (we omit the dependence on  $y$  if  $f$  is holomorphic), we define

$$f|U_\ell(z) := \sum_{n \in \mathbb{Z}} c_{f, \frac{y}{\ell}}(\ell n) q^n, \quad f|V_\ell(z) := f(\ell z).$$

Moreover let for  $M \in \mathbb{N}$ ,  $m \in \mathbb{N}_0$  the *sieving operator*

$$f | S_{M,m}(z) := \sum_{\substack{n \in \mathbb{Z} \\ n \equiv m \pmod{M}}} c_{f,y}(n) q^n$$

and for a character  $\psi$  we define the *twist of  $f$  by  $\psi$*  as

$$f \otimes \psi(z) := \sum_{n \in \mathbb{Z}} \psi(n) c_{f,y}(n) q^n.$$

The *conductor* of a character  $\chi \pmod{N}$  is the smallest  $M \in \mathbb{N}$  such that for all  $n \in \mathbb{Z}$  with  $\gcd(n, N) = 1$  we have  $\chi(n + M) = \chi(n)$ . To state the modular properties of these functions, we also require the *radical*  $\text{rad}(n) := \prod_{p|n} p$  and the character  $\chi_D(n) := (\frac{D}{n})$ . The following properties of these operators are well-known; for a proof see for example [6, Lemma 2.3].

**Lemma 2.1.** *Suppose that  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfies modularity of weight  $k \in \mathbb{Z} + \frac{1}{2}$  on  $\Gamma_0(N)$  ( $4 \mid N$ ) with character  $\chi$  of conductor  $N_\chi \mid N$ .*

- (1) *For  $\delta \in \mathbb{N}$ , the function  $f|U_\delta$  satisfies modularity of weight  $k$  on  $\Gamma_0(4 \text{lcm}(\frac{N}{4}, \text{rad}(\delta)))$  with character  $\chi\chi_{4\delta}$ .*
- (2) *Suppose that  $M \mid 24$  and  $M \not\equiv 2 \pmod{4}$ . Then  $f|S_{M,m}$  satisfies modularity of weight  $k$  on  $\Gamma_0(\text{lcm}(N, M^2, MN_\chi))$  with character  $\chi$ .*
- (3) *For  $\delta \in \mathbb{N}$ ,  $f|V_\delta$  satisfies modularity of weight  $k$  on  $\Gamma_0(N\delta)$  with character  $\chi\chi_{4\delta}$ .*

The following lemma may be shown similarly to Lemma 2.1.

**Lemma 2.2.** *Let  $N \in \mathbb{N}$ ,  $\chi$  a character  $\pmod{N}$  with conductor  $N_\chi \mid N$ , and  $k \in \mathbb{N}$ , and suppose that  $f$  satisfies weight  $k$  modularity on  $\Gamma_0(N)$  with character  $\chi$ .*

- (1) *For  $\delta \in \mathbb{N}$  the function  $f|V_\delta$  satisfies weight  $k$  modularity on  $\Gamma_0(N\delta)$  with character  $\chi$ .*
- (2) *If  $M \in \mathbb{N}$  with  $M \mid 24$  and  $m \in \mathbb{Z}$ , then the function  $f|S_{M,m}$  satisfies weight  $k$  modularity on  $\Gamma_0(\text{lcm}(N, M^2, MN_\chi))$  with character  $\chi$ .*
- (3) *For  $M \in \mathbb{N}$  and  $\psi$  a character  $\pmod{M}$ , the function  $f \otimes \psi$  satisfies weight  $k$  modularity for  $\Gamma_0(\text{lcm}(N, M^2, MN_\chi))$  with character  $\chi\psi^2$ .*
- (4) *If  $\delta \mid N$ , then  $f|U_\delta$  satisfies weight  $k$  modularity on  $\Gamma_0(N)$  with character  $\chi$ .*

**2.3. Eisenstein series.** For  $k \in \mathbb{N}$  with  $k \geq 2$  and primitive characters  $\chi, \psi$ , we define the *Eisenstein series*

$$E_{k,\chi,\psi}(z) := \delta_{\chi=\chi_1} L(1-k, \psi) + 2 \sum_{n \geq 1} \sum_{d|n} \chi\left(\frac{n}{d}\right) \psi(d) d^{k-1} q^n, \quad (2.1)$$

where the *L-function* for the character  $\psi$  is defined for  $\text{Re}(s) > 1$  by  $L(s, \psi) := \sum_{n \geq 1} \frac{\psi(n)}{n^s}$  and it is meromorphically continued to  $s \in \mathbb{C}$ . The modular properties of these Eisenstein series were given in [12, Theorem 4.5.2] and [12, Theorem 4.6.2].

**Lemma 2.3.** *Suppose that  $\chi$  and  $\psi$  are primitive characters of conductors  $N_\chi$  and  $N_\psi$ , respectively. If  $k > 2$  or ( $k = 2$  and either  $\chi$  or  $\psi$  is non-trivial), then  $E_{k,\chi,\psi} \in M_k(\Gamma_0(N_\chi N_\psi), \chi\psi)$ .*

For  $k, N \in \mathbb{N}$  and a character  $\varrho \pmod{N}$ , we call the space spanned by  $E_{k,\chi,\psi}|V_d$  with characters  $\chi, \psi$  and  $d \in \mathbb{N}$  such that  $N_\chi N_\psi d \mid N$  and  $\chi\psi = \varrho$  the *Eisenstein series subspace* of  $M_k(\Gamma_0(N), \varrho)$ . Here  $\chi\psi = \varrho$  means that they agree as characters  $\pmod{N}$ . One can split  $f \in M_k(\Gamma_0(N), \varrho)$  uniquely as  $f = E + g$  with  $E$  contained in the Eisenstein series subspace and  $g$  a cusp form. We call  $E$  the *Eisenstein series part* of  $f$  and  $g$  the *cuspidal part* of  $f$ .

If  $\chi = \chi_1$  and  $\psi = \chi_{\left(\frac{-1}{p}\right)_p}$  for an odd prime  $p$ , then we normalize the Eisenstein series by multiplying by

$$L_{k,p} := \frac{1}{L\left(1-k, \chi_{\left(\frac{-1}{p}\right)_p}\right)}.$$

One can use the Euler–Maclaurin summation formula (see [27, (44)]) to determine the behavior of the Eisenstein series at 0.

**Lemma 2.4.** *Suppose that  $k \geq 2$  and  $p$  is an odd prime. Then  $L_{k,p}E_{k,\chi_1,\chi_{\left(\frac{-1}{p}\right)_p}}$  has constant term 1 at the cusp  $i\infty$  and vanishes at the cusp 0.*

In addition to the holomorphic Eisenstein series, we also require the quasimodular Eisenstein series of weight 2, with  $\sigma(n) := \sum_{d|n} d$ ,

$$E_2(z) := 1 - 24 \sum_{n \geq 1} \sigma(n)q^n.$$

The Eisenstein series  $E_2$  is not modular, but it has a natural “modular completion”

$$\widehat{E}_2(z) := E_2(z) - \frac{3}{\pi y}.$$

Although  $\widehat{E}_2$  is not holomorphic, it satisfies for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$\widehat{E}_2|_2\gamma = \widehat{E}_2.$$

**2.4. Hecke operators.** In this subsection, we restrict ourselves to integral weight modular forms. For  $N, k \in \mathbb{N}$ ,  $\chi$  a character (mod  $N$ ), and  $p$  a prime, we define the *Hecke operator*  $T_p$  acting on  $f \in M_k(\Gamma_0(N), \chi)$  by (see [22, Definition 2.1])

$$f|T_p(z) := \sum_{n \geq 0} \left( c_f(pn) + \chi(p)p^{k-1}c_f\left(\frac{n}{p}\right) \right) q^n,$$

where  $c_f(\alpha) := 0$  for  $\alpha \in \mathbb{Q}^+ \setminus \mathbb{N}_0$ . There is a natural basis of cusp forms which are simultaneous eigenfunctions under all Hecke operators  $T_p$  with  $p \nmid N$ . We call these simultaneous eigenfunctions *Hecke eigenforms*. If  $f \in M_k(\Gamma_0(N), \chi)$  is a Hecke eigenform, then  $f|V_d \in M_k(\Gamma_0(Nd), \chi)$  is also a Hecke eigenform. For  $M \in \mathbb{N}$ , the subspace of  $M_k(\Gamma_0(M), \chi)$  spanned by the eigenforms  $f|V_d$  with  $f \in M_k(\Gamma_0(N), \chi)$  for  $1 \leq N < M$  is called the *old space*, and the orthogonal complement of these is the *new space*. The Hecke eigenforms in the new spaces are called *newforms*, and we normalize the Fourier expansions so that  $c_f(1) = 1$ . Letting  $d(n)$  denote the number of divisors of  $n$ , a celebrated result of Deligne [11] gives an explicit bound on  $c_f(n)$ .

**Theorem 2.5.** *(Deligne) Suppose that  $k, N \in \mathbb{N}$ ,  $\chi$  is a character (mod  $N$ ), and  $f \in S_k(\Gamma_0(N), \chi)$  is a normalized newform. Then*

$$|c_f(n)| \leq d(n)n^{\frac{k-1}{2}}.$$

Letting  $\|f\| := \sqrt{\langle f, f \rangle}$  denote the Petersson norm, Schulze-Pillot–Yenirce [25, Theorem 12] constructed an explicit orthonormal basis for  $S_k(\Gamma_0(N), \chi)$  and used Theorem 2.5 to obtain a bound for the Fourier coefficients of any  $f \in S_k(\Gamma_0(N), \chi)$ .

**Lemma 2.6.** *Suppose that  $k, N \in \mathbb{N}$ ,  $\chi$  is a character  $(\bmod N)$ , and  $f \in S_k(\Gamma_0(N), \chi)$ . Then*

$$|c_f(n)| \leq 2\sqrt{\pi}e^{2\pi} \sqrt{\dim_{\mathbb{C}}(S_k(\Gamma_0(N), \chi))} \sqrt{N} \prod_{p|N} \frac{\left(1 + \frac{1}{p}\right)^3}{\sqrt{1 - \frac{1}{p^4}}} \|f\| d(n) n^{\frac{k-1}{2}}.$$

**2.5. Valence formula.** In order to show identities between modular forms, we use the following consequence of the valence formula.

**Lemma 2.7.** *Let  $k \in \frac{1}{2}\mathbb{N}$ ,  $N \in \mathbb{N}$ , and  $\chi$  be a character  $(\bmod N)$ . If  $f \in M_k(\Gamma_0(N), \chi)$  satisfies  $c_f(n) = 0$  for every  $0 \leq n \leq N \frac{k}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right)$ , then  $f \equiv 0$ .*

**2.6.  $L$ -functions of Dirichlet characters.** We require the following lemma for evaluating  $L$ -functions of Dirichlet characters at non-positive integers. The following identity is well-known and can be easily concluded from [3, p. 249] and [3, Theorem 12.13].

**Lemma 2.8.** *For a discriminant  $D$  and  $k \in \mathbb{N}$ , we have*

$$L(1-k, \chi_D) = -\frac{|D|^{k-1}}{k} \sum_{r=1}^{|D|} \chi_D(r) B_k\left(\frac{r}{|D|}\right),$$

where  $B_k(x)$  denotes the  $k$ -th Bernoulli polynomial.

For the specific case  $D = \left(\frac{-1}{p}\right)p$  and  $k \equiv k_p \pmod{2}$  with  $k_p := \frac{p-1}{2}$ , one may use the functional equation for the Dirichlet  $L$ -function and Gauss's evaluation of the quadratic Gauss sum to obtain the sign of the  $L$ -value in Lemma 2.8.

**Lemma 2.9.** *Let  $p$  be an odd prime and  $k \geq 2$  with  $k \equiv k_p \pmod{2}$  be an integer. Then*

$$\operatorname{sgn}(L_{k,p}) = (-1)^{\frac{k}{2} + \frac{p-1}{4}} \left(\frac{-2}{p}\right).$$

In particular, we have

$$\operatorname{sgn}(L_{k_p,p}) = \left(\frac{-2}{p}\right) \quad \text{and} \quad \operatorname{sgn}(L_{pk_p,p}) = \left(\frac{2}{p}\right).$$

**2.7. Hurwitz class numbers.** For  $D \in \mathbb{N}$ , denote by  $H(D)$  the *Hurwitz class number*, which counts the class number of positive-definite integral binary quadratic forms of discriminant  $-D$ , where each class is weighted by the inverse of the size of its automorphism group in  $\operatorname{PSL}_2(\mathbb{Z})$ . We note that if  $-D < 0$  is a discriminant, then  $H(D) > 0$ , while  $H(D) = 0$  if  $-D$  is not a discriminant. We extend the definition by setting  $H(0) := -\frac{1}{12}$  and  $H(r) := 0$  for  $r \in \mathbb{Q} \setminus \mathbb{N}_0$ . For  $-D$  a fundamental discriminant we have [9, p. 273]

$$H(Df^2) = H(D)S_D(f), \tag{2.2}$$

where, letting  $\mu$  denote the Möbius  $\mu$ -function,

$$S_D(f) := \sum_{d|f} \mu(d) \chi_{-D}(d) \sigma\left(\frac{f}{d}\right). \tag{2.3}$$

For  $\ell_1, \ell_2 \in \mathbb{N}$ , let

$$\mathcal{H}(z) := \sum_{D \geq 0} H(D) q^D, \quad \mathcal{H}_{\ell_1, \ell_2} := \mathcal{H} \mid (U_{\ell_1 \ell_2} - \ell_2 U_{\ell_1} \circ V_{\ell_2}). \tag{2.4}$$

With  $\Gamma(s, y) := \int_y^\infty e^{-t} t^{s-1} dt$  for  $y > 0$  the *incomplete gamma function*, the modularity of

$$\widehat{\mathcal{H}}(z) := \mathcal{H}(z) + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n \geq 1} n \Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}$$

was proven by Zagier [26] (see also [13, Chapter 2, Theorem 2]). Using Zagier's result, the modularity of  $\mathcal{H}_{\ell_1, \ell_2}$  was given in [6, Lemma 2.6].

**Lemma 2.10.** *For  $\ell_1, \ell_2 \in \mathbb{N}$ , with  $\gcd(\ell_1, \ell_2) = 1$  and  $\ell_2$  squarefree, we have that*

$$\mathcal{H}_{\ell_1, \ell_2} \in M_{\frac{3}{2}}(\Gamma_0(4 \operatorname{rad}(\ell_1)\ell_2), \chi_{4\ell_1\ell_2}).$$

### 3. PROOF OF THEOREM 1.1

We abbreviate  $b_1(n) := C_{1-2_23_4^2}(n)$ .

*Proof of Theorem 1.1.* By [22, Theorem 1.60], we have

$$\frac{\eta(16z)^2}{\eta(8z)} = \sum_{n \geq 0} q^{(2n+1)^2}.$$

Thus<sup>1</sup>

$$\sum_{n \geq 0} b_1(n) q^{8n+4} = \sum_{\mathbf{n} \in \mathbb{N}_0^3} q^{8(T_{n_1} + T_{n_2} + 2T_{n_3}) + 4},$$

where  $T_n := \frac{n(n+1)}{2}$ . Comparing the  $(8n+4)$ -th Fourier coefficient on both sides, we see that

$$b_1(n) = \# \{ \mathbf{n} \in \mathbb{N}_0^3 : T_{n_1} + T_{n_2} + 2T_{n_3} = n \}. \quad (3.1)$$

It was proven by Liouville [21] (also see the statement in [5] and its generalization in [5, Theorem 1.1]) that, for  $n \in \mathbb{N}_0$ ,

$$\# \{ \mathbf{n} \in \mathbb{N}_0^3 : T_{n_1} + T_{n_2} + 2T_{n_3} = n \} > 0,$$

which together with (3.1) immediately implies Theorem 1.1.  $\square$

### 4. PROOF OF THEOREM 1.2

Abbreviating  $b_2(n) := C_{1^3_2-2_3^3}(n)$ ,  $s_2(n) := \operatorname{sgn}(b_2(n))$ , we first show the following.

**Lemma 4.1.** *We have*

$$b_2(n) = \begin{cases} \sum_{d|(3n+1)} \left(\frac{3}{d}\right) d & \text{if } n \equiv 0 \pmod{4}, \\ -\frac{1}{3} \sum_{d|(3n+1)} \left(\frac{3}{d}\right) d & \text{if } n \equiv 2 \pmod{4}, \\ -\frac{1}{3} \sum_{d|(3n+1)} \left(\frac{12}{d}\right) d - \frac{2}{3} \sum_{d|(3n+1)} \left(\frac{12}{\frac{3n+1}{d}}\right) d & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

*Proof.* By [22, Theorem 1.64],  $\frac{\eta(3z)^3 \eta(9z)^3}{\eta(6z)^2} \in M_2(\Gamma_0(72), \chi_{12})$ .

The generating function of the right-hand side of Lemma 4.1 is

$$\frac{1}{2} E_{2, \chi_1, \chi_{12}} | S_{12,1} - \frac{1}{6} E_{2, \chi_1, \chi_{12}} | S_{12,7} - \frac{1}{6} E_{2, \chi_1, \chi_{12}} | S_{6,4} - \frac{1}{3} E_{2, \chi_{12}, \chi_1} | S_{6,4}.$$

By Lemma 2.3,  $E_{2, \chi_1, \chi_{12}}, E_{2, \chi_{12}, \chi_1} \in M_2(\Gamma_0(12), \chi_{12})$ . Hence, for  $m \in \mathbb{Z}$  and  $M \in \mathbb{N}$  with  $M \mid 12$ , Lemma 2.2 (2) implies that  $E_{2, \chi_1, \chi_{12}} | S_{M,m}, E_{2, \chi_{12}, \chi_1} | S_{M,m} \in M_2(\Gamma_0(144), \chi_{12})$ . Thus

<sup>1</sup>Throughout we use boldface letters for vectors.



the generating functions of both sides of the claimed identity lie in  $M_2(\Gamma_0(144), \chi_{12})$ . By Lemma 2.7, the identity holds if it holds for the first 48 Fourier coefficients. The identity was checked with a computer for  $3n + 1 \leq 301$ , verifying the claim.  $\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Writing  $3n + 1 = 2^\ell m$  with  $\gcd(6, m) = 1$ , we have

$$\sum_{d|(3n+1)} \left(\frac{12}{d}\right) d = \sum_{d|m} \left(\frac{3}{d}\right) d \sum_{r=0}^{\ell} \left(\frac{12}{2^r}\right) 2^r = \prod_{p|m} \sum_{j=0}^{\text{ord}_p(m)} \left(\frac{3}{p^j}\right) p^j = \prod_{p|m} \frac{1 - \left(\left(\frac{3}{p}\right) p\right)^{\text{ord}_p(m)+1}}{1 - \left(\frac{3}{p}\right) p}.$$

Now

$$\frac{1 - \left(\left(\frac{3}{p}\right) p\right)^{\text{ord}_p(m)+1}}{1 - \left(\frac{3}{p}\right) p} = \begin{cases} \frac{1-p^{\text{ord}_p(m)}}{1-p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ \frac{1-p^{\text{ord}_p(m)}}{1+p} & \text{if } p \equiv \pm 5 \pmod{12}, \text{ ord}_p(m) \text{ odd}, \\ \frac{1+p^{\text{ord}_p(m)}}{1+p} & \text{if } p \equiv \pm 5 \pmod{12}, \text{ ord}_p(m) \text{ even}. \end{cases}$$

This implies that

$$\text{sgn} \left( \sum_{d|m} \left(\frac{3}{d}\right) d \right) = \left(\frac{3}{m}\right) = (-1)^{\frac{m-1}{2}} \left(\frac{m}{3}\right). \quad (4.1)$$

For  $n$  even, we have  $3n + 1 = m$ , so (4.1) implies that

$$\text{sgn} \left( \sum_{d|(3n+1)} \left(\frac{3}{d}\right) d \right) = (-1)^{\frac{m-1}{2}} \left(\frac{m}{3}\right). \quad (4.2)$$

If  $n \equiv 0 \pmod{4}$ , then  $m = 3n + 1 \equiv 1 \pmod{12}$ , so plugging (4.2) into Lemma 4.1 yields that  $s_2(n) = 1$ . If  $n \equiv 2 \pmod{4}$ , then  $m = 3n + 1 \equiv 7 \pmod{12}$ , so inserting (4.2) into Lemma 4.1 implies that  $s_2(n) = 1$ .

For  $2 \nmid n$ , we write  $3n + 1 = 2^\ell m$  with  $m$  odd. Noting that  $\left(\frac{12}{\frac{3n+1}{d}}\right) = 0$  unless  $2^\ell \mid d$ , Lemma 4.1 gives that

$$b_2(n) = -\frac{1}{3} \left( 1 + 2^{\ell+1} \left(\frac{3}{m}\right) \right) \sum_{d|m} \left(\frac{3}{d}\right) d.$$

By (4.1), we have  $\text{sgn}(\sum_{d|m} \left(\frac{3}{d}\right) d) = \left(\frac{3}{m}\right)$ . Since  $2^{\ell+1} > 1$ ,  $\text{sgn}(1 + 2^{\ell+1} \left(\frac{3}{m}\right)) = \text{sgn}\left(\left(\frac{3}{m}\right)\right)$ . So overall we obtain  $s_2(n) = -1$  for  $n$  odd. Combining gives the claim.  $\square$

## 5. PROOF OF THEOREM 1.3

The goal of this section is to prove Theorem 1.3.

**5.1. Decomposition of the eta-quotient.** We now explicitly decompose  $\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2}$  into an Eisenstein series and a cusp form. For this, we define the Eisenstein series

$$\begin{aligned}\mathcal{E}(z) &:= \sum_{\substack{n \geq 1 \\ n \equiv 1 \pmod{12}}} \sum_{d|n} \left(\frac{-1}{d}\right) d^2 q^n - \frac{1}{2} \sum_{\substack{n \geq 1 \\ n \equiv 5 \pmod{12}}} \sum_{d|n} \left(\frac{-1}{d}\right) d^2 q^n \\ &\quad - 2 \sum_{\substack{n \geq 1 \\ n \equiv 9 \pmod{12}}} \left( \sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6 \sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2 \right) q^n \\ &= \frac{1}{2} E_{3,\chi_1,\chi_{-4}}|S_{12,1} - \frac{1}{4} E_{3,\chi_1,\chi_{-4}}|S_{12,5} - E_{3,\chi_1,\chi_{-4}}|S_{12,9} - 6 E_{3,\chi_1,\chi_{-4}}|V_3|S_{4,1}.\end{aligned}$$

For the cuspidal part, we let  $g_1$  denote the normalized newform in  $S_3(\Gamma_0(12), \chi_{-4})$  with

$$g_1(z) = q - (1 + \sqrt{3}i)q^2 + \sqrt{3}iq^3 - 2(1 - \sqrt{3}i)q^4 - 2q^5 + (3 - \sqrt{3}i)q^6 - 4\sqrt{3}iq^7 + 8q^8 - 3q^9 + O(q^{10})$$

and let  $g_2$  be the normalized newform in  $S_3(\Gamma_0(36), \chi_{-4})$  with

$$g_2(z) = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} - 16q^{17} + O(q^{20}).$$

We then obtain the following decomposition of the eta-quotient.

**Lemma 5.1.** *We have*

$$\begin{aligned}\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2} &= \frac{1}{7}\mathcal{E}(z) - \frac{27}{14}g_1|S_{12,9}(z) + \frac{3}{14}g_1|S_{4,1}(z) + \frac{9}{14}g_1 \otimes \chi_{-3}|S_{4,1}(z) \\ &\quad - \frac{3}{16}g_2|S_{4,1}(z) + \frac{3}{16}g_2 \otimes \chi_{-3}|S_{4,1}(z).\end{aligned}$$

*Proof.* By [22, Theorem 1.64],  $\frac{\eta(4z)^4\eta(8z)^4}{\eta(12z)^2} \in M_3(\Gamma_0(72), \chi_{-4})$ . By Lemma 2.3, we have that  $E_{3,\chi_1,\chi_{-4}} \in M_3(\Gamma_0(4), \chi_{-4})$ . Using Lemma 2.2 (2),  $E_{3,\chi_1,\chi_{-4}}|S_{12,m} \in M_3(\Gamma_0(144), \chi_{-4})$  for  $m \in \mathbb{Z}$ . Lemma 2.2 (1) implies that  $E_{3,\chi_1,\chi_{-4}}|V_3 \in M_3(\Gamma_0(12), \chi_{-4})$ . Lemma 2.2 (2) then gives that  $E_{3,\chi_1,\chi_{-4}}|V_3|S_{4,1} \in M_3(\Gamma_0(48), \chi_{-4})$ . Thus  $\mathcal{E} \in M_3(\Gamma_0(144), \chi_{-4})$ .

By Lemma 2.2 (2), (3), we have

$$-\frac{27}{14}g_1|S_{12,9} + \frac{3}{14}g_1|S_{4,1} + \frac{9}{14}g_1 \otimes \chi_{-3}|S_{4,1} - \frac{3}{16}g_2|S_{4,1} + \frac{3}{16}g_2 \otimes \chi_{-3}|S_{4,1} \in M_3(\Gamma_0(144), \chi_{-4}).$$

Thus both sides of the claimed identity lie in  $M_3(\Gamma_0(144), \chi_{-4})$ . To prove the identity, it suffices to verify the identity for the first 72 Fourier coefficients. This was done with a computer.  $\square$

**5.2. Fourier coefficients of the Eisenstein series part.** We now determine the sign of the  $n$ -th Fourier coefficients  $A(n)$  of the Eisenstein series part  $\mathcal{E}$ .

**Lemma 5.2.** *Suppose that  $n \equiv 1 \pmod{4}$ . Then we have*

$$\sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6 \sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2 > 0.$$

In particular,

$$\operatorname{sgn}(A(4n+1)) = \begin{cases} 1 & \text{if } 3 \mid n, \\ -1 & \text{otherwise.} \end{cases}$$

*Proof.* If  $3 \nmid n$ , then the second sum on the left-hand side of the first claim vanishes. Since  $n \equiv 1 \pmod{4}$ , we have that  $n$  (and thus  $d$ ) is odd. Thus the above becomes

$$\prod_{p|n} \sum_{j=0}^{\operatorname{ord}_p(n)} \left(\frac{-1}{p^j}\right) p^{2j} = \prod_{p|n} \begin{cases} \frac{1-p^{2\operatorname{ord}_p(n)+2}}{1-p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{1-p^{2\operatorname{ord}_p(n)+2}}{1+p^2} & \text{if } p \equiv 3 \pmod{4}, 2 \nmid \operatorname{ord}_p(n), \\ \frac{1+p^{2\operatorname{ord}_p(n)+2}}{1+p^2} & \text{if } p \equiv 3 \pmod{4}, 2 \mid \operatorname{ord}_p(n). \end{cases}$$

Now only the second case yields a negative sign, so

$$\operatorname{sgn} \left( \sum_{d|n} \left(\frac{-1}{d}\right) \right) = (-1)^{\#\{p|n : p \equiv 3 \pmod{4}, 2 \nmid \operatorname{ord}_p(n)\}} \equiv n \pmod{4} \quad (5.1)$$

since  $n$  is odd. Since  $n \equiv 1 \pmod{4}$ , we conclude that the sign in this case is 1.

Finally, suppose that  $3 \mid n$ . Write  $n = 3^\ell m$  ( $\ell \in \mathbb{N}$ ),  $3 \nmid m$ . Then we need to rewrite the left-hand side of the first claim of the lemma as

$$\begin{aligned} \sum_{d|n} \left(\frac{-1}{d}\right) d^2 \sum_{j=0}^{\ell} \left(\frac{-1}{3^j}\right) 3^{2j} + 6 \sum_{d|m} \left(\frac{-1}{d}\right) d^2 \sum_{j=0}^{\ell-1} \left(\frac{-1}{3^j}\right) 3^{2j} \\ = \sum_{d|m} \left(\frac{-1}{d}\right) d^2 \left( 7 \sum_{j=0}^{\ell-1} (-1)^j 3^{2j} + (-1)^\ell 3^{2\ell} \right). \end{aligned} \quad (5.2)$$

The sign of the first sum is evaluated in (5.1). The parenthesis equals  $\frac{7+(-1)^\ell 3^{2\ell+1}}{10}$ . Note that the sign of this factor is  $(-1)^\ell$  (because  $\ell \in \mathbb{N}$ ). Combining with above, we have for  $3 \mid n$

$$\operatorname{sgn} \left( \sum_{d|n} \left(\frac{-1}{d}\right) d^2 + 6 \sum_{d|\frac{n}{3}} \left(\frac{-1}{d}\right) d^2 \right) = (-1)^{\#\{p|n : p \equiv 3 \pmod{4}, 2 \nmid \operatorname{ord}_p(n)\}}.$$

Since  $n \equiv 1 \pmod{4}$ , (5.1) again implies that the sign is 1. Plugging the first claim into the definition of  $\mathcal{E}$  yields the second claim.  $\square$

**5.3. Finishing the proof of Theorem 1.3.** By Lemma 5.2, the signs of the Fourier coefficients of  $\mathcal{E}$  match those of the signs of the Fourier coefficients claimed in Theorem 1.3. We are now ready to prove Theorem 1.3. Let  $b_3(n) := C_{14223-2}(n)$  and  $s_3(n) := \operatorname{sgn}(b_3(n))$ .

*Proof of Theorem 1.3.* We claim that for  $n \in \mathbb{N}$ ,  $s_3(n) = \operatorname{sgn}(A(4n+1))$ . Lemma 5.2 then gives the claim. To show this, we explicitly compare the growth of the Fourier coefficients of  $\mathcal{E}$  with the growth of the Fourier coefficients of the cuspidal part from Lemma 5.1. We begin by bounding the Fourier coefficients of the cuspidal part of  $\frac{\eta(4z)^4 \eta(8z)^4}{\eta(12z)^2}$ . Write

$$g_1(z) =: \sum_{n \geq 1} a_1(n) q^n, \quad g_2(z) =: \sum_{n \geq 1} a_2(n) q^n.$$

Since  $g_1$  and  $g_2$  are weight 3 newforms,  $|a_1(n)|, |a_2(n)| \leq d(n)n$  by Theorem 2.5. Moreover, by Lemma 5.1, for  $n \equiv 1 \pmod{4}$  the  $n$ -th Fourier coefficient of the cuspidal part of the function on the right-hand side of Lemma 5.1 is

$$-\frac{12}{7}\delta_{3|n}a_1(n) + \delta_{3\nmid n}\frac{3}{14}\left(1 + 3\left(\frac{-3}{n}\right)\right)a_1(n) + \frac{3}{16}\left(-1 + \left(\frac{-3}{n}\right)\right)a_2(n). \quad (5.3)$$

Note that  $\left(\frac{-3}{n}\right) = \left(\frac{n}{3}\right)$ . We now look at the various residue classes of  $n \pmod{12}$ .

Assume first that  $n \equiv 1 \pmod{12}$ . Then (5.3) equals  $\frac{6}{7}a_1(n)$  and its absolute value can be bounded against  $\frac{6}{7}d(n)n$ . From Lemma 5.1, the absolute value of the  $n$ -th Fourier coefficient of the Eisenstein series part  $\mathcal{E}$  is

$$\frac{1}{7}\left|\sum_{d|n}\left(\frac{-1}{d}\right)d^2\right| \geq \frac{1}{7}\prod_{p|n}\frac{p^{2\text{ord}_p(n)+2}-1}{p^2+1}.$$

We conclude that  $s_3(n)$  agrees with the claimed value if

$$d(n) \leq \frac{1}{6}\prod_{p|n}\frac{p^{2\text{ord}_p(n)+2}-1}{p^{\text{ord}_p(n)}(p^2+1)}.$$

Writing  $n = \prod_{j=1}^r p_j^{\ell_j}$ , we have  $d(n) = \prod_{j=1}^r (\ell_j + 1)$ . Hence the above is equivalent to

$$\prod_{j=1}^r (\ell_j + 1) \leq \frac{1}{6}\prod_{j=1}^r \frac{p_j^{2\ell_j+2}-1}{p_j^{\ell_j}(p_j^2+1)}. \quad (5.4)$$

We claim that, for a prime  $p$  and  $\ell \in \mathbb{N}$ ,

$$\frac{p^{2\ell+2}-1}{p^\ell(p^2+1)} \geq \begin{cases} 2.4(\ell+1) & \text{for } p=5 \text{ and } \ell=1, \\ 3.4(\ell+1) & \text{for } p=7 \text{ and } \ell=1, \\ 5(\ell+1) & \text{for } p=11 \text{ and } \ell=1, \\ 6(\ell+1) & \text{for } p \geq 13 \text{ or } (p \in \{5, 7, 11\} \text{ and } \ell \geq 2). \end{cases} \quad (5.5)$$

This clearly implies the claim unless  $n$  is one of the problem cases, which are  $n \in \{5, 7, 11\}$ . But none of these satisfy  $n \equiv 1 \pmod{12}$ .

We now prove (5.5). We first check the cases  $\ell = 1$  and  $p \in \{5, 7, 11\}$  directly by computing both sides of (5.3) and confirming the claim. The claim for the remaining cases in (5.5) follows after showing that for  $x \geq 13$  or  $(x \in \{5, 7, 11\} \text{ and } \ell \geq 2)$

$$f_\ell(x) := x^{2\ell+2} - 1 - 6(\ell+1)x^\ell(x^2+1) > 0.$$

We do so by induction on  $\ell$  for each  $x$ . First, for  $x \geq 13$ , we have

$$f_1(x) = x^4 - 1 - 12x(x^2+1) > 0.$$

We also check directly that  $f_2(5) = 3924 > 0$ ,  $f_2(7) = 73548 > 0$ , and  $f_2(11) = 1505844 > 0$ , so we see that the base case for the induction holds for each  $p$ . Next

$$f_{\ell+1}(x) = x^2 f_\ell(x) + 6(\ell+1)x^{\ell+1}(x^2+1)(x-1) + x^2 - 1 - 6x^{\ell+1}(x^2+1).$$

The first term is positive by induction. Using  $x \geq 2$ , we show that the remaining terms are non-negative. This proves (5.5) for  $n \equiv 1 \pmod{12}$ .

Next assume that  $n \equiv 5 \pmod{12}$ . Simplifying (5.3) in this case, the  $n$ -th Fourier coefficient of the cuspidal part is  $-\frac{3}{7}a_1(n) - \frac{3}{8}a_2(n)$ . Then Theorem 2.5 implies that the absolute value is bounded by  $\frac{45}{56}d(n)n$ . With the same argument as before, we may conclude the claim if

$$\prod_{j=1}^r (\ell_j + 1) \leq \frac{56}{315} \prod_{j=1}^r \frac{p_j^{2\ell_j+2} - 1}{p_j^{\ell_j} (p_j^2 + 1)}.$$

If the above inequality fails, then, since  $\frac{56}{315} > \frac{1}{6}$ , (5.4) would also fail. As computed for  $n \equiv 1 \pmod{12}$ , if (5.4) fails, then  $n \in \{5, 7, 11\}$ . Thus for  $n > 5$  with  $n \equiv 5 \pmod{12}$ , we have  $s_3(\frac{n-1}{4}) = \text{sgn}(A(n))$ , as claimed. For  $n = 5$ , we directly evaluate  $s_3(1) = -1 = \text{sgn}(A(5))$  with a computer, as this is the only possible exceptional case satisfying  $n \equiv 5 \pmod{12}$ .

We finally consider the case  $n \equiv 9 \pmod{12}$ . Then, by (5.3), we have as Fourier coefficient of the cusp form  $-\frac{12}{7}a_1(n) - \frac{3}{16}a_2(n)$ . By Theorem 2.5, the absolute value of this can be bounded against  $\frac{213}{112}d(n)n$ . Writing  $n = 3^\ell m$  and comparing (5.2) with the Fourier coefficients of the Eisenstein series in the other congruence classes and simplifying, the Fourier coefficients of the Eisenstein series have the extra factor

$$\frac{3^{2\ell+1} + 7(-1)^{\ell+1}}{5}.$$

Bounding as before, we have  $s_3(\frac{n-1}{4}) = \text{sgn}(A(n))$  if

$$\frac{3^{2\ell+1} + 7(-1)^\ell}{35} \prod_{p|m} \frac{p^{2 \text{ord}_p(n)+2} - 1}{p^2 + 1} \geq \frac{213}{112}d(n)n.$$

Writing  $d(n) = (\ell + 1)d(m)$  and plugging in  $n = 3^\ell m$ , this is equivalent to

$$\prod_{p|m} \frac{p^{2 \text{ord}_p(n)+2} - 1}{p^2 + 1} \geq \frac{213}{112} \frac{35(\ell + 1)3^\ell}{3^{2\ell+1} + 7(-1)^\ell} d(m)m.$$

Note that

$$\frac{(\ell + 1)3^\ell}{3^{2\ell+1} + 7(-1)^\ell} \leq \begin{cases} \frac{3}{10} & \text{if } \ell = 1, \\ \frac{27}{250} & \text{if } \ell = 2, \\ \frac{27}{545} & \text{if } \ell = 3, \\ \frac{81}{3938} & \text{if } \ell = 4, \\ \frac{729}{88750} & \text{if } \ell \geq 5. \end{cases}$$

We then extend (5.5) with

$$\frac{p^{2\ell+2} - 1}{p^\ell(p^2 + 1)} \geq \begin{cases} 6.4(\ell + 1) & \text{for } p = 13 \text{ and } \ell = 1, \\ 8(\ell + 1) & \text{for } p = 5 \text{ and } \ell = 2, \\ 8.4(\ell + 1) & \text{for } p = 17 \text{ and } \ell = 1, \\ 9.4(\ell + 1) & \text{for } p = 19 \text{ and } \ell = 1, \\ 11.4(\ell + 1) & \text{for } p = 23 \text{ and } \ell = 1, \\ 14.4(\ell + 1) & \text{for } p = 29 \text{ and } \ell = 1, \\ 15.4(\ell + 1) & \text{for } p = 31 \text{ and } \ell = 1, \\ 16(\ell + 1) & \text{for } p = 7 \text{ and } \ell = 2, \\ 18.4(\ell + 1) & \text{for } p = 37 \text{ and } \ell = 1, \\ 20(\ell + 1) & \text{for } p \geq 41 \text{ or } (11 \leq p \leq 37 \text{ and } \ell \geq 2) \\ & \text{or } (p \in \{5, 7\} \text{ and } \ell \geq 3). \end{cases} \quad (5.6)$$

Combining (5.6) with (5.5), we then conclude that for  $n \equiv 9 \pmod{12}$  with

$$n \notin \{9, 21, 33, 45, 57, 69, 81, 93, 105, 117, 165\},$$

we have  $s_3(\frac{n-1}{4}) = \text{sgn}(A(n))$ . Computing  $s_3(\frac{n-1}{4})$  for  $n \leq 165$  by computer yields the claim.  $\square$

## 6. PROOF OF THEOREM 1.4

**6.1. Proof of Theorem 1.4 (1).** Let  $b_4(n) := C_{142242}(n)$  and  $s_4 := \text{sgn}(b_4(n))$ . We first obtain a formula for the generating function of  $b_4(n)$ .

**Lemma 6.1.** *We have*

$$\frac{\eta(z)^4 \eta(2z)^2}{\eta(4z)^2} = 1 - 4 \sum_{n \geq 1} \left( \frac{-4}{n} \right) \sigma(n) q^n + 8 \sum_{n \geq 1} (-1)^n \sigma(n) q^{4n} - 32 \sum_{n \geq 1} \sigma(n) q^{16n}.$$

*Proof.* By [22, Theorem 1.64],  $\frac{\eta(z)^4 \eta(2z)^2}{\eta(4z)^2} \in M_2(\Gamma_0(16))$ .

Next note that the right-hand side of Lemma 6.1 is

$$-2E_{2, \chi_{-4}, \chi_{-4}} + \frac{1}{3}E_2|V_4 - \frac{2}{3}E_2|U_2 \circ V_8 + \frac{4}{3}E_2|V_{16}. \quad (6.1)$$

It is not hard to see that (6.1) equals the identity with  $E_2$  replaced by  $\widehat{E}_2$ , so (6.1) is modular. By Lemmas 2.3 and 2.2 (1), (4), (6.1) lies in  $M_2(\Gamma_0(16))$ . Thus both sides of the claimed identity are elements of  $M_2(\Gamma_0(16))$ . By Lemma 2.7, the claim holds if it holds for the first 4 Fourier coefficients, which has been verified with a computer.  $\square$

We are now ready to prove Theorem 1.4 (1).

*Proof of Theorem 1.4 (1).* For  $n = 0$ , Lemma 6.1 gives  $b_4(0) = 1 > 0$ . We next suppose that  $n \in \mathbb{N}$  and split into cases depending on  $\text{ord}_2(n)$ . By Lemma 6.1, if  $n$  is odd, then

$$b_4(n) = -4 \left( \frac{-1}{n} \right) \sigma(n).$$

Since  $4\sigma(n) > 0$ , this gives

$$s_4(n) = - \left( \frac{-1}{n} \right) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod{4}, \\ -1 & \text{if } n \equiv 1 \pmod{4}. \end{cases}$$

This gives the claim for  $n$  odd.

For  $n \equiv 2 \pmod{4}$ , we see directly that none of the sums in Lemma 6.1 contributes to the  $n$ -th Fourier coefficients. This gives the claim in this case.

Next suppose that  $4 \mid n$  but  $16 \nmid n$ . In this case, only the second sum in Lemma 6.1 contributes to the  $n$ -th Fourier coefficient, so we obtain

$$b_4(n) = -8\sigma\left(\frac{n}{4}\right),$$

which implies that  $s_4(n) = -1$ . This gives the claim in this case.

Finally, if  $16 \mid n$ , then the final two sums in Lemma 6.1 contribute to the  $n$ -th Fourier-coefficient, and we have

$$b_4(n) = 8\left(\sigma\left(\frac{n}{4}\right) - 4\sigma\left(\frac{n}{16}\right)\right).$$

We next write  $n = 2^\ell m$  with  $\ell \in \mathbb{N}_{\geq 3}$  and  $m$  odd and use multiplicativity of  $\sigma(n)$  to simplify

$$\sigma\left(\frac{n}{4}\right) - 4\sigma\left(\frac{n}{16}\right) = 3\sigma(m).$$

Thus, for  $16 \mid n$ , we obtain

$$b_4(n) = 24\sigma(m) > 0.$$

Hence  $s_4(n) = 1$  in this case, completing the proof.  $\square$

**6.2. Proof of Theorem 1.4 (2).** For a vector  $\mathbf{a} \in \mathbb{N}^\ell$ , set

$$r_{\mathbf{a}}(n) := \#\left\{ \mathbf{n} \in \mathbb{Z}^\ell : \sum_{j=1}^{\ell} a_j n_j^2 = n \right\}.$$

Using [22, Theorem 1.62, Propositions 1.41 and 3.7 (1)], Lemmas 2.1 and 2.7, we obtain the following identity.

**Lemma 6.2.** *We have*

$$\begin{aligned} \frac{\eta(z)^4 \eta(2z)^4}{\eta(4z)^3} &= - \sum_{n \equiv 1 \pmod{4}} r_{(1,1,2,2,2)}(n) q^n + \sum_{n \equiv 3 \pmod{4}} r_{(1,1,2,2,2)}(n) q^n \\ &- \frac{1}{5} \sum_{n \equiv 2 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n - \frac{1}{5} \sum_{n \equiv 4 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n + \frac{1}{5} \sum_{n \equiv 6 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n \\ &- \frac{3}{7} \sum_{n \equiv 10 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n - \frac{1}{5} \sum_{n \equiv 12 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n + \frac{1}{5} \sum_{n \equiv 14 \pmod{16}} r_{(1,1,2,2,2)}(n) q^n \\ &+ \frac{1}{5} \sum_{n \equiv 0 \pmod{8}} \left( r_{(1,1,2,2,2)}(n) + 4r_{(1,1,2,2,2)}\left(\frac{n}{4}\right) \right) q^n. \end{aligned}$$

We are now ready to prove Theorem 1.4 (2).

*Proof of Theorem 1.4 (2).* Liouville [20] proved that  $r_{(1,1,2,2)}(n) > 0$  for every  $n \in \mathbb{N}$ . Since  $r_{(1,1,2,2,2)}(n) \geq r_{(1,1,2,2)}(n)$  holds trivially by taking 0 for the fifth variable, we have  $r_{(1,1,2,2,2)}(n) > 0$  for all  $n \in \mathbb{N}$ . Theorem 1.4 (2) now follows immediately from Lemma 6.2.  $\square$

7. PROOF OF THEOREM 1.5, COROLLARY 1.6, AND THEOREM 1.7

7.1. **Proof of Theorem 1.5.** We next bound and determine the sign of  $\sum_{d|n} \left(\frac{d}{p}\right) d^k$ .

**Lemma 7.1.** *For  $n = p^a m$  with  $p \nmid m$  and  $k \in \mathbb{N}$ , we have*

$$\left| \sum_{d|n} \left(\frac{d}{p}\right) d^k \right| \gg_{\varepsilon} m^{k-\varepsilon}, \quad \text{sgn} \left( \sum_{d|n} \left(\frac{d}{p}\right) d^k \right) = \left(\frac{m}{p}\right).$$

*Proof.* By multiplicativity, we have

$$\sum_{d|n} \left(\frac{d}{p}\right) d^k = \prod_{\ell|n} \sum_{j=0}^{\text{ord}_{\ell}(n)} \left(\frac{\ell}{p}\right)^j \ell^{kj}.$$

Since  $k \geq 1$ , for  $\ell \neq p$  and  $r \in \mathbb{N}$  we have

$$\left| \sum_{j=0}^r \left(\frac{\ell}{p}\right)^j \ell^{kj} \right| \geq \ell^{kr} - \ell^{k(r-1)}.$$

Thus, for any  $\varepsilon > 0$ , we have, using [23, (3.27)] in the final step,

$$\left| \sum_{d|n} \left(\frac{d}{p}\right) d^k \right| \geq m^k \prod_{\ell|m} (1 - \ell^{-k}) \gg_{\varepsilon} m^{k-\varepsilon}.$$

This proves the first claim. For the second claim, we compute

$$\sum_{j=0}^r \left(\frac{\ell}{p}\right)^j \ell^{kj} = \begin{cases} \frac{\ell^{(r+1)k}-1}{\ell^k-1} & \text{if } \left(\frac{\ell}{p}\right) = 1, \\ \frac{\ell^{(r+1)k}+1}{\ell^k+1} & \text{if } \left(\frac{\ell}{p}\right) = -1, 2 \mid r, \\ -\frac{\ell^{(r+1)k}-1}{\ell^k+1} & \text{if } \left(\frac{\ell}{p}\right) = -1, 2 \nmid r, \\ 1 & \text{if } \ell = p. \end{cases}$$

Since  $k \geq 1$ , this is positive unless  $\left(\frac{\ell}{p}\right) = -1$  and  $r$  is odd in which case it is negative. Thus

$$\text{sgn} \left( \sum_{d|n} \left(\frac{d}{p}\right) d^k \right) = (-1)^{\#\{\ell \text{ prime: } \left(\frac{\ell}{p}\right) = -1, 2 \nmid \text{ord}_{\ell}(m)\}}.$$

Now write  $n = p^a m$  with  $a \in \mathbb{N}_0$  and  $\gcd(p, m) = 1$ . The claim follows since

$$\left(\frac{m}{p}\right) = \prod_{\ell^r \parallel m} \left(\frac{\ell^r}{p}\right) = (-1)^{\#\{\ell \text{ prime: } \left(\frac{\ell}{p}\right) = -1, 2 \nmid \text{ord}_{\ell}(m)\}}. \quad \square$$

We are now ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* By [22, Theorem 1.64],  $Q_p \in M_{\frac{p^2-p}{2}}(\Gamma_0(p), \chi_{\left(\frac{-1}{p}\right)_p})$ , and  $P_p \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_{\left(\frac{-1}{p}\right)_p})$ , Lemma 2.4 yields that there exist  $f_p \in S_{pk_p}(\Gamma_0(p), \chi_{\left(\frac{-1}{p}\right)_p})$  and  $g_p \in S_{k_p}(\Gamma_0(p), \chi_{\left(\frac{-1}{p}\right)_p})$  such that

$$Q_p = L_{pk_p, p} E_{pk_p, 1, \chi_{\left(\frac{-1}{p}\right)_p}} + f_p, \quad P_p = L_{k_p, p} E_{k_p, 1, \chi_{\left(\frac{-1}{p}\right)_p}} + g_p. \quad (7.1)$$



We naturally split the Fourier expansions

$$Q_p(z) =: \sum_{n \geq 0} c_{Q_p}(n)q^n =: \sum_{n \geq 0} A_E(n)q^n + \sum_{n \geq 1} c_{f_p}(n)q^n, \quad (7.2)$$

$$P_p(z) =: \sum_{n \geq 0} c_{P_p}(n)q^n =: \sum_{n \geq 0} B_E(n)q^n + \sum_{n \geq 1} c_{g_p}(n)q^n \quad (7.3)$$

into the two pieces corresponding to the right-hand sides of (7.1).

(1) Employing Lemma 2.6 and noting that  $f_p$  is uniquely determined by  $p$ , we have, recalling that  $n = p^a m$  with  $p \nmid m$ ,

$$|c_{f_p}(n)| \ll_{p,\varepsilon} \|f_p\| n^{\frac{pk_p-1}{2}+\varepsilon} \ll_p n^{\frac{pk_p-1}{2}+\varepsilon} \ll_{a,p,\varepsilon} m^{\frac{pk_p-1}{2}+\varepsilon}.$$

Using the bound in Lemma 7.1, we conclude from (7.2) that for  $m$  sufficiently large we have

$$\operatorname{sgn}(c_{Q_p}(n)) = \operatorname{sgn}(A_E(n)). \quad (7.4)$$

By the evaluation of the sign in Lemma 7.1, we then obtain

$$\operatorname{sgn}(A_E(n)) = \operatorname{sgn}(L_{pk_p,p}) \left( \frac{m}{p} \right).$$

Finally, using Lemma 2.9, we conclude that

$$\operatorname{sgn}(A_E(n)) = \left( \frac{2}{p} \right) \left( \frac{m}{p} \right). \quad (7.5)$$

This finishes the proof of part (1).

(2) By Lemma 2.6, we have  $|c_{g_p}(n)| \ll_{a,p,\varepsilon} m^{\frac{k_p-1}{2}+\varepsilon}$ . As in part (1), Lemma 7.1 and (7.3) hence imply that, for  $m$  sufficiently large, we have

$$\operatorname{sgn}(c_{P_p}(n)) = \operatorname{sgn}(B_E(n)) = \operatorname{sgn}(L_{k_p,p}) \left( \frac{m}{p} \right). \quad (7.6)$$

Finally, Lemma 2.9 implies that

$$\operatorname{sgn}(B_E(n)) = \left( \frac{-2}{p} \right) \left( \frac{m}{p} \right), \quad (7.7)$$

finishing the proof.  $\square$

**7.2. Proof of Corollary 1.6.** Since (see [18])

$$S_3(\Gamma_0(3), \chi_{-3}) = \{0\}, \quad (7.8)$$

(7.1) and Lemma 2.8 yield the following identity for  $b_5(n) := C_{1^9 3^{-3}}(n)$ .

**Lemma 7.2.** *We have*

$$\frac{\eta(z)^9}{\eta(3z)^3} = 1 - 9 \sum_{n \geq 1} \sum_{d|n} \left( \frac{d}{3} \right) d^2 q^n.$$

*In particular, for  $n \in \mathbb{N}$  we have*

$$b_5(n) = -9 \sum_{d|n} \left( \frac{d}{3} \right) d^2.$$

We next prove Corollary 1.6 (1). Let  $s_5(n) := \operatorname{sgn}(b_5(n))$  and  $s_6(n) := \operatorname{sgn}(C_{1^5 5^{-1}}(n))$ .

*Proof of Corollary 1.6.* (1) Since  $f_3 \equiv 0$  in (7.1) by (7.8), (7.2) implies that (7.4) holds for  $n \in \mathbb{N}$ , and hence (7.5) gives

$$s_5(n) = \operatorname{sgn}(c_{Q_3}(n)) = \operatorname{sgn}(A_E(n)) = \left(\frac{2}{3}\right) \left(\frac{m}{3}\right) = -\left(\frac{m}{3}\right).$$

(2) Since  $S_2(\Gamma_0(5), \chi_5) = \{0\}$  (see [18]), we similarly conclude from (7.3) that (7.6) holds for  $n \in \mathbb{N}$  and thus (7.7) yields that  $s_6(n) = -\left(\frac{m}{5}\right)$ .  $\square$

**7.3. Proof of Theorem 1.7.** The first step to prove Theorem 1.7 is to write our functions in terms of class number generating function. We have the following.

**Lemma 7.3.** *We have*

$$\frac{\eta(8z)^2 \eta(16z)^2}{\eta(24z)} = (\mathcal{H}_{4,3} - \mathcal{H}_{1,3}) | S_{24,1}(z) - \frac{1}{2}(\mathcal{H}_{4,3} - \mathcal{H}_{1,3}) | S_{24,17}(z) - (\mathcal{H}_{4,3} + 2\mathcal{H}_{1,3}) | S_{24,9}(z).$$

*Proof.* By Lemma 2.10,  $\mathcal{H}_{1,3} \in M_{\frac{3}{2}}(\Gamma_0(12), \chi_{12})$  and  $\mathcal{H}_{4,3} \in M_{\frac{3}{2}}(\Gamma_0(24), \chi_{12})$ . We then use Lemma 2.1 (2) to conclude that the right-hand side of the lemma lies in  $M_{\frac{3}{2}}(\Gamma_0(576), \chi_{12})$ . Moreover,  $\eta(24z) \in M_{\frac{1}{2}}(\Gamma_0(576), \chi_{12})$  (see [22, Corollary 1.62]). Thus  $\eta(24z)$  times the right-hand side is in  $M_2(\Gamma_0(576))$ .

Next, by [22, Theorem 1.64],  $\eta(8z)^2 \eta(16z)^2 \in M_2(\Gamma_0(64))$ . Thus  $\eta(24z)$  times the difference of the left- and right-hand sides lies in  $M_2(\Gamma_0(576))$ . By Lemma 2.7, we have to check 192 Fourier coefficients. The claim was verified by checking the identity for the first 192 Fourier coefficients with a computer.  $\square$

Let  $b_7(n) := C_{12^2 2^3 - 1}(n)$  and  $s_7(n) := \operatorname{sgn}(b_7(n))$ . We are now ready to prove Theorem 1.7.

*Proof of Theorem 1.7.* Let

$$\frac{\eta(8z)^2 \eta(16z)^2}{\eta(24z)} =: \sum_{n \geq 0} C(n) q^n.$$

Then  $b_7(n) = C(8n + 1)$ . Thus

$$s_7(n) = \operatorname{sgn}(C(8n + 1)).$$

By (2.4), we have

$$\mathcal{H}_{\ell_1, \ell_2}(z) = \sum_{n \geq 0} \left( H(\ell_1 \ell_2 n) - \ell_2 H\left(\frac{\ell_1 n}{\ell_2}\right) \right) q^n.$$

Hence Lemma 7.3 implies that

$$b_7(3n) = H(12(24n + 1)) - H(3(24n - 1)).$$

We write  $3(24n + 1) = Df^2$  and  $12(24n + 1) = D(2f)^2$  with  $-D$  a fundamental discriminant and  $f \in \mathbb{N}$ . Then (2.2) implies that

$$H(3(24n + 1)) = H(D)S_D(f), \quad H(12(24n + 1)) = H(D)S_D(2f).$$

Since  $S_D(f)$  is multiplicative and  $f$  is odd, we have

$$S_D(2f) - S_D(f) = S_D(f)S_D(2) - S_D(f) = S_D(f) \left( \sigma(2) - \left(\frac{-D}{2}\right) - 1 \right) > 0.$$

So  $s_7(3n) = 1 = \left(\frac{8(3n)+1}{3}\right)$ , and we are done in this case. Similarly  $b_7(3n + 2) = C(24n + 17)$ . The proof goes exactly in the same way and gives  $s_7(3n + 2) = -1 = \left(\frac{8(3n+2)+1}{3}\right)$ . This finishes

the case  $a = 0$ , as  $3 \mid (8n + 1)$  if and only if  $n \equiv 1 \pmod{3}$ , where we write  $8n + 1 = 3^a m$  with  $3 \nmid m$  as in the theorem statement.

Next, Lemma 7.3 implies that

$$b_7(3n + 1) = C(24n + 9) = -H(36(8n + 3)) + 3H(4(8n + 3)) - 2H(9(8n + 3)) + 6H(8n + 3).$$

Write  $3^{a-1}m = 8n + 3 = Df^2$  with  $-D$  fundamental and  $f \in \mathbb{N}$ . Using (2.2) gives

$$C(24n + 9) = -H(D)(S_D(6f) - 3S_D(2f) + 2S_D(3f) - 6S_D(f)). \quad (7.9)$$

Note that  $2 \nmid f$ . Since  $D$  is a fundamental discriminant, we have  $9 \nmid D$ , so  $a \geq 3$  if and only if  $3 \mid f$ . Moreover,  $9 \mid (8n + 3)$  if and only if  $3 \mid f$ . Therefore

$$a \geq 3 \Leftrightarrow 3 \mid f \Leftrightarrow 8n + 3 \equiv 0 \pmod{9} \Leftrightarrow n \equiv 3 \pmod{9}.$$

We assume next that  $a \in \{1, 2\}$ , and hence  $n \not\equiv 3 \pmod{9}$ . Since  $S_D(f)$  is multiplicative, using the definition (2.3) and (2.2), we have, by (7.9),

$$\begin{aligned} C(24n + 9) &= -H(D)S_D(f)(S_D(6) - 3S_D(2) + 2S_D(3) - 6) \\ &= -H(8n + 3) \left( \sigma(6) - \left(\frac{-D}{3}\right) \sigma(2) - \left(\frac{-D}{2}\right) \sigma(3) + \left(\frac{-D}{6}\right) - 3\sigma(2) \right. \\ &\quad \left. + 3 \left(\frac{-D}{2}\right) + 2\sigma(3) - 2 \left(\frac{-D}{3}\right) - 6 \right). \end{aligned}$$

Note that  $\left(\frac{-D}{2}\right) = \left(\frac{-3}{2}\right) = -1$  since  $D \equiv 3 \pmod{8}$  and  $\left(\frac{-D}{3}\right) = \left(\frac{-8n}{3}\right) = \left(\frac{n}{3}\right)$ . Thus

$$C(24n + 9) = -6H(8n + 3) \left( 1 - \left(\frac{n}{3}\right) \right).$$

This finishes the claim for  $a \in \{1, 2\}$ .

Finally, suppose that  $a \geq 3$ , which is equivalent to  $n \equiv 3 \pmod{9}$ . We write  $f = 3^r g$  with  $3 \nmid g$ . By (7.9) and the multiplicativity of  $S_D(f)$  (splitting off the 3-powers), we have

$$C(24n + 9) = -H(D)S_D(g) (S_D(2 \cdot 3^{r+1}) - 3S_D(2 \cdot 3^r) + 2S_D(3^{r+1}) - 6S_D(3^r)).$$

We compute

$$S_D(3^\ell) = \sigma(3^\ell) - \left(\frac{-D}{3}\right) \sigma(3^{\ell-1}), \quad S_D(2) = \sigma(2) - \left(\frac{-D}{2}\right) = 3 - (-1) = 4.$$

Simplifying gives

$$-H(D)S_D(g) (6S_D(3^{r+1}) - 18S_D(3^r)) = -6H(D)S_D(g) \left( 1 - \left(\frac{-D}{3}\right) \right).$$

Noting that the right-hand side is independent of  $r$ , the result follows.  $\square$

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