A VARIANT OF THE LINNIK-SPRINDŽUK THEOREM FOR SIMPLE ZEROS OF DIRICHLET L-FUNCTIONS

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ABSTRACT. For a primitive Dirichlet character $\mathfrak{X}$, a new hypothesis $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is introduced, which asserts that (1) all *simple* zeros of $L(s,\mathfrak{X})$ in the critical strip are located on the critical line, and (2) these zeros satisfy some specific conditions on their vertical distribution. Hypothesis $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is likely to be true since it is a consequence of the *generalized Riemann hypothesis*.

Assuming only the *generalized Lindelöf hypothesis*, we show that if $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ holds for one primitive character $\mathfrak{X}$, then it holds for *every* such $\mathfrak{X}$. If this occurs, then for every character χ (primitive or not), all simple zeros of $L(s,\chi)$ in the critical strip are located on the critical line. In particular, Siegel zeros cannot exist in this situation.

CONTENTS

1. INTRODUCTION

An old result of Sprindžuk [**13**, **14**], which he obtained by developing ideas of Linnik [**11**], asserts that the *generalized Riemann hypothesis* (GRH) holds for all Dirichlet L-functions provided that the *Riemann hypothesis* (RH) is true and that certain conditions on the vertical distribution of the zeros of $\zeta(s)$ are met. Specifically, Sprindžuk showed under RH that every L-function $L(s,\chi)$ satisfies GRH provided that the asymptotic formula

$$\sum_{\gamma} |\gamma|^{i\gamma} \mathrm{e}^{-i\gamma-\pi|\gamma|/2}\Big(x+2\pi i\frac{h}{k}\Big)^{-1/2-i\gamma} = -\frac{\mu(k)}{x\sqrt{2\pi}\,\phi(k)} + O(x^{-1/2-\varepsilon})$$

holds as $x\to 0^+$ for any coprime integers h,k with $0<|h|\leqslant k/2$, where the sum runs over the imaginary parts γ of the nontrivial zeros of $\zeta(s)$. This is known

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as the *Linnik–Sprindžuk theorem*. Some similar results have been obtained by Fujii [**6**–**8**], Suzuki [**15**], Kaczorowski and Perelli [**10**], and the author [**1**–**3**].

In the present paper, we establish a variant of the Linnik–Sprindžuk theorem focused on the *simple* zeros of Dirichlet L-functions. Assuming the *generalized Lindelöf hypothesis*, we show that the horizontal and vertical distribution of the simple zeros of any single L-function $L(s,\chi)$ can strongly influence the horizontal and vertical distribution of the simple zeros of any other Dirichlet L-function. In particular, we state a new criterion for the *nonexistence of Siegel zeros*.

Our main results are formulated in §2 after the necessary notation has been introduced. To prove the main theorem, we study the interplay and consequences of the following hypotheses on L-functions attached to Dirichlet characters χ.

Hypothesis $\mathtt{RH}[\chi]$: *If $\rho=\beta+i\gamma$ is a zero of $L(s,\chi)$ with $\beta>0$, then $\beta=\frac12$.*

In particular, $\mathtt{RH}[\mathbb{1}]$ is the Riemann hypothesis, where $\mathbb{1}$ is the trivial character defined by $\mathbb{1}(n)\coloneqq 1$ for all n. Also, GRH is equivalent to the truth of $\mathtt{RH}[\chi]$ for all characters χ.

Hypothesis $\mathtt{RH}_{\mathtt{sim}}[\chi]$: *If $\rho=\beta+i\gamma$ is a* `simple` *zero of $L(s,\chi)$ such that $\beta>0$, then $\beta=\frac12$.*

Hypothesis $\mathtt{RH}_{\mathtt{sim}}[\star]$: *The hypothesis $\mathtt{RH}_{\mathtt{sim}}[\chi]$ holds for* `every` *character χ.*

These two hypotheses lie at the heart of our work. $\mathtt{RH}_{\mathtt{sim}}[\chi]$ is a weak form of $\mathtt{RH}[\chi]$ which asserts that the simple zeros of $L(s,\chi)$ in the critical strip all lie on the critical line; nothing is assumed about trivial zeros of $L(s,\chi)$ (which are all simple) or zeros of multiplicity two or more in the critical strip (that is, *non-simple zeros*).

Hypothesis $\mathtt{LH}[\chi]$: *The function $L(s,\chi)$ satisfies the Lindelöf bound*

$$L(\tfrac12+it,\chi)\ll_q \tau^{\varepsilon}\qquad(t\in\mathbb{R}),\tag{1.1}$$

where $q\geqslant 1$ is the modulus of χ, and $\tau=\tau(t)\coloneqq|t|+10$.

Hypothesis $\mathtt{LH}[\star]$: *The hypothesis $\mathtt{LH}[\chi]$ holds for* `every` *character χ.*

In particular, note that $\mathtt{LH}[\mathbb{1}]$ is the classical *Lindelöf hypothesis* for $\zeta(s)$. More generally, $\mathtt{LH}[\chi]$ is (a weak form of) the *generalized Lindelöf hypothesis* for $L(s,\chi)$.

Remark. *The most important hypothesis from the perspective of the present paper, namely $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$, can only be formulated after the necessary notation has been introduced; see §2.*

2. Notation and statement of results

Following Riemann, the letter s always denotes a complex variable, and we write $\sigma := \Re(s)$, and $t := \Im(s)$. As in (1.1), we put $\tau := |t| + 10$ for all $t \in \mathbb{R}$.

Any implied constants in the symbols $\ll$, O, etc., may depend (where obvious) on the small parameter $\varepsilon > 0$; any dependence on other parameters is indicated explicitly by the notation. For example, (1.1) asserts that, for any $\varepsilon > 0$, the bound $|L(\frac{1}{2} + it, \chi)| \leqslant C\tau^{\varepsilon}$ holds for all $t \in \mathbb{R}$ with some constant $C > 0$ that depends only on q and ε.

For an arbitrary character χ, we make extensive use of the function $D(s, \chi)$, which is defined in the half-plane $\{\sigma > 1\}$ by

$$D(s,\chi) := \frac{L'(s,\chi)^2}{L(s,\chi)} = \sum_{n\in\mathbb{N}} \frac{\ell(n)\chi(n)}{n^s}$$

and extended analytically to the complex plane. Here, ℓ is the arithmetical function given by

$$\ell(n) := (\Lambda * \log)(n) = \sum_{\substack{a,b\in\mathbb{N}\\ ab=n}} \Lambda(a)\log b.$$

Note that $0 \leqslant \ell(n) \leqslant (\log n)^2$. The function $D(s,\chi)$ is meromorphic with a simple pole of residue $L'(\rho,\chi)$ at every simple zero ρ of $L(s,\chi)$, and a pole of order three at $s = 1$ when χ is principal. On the other hand, $D(s,\chi)$ is analytic in a neighborhood of any non-simple zero ρ of $L(s,\chi)$.

For a *primitive* character $\mathfrak{X}$ mod $\mathfrak{q}$, we denote

$$\kappa := \begin{cases} 0 & \text{if } \mathfrak{X}(-1) = +1, \\ 1 & \text{if } \mathfrak{X}(-1) = -1, \end{cases} \qquad \tau(\mathfrak{X}) := \sum_{a \bmod \mathfrak{q}} \mathfrak{X}(a)\mathsf{e}(a/\mathfrak{q}), \qquad \epsilon := \frac{\tau(\mathfrak{X})}{i^{\kappa}\sqrt{\mathfrak{q}}}, \tag{2.1}$$

where $\mathsf{e}(u) := \mathrm{e}^{2\pi i u}$ for all $u \in \mathbb{R}$. The function defined by

$$\mathcal{M}_{\mathfrak{X}}(s) := \epsilon\, 2^s \pi^{s-1} \mathfrak{q}^{1/2-s}\Gamma(1-s)\sin\tfrac{\pi}{2}(s+\kappa)$$

is familiar and plays an important rôle in analytic number theory, appearing as it does in the asymmetric form of the functional equation:

$$L(s,\mathfrak{X}) = \mathcal{M}_{\mathfrak{X}}(s) L(1-s,\overline{\mathfrak{X}}).$$

Lemma 2.1. *Let $\mathcal{I}$ be a compact interval in $\mathbb{R}$. Uniformly for $c \in \mathcal{I}$ and $t \geqslant 1$, we have*

$$\mathcal{M}_{\mathfrak{X}}(1-c-it) = \tau(\mathfrak{X})\mathfrak{q}^{c-1}\mathrm{e}^{-\pi i/4}\exp\Big(it\log\Big(\frac{\mathfrak{q}t}{2\pi \mathrm{e}}\Big)\Big)\Big(\frac{t}{2\pi}\Big)^{c-1/2}\big\{1 + O_{\mathcal{I}}(t^{-1})\big\}.$$

Proof. See Banks [**3**, Lemma 2.1]. □

The next result is a variant of Conrey, Ghosh, and Gonek [**4**, Lemma 1]. The proof relies on Gonek [**9**, Lemma 2]; it is based on the stationary phase method.

LEMMA 2.2. *Uniformly for $v > 0$ and $c \in [\frac{1}{10}, 2]$, we have*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} v^{-s}\mathcal{M}_{\mathfrak{X}}(1-s)\,ds = \begin{cases} \tau(\mathfrak{X})\mathfrak{q}^{-1}\mathsf{e}(-v/\mathfrak{q}) + E(\mathfrak{q},T,v) & \text{if } \frac{\mathfrak{q}}{2\pi} < v \leqslant \frac{\mathfrak{q}T}{2\pi}, \\ E(\mathfrak{q},T,v) & \text{otherwise}, \end{cases}$$

where

$$E(\mathfrak{q},T,v) \ll \frac{\mathfrak{q}^{c-1/2}}{v^c}\left(T^{c-1/2} + \frac{T^{c+1/2}}{|T-2\pi v/\mathfrak{q}| + T^{1/2}} + 1\right).$$

Proof. See Banks [3, Lemma 2.3]. □

Recall the Laurent series expansion of $\zeta(s)$ at $s = 1$ (see, e.g., [5, Prop. 10.3.19]):

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\gamma_n(s-1)^n,$$

where $\{\gamma_n\}$ are the *Stieltjes constants* given by

$$\gamma_n := \lim_{x\to\infty}\left(\sum_{k\leqslant x}\frac{(\log k)^n}{k} - \frac{(\log x)^{n+1}}{n+1}\right) \qquad (n \geqslant 0).$$

In particular, γ_0 is the *Euler-Mascheroni constant.* For any q, let

$$g(s) := \prod_{p\,|\,q}(1-p^{-s}),$$

and denote

$$\begin{aligned} P_q(X) := {}& \tfrac{1}{2}g_q(1)X(\log X)^2 - (1+\gamma_0)g_q(1)X\log X \\ &+ \big((1+\gamma_0+\gamma_0^2+3\gamma_1)g_q(1) - \gamma_0 g_q'(1) - \tfrac{1}{2}g_q''(1)\big)X. \end{aligned}$$

This function is defined so that

$$\operatorname*{Res}_{s=1} D(s,\chi_{0,q})\frac{X^s}{s} = P_q(X) \qquad (X > 0), \tag{2.2}$$

where $\chi_{0,q}$ is the principal character mod q.

The following hypothesis on a primitive character $\mathfrak{X}$, despite its quite technical formulation, is crucial for understanding the relationship between simple zeros of different Dirichlet L-functions.

HYPOTHESIS $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$: *The primitive character $\mathfrak{X}$ mod $\mathfrak{q}$ satisfies $\mathtt{RH}_{\mathtt{sim}}[\mathfrak{X}]$, and for any rational number $\xi = h/k$ with $h,k > 0$ and $(h,k) = 1$, any $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, and $X \geqslant 10$, we have*

$$\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma>0}} \xi^{-\rho}L'(\rho,\mathfrak{X}\cdot\chi_{0,\mathfrak{q}k})\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\mathcal{B}\left(\frac{\gamma}{2\pi\xi X}\right) - C_{\mathfrak{X},\xi}\cdot F_{\mathfrak{q},k}(X) \ll_{\mathfrak{q},\xi,\mathcal{B}} X^{1/2+\varepsilon}, \tag{2.3}$$

where the sum runs over simple zeros of $L(s,\mathfrak{X})$ in the critical strip (counted with multiplicity),

$$C_{\mathfrak{X},\xi} := \begin{cases} \dfrac{\overline{\mathfrak{X}}(h)\mathfrak{X}(k)\mu(k)}{\phi(\mathfrak{q}k)} & \text{if } (h,\mathfrak{q}k) = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

and

$$F_{\mathfrak{q},k}(X) := \int_0^\infty \mathcal{B}(u/(\mathfrak{q}X))P'_{\mathfrak{q}k}(u)\,du. \tag{2.5}$$

REMARKS. *Hypothesis* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}]$ *is true for all primitive characters* $\mathfrak{X}$ *under the generalized Riemann hypothesis. It is important to observe that* $\mathtt{RH}^\dagger[\chi]$ *has been formulated entirely in terms of the simple zeros of a single L-function* $L(s,\mathfrak{X})$*; its validity depends only on the horizontal and vertical distribution of the simple zeros. If* $\mathfrak{X}$ *and* $\mathfrak{X}_\circ$ *are* different *primitive characters, then there is no reason, a priori, that the hypotheses* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}]$ *and* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}_\circ]$ *should be related to one another. Nevertheless, we show that these hypotheses are equivalent if one assumes* $\mathtt{LH}[\star]$.

Our main results are the following.

THEOREM 2.3. *Assume* $\mathtt{LH}[\star]$*. For each primitive character* $\mathfrak{X}$*, the hypotheses* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}]$ *and* $\mathtt{RH}_{\mathtt{sim}}[\star]$ *are equivalent. Thus, If* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}]$ *holds for some primitive character* $\mathfrak{X}$*, then it holds for every primitive character* $\mathfrak{X}$*.*

COROLLARY 2.4. *Assume* $\mathtt{LH}[\star]$*. If there is a primitive character* $\mathfrak{X}$ *such that* $\mathtt{RH}^\dagger_{\mathtt{sim}}[\mathfrak{X}]$ *is true, then Siegel zeros do not exist.*

Indeed, under the conditions of the corollary, Theorem 2.3 shows that $\mathtt{RH}_{\mathtt{sim}}[\star]$ is true. Thus, for every character χ, all simple zeros of $L(s,\chi)$ in the critical strip must lie on the critical line.

3. BOUNDING $D(s,\chi)$

For any real $\sigma_0 > 0$, let $\mathcal{R}_\bullet(\sigma_0)$ and $\mathcal{R}(\sigma_0)$ be the closed regions defined by

$$\mathcal{R}_\bullet(\sigma_0) := \{s : \sigma \geqslant \sigma_0,\ |s-1| \geqslant \tfrac{1}{25}\} \qquad \text{and} \qquad \mathcal{R}(\sigma_0) := \{s : \sigma \geqslant \sigma_0\};$$

see Figure 1 below. Let $\mathcal{V}_\bullet(\sigma_0)$ [resp. $\mathcal{V}(\sigma_0)$] be the vector space of consisting of all meromorphic functions F that satisfy, for every $\varepsilon > 0$, the bound

$$F(s) \underset{F,\sigma_0}{\ll} \tau^{\lambda(\sigma)+\varepsilon}, \qquad \lambda(\sigma) := \max\{0, \tfrac{1}{2} - \sigma\}, \tag{3.1}$$

uniformly at all points s in $\mathcal{R}_\bullet(\sigma_0)$ [resp. $\mathcal{R}(\sigma_0)$].

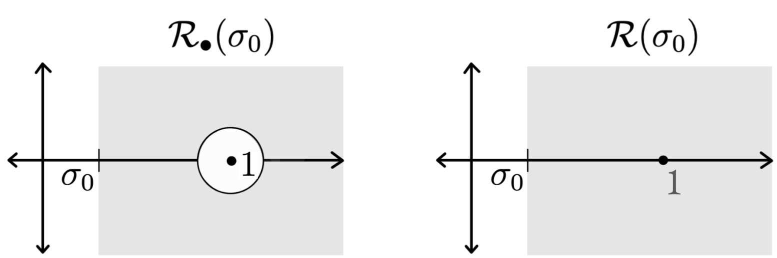

FIGURE 1.

In this section, we consider the problem of bounding the function $D(s,\chi)$ introduced in §2, where χ is an arbitrary character mod q. We denote by $\chi_{0,q}$

the principal character mod q, which is also the indicator function of integers coprime to q. We also denote

$$\widetilde{L}(s,\chi) := L(s,\chi) - \frac{\delta_\chi c_q}{s-1} \tag{3.2}$$

with

$$\delta_\chi := \begin{cases} 1 & \text{if } \chi \text{ is principal,} \\ 0 & \text{if } \chi \text{ is nonprincipal,} \end{cases} \qquad \text{and} \qquad c_q := \frac{\phi(q)}{q},$$

where ϕ is the Euler totient function. For any choice of χ, the function $\widetilde{L}(s,\chi)$ extends to an entire function. Since (3.2) implies the estimate

$$\widetilde{L}(s,\chi) = L(s,\chi) + O(1) \qquad (s \in \mathcal{R}_\bullet(\tfrac{1}{25})), \tag{3.3}$$

the hypothesis $\mathtt{LH}[\chi]$ admits the following equivalent formulation.

HYPOTHESIS $\mathtt{LH}[\chi]$: *The function $\widetilde{L}(s,\chi)$ satisfies the Lindelöf bound*

$$\widetilde{L}(\tfrac12 + it,\chi) \ll_q \tau^\varepsilon \qquad (t \in \mathbb{R}). \tag{3.4}$$

LEMMA 3.1. *The following statements are equivalent:*

(i) $\widetilde{L}(s,\chi)$ *belongs to* $\mathcal{V}(\frac{1}{25})$*;*

(ii) $\mathtt{LH}[\chi]$ *is true.*

REMARK. *The number $\frac{1}{25}$ can be replaced by any positive absolute constant.*

Proof. If $\widetilde{L}(s,\chi)$ belongs to $\mathcal{V}(\frac{1}{25})$, then applying (3.1) with $\sigma := \frac12$ and $\sigma_0 := \frac{1}{25}$, we obtain (3.4) at once. This shows that $(i) \Longrightarrow (ii)$.

Conversely, when $\mathtt{LH}[\chi]$ is true, (3.4) holds. Also, $\widetilde{L}(2+it,\chi) \ll 1$ holds unconditionally. Since $\widetilde{L}(s,\chi)$ is entire, the Phragmen-Lindelöf theorem gives

$$\widetilde{L}(s,\chi) \ll_q \tau^\varepsilon \qquad (\sigma \geqslant \tfrac12). \tag{3.5}$$

Next, suppose that $\sigma \in [\frac{1}{25}, \frac12]$. Replacing s by $1-s$ in (3.5), we have

$$L(1-s,\chi) \ll_q \tau^\varepsilon \qquad \text{and} \qquad L(1-s,\overline{\chi}) \ll_q \tau^\varepsilon. \tag{3.6}$$

Moreover, we have

$$L(s,\chi) \ll_q \tau^{1/2-\sigma} L(1-s,\overline{\chi}). \tag{3.7}$$

Indeed, for a primitive character $\mathfrak{X}$ mod $\mathfrak{q}$, the bound

$$L(s,\mathfrak{X}) \ll_{\mathfrak{q}} \tau^{1/2-\sigma} |L(1-s,\overline{\mathfrak{X}})|$$

follows from [**12**, Cors. 10.5 and 10.10]. More generally, if χ is induced from $\mathfrak{X}$, then (with s as above) we have

$$L(s,\chi) \asymp_q L(s,\mathfrak{X}) \qquad \text{and} \qquad L(1-s,\overline{\chi}) \asymp_q L(1-s,\overline{\mathfrak{X}}),$$

and (3.7) follows. Using (3.3), (3.6), and (3.7), we get that

$$\widetilde{L}(s,\chi) = L(s,\chi) + O(1) \ll_q \tau^{1/2-\sigma} \big|L(1-s,\overline{\chi})\big| + O(1) \ll_q \tau^{1/2-\sigma+\varepsilon} \quad (\sigma \in [\tfrac{1}{25}, \tfrac12]).$$

Combining this with (3.5), we see that $\widetilde{L}(s,\chi) \in \mathcal{V}(\frac{1}{25})$, and so $(ii) \Longrightarrow (i)$. □

LEMMA 3.2. *If* $\sigma_0 > 0$ *and* $F \in \mathcal{V}(\sigma_0)$, *then* $F' \in \mathcal{V}(\sigma_0 + \frac{1}{25})$.

Proof. Let $s \in \mathcal{R}(\sigma_0 + \frac{1}{25})$. For any $\varepsilon \in (0, \frac{2}{25})$, let $\mathcal{C}$ be the circle in the complex plane with center s and radius $\frac{\varepsilon}{2}$. Since $F \in \mathcal{V}(\sigma_0)$ and each $z \in \mathcal{C}$ satisfies $\Re(z) \geqslant \sigma - \frac{\varepsilon}{2} \geqslant \sigma_0$ (and thus $z \in \mathcal{R}(\sigma_0)$), we have

$$F(z) \ll_{F,\sigma_0} \tau^{\max\{0,1/2-\Re(z)\}+\varepsilon/2} \leqslant \tau^{\lambda(\sigma)+\varepsilon}.$$

By the Cauchy integral formula,

$$\big|F'(s)\big| = \left|\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{F(z)\,dz}{(z-s)^2}\right| \leqslant \frac{2}{\varepsilon}\max_{z\in\mathcal{C}}\big|F(z)\big| \ll_{F,\sigma_0} \tau^{\lambda(\sigma)+\varepsilon},$$

and the lemma follows. □

Combining Lemmas 3.1 and 3.2, the next result is immediate.

COROLLARY 3.3. *Under* $\mathtt{LH}[\chi]$, *we have*

$$\begin{aligned} L(s,\chi) &= \delta_\chi c_q (s-1)^{-1} + O_q(\tau^{\lambda(\sigma)+\varepsilon}) &\qquad& (s \in \mathcal{R}(\tfrac{1}{25})),\\ L'(s,\chi) &= -\delta_\chi c_q (s-1)^{-2} + O_q(\tau^{\lambda(\sigma)+\varepsilon}) && (s \in \mathcal{R}(\tfrac{2}{25})),\\ L''(s,\chi) &= 2\delta_\chi c_q (s-1)^{-3} + O_q(\tau^{\lambda(\sigma)+\varepsilon}) && (s \in \mathcal{R}(\tfrac{3}{25})). \end{aligned} \tag{3.8}$$

In particular, if $\rho = \beta + i\gamma$ *is a nontrivial zero of* $L(s,\chi)$, *then*

$$L'(\rho,\chi) \ll_q \tau^{\lambda(\beta)+\varepsilon} \qquad (\rho \in \mathcal{R}(\tfrac{2}{25})), \tag{3.9}$$

$$L''(\rho,\chi) \ll_q \tau^{\lambda(\beta)+\varepsilon} \qquad (\rho \in \mathcal{R}(\tfrac{3}{25})). \tag{3.10}$$

LEMMA 3.4. *Assume* $\mathtt{RH}_{\mathtt{sim}}[\chi]$ *and* $\mathtt{LH}[\chi]$. *For any nontrivial zero* ρ *of* $L(s,\chi)$, *the function* f_ρ *defined by*

$$f_\rho(s) := \begin{cases} \dfrac{L'(s,\chi) - L'(\rho,\chi)}{s-\rho} & \text{if } s \neq \rho,\\ L''(\rho,\chi) & \text{if } s = \rho, \end{cases}$$

satisfies the bound

$$f_\rho(s) \ll_q \tau^{\lambda(\sigma)+\varepsilon} + \delta_\chi c_q |s-1|^{-2}\tau^\varepsilon \tag{3.11}$$

uniformly for $s \in \mathcal{R}(\frac{4}{25})$.

Proof. We start with the fact that

$$L'(\rho,\chi) \ll_q \tau^{\varepsilon/2} \tag{3.12}$$

for any nontrivial zero ρ. Indeed, if ρ is non-simple, then we have $L'(\rho,\chi) = 0$. On the other hand, if $\rho = \beta + i\gamma$ is a simple (and nontrivial) zero, then $\beta = \frac{1}{2}$ under $\mathtt{RH}_{\mathtt{sim}}[\chi]$, whence (3.9) immediately implies (3.12).

Now, let $s \in \mathcal{R}(\frac{4}{25})$, $s \neq 1$. We consider three different cases.

First, suppose $s = \rho$. Then $f_\rho(s) = L''(\rho,\chi)$, and $\rho = s \in \mathcal{R}(\frac{4}{25})$, hence (3.11) follows directly from (3.10).

Next, suppose $0 < |s-\rho| \leqslant \frac{1}{50\log\tau}$, and write

$$f_\rho(s) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f_\rho(z)}{(z-s)}\,dz = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{L'(z,\chi)-L'(\rho,\chi)}{(z-\rho)(z-s)}\,dz$$

where $\mathcal{C}$ is the circle in the complex plane with center s and radius $\frac{1}{25\log\tau}$, oriented counterclockwise. It is straightforward to check that

$$\min\{|z-s|,|z-\rho|\} \geqslant \tfrac{1}{50\log\tau} \qquad (z\in\mathcal{C}),$$

and therefore

$$f_\rho(s) \ll (\log\tau)\Big\{\max_{z\in\mathcal{C}}\big|L'(z,\chi)\big| + \big|L'(\rho,\chi)\big|\Big\} \ll_q (\log\tau)\max_{z\in\mathcal{C}}\big|L'(z,\chi)\big| + \tau^\varepsilon,$$

where we used (3.12) in the second step. To prove (3.11) in this case, it is enough to show that

$$L'(z,\chi) \ll_q \tau^{\lambda(\sigma)+\varepsilon/2} \qquad (z\in\mathcal{C}). \tag{3.13}$$

Let $z = x+iy$ be a number in $\mathcal{C}$. Since $|x-\sigma| \leqslant |z-s| = \frac{1}{25\log\tau}$, we have $\tau^{\lambda(x)} \asymp \tau^{\lambda(\sigma)}$ and also $z\in\mathcal{R}(\frac15)$; hence (3.13) follows from (3.8).

Finally, suppose $|s-\rho| \geqslant \frac{1}{50\log\tau}$. In this case,

$$f_\rho(s) \ll (\log\tau)\Big\{\big|L'(s,\chi)\big| + \big|L'(\rho,\chi)\big|\Big\} \ll_q (\log\tau)\big|L'(s,\chi)\big| + \tau^\varepsilon.$$

Since $s\in\mathcal{R}(\frac{4}{25})$, the required bound (3.11) follows from (3.8). □

Lemma 3.5. *Under* $\mathtt{RH_{sim}}[\chi]$ *and* $\mathtt{LH}[\chi]$, *we have*

$$D(s,\chi) + \delta_\chi\frac{L'(s,\chi)}{s-1} \ll_q \tau^{\lambda(\sigma)+\varepsilon} + \delta_\chi|s-1|^{-2}\tau^\varepsilon + |\sigma-\tfrac12|^{-1}\tau^\varepsilon \tag{3.14}$$

uniformly for $s\in\mathcal{R}(\frac15)$.

Proof. Suppose χ is induced from the primitive character $\mathfrak{X}$. Unconditionally, we have (see, e.g., [12, Lems. 12.1 and 12.6]):

$$\frac{L'}{L}(s,\mathfrak{X}) = -\frac{\delta_{\mathfrak{X}}}{s-1} + \sum_{\substack{\rho\\ |\gamma-t|\leqslant 1}}\frac{1}{s-\rho} + O_{\mathfrak{q}}(\log\tau) \qquad (\sigma\in[-1,2]),$$

where the sum runs over nontrivial zeros ρ of $L(s,\mathfrak{X})$. Taking into account that

$$\frac{L'}{L}(s,\chi) = \frac{L'}{L}(s,\mathfrak{X}) + \sum_{p\,|\,q}\frac{\mathfrak{X}(p)\log p}{p^s-\mathfrak{X}(p)},$$

it follows that

$$\frac{L'}{L}(s,\chi) = -\frac{\delta_\chi}{s-1} + \sum_{\substack{\rho\\ |\gamma-t|\leqslant 1}}\frac{1}{s-\rho} + O_q(\log\tau) \qquad (\sigma\in[\tfrac15,2]).$$

Multiplying by $L'(s,\chi)$, we get that

$$\widetilde{D}(s,\chi) = \sum_{\substack{\rho\\ |\gamma-t|\leqslant 1}}\frac{L'(s,\chi)}{s-\rho} + O_q(|L'(s,\chi)|\log\tau) \qquad (\sigma\in[\tfrac15,2]), \tag{3.15}$$

where $\widetilde{D}(s,\chi)$ is the function defined on the left side of (3.14). The error term in (3.15) is acceptable in view of (3.8). To bound the sum in (3.15), observe that

$$\sum_{\substack{\rho \\ |\gamma-t|\leqslant 1}} \frac{L'(s,\chi)}{s-\rho} = \sum_{\substack{\rho\ \texttt{non-simple} \\ |\gamma-t|\leqslant 1}} f_\rho(s) + \sum_{\substack{\rho\ \texttt{simple} \\ |\gamma-t|\leqslant 1}} \frac{L'(\rho,\chi)}{s-\rho}.$$

As these sums all involve $\ll \log q\tau$ zeros, we obtain (3.14) by applying Lemma 3.4 together with (3.12), taking into account that $|s-\rho| \geqslant |\sigma - \frac{1}{2}|$ for any simple zero ρ (under $\texttt{RH}_{\texttt{sim}}[\chi]$). This completes the proof. □

The next result, used in the proof of Theorem 5.1 below, is conditional on $\texttt{LH}[\chi]$ but not on $\texttt{RH}_{\texttt{sim}}[\chi]$.

Lemma 3.6. *Assume* $\texttt{LH}[\chi]$. *For any* $t \geqslant 2$, *there is a real number* $t_* \in [t, t+1]$ *such that*

$$D(\sigma \pm it_*, \chi) \ll_q \tau^{\lambda(\sigma)+\varepsilon} \qquad (\sigma \in [\tfrac{2}{25}, 2]).$$

Proof. By [**12**, Lemmas 12.2 and 12.7], there is a number $t_* \in [t, t+1]$ such that

$$\frac{L'}{L}(\sigma \pm it_*, \chi) \ll (\log qt)^2 \qquad (\sigma \in [-1, 2]).$$

Multiplying by $L'(\sigma \pm it_*, \chi)$ and using (3.8), the result follows. □

4. Criteria for $\texttt{RH}_{\texttt{sim}}[\chi]$

Theorem 4.1. *Let* χ *be a character mod* q *that satisfies* $\texttt{LH}[\chi]$. *Then the following are equivalent:*

(*i*) $\texttt{RH}_{\texttt{sim}}[\chi]$ *is true;*

(*ii*) *For any* $X \geqslant 10$, *we have*

$$\sum_{n\leqslant X} \ell(n)\chi(n) = \delta_\chi P_q(X) + O_q(X^{1/2+\varepsilon}); \tag{4.1}$$

(*iii*) *For any* $\mathcal{B} \in C_c^\infty(\mathbb{R}^+)$ *and* $X \geqslant 10$, *we have*

$$\sum_n \ell(n)\chi(n)\mathcal{B}(n/X) = \delta_\chi \int_0^\infty \mathcal{B}(u/X)P_q'(u)\,du + O_{q,\mathcal{B}}(X^{1/2+\varepsilon}). \tag{4.2}$$

Proof. (*i*) $\Longrightarrow$ (*ii*). Let $T \coloneqq \sqrt{X}$, $\sigma_0 \coloneqq 1 + \frac{1}{\log X}$, and $\sigma_1 \coloneqq \frac{1}{2} + \frac{1}{\log X}$. Let $\mathcal{C}$ be the rectangular contour in $\mathbb{C}$ consisting of the four directed line segments:

$$\begin{aligned} \mathcal{C}_1: &\quad \sigma_0 - iT \longrightarrow \sigma_0 + iT, \\ \mathcal{C}_2: &\quad \sigma_0 + iT \longrightarrow \sigma_1 + iT, \\ \mathcal{C}_3: &\quad \sigma_1 + iT \longrightarrow \sigma_1 - iT, \\ \mathcal{C}_4: &\quad \sigma_1 - iT \longrightarrow \sigma_0 - iT. \end{aligned}$$

Using Perron's formula (see, e.g., [**12**, Thm. 5.2 and Cor. 5.3]), it follows that

$$\sum_{n\leqslant X} \ell(n)\chi(n) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} D(s,\chi)\,\frac{X^s}{s}\,ds + O(X^{1/2}(\log X)^3),$$

where the implied constant is absolute. Under $\mathtt{RH_{sim}}[\chi]$, the function $D(s,\chi)$ is analytic in the half-plane $\{\sigma > \frac{1}{2}\}$ unless $\chi = \mathbb{1}$, in which case there is a triple pole at $s = 1$. Using Cauchy's theorem and (2.2), we see that

$$\frac{1}{2\pi i}\left(\int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4}\right) D(s,\chi)\,\frac{X^s}{s}\,ds = \frac{1}{2\pi i}\oint_{\mathcal{C}} D(s,\chi)\,\frac{X^s}{s}\,ds = \delta_\chi P_q(X);$$

consequently,

$$\sum_{n\leqslant X} \ell(n)\chi(n) = \delta_\chi P_q(X) - \frac{1}{2\pi i}\left(\int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4}\right) D(s,\chi)\,\frac{X^s}{s}\,ds + O_\varepsilon(X^{1/2+\varepsilon}).$$

By Corollary 3.3 (bound (3.8)) and Lemma 3.5 (bound (3.14)), the bound

$$D(s,\chi) \ll_q \tau^{\varepsilon/2}\log X$$

holds uniformly for any s on the segments $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$; consequently,

$$\int_{\mathcal{C}_j} D(s,\chi)\,\frac{X^s}{s}\,ds \ll_q \frac{\log X}{T^{1-\varepsilon/2}}\int_{\sigma_1}^{\sigma_0} X^\sigma\,d\sigma \ll X^{1/2+\varepsilon} \qquad (j = 2 \text{ or } 4),$$

and

$$\int_{\mathcal{C}_3} D(s,\chi)\,\frac{X^s}{s}\,ds \ll_q X^{1/2}\log X\int_{-T}^{T}\frac{\tau^{\varepsilon/2}}{1+|t|}\,dt \ll X^{1/2+\varepsilon}.$$

Putting everything together, we obtain (4.1).

(ii) $\Rightarrow$ *(i)*. Let

$$S(u) := \sum_{n\leqslant u}\ell(n)\chi(n) = \delta_\chi P_q(u) + E(u), \qquad \text{where} \quad E(u) \ll_q u^{1/2+\varepsilon}. \tag{4.3}$$

In the region $\{\sigma > 1\}$ we have

$$D(s,\chi) = \sum_{n=1}^{\infty}\frac{\ell(n)\chi(n)}{n^s} = \delta_\chi\int_1^\infty u^{-s}\,P_q'(u)\,du + \int_{1^-}^\infty u^{-s}\,dE(u). \tag{4.4}$$

The first integral in (4.4) evaluates to

$$D_\infty(s,\chi) := \frac{g_q(1)}{(s-1)^3} - \frac{\gamma_0 g_q(1)}{(s-1)^2} + \frac{(\gamma_0^2 + 3\gamma_1)g_q(1) - \gamma_0 g_q'(1) - \frac{1}{2}g_q''(1)}{s-1}, \tag{4.5}$$

which is the singular part of $D(s,\chi)$ at $s = 1$ in the case that $\chi = \chi_{0,q}$. Integrating by parts, we see that the second integral in (4.4) is equal to

$$\int_{1^-}^\infty u^{-s}\,dE(u) = -P_q(1) + s\int_1^\infty u^{-s-1}E(u)\,du.$$

Since $E(u) \ll_q u^{1/2+\varepsilon}$, the right side continues analytically to the half-plane $\{\sigma > \frac{1}{2} + \varepsilon\}$; consequently, the function

$$D(s,\chi) - \delta_\chi D_\infty(s,\chi)$$

continues analytically to the same half-plane. In particular, $L(s,\chi)$ has no simple zeros in that region. Taking $\varepsilon \to 0^+$, this verifies $\mathtt{RH_{sim}}[\chi]$.

$(ii) \Rightarrow (iii)$. Using (4.1) and (4.3), we have by partial summation:

$$\sum_n \ell(n)\chi(n)\mathcal{B}(n/X) = \delta_\chi \int_0^\infty \mathcal{B}(u/X)P_q'(u)\,du + \int_0^\infty \mathcal{B}(u/X)\,dE(u).$$

Using integration by parts and the bound $E(u) \ll_q u^{1/2+\varepsilon}$, it is easily shown that the second integral is $\ll_{q,\mathcal{B}} X^{1/2+\varepsilon}$, hence we have (4.2).

$(iii) \Rightarrow (i)$. For each $\mathcal{B} \in C_c^\infty(\mathbb{R}^+)$, we define

$$\widehat{\mathcal{B}}(s) = \int_0^\infty \mathcal{B}(u)u^{s-1}\,du \qquad (s \in \mathbb{C}),$$

and we fix a number $d_\mathcal{B} < 1$ such that $\mathcal{B}(u) = 0$ if $u \leqslant d_\mathcal{B}$ or $u \geqslant d_\mathcal{B}^{-1}$. Let

$$f_\mathcal{B}(x) := \sum_n \ell(n)\chi(n)\mathcal{B}(n/x), \qquad g_\mathcal{B}(x) := \delta_\chi \int_{xd_\mathcal{B}}^{x/d_\mathcal{B}} \mathcal{B}(u/x)P_q'(u)\,du,$$

and observe that $f_\mathcal{B}(x) = 0$ for all $x \leqslant d_\mathcal{B}$. By (4.2), we have

$$f_\mathcal{B}(x) - g_\mathcal{B}(x) \underset{q,\mathcal{B}}{\ll} x^{1/2+\varepsilon} \qquad (x \geqslant 10). \tag{4.6}$$

In the half-plane $\{\sigma > 1\}$, it is straightforward to show that

$$\int_{d_\mathcal{B}}^\infty f_\mathcal{B}(x)x^{-s-1}\,dx = \widehat{\mathcal{B}}(s)\cdot D(s,\chi),$$
$$\int_{d_\mathcal{B}}^\infty g_\mathcal{B}(x)x^{-s-1}\,dx = \widehat{\mathcal{B}}(s)\cdot \delta_\chi \int_{d_\mathcal{B}}^\infty u^{-s}P_q'(u)\,du.$$

The latter integral can be split as $D_0(s,\chi) + D_\infty(s,\chi)$, where

$$D_0(s,\chi) := \int_{d_\mathcal{B}}^1 u^{-s}P_q'(u)\,du \qquad \text{and} \qquad D_\infty(s,\chi) := \int_1^\infty u^{-s}P_q'(u)\,du.$$

The first integral defines $D_0(s,\chi)$ as an entire function in the complex plane. For $\sigma > 1$, $D_\infty(s,\chi)$ is explicitly given by (4.5). We see that $D_\infty(s,\chi)$ analytically continues to the complex plane except for a possible pole at $s = 1$. As mentioned above, $D_\infty(s,\chi)$ is the singular part of $D(s,\chi)$ in $s = 1$ when $\chi = \chi_{0,q}$. As $\widehat{\mathcal{B}}(s)$ is entire, it follows that any pole of the function

$$I(s,\chi) := \int_{d_\mathcal{B}}^\infty \big\{f_\mathcal{B}(x) - g_\mathcal{B}(x)\big\}x^{-s-1}\,dx = \widehat{\mathcal{B}}(s)\cdot\big\{D(s,\chi) - D_0(s,\chi) - D_\infty(s,\chi)\big\}$$

with real part $\sigma > 0$ must occur at a *simple* zero ρ of $L(s,\chi)$. Moreover, if ρ is such a zero, then choosing $\mathcal{B}$ so that $\widehat{\mathcal{B}}(\rho) \neq 0$, we see that $I(s,\chi)$ does indeed have a pole at $s = \rho$.

On the other hand, in view of (4.6), the integral defining $I(s,\chi)$ converges absolutely in the half-plane $\{\sigma > \frac{1}{2} + \varepsilon\}$, so $I(s,\chi)$ cannot have a pole there for any choice of $\mathcal{B}$. Taking $\varepsilon \to 0^+$, we deduce that $L(s,\chi)$ has no simple zeros in $\{\sigma > \frac{1}{2}\}$, and $\mathtt{RH}_{\mathtt{sim}}[\chi]$ is verified. □

5. Twisted sums with $\ell(n)$

Theorem 5.1. *Let χ be a character mod q induced from a primitive character $\mathfrak{X}$ mod $\mathfrak{q}$ that satisfies $\mathtt{LH}[\mathfrak{X}]$. For any $\xi \in \mathbb{R}^+$ and $T > 0$,*

$$\sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12\\ 0<\gamma\leqslant T}} \xi^{-\rho} L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho) - \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}} \sum_{n\leqslant \mathfrak{q}T/(2\pi\xi)} \ell(n)\chi(n)\mathsf{e}(-n\xi) \ll_{q,\xi} T^{1/2+\varepsilon}+1.$$

Proof. The result is trivial for $T < 100$, so we assume $T \geqslant 100$ in what follows. For any $u > 0$, let

$$\Sigma_1(u) := \sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12\\ 0<\gamma\leqslant u}} \xi^{-\rho} L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho), \qquad \Sigma_2(u) := \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}} \sum_{n\leqslant \mathfrak{q}u/(2\pi\xi)} \ell(n)\chi(n)\mathsf{e}(-n\xi).$$

In this notation, the theorem (for $T \geqslant 100$) asserts that

$$\Sigma_1(T) - \Sigma_2(T) \ll_{q,\xi} T^{1/2+\varepsilon}. \tag{5.1}$$

In fact, to prove (5.1) for any $T \geqslant 100$, it suffices to prove that

$$\Sigma_1(T_*) - \Sigma_2(T_*) \ll_{q,\xi} T^{1/2+\varepsilon} \tag{5.2}$$

holds for *at least one* number $T_* \in [T, T+1]$. Indeed, by Lemma 2.1, we have $\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho) \ll (\mathfrak{q}\gamma)^{1/2}$ uniformly for all nontrivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ such that $\gamma \geqslant 10$ (say). Consequently,

$$\big|\Sigma_1(T_*) - \Sigma_1(T)\big| \leqslant \sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12\\ T<\gamma\leqslant T_*}} \big|\xi^{-\rho}\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\big| \ll_{q,\xi} T^{1/2}\log T$$

since there at most $O(\log qT)$ zeros with $T < \gamma \leqslant T_*$. Furthermore,

$$\big|\Sigma_2(T_*) - \Sigma_2(T)\big| \leqslant \frac{|\tau(\overline{\mathfrak{X}})|}{\mathfrak{q}} \sum_{\mathfrak{q}T/(2\pi\xi)<n\leqslant \mathfrak{q}T_*/(2\pi\xi)} \big|\ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q})\big| \ll_{\mathfrak{q},\xi} (\log T)^2$$

since $0 \leqslant \ell(n) \leqslant (\log n)^2$. These bounds make it clear that (5.1) and (5.2) are equivalent, and the claim is proved.

By the preceding argument, and recalling Lemma 3.6, for the proof of (5.1) we can assume without loss of generality that

$$D(\sigma \pm iT,\chi) \ll_q T^{\lambda(\sigma)+\varepsilon} \qquad (\sigma \in [\tfrac{2}{25}, 2]). \tag{5.3}$$

Moreover, by Lemma 3.6, there is a number $t_\circ \in [2,3]$ such that

$$D(\sigma \pm it_\circ,\chi) \ll_q 1 \qquad (\sigma \in [\tfrac{2}{25}, 2]).$$

For such $t_\circ$, it can be shown that

$$\Sigma_1(t_\circ) \ll_{q,\xi} 1 \qquad \text{and} \qquad \Sigma_2(t_\circ) \ll_{q,\xi} 1; \tag{5.4}$$

see, e.g., the proof of [**3**, Thm. 3.1]. We fix $t_\circ$ and T with these properties. Put $c := 1 + \frac{1}{\log qT}$, and let b be any number in the open interval $(\frac{1}{2} - \frac{1}{\log qT}, \frac{1}{2})$ such that $L(s,\chi) \neq 0$ for $\sigma \in [b, \frac{1}{2})$ and $t \in [t_\circ, T]$. Finally, let $\mathcal{C}$ be the rectangular contour consisting of the four directed line segments:

$$\begin{aligned}
\mathcal{C}_1 :&\quad c + it_\circ \longrightarrow c + iT,\\
\mathcal{C}_2 :&\quad c + iT \longrightarrow b + iT,\\
\mathcal{C}_3 :&\quad b + iT \longrightarrow b + it_\circ,\\
\mathcal{C}_4 :&\quad b + it_\circ \longrightarrow c + it_\circ.
\end{aligned}$$

Our choices of T, $t_\circ$, b, and c guarantee that $D(s,\chi)$ has no singularity on the contour $\mathcal{C}$. By Cauchy's theorem, we have

$$\begin{aligned}
\Sigma_1(T) - \Sigma_1(t_\circ) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} D(s,\chi)\, \xi^{-s} \mathcal{M}_{\overline{\mathfrak{X}}}(1-s)\, ds\\
&= \frac{1}{2\pi i}\bigg(\int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4}\bigg) D(s,\chi)\, \xi^{-s} \mathcal{M}_{\overline{\mathfrak{X}}}(1-s)\, ds\\
&= I_1 + I_2 + I_3 + I_4 \quad \text{(say)},
\end{aligned}$$

and thus by (5.4), we have

$$\Sigma_1(T) = I_1 + I_2 + I_3 + I_4 + O_q(1). \tag{5.5}$$

We estimate the four integrals I_j separately.

First, recalling the Dirichlet expansion

$$D(s,\chi) := \frac{L'(s,\chi)^2}{L(s,\chi)} = \sum_{n\in\mathbb{N}} \frac{\ell(n)\chi(n)}{n^s} \qquad (\sigma > 1),$$

it is immediate that

$$I_1 = \sum_{n\in\mathbb{N}} \ell(n)\chi(n) \cdot \frac{1}{2\pi i}\int_{c+it_\circ}^{c+iT} (n\xi)^{-s} \mathcal{M}_{\overline{\mathfrak{X}}}(1-s)\, ds.$$

Applying Lemma 2.2 with both T and $t_\circ$, we derive the estimate

$$\begin{aligned}
I_1 &= \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}} \sum_{\mathfrak{q}t_\circ/(2\pi\xi) < n \leqslant \mathfrak{q}T/(2\pi\xi)} \ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q}) + O\bigg(\sum_n \ell(n)\big|E(\mathfrak{q},T,n\xi)\big|\bigg)\\
&= \Sigma_2(T) - \Sigma_2(t_\circ) + O_{q,\xi}\big(T^{c-1/2}(E_1 + E_2)\big),
\end{aligned}$$

where

$$E_1 := \sum_n \frac{\ell(n)}{n^c} \qquad \text{and} \qquad E_2 := \sum_n \frac{\ell(n)}{n^c}\frac{T}{|T - 2\pi n\xi/\mathfrak{q}| + T^{1/2}}.$$

Clearly,

$$E_1 = \frac{\zeta'(c)^2}{\zeta(c)} \ll \frac{1}{(c-1)^3} = (\log T)^3. \tag{5.6}$$

Also, setting $T_\circ := T/(2\pi\xi)$, we have

$$E_2 \ll_\xi T^{3/2} \sum_n \frac{\ell(n)}{n^c}\frac{1}{|n - T_\circ| + T^{1/2}}. \tag{5.7}$$

Modifying slightly the proof of [3, Thm. 3.1], we find that the sum in (5.7) is $\ll T^{-1}(\log T)^3$. Therefore, using (5.6) and (5.7) along with (5.4), we see that

$$I_1 = \Sigma_2(T) + O_{q,\xi}(T^{1/2+\varepsilon}).$$

Next, by (5.3) and Lemma 2.1, we have

$$D(\sigma+it,\chi) \ll_q \tau^\varepsilon \qquad \text{and} \qquad \mathcal{M}_{\overline{\mathfrak{X}}}(1-\sigma-it) \ll_{\mathfrak{q}} \tau^{\sigma-1/2} \qquad (\sigma\in[b,c],\ t\geqslant 1),$$

from which we derive that

$$I_2 = \frac{1}{2\pi i}\int_{c+iT}^{b+iT} D(s,\chi)\,\xi^{-s}\mathcal{M}_{\overline{\mathfrak{X}}}(1-s)\,ds \ll_{q,\xi} T^{1/2+\varepsilon}.$$

Similarly, we have

$$I_3 \ll_{q,\xi} T^\varepsilon \qquad \text{and} \qquad I_4 \ll_{q,\xi} 1.$$

Combining (5.5) with the above estimates for the integrals I_j, we obtain (5.1), finishing the proof. □

Corollary 5.2. *Let χ be a character mod q induced from a primitive character $\mathfrak{X}$ mod $\mathfrak{q}$ satisfying* $\mathtt{LH}[\mathfrak{X}]$*. For any $\xi\in\mathbb{R}^+$, $\mathcal{B}\in C_c^\infty(\mathbb{R}^+)$, and $X\geqslant 10$,*

$$\sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12,\ \gamma>0}} \xi^{-\rho}L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\mathcal{B}\Big(\frac{\gamma}{2\pi\xi X}\Big) - \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}}\sum_n \ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q})\mathcal{B}(n/(\mathfrak{q}X)) \ll_{q,\xi,\mathcal{B}} X^{1/2+\varepsilon}.$$

Proof. As in the proof of Theorem 5.1, we define for $u>0$:

$$\Sigma_1(u) \coloneqq \sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12\\ 0<\gamma\leqslant u}} \xi^{-\rho}L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho), \qquad \Sigma_2(u) \coloneqq \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}}\sum_{n\leqslant \mathfrak{q}u/(2\pi\xi)} \ell(n)\chi(n)\mathsf{e}(-n\xi).$$

By Theorem 5.1 with $T\coloneqq 2\pi\xi Xu$, we have

$$\Sigma_1(2\pi\xi Xu) - \Sigma_2(2\pi\xi Xu) \ll_{q,\xi} (Xu)^{1/2+\varepsilon}+1 \qquad (u>0). \tag{5.8}$$

Next, we denote

$$\Sigma_3(u) \coloneqq \Sigma_2(2\pi\xi u/\mathfrak{q}) = \frac{\tau(\overline{\mathfrak{X}})}{q}\sum_{n\leqslant u}\ell(n)\chi(n)\mathsf{e}(-n\xi),$$

$$\Sigma_4(X) \coloneqq \sum_{\substack{\rho=\beta+i\gamma\\ \beta\geqslant\frac12,\ \gamma>0}} \xi^{-\rho}L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\mathcal{B}\Big(\frac{\gamma}{2\pi\xi X}\Big),$$

$$\Sigma_5(X) \coloneqq \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}}\sum_n \ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q})\mathcal{B}(n/(\mathfrak{q}X)).$$

Using Riemann-Stieltjes integration, we have

$$\Sigma_4(X) = \int_0^\infty \mathcal{B}\Big(\frac{u}{2\pi\xi X}\Big)\,d\Sigma_1(u) = \int_0^\infty \mathcal{B}(u)\,d\Sigma_1(2\pi\xi Xu)$$
$$= -\int_0^\infty \mathcal{B}'(u)\Sigma_1(2\pi\xi Xu)\,du,$$

and

$$\Sigma_5(X) = \int_0^\infty \mathcal{B}(u/(\mathfrak{q}X))\,d\Sigma_3(u) = \int_0^\infty \mathcal{B}(u)\,d\Sigma_3(\mathfrak{q}Xu)$$
$$= -\int_0^\infty \mathcal{B}'(u)\Sigma_3(\mathfrak{q}Xu)\,du = -\int_0^\infty \mathcal{B}'(u)\Sigma_2(2\pi\xi Xu)\,du.$$

Hence, using (5.8), we get that

$$\Sigma_4(X) - \Sigma_5(X) \ll_{q,\xi} \int_0^\infty \big|\mathcal{B}'(u)\big|\big((Xu)^{1/2+\varepsilon}+1\big)\,du \ll_{\mathcal{B}} X^{1/2+\varepsilon},$$

which finishes the proof. □

6. Computation under $\mathtt{RH}_{\mathtt{sim}}[\star]$ and $\mathtt{LH}[\star]$

Lemma 6.1. *Assume $\mathtt{RH}_{\mathtt{sim}}[\star]$ and $\mathtt{LH}[\star]$. Let $\mathfrak{X}$ be a primitive character mod $\mathfrak{q}$. Let $\xi = h/k$ be a rational number with $h, k > 0$ and $(h,k)=1$. Put $q := \mathfrak{q}k$, and let $\chi := \mathfrak{X}\cdot\chi_{0,q}$ be the character mod q induced from $\mathfrak{X}$. For all $\mathcal{B} \in C_c^\infty(\mathbb{R}^+)$ and $X \geqslant 10$, we have*

$$\frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}}\sum_n \ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q})\mathcal{B}(n/(\mathfrak{q}X)) = C_{\mathfrak{X},\xi}\cdot F_{\mathfrak{X},\xi}(X) + O_{q,\xi,\mathcal{B}}(X^{1/2+\varepsilon}), \quad (6.1)$$

where $q := \mathfrak{q}k$, χ is the character mod q induced from $\mathfrak{X}$, $C_{\mathfrak{X},\xi}$ is given by (2.4), and $F_{\mathfrak{X},\xi}(X)$ is given by (2.5).

Proof. The character χ is supported on integers coprime to q, hence the sum in (6.1) is equal to

$$\sum_{\substack{a \bmod q\\(a,q)=1}} \mathsf{e}(-ah/q)\mathfrak{X}(a) \sum_{n\equiv a \bmod q} \ell(n)\mathcal{B}(n/(\mathfrak{q}X))$$
$$= \frac{1}{\phi(q)}\sum_{\substack{a \bmod q\\(a,q)=1}} \mathsf{e}(-ah/q)\mathfrak{X}(a)\sum_{\chi' \bmod q}\overline{\chi'}(a)\sum_n \ell(n)\chi'(n)\mathcal{B}(n/(\mathfrak{q}X)),$$

where the middle sum runs over all characters χ' mod q. By Theorem 4.1 (*iii*) the total contribution from all nonprincipal characters χ' is $O_{\chi,\xi,\mathcal{B}}(X^{1/2+\varepsilon})$. For the principal character $\chi' = \chi_{0,q}$, the contribution is

$$\frac{c}{\phi(q)}\int_0^\infty \mathcal{B}(u/(\mathfrak{q}X))P_q'(u)\,du + O_{\chi,\xi,\mathcal{B}}(X^{1/2+\varepsilon}), \quad (6.2)$$

where we have used Theorem 4.1 (*iii*) again, and

$$c := \sum_{\substack{a \bmod q\\(a,q)=1}} \mathsf{e}(-ah/q)\mathfrak{X}(a);$$

note that the integral in (6.2) is $F_{\mathfrak{X},\xi}(X)$ by definition. By [**12**, Theorem 9.12], we have

$$c = \begin{cases} \overline{\mathfrak{X}}(-h)\mathfrak{X}(k)\mu(k)\tau(\mathfrak{X}) & \text{if } (h,\mathfrak{q}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the well known relation

$$\tau(\mathfrak{X})\tau(\overline{\mathfrak{X}}) = \mathfrak{X}(-1)\mathfrak{q} \tag{6.3}$$

for the Gauss sums defined in (2.1), and the fact that $(h,q) = (h,\mathfrak{q})$, we obtain the stated result. □

7. Proof of Theorem 2.3

Throughout the proof, $\mathtt{LH}[\star]$ is assumed to hold. Once and for all, let $\mathfrak{X}$ be a fixed primitive character mod $\mathfrak{q}$.

In one direction, suppose that $\mathtt{RH}_{\mathtt{sim}}[\star]$ is true. In particular, $\mathtt{RH}_{\mathtt{sim}}[\mathfrak{X}]$ holds, and the first condition of $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is verified. Let $\xi := h/k$ with $h,k > 0$ and $(h,k) = 1$, $\mathcal{B} \in C_c^{\infty}(\mathbb{R}^+)$, and $X \geqslant 10$. As $\mathtt{RH}_{\mathtt{sim}}[\mathfrak{X}]$ holds, Corollary 5.2 gives

$$\sum_{\substack{\rho=\frac12+i\gamma\\ \gamma>0}} \xi^{-\rho} L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\mathcal{B}\Big(\frac{\gamma}{2\pi\xi X}\Big) - \frac{\tau(\overline{\mathfrak{X}})}{\mathfrak{q}} \sum_n \ell(n)\chi(n)\mathsf{e}(-n\xi/\mathfrak{q})\mathcal{B}(n/(\mathfrak{q}X)) \ll_{\mathfrak{q},\xi,\mathcal{B}} X^{1/2+\varepsilon}.$$

Using estimate (6.1) of Lemma 6.1, we obtain (2.3). Since ξ is arbitrary, the second condition of $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is verified. Thus, $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is true, and the proof is complete in this direction.

In the other direction, suppose that $\mathtt{RH}^{\dagger}_{\mathtt{sim}}[\mathfrak{X}]$ is true. To prove the theorem, we show that $\mathtt{RH}_{\mathtt{sim}}[\mathfrak{X}_\circ]$ holds for an arbitrary primitive character $\mathfrak{X}_\circ$ mod k.

Observe that if $\chi_\circ$ is any character induced from $\mathfrak{X}_\circ$, then $\mathtt{RH}_{\mathtt{sim}}[\chi_\circ]$ and $\mathtt{RH}_{\mathtt{sim}}[\mathfrak{X}_\circ]$ are equivalent since $L(s,\chi_\circ)$ and $L(s,\mathfrak{X}_\circ)$ have the same zeros in the critical strip. Therefore, it suffices to show that $\mathtt{RH}_{\mathtt{sim}}[\chi_\circ]$ holds for *some* character $\chi_\circ$ induced from $\mathfrak{X}_\circ$. For this purpose, we define

$$\chi_\circ := \mathfrak{X}_\circ \cdot \chi_{0,\mathfrak{q}k}, \qquad \chi := \mathfrak{X} \cdot \chi_{0,\mathfrak{q}k}, \qquad \vartheta := \overline{\chi} \cdot \chi_\circ,$$

and turn our attention to the sum

$$\mathcal{W} := \sum_n \ell(n)\chi_\circ(n)\mathcal{B}(n/(\mathfrak{q}X)) = \sum_{(n,\mathfrak{q}k)=1} \ell(n)\vartheta(n)\chi(n)\mathcal{B}(n/(\mathfrak{q}X)). \tag{7.1}$$

If $(n,\mathfrak{q}k) = 1$, then (cf. [**12**, Theorem 9.5])

$$\vartheta(n)\tau(\overline{\vartheta}) = \sum_{h \bmod \mathfrak{q}k} \overline{\vartheta}(h)\mathsf{e}(hn/(\mathfrak{q}k)) = \sum_{\substack{1\leqslant h\leqslant \mathfrak{q}k\\ (h,\mathfrak{q}k)=1}} \overline{\vartheta}(-h)\mathsf{e}(-hn/(\mathfrak{q}k)),$$

and it follows that

$$\mathcal{W} = \frac{1}{\tau(\overline{\vartheta})} \sum_{\substack{1\leqslant h\leqslant \mathfrak{q}k\\ (h,\mathfrak{q}k)=1}} \overline{\vartheta}(-h) \sum_{(n,\mathfrak{q}k)=1} \ell(n)\mathsf{e}(-hn/(\mathfrak{q}k))\chi(n)\mathcal{B}(n/(\mathfrak{q}X)).$$

Applying Corollary 5.2 and then (2.3) to each inner sum, we have (with $\xi := h/k$)

$$\sum_{(n,\mathfrak{q}k)=1} \ell(n)\mathsf{e}(-hn/(\mathfrak{q}k))\chi(n)\mathcal{B}(n/(\mathfrak{q}X))$$
$$= \frac{\mathfrak{q}}{\tau(\overline{\mathfrak{X}})} \sum_{\substack{\rho=\beta+i\gamma \\ \beta\geqslant\frac{1}{2},\,\gamma>0}} \xi^{-\rho} L'(\rho,\chi)\mathcal{M}_{\overline{\mathfrak{X}}}(1-\rho)\mathcal{B}\Big(\frac{\gamma}{2\pi\xi X}\Big) + O_{\mathfrak{q},k,\mathcal{B}}(X^{1/2+\varepsilon})$$
$$= \frac{\mathfrak{q}}{\tau(\overline{\mathfrak{X}})} C_{\mathfrak{X},h/k} \cdot F_{\mathfrak{q},k}(X) + O_{\mathfrak{q},k,\mathcal{B}}(X^{1/2+\varepsilon});$$

thus,

$$\mathcal{W} = \frac{\mathfrak{q}}{\tau(\overline{\vartheta})\tau(\overline{\mathfrak{X}})} F_{\mathfrak{q},k}(X) \sum_{\substack{1\leqslant h\leqslant \mathfrak{q}k \\ (h,\mathfrak{q}k)=1}} \overline{\vartheta}(-h) C_{\mathfrak{X},h/k} + O_{\mathfrak{q},k,\mathcal{B}}(X^{1/2+\varepsilon}).$$

Finally, if $(h,\mathfrak{q}k)=1$, then (see (2.4))

$$\overline{\vartheta}(-h)\cdot C_{\mathfrak{X},h/k} = \mathfrak{X}(-h)\overline{\chi_\circ}(-h)\cdot \frac{\overline{\mathfrak{X}}(h)\mathfrak{X}(k)\mu(k)}{\phi(\mathfrak{q}k)} = \overline{\chi_\circ}(-h)\cdot\frac{\mathfrak{X}(-k)\mu(k)}{\phi(\mathfrak{q}k)},$$

and so

$$\mathcal{W} = \frac{\mathfrak{q}\,\mathfrak{X}(-k)\mu(k)}{\tau(\overline{\vartheta})\tau(\overline{\mathfrak{X}})\phi(\mathfrak{q}k)} F_{\mathfrak{q},k}(X) \sum_{\substack{1\leqslant h\leqslant \mathfrak{q}k \\ (h,\mathfrak{q}k)=1}} \overline{\chi_\circ}(-h) + O_{\mathfrak{q},k,\mathcal{B}}(X^{1/2+\varepsilon}). \tag{7.2}$$

We are now in a position to complete the proof. Using orthogonality, we evaluate the sum in (7.2) as follows:

$$\sum_{\substack{1\leqslant h\leqslant \mathfrak{q}k \\ (h,\mathfrak{q}k)=1}} \overline{\chi_\circ}(-h) = \begin{cases} \phi(\mathfrak{q}k) & \text{if } \mathfrak{X}_\circ = \mathbb{1}, \\ 0 & \text{if } \mathfrak{X}_\circ \neq \mathbb{1}. \end{cases} \tag{7.3}$$

In the case that $\chi_\circ$ is nonprincipal, we have $\mathfrak{X}_\circ \neq \mathbb{1}$. Hence, combining (7.1)–(7.3), we derive the bound

$$\sum_n \ell(n)\chi_\circ(n)\mathcal{B}(n/(\mathfrak{q}X)) \ll_{\mathfrak{q},k,\mathcal{B}} X^{1/2+\varepsilon}.$$

The dependence on $\mathfrak{q}$ can be ignored since the character $\mathfrak{X}$ mod $\mathfrak{q}$ is *fixed*. Taking into account that $\delta_{\chi_\circ}=0$, the estimate (4.2) of Theorem 4.1 is verified, and thus $\mathtt{RH}_{\mathtt{sim}}[\chi_\circ]$ is true. On the other hand, if $\chi_\circ$ is principal, then

$$\chi_\circ = \chi_{0,q}, \qquad \delta_{\chi_\circ}=1, \qquad \mathfrak{X}_\circ = \mathbb{1}, \qquad k=1, \qquad \chi := \mathfrak{X}, \qquad \vartheta := \overline{\mathfrak{X}}.$$

Combining (6.3) and (7.1)–(7.3), we get that

$$\sum_n \ell(n)\chi_\circ(n)\mathcal{B}(n/(\mathfrak{q}X)) = F_{\mathfrak{X},1}(X) + O_{\mathfrak{q},\mathcal{B}}(X^{1/2+\varepsilon})$$
$$= \delta_{\chi_\circ}\int_0^\infty \mathcal{B}(u/(\mathfrak{q}X))P_{\mathfrak{q}}'(u)\,du + O_{\mathfrak{q},\mathcal{B}}(X^{1/2+\varepsilon}).$$

Replacing $\mathfrak{q}X$ by X, we again verify estimate (4.2) of Theorem 4.1, hence $\mathtt{RH}_{\mathtt{sim}}[\chi_\circ]$ is true in this case as well.

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