IWASAWA THEORY OF GRAPHS AND THEIR DUALS

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ABSTRACT. In this article, we study questions pertaining to ramified \mathbb{Z}_p^d -extensions of a finite connected graph X. We also study the Iwasawa theory of dual graphs.

1. INTRODUCTION

Let $F = F_0$ be a number field and fix a prime p. Consider a tower of fields

$$
F = F_0 \subset F_1 \subset F_2 \subset F_3 \subset \cdots \subset \cdots F_{\infty}
$$

such that F_n/F is Galois with $Gal(F_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and $Gal(F_{\infty}/F) \simeq \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$. K. Iwasawa proved an asymptotic formula for the growth of the p -part of the class group in such a tower. More precisely, he showed that for $n \gg 0$,

$$
v_p(|\text{Cl}(K_n)|) = \mu p^n + \lambda n + \nu,
$$

where μ , λ are non-negative integers and ν is an integer, all of which are independent of n.

In the last couple of years this theory was generalized to unramified coverings of finite connected graphs; see for example [\[Gon22,](#page-22-0) [Val21,](#page-22-1) [DV23,](#page-22-2) [Kat23,](#page-22-3) [KM23,](#page-22-4) [MV23,](#page-22-5) [MV24,](#page-22-6) [DLRV24,](#page-22-7) [KM24\]](#page-22-8). More recently R. Gambherra–D. Vallierés $\boxed{GV24}$ considered *branched* \mathbb{Z}_p -coverings of finite connected graphs, i.e. coverings where at least one of the vertices is ramified. Let

$$
X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots
$$

be a tower of branched coverings such that $\mathbb{Z}/p^n\mathbb{Z}$ acts without inversion on X_n/X . Assume that all graphs X_n are connected and denote the number of spanning trees of X_n by $\kappa(X_n)$, then for all n large enough

$$
v_p(\kappa(X_n)) = \mu p^n + \lambda n + \nu.
$$

For a finite connected graph, the Kirchhoff Matrix Theorem asserts that the number of spanning trees coincides with the size of the Jacobian; see Section [2.2](#page-2-0) for a precise definition. If a group G acts on X_n/X it also acts naturally on the Jacobian, denoted by $Jac(X_n)$, and the Picard group, denoted by Pic (X_n) , turning both these groups into Iwasawa modules. Assume that X_{∞}/X is a branched covering obtained from a voltage assignment; see Section [2.4](#page-3-0) for details on how such a graph is constructed. Each such voltage assignment also induces an unramified tower X_∞^{unr}/X together with an immersion

$$
\iota\colon X_\infty^{\mathrm{unr}}\to X_\infty.
$$

In [\[GV24\]](#page-22-9) the authors describe the characteristic ideal of $\lim_{n \to \infty} \text{Pic}(X_n) \otimes \mathbb{Z}_p$ in terms of a linear operator defined on $\lim_{\Delta n \to \infty} \text{Div}(X_n) \otimes \mathbb{Z}_p$. In the special case when $X_\infty = X_\infty^{\text{unr}}$ is an unramified cover of X their result recovers the Iwasawa Main Conjecture (IMC) proved by the second named author and S. Kleine in [\[KM24\]](#page-22-8).

Our results. The first objective of the present article is to generalize the aforementioned theorem of branched coverings to \mathbb{Z}_p^d -coverings of X where $d \geq 1$; see Theorem [4.7.](#page-12-0)

Theorem. Let $X_\infty = X(\mathbb{Z}_p^d, \mathcal{I}, \alpha)$ denote a (ramified) derived graph of X. Then there exists an explicit linear operator Δ on $Pic_\Lambda(\hat{X}_{\infty}^{\text{unr}})$ such that

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = det(\Delta).
$$

Furthermore, we prove an asymptotic formula for the number of spanning trees of the intermediate graphs of \mathbb{Z}_p^d -towers in Theorem [4.16](#page-16-0) when $d \geq 2$.

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Theorem. Let X_∞ be as in the previous theorem and let X_n/X be the intermediate graphs. Then there exist non-negative constants μ and λ (independent of n) such that

$$
v_p(\kappa(X_n)) = \mu p^{nd} + \lambda n p^{n(d-1)} + O(p^{n(d-1)}).
$$

In the last section of this paper, we initiate the study of towers of dual (planar) graphs; see Theorem [5.3.](#page-17-0)

Theorem. Let X_{∞}/X be a branched G-tower, where G is a uniform pro-p p-adic Lie group. Assume that all X_n are planar. If X_n^{\vee}/X^{\vee} is a branched covering for all n, then $X_{\infty}^{\vee}/X^{\vee}$ is also a G-tower.

In Section [5.2](#page-20-0) we provide conditions to ensure that the hypothesis of the above theorem are satisfied.

Future Direction. This topic of Iwasawa theory of graphs, and in particular branched coverings of finite connected graphs leaves plenty of room for further investigations.

- Even though the operator Δ describing the characteristic ideal of Pic_Λ(X_{∞}) generalizes the linear operator appearing in the IMC for the unramified case, it is not clear whether this operator can be interpreted as a p -adic L -function in the branched setting as well. In other words, so far the analytic side of an IMC does not exist in the setting of branched covers.
- The asymptotic formula for the number of spanning trees in a branched \mathbb{Z}_p^d -covering involves an error term of size $O(p^{n(d-1)})$. This error term does not occur in the unramified setting: for an unramified \mathbb{Z}_p^d -covering there exists a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ of total degree d and degree at most 1 in Y such that

$$
v_p(\kappa(X_n)) = P(p^n, n).
$$

In a future project the authors intend to investigate whether it is possible to improve the error term.

• It would also be interesting to study a Kida-type formula for the branched \mathbb{Z}_p^d -covers.

Organization. Section [2](#page-1-0) is preliminary in nature. In this section we collect all the definitions and notations which will be in used throughout the article. In Section [3](#page-5-0) we prove our first result which provides sufficient conditions for the derived graph of a planar graph to be planar. Section [4](#page-10-0) concerns proving Iwasawa theory results for \mathbb{Z}_p^d towers of branched covers of X. We prove an asymptotic growth formula in the general setting. We also show that it is possible to interpret the characteristic ideal of $Pic_\Lambda(X_\infty)$ as an abstract determinant of an operator on $Div_{\Lambda}(X_{\infty}^{\text{unr}})$. In Section [5](#page-16-1) we introduce the study of towers of dual graphs. Throughout this article, we elucidate our results with explicit examples.

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2. Preliminaries

In this section we collect definitions and set notations that is required for the remainder of the article. We also record some elementary lemmas in this section.

2.1. Basic Definitions in Graph Theory. A graph X consists of a vertex set $V(X)$ and a set of directed edges $\mathbb{E}(X)$ along with an *incidence function*

inc:
$$
\mathbb{E}(X) \longrightarrow V(X) \times V(X)
$$

 $e \mapsto (o(e), t(e))$

and an inversion function

$$
inv: \mathbb{E}(X) \longrightarrow \mathbb{E}(X)
$$

$$
e \mapsto \overline{e}
$$

satisfying the following conditions for all $e \in \mathbb{E}(X)$

(i)
$$
\overline{e} \neq e
$$

\n(ii) $\overline{\overline{e}} = e$
\n(iii) $o(\overline{e}) = t(e)$ and $t(\overline{e}) = o(e)$.

$$
\mathbb{E}_v^o(X) = \{e \in \mathbb{E}(X) : o(e) = v\}
$$

$$
\mathbb{E}_v^t(X) = \{e \in \mathbb{E}(X) : t(e) = v\}.
$$

The inversion map induces the bijection

$$
\text{inv} \colon \mathbb{E}^o_v(X) \longrightarrow \mathbb{E}^t_v(X) \text{ for all } v \in V(X).
$$

The set of undirected edges is obtained by identifying e with \overline{e} and is denoted by $E(X)$. Finally, write $E_v(X)$ to denote the set of all undirected edges adjacent to v. When it is clear from the context which graph is being referred to, we often drop it from the notation of $E, E_v, \mathbb{E}_v^o, \mathbb{E}_v^t$, and V.

Definition 2.1. Let X and Y be (directed) graphs. A morphism $f: Y \to X$ of graphs consists of two functions $f_V: V(Y) \to V(X)$ and $f_{\mathbb{E}}: \mathbb{E}(Y) \to \mathbb{E}(X)$ which satisfy the following properties for all $e \in \mathbb{E}(Y)$

- (i) $f_V(o(e)) = o(f_{\mathbb{E}}(e)),$
- (*ii*) $f_V(t(e)) = t(f_{\mathbb{E}}(e)),$
- (*iii*) $\overline{f_{\mathbb{E}}(e)} = f_{\mathbb{E}}(\overline{e})$.

We write f to denote both f_V and f_E when there is no chance of confusion. Given a morphism of graphs $f: Y \to X$ and any vertex $v \in V(Y)$, the restriction $f|_{\mathbb{E}^o_v(Y)}$ induces a function

$$
f|_{\mathbb{E}^o_v(Y)} \colon \mathbb{E}^o_v(Y) \longrightarrow \mathbb{E}^o_{f(v)}(X).
$$

2.2. Jacobian of Graphs.

Definition 2.2. A divisor on a (possibly infinite) graph X is an element of the free abelian group on the vertices $V = V(X)$ defined as

$$
Div(X) = \left\{ \sum_{v \in V(X)} a_v v \mid a_v \in \mathbb{Z} \right\}
$$

where each $\sum_{v \in V(X)} a_v v$ is a formal linear combination of the vertices of X with integer coefficients and only finitely many a_v are non-zero (when X is an infinite graph).

The degree homomorphism is defined as

$$
\deg\colon \operatorname{Div}(X) \longrightarrow \mathbb{Z}
$$

$$
\sum_{v \in V(X)} a_v v \mapsto \sum_{v \in V(X)} a_v.
$$

The kernel of this degree map is the subgroup of divisors of degree 0, and is denoted as $Div^0(X)$.

Definition 2.3. A firing move based at a vertex v takes a divisor D to D' where for all $w \in V(X)$, the coefficient corresponding to vertex w is given by

$$
a_w = \begin{cases} a_v - \deg(v) & \text{if } w = v \\ a_v + 1 & \text{if } w \text{ is adjacent to } v \\ 0 & \text{if } w \text{ is not adjacent to } v. \end{cases}
$$

Two divisors $D, D' \in Div(X)$ are said to be linearly equivalent if D' may be obtained from D by a sequence of firing moves for various vertices $v \in V(X)$.

Definition 2.4. Let O denote the zero element of the additive group $\text{Div}(X)$. The divisors that are equivalent to O are called the principal divisors on X and is denoted by $Pr(X)$. The Picard group of X is defined as

$$
Pic(X) = Div(X)/Pr(X).
$$

The Jacobian group of X is defined as

$$
Jac(X) = Div^{0}(X)/Pr(X).
$$

2.3. Group action on graphs, Galois theory, and coverings of graphs. Write $Aut(X)$ to denote the group of automorphisms of a graph X.

Definition 2.5. Let G be a group and X be a graph. The group G is said to act on X if there exists a group morphism $G \to \text{Aut}(X)$. The group G is said to act without inversion if for all $e \in \mathbb{E}(X)$ and all $g \in G$, one has $q \cdot e \neq \overline{e}$.

When a group G acts on a graph X , it is understood that G acts on both the set of vertices and edges. If G acts on a graph X such that the action on $V(X)$ is free, then it acts freely on the set of edges as well.

Lemma 2.6. Let X be a locally finite graph, and let G be a group acting on X without inversion. Then, $Jac(X)$ is a $\mathbb{Z}[G]$ -module.

Proof. See [\[GV24,](#page-22-9) Corollary 2.6].

Definition 2.7. Let $f: Y \to X$ be morphism of directed graphs, that is surjective on the set of edges and on the set of vertices. The morphism f is a branched cover if for each vertex $v \in V(Y)$, the cardinality $m_v = (f|_{\mathbb{E}^o_v}^{-1}(e))$ does not depend on the choice of $e \in \mathbb{E}^o_{f(v)}(X)$. The cardinality m_v is called the ramification index at v. If $m_v = 1$, the morphism f is said to be an unramified cover.

The function $f|_{\mathbb{E}_{v}^o} : \mathbb{E}_{v}^o(Y) \to \mathbb{E}_{f(v)}^o(X)$ is $m_v\text{-}to\text{-}1$. The graph Y is called a d-sheeted covering of a connected graph X if for all $w \in V(X)$

$$
d = [Y : X] = \sum_{v \in f^{-1}(w)} m_v
$$

When Y is a d-sheeted unramified cover of a connected graph X, then for all $w \in V(X)$

$$
d = [Y : X] = \sum_{v \in f^{-1}(w)} 1 = \# \{ v \in V(Y) \mid f(v) = w \}.
$$

Definition 2.8. If Y/X is a d-sheeted unramified covering with projection map $\pi: Y \to X$ and if there are exactly d graph automorphisms $\sigma: Y \to Y$ such that $\pi \circ \sigma = \pi$, then it is normal or Galois. In this case, the Galois group is $G = \text{Gal}(Y/X) = \{\sigma: Y \to Y \mid \pi \circ \sigma = \pi\}.$

If Y/X is a branched covering and G is a group that acts freely without inversion on Y, trivially on X, and is compatible with the covering $\pi: Y \to X$, then Y/X is Galois with Galois group G.

Let Y/X be an abelian covering of connected graphs; i.e., we assume that its Galois group G is abelian. Then the Jacobian group $Jac(Y)$ is naturally equipped with a $\mathbb{Z}[G]$ -module structure.

2.4. Iwasawa theory of pro- p tower of graphs obtained from a voltage assignment. We describe the construction of uniform pro-p p-adic Lie extensions of connected graphs. Let $\mathcal G$ be a (possibly non-Abelian) uniform pro-p p-adic Lie group of dimension d. This means $\mathcal G$ is topologically generated by d elements and has a filtration

$$
\mathcal{G} \supseteq \mathcal{G}^p \supseteq \mathcal{G}^{p^2} \supseteq \cdots \supseteq \mathcal{G}^{p^n} \supseteq \cdots
$$

such that $\mathcal{G}^{p^{n+1}} \trianglelefteq \mathcal{G}^{p^n}$ and each subsequent quotient is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^d$.

Let X be a finite connected graph with vertex set $V(X)$ and set of edges $\mathbb{E}(X)$. Define a voltage assignment function

$$
\alpha \colon \mathbb{E}(X) \longrightarrow \mathcal{G}
$$
 satisfying $\alpha(\overline{e}) = \alpha(e)^{-1}$.

For each vertex $v \in V(X)$, choose a closed subgroup I_v of $\mathcal G$, and define the set

$$
\mathcal{I} = \{(v, I_v) : v \in V(X)\}.
$$

This gives a graph $X_{\infty} = X(\mathcal{G}, \mathcal{I}, \alpha)$ with vertex set equal to the disjoint union

$$
V(X_{\infty}) = \bigsqcup_{v \in V(X)} \{v\} \times \mathcal{G}/I_v
$$

and the collection of directed edges is given by

$$
\mathbb{E}(X_{\infty}) = \mathbb{E}(X) \times \mathcal{G}.
$$

Let $g \in \mathcal{G}$. The directed edge (e, g) connects the vertex $(o(e), gl_{o(e)})$ to the vertex $(t(e), g\alpha(e)I_{t(e)})$. Finally, let $\overline{(e,g)} = (\overline{e}, g\alpha(e))$. This is a branched cover $X_{\infty} \to X$. Moreover, the graph X_{∞} is infinite.

In order to get finite graphs, for each positive integer n , consider the natural surjective group morphism $\pi_n: \mathcal{G} \to \mathcal{G}_n$, where $\mathcal{G}_n = \mathcal{G}/\mathcal{G}^{p^n}$. Set $\alpha_n = \pi_n \circ \alpha: \mathbb{E}(X) \to \Gamma_n$ and define subgroups

$$
\mathcal{I}_n = \{(v, \pi_n(I_v)) \ : \ v \in V(X)\}
$$

of \mathcal{G}_n indexed by the vertices of X. This gives a family of graphs $X_n = X(\mathcal{G}_n, I_n, \alpha_n)$ which are finite. The natural surjective group morphisms $\mathcal{G}_{n+1} \to \mathcal{G}_n$ induce branched covers $X_{n+1} \to X_n$ for each non-negative integer n . Therefore, we obtain a (Galois) tower of graphs

$$
X = X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots \longleftarrow X_{\infty},
$$

where each map $X_{n+1} \to X_n$ is a branched cover satisfying $[X_{n+1} : X_n] = p^d$. Such a tower will be referred to as a branched G-tower of finite graphs provided that all X_n are connected.

When the closed subgroups I_v are all trivial, the graph $X_\infty = X(\mathcal{G}, \alpha)$ is the unramified $\mathcal{G}\text{-}tower$.

Lemma 2.9. Let $(X_n)_{n>0}$ be a G-tower of graphs constructed as above and assume that all X_n are connected. Suppose that there exists an index n_0 such that the number of ramified vertices in X_{∞}/X_{n_0} stabilizes. Let v_r be the number of ramified vertices and let v_u be the number of unramified vertices of $X_\infty/X_{n_0}.$ Then the number of vertices of X_n for $n \geq n_0$ is given by

$$
v_r + \frac{|\mathcal{G}_n|}{|\mathcal{G}_{n_0}|} v_u.
$$

Proof. When $n \geq n_0$, the vertices of X_n ramified in X_∞/X_n and the ones ramified in X_∞/X_{n_0} are in one-to-one correspondence by the definition of n_0 . The pre-images of the unramified vertices are given by $\{(v,g) \mid g \in \mathcal{G}^{p^{n_0}} / \mathcal{G}^{p^n}\}.$ The claim follows.

2.5. Matrices associated to graphs. We now remind the reader of some matrices associated to graphs that will be important throughout the discussion.

Definition 2.10. Let X be a finite connected graph.

(i) The adjacency matrix $A = (\alpha_{i,j})$ is defined as one where

$$
\alpha_{i,j} = \begin{cases} \text{twice the number of undirected loops at } v_i & \text{when } i = j \\ \text{number of undirected edges connecting the } v_i \text{ to } v_j & \text{when } i \neq j. \end{cases}
$$

(ii) The valency matrix $D = (d_{i,j})$ is a diagonal matrix where

$$
d_{i,j} = \begin{cases} \left| \mathbb{E}^o_{v_i}(X) \right| & when i = j \text{ and } v_i \text{ is an unramified vertex} \\ 0 & when i \neq j \text{ or } v_i \text{ is a ramified vertex.} \end{cases}
$$

- (iii) The matrix $D A$ is called the Laplacian matrix.
- (iv) Consider a covering Y/X. Suppose that there are s many vertices in X arranged such that v_1, \ldots, v_r are unramified vertices and v_{r+1}, \ldots, v_s are the ramified vertices. Further suppose that the ramified vertices are totally ramified^{[1](#page-4-0)}. The matrix $B = (b_{i,j}) \in M_{s \times s}(T)$ is defined as follows

$$
b_{i,j} = \begin{cases}\n-T & \text{when } r+1 \leq i = j \leq s \\
\sum_{\substack{e \in \mathbb{E}(X) \\ \text{inc}(e) = (v_j, v_i)}} (1+T)^{\alpha(e)} & \text{when } 1 \leq i \leq s \text{ and } 1 \leq j \leq r \\
0 & \text{otherwise.} \n\end{cases}
$$

2.6. Growth patterns in \mathbb{Z}_p^d -towers of graphs. A strong analogy between number theory and graph theory has been observed by mathematicians. This suggests that it might be possible to study the variation of the p-part of the number of spanning trees as one goes up a \mathbb{Z}_p^d -tower of graphs. In fact, there is a perfect analogue of Iwasawa's asymptotic class number formula in the setting of graph theory [\[Val21,](#page-22-1) [Gon22\]](#page-22-0). For example, when $d = 1$ there exist non-negative integers μ, λ, n_0 , and $\nu \in \mathbb{Z}$ such that for $n \geq n_0$

$$
\mathrm{ord}_p(\kappa(X_n)) = \mu p^n + \lambda n + \nu
$$

where $\kappa(X_n)$ is the number of spanning trees of X_n .

¹For a more general version of this matrix, see $\sqrt{GV24}$, Section 6

6 D. KUNDU AND K. MÜLLER

In the following paragraphs we summarize some relevant notions from [\[KM24\]](#page-22-8). Henceforth, let $\mathcal{G} = \mathbb{Z}_p^d$ for some $d \geq 1$. As discussed previously, to every finite connected graph X we can attach a finite abelian group called the Jacobian of X. Moreover, any cover of finite connected graphs $f: Y \to X$ induces a surjective group morphism of their respective Jacobian, i.e., there is a surjective map

$$
f_*\colon \operatorname{Jac}(Y) \longrightarrow \operatorname{Jac}(X).
$$

Starting with a \mathbb{Z}_p^d -tower of a finite connected graph X, we obtain a compatible system of maps between the Sylow-p subgroups of the Jacobians, i.e.,

$$
Jac(X_{n+1})[p^{\infty}] \longrightarrow Jac(X_n)[p^{\infty}].
$$

Recall that $\text{Jac}(X_n)[p^{\infty}]$ is a $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^d]$ -module. Since X is assumed to be finite and connected,

$$
Jac(X)[p^{\infty}] \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}} Jac(X);
$$

the same holds for each subsequent layer X_n . We then define the inverse limit

$$
Jac_{\Lambda}(X_{\infty}) := \varprojlim_{n} Jac(X_{n})[p^{\infty}] = \varprojlim_{n} (\mathbb{Z}_{p} \otimes_{\mathbb{Z}} Jac(X_{n}));
$$

this is a finitely generated torsion module over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathcal{G}]\!]$.

It is well-known [\[Gre73\]](#page-22-10) that there is a non-canonical isomorphism

$$
\mathbb{Z}_p[\hspace{-0.04cm}[\mathcal{G}]\hspace{-0.04cm}]\simeq \mathbb{Z}_p[\hspace{-0.04cm}[T_1,\ldots,T_d]\hspace{-0.04cm}]
$$

$$
\gamma_i\mapsto 1+T_i,
$$

where $\gamma_1, \ldots, \gamma_d$ are topological generators of G. Viewing Jac_Λ(X_∞) as a $\mathbb{Z}_p[[T_1, \ldots, T_d]]$ -module, and using the structure theorem we can define the *characteristic ideal* char_Λ(Jac_Λ(X_{∞})). In other words,

$$
char_{\Lambda}(Jac_{\Lambda}(X_{\infty})) = \langle f_X(T_1, \ldots, T_d) \rangle,
$$

where $f_X(T_1, \ldots, T_d)$ is an element of Λ and is a generator of the characteristic ideal; it is called the *characteristic element* of $Jac_{\Lambda}(X_{\infty}).$

Notation: We write $\text{Div}_{\Lambda}(X_{\infty})$ to denote $\text{Div}(X_{\infty}) \otimes \Lambda$, and similarly for $\text{Div}_{\Lambda}^{0}(X_{\infty})$, $\text{Pr}_{\Lambda}(X_{\infty})$.

3. Planar Derived Graphs

We start by recalling the construction of an unramified derived graph, which is denoted by $Y = X(G, \alpha)$.

Construction: Let X be a finite and connected graph. Suppose that the voltage group G is finite of size N . Label the elements in G as $\tau_0, \cdots, \tau_{N-1}$.

To get the vertices of Y, consider N copies of each vertex $x \in V(X)$ and label them as $x_{\tau_0}, \dots, x_{\tau_{N-1}}$. Therefore, the derived graph Y has $N|X|$ many vertices. Suppose that $x \to x'$ is a directed edge in X with voltage assignment $\alpha(e_{x,x'}) = \tau \in G$. For every τ_i in G, there is an edge in Y joining x_{τ_i} to $x'_{\tau_i \cdot \tau}$. This process is then repeated for every $x \in V(X)$.

When G is infinite (as will be in our case) we can repeat the same construction formally.

Intuition: A voltage graph takes $|G|$ -many copies of the base graph X, stacked vertically above each other. Then for each directed edge $x \to x'$ with voltage τ , we do the following: for each sheet indexed by $\sigma \in G$, we join the vertex x_{σ} to the vertex x'_{σ} . So if we "look at the layer of sheets from above" we see vertex x joined to vertex x' exactly $|G|$ times.

3.1. Main result and examples. In this section, we provide a sufficient condition for the derived graph of a planar graph to be planar. It is easy to construct examples of planer graphs and voltage assignments, such that the derived graph is not planar. We provide an example below

Example 3.1. Let X be a bouquet graph with two loops, i.e. a graph consisting of a single vertex with two loops e_1, e_2 . Let $G = (\mathbb{Z}/5\mathbb{Z}, +)$ and define $\alpha(e_1) = 1$, $\alpha(e_2) = 2$. Let $Y = X(G, \alpha)$. Then Y is the complete graph on 5 vertices which is not planar.

Definition 3.2. Let X be a finite undirected graph and let $W(X)$ be the \mathbb{F}_2 -vector space on the edges, i.e., each edge e defines a basis element v_e of $W(X)$.

Let c be a cycle in X. Associated to c, there is a vector $w_c = \sum_{e \in c} v_e$. Define $C(X) \subset W(X)$ to be the subspace generated by $\{w_c \mid c \text{ cycle in X}\}.$

The vector (sub)space $C(X)$ is said to have a two basis if there is a basis $\{b_1, \ldots, b_n\}$ with the property that for each e there are at most two b_i 's that have a non-zero entry at the position v_e .

Theorem 3.3. A finite undirected graph is planar if and only if there exists a two basis of $C(X)$.

Proof. This is $\left[\text{Mac37}, \text{Theorem 1}\right]$.

Theorem 3.4. Fix a finite group G. Let X be a finite undirected planar graph and Y be the undirected

Proof. Let e_0 be the edge with the non-trivial voltage assignment α . Let v and w be the vertices joined by e_0 . It suffices to show that each connected component of Y is planar. We can therefore without loss of generality assume that G is generated by $\alpha(e_0)$ where α is the voltage assignment.

For each element $g \in G$, define

$$
E^g = \{(e_0, g\alpha^{-1}(e_0)\} \cup \bigcup_{e \in E(X) \setminus e_0} (e, g)
$$

and let W^g be the free \mathbb{F}_2 -vector space on the elements in E^g . We obtain a decomposition

derived graph of X of a single^{[2](#page-6-0)} unramified voltage assignment. Then Y is planar.

$$
W(Y) = \bigoplus_{g \in G} W^g.
$$

Consider a cycle c in Y. Let c^g be the projection of w_c to W^g . Let $\pi: Y \to X$ be the natural projection. Then $\pi(c^g)$ is a cycle in X.

A cycle in X that contains e_0 is referred to as a special cycle. Similarly, a special cycle of Y is one that contains edges of more than one sheet^{[3](#page-6-1)}. Observe that there is a one-to-one correspondence

{special cycles of Y } \longleftrightarrow {tuples of special cycles $(c_1, \ldots, c_{|G|}) \in C(X)^{|G|}$ }.

Fix a basis B for $C(X)$. For clarity of exposition, we often represent the basis elements (cycles) $b \in B$ in terms of the their edges (or vertices).

Claim: It suffices to assume that B only contains one basis element that has a non-zero entry at e_0 .

Justification: Suppose that there are two cycles $b_0 = e_0, e_1, \ldots, e_s$ and $b'_0 = e_0, e'_1, \ldots, e'_{s'}$ in B, we can substitute b'_0 by $b''_0 = \overline{e'_{s'}}, \ldots, \overline{e'_{1}}, e_1, \ldots, e_s$.

Denote this basis element in B as b_0 . Let c_0 in Y be the cycle obtained by composing all the lifts of b_0 . Define $\vec{c} = (c_1, \ldots, c_{|G|})$, this is a tuple of special cycles in $C(X)^{|G|}$. It will be convenient to index the components of \vec{c} by $g \in G$. Each special cycle c_g can uniquely be written as $c_g = b_0 + b_g$, where b_g is a linear combination of non-special cycles in the basis B.

It follows that \vec{c} corresponds to $c_0 + \sum_{g \in G} b_g$, where b_g denotes the unique lift of b_g in W^g . Let $C'_g \subset W^g$ be the subspace generated by non-special cycles. It follows that

$$
C(Y) = \mathbb{F}_2 c_0 + \bigoplus_{g \in G} C'_g.
$$

²By this we mean that exactly one of the edges has a non-trivial voltage assignment.

 3 In the case of a single unramified voltage assignment, a special cycle has to run through all sheets.

As X is assumed to be planar, C_g' is either empty or has a two basis. In addition, none of its basis vectors have a non-zero entry at (e_0, g) .

Therefore, $C(Y)$ has a two basis and is planar by Theorem [3.3.](#page-6-2)

We explain the above theorem via two explicit examples. The first example is rather simple but might be helpful for the reader to understand the terms defined in the proof of the above theorem. The nuances of the proof become more clear with the second example.

Example 3.5. Let the voltage group $G = \{1, \tau\}$. Consider the finite planar graph X where exactly one edge (namely AB) has a non-trivial voltage assignment $\alpha(e_{AB}) = \tau$.

Here $e_0 = e_{AB}$. By definition, the basis elements of $W(X)$ are v_{AB}, v_{BC} , and v_{CA} . There is a unique cycle c_X in X and this is a special cycle of X. Associated to c_X , there is a vector

$$
w_{c_X} = v_{AB} + v_{BC} + v_{CA}.
$$

Now, in our example $C(X) \subseteq W(X)$ is the subspace generated by $B = \{w_{c,x}\}\$. In the notation, introduced in the proof $b_0 = w_{cX} = v_{AB} + v_{BC} + v_{CA}$. It is trivially true in this example that $C(X)$ has a two-basis.

For the first layer of the tower of derived graphs, the voltage group $G = \{1, \tau\}$ we can write

$$
E^{1} = (e_{BC}, 1) \cup (e_{CA}, 1) \cup (e_{AB}, \tau^{-1})
$$

$$
E^{g} = (e_{BC}, \tau) \cup (e_{CA}, \tau) \cup (e_{AB}, 1)
$$

Let W^1 and W^{τ} be the free \mathbb{F}_2 -vector space on the elements of E^1 and E^{τ} , respectively. The derived graph $Y = X_1 = X(\mathbb{Z}/2\mathbb{Z}, \alpha)$ in this case is a 6-gon. We draw this graph in two ways for ease of visualization.

There is only one cycle c_{X_1} in $Y = X_1$ and it is a special cycle. If required, we write $c_{X_1} = A_1C_1B_1A_2C_2B_2A_1$. (Since there is a unique edge between any two vertices this notation is self explanatory). Therefore,

$$
w_{c_{X_1}} = v_{A_1C_1} + v_{C_1B_1} + v_{B_1A_2} + v_{A_2C_2} + v_{C_2B_2} + v_{B_2A_1}
$$

where each v_e is a basis element of $W(X_1)$ corresponding to the edge e. Here, since there is only one cycle, it is trivially true that $c_0 = w_{c_{X_1}}$. Write $c_{X_1}^1$ (resp. $c_{X_1}^{\tau}$) to denote the projection of $w_{c_{X_1}}$ to W^1 (resp. W^{τ}).

Let $\vec{c} = (c_X, c_X) = (c_1, c_\tau)$ be the unique tuple of special cycles in $C(X) \times C(X)$. This is clearly in one-to-one correspondence with the unique special cycle of C_{X_1} . In this example,

$$
c_1=c_\tau=b_0.
$$

Since there are no non-special cycles in this example, we have that $C(X_1) = \mathbb{F}_2 c_0$. This graph is planar.

The next layer is the derived graph $Y = X_2 = X(\mathbb{Z}/4\mathbb{Z}, \tau) = X_1(\mathbb{Z}/2\mathbb{Z}, \tau)$ which is essentially a 12-gon and can be seen to planar in the same way as before.

Example 3.6. Let the voltage group $G = \{1, \tau\}$. Once again there is a unique edge (which we call e_0) with non-trivial voltage assignment, i.e., $\alpha(e_0) = \tau$. The tower of derived graphs $Y = X_1 = X(\mathbb{Z}/2\mathbb{Z}, \tau)$ and $Y = X_2 = X(\mathbb{Z}/4\mathbb{Z}, \tau)$ is drawn below

Denote the edges of the base graph X by e_b (for the black edge), e_r (for the red edge), and e_0 (for the dashed edge with the non-trivial voltage assignment). The basis elements of $W(X)$ are v_{e_b} , v_{e_r} , and v_{e_0} and for ease of notation, we denote them by v_b, v_r , and v_0 . In the base graph X, there are three cycles

$$
c_{br} = e_b e_r
$$

$$
c_{0r} = e_0 e_r
$$

$$
c_{0b} = e_0 e_b.
$$

Note that c_{0b} and c_{0r} are special cycles of X. Associated to each cycle above, we can associate the vectors

$$
w_{br} = v_b + v_r
$$

$$
w_{0r} = v_0 + v_r
$$

$$
w_{0b} = v_0 + v_b.
$$

By definition, $C(X)$ is the subspace of $W(X)$ generated by the vectors w_{br}, w_{0r} , and w_{0b} . Choose the basis $B = \{w_{0b}, w_{br}\}\$ of $C(X)$; by the notation introduced in the theorem write $b_0 = w_{0b}$. We check that $C(X)$ is a two-basis from the description of the basis elements w_* above.

For the first layer of the tower the voltage group $G = \{1, \tau\}$, define

$$
E^{\mathbf{1}} = (e_b, \mathbf{1}) \cup (e_r, \mathbf{1}) \cup (e_0, \tau^{-1})
$$

$$
E^{\tau} = (e_b, \tau) \cup (e_r, \tau) \cup (e_0, \mathbf{1}).
$$

In $Y = X_1$, label the black (resp. red) edges joining $A_i B_i$ as e_{b_i} (resp, e_{r_i}) and the dashed edge $B_1 A_2$ (resp. B_2A_1) as e_{0_1} (resp. e_{0_2}). By definition, the basis elements of $W(X_1)$ are $v_{b_1}, v_{b_2}, v_{r_1}, v_{r_2}, v_{0_1}$, and v_{0_2} .

Now, suppose that W^1 (resp. W^{τ}) are the free \mathbb{F}_2 -vector space on the elements in E^1 (resp. E^{τ}). From here, it is now immediate that

$$
W(X_1) = W^1 \oplus W^{\tau}.
$$

Consider the following cycle in $Y = X_1$ obtained using the black and dashed edges, i.e., consider the cycle

$$
c_{X_1} = e_{0_1} e_{b_2} e_{0_2} e_{b_1} =: c_{0b0b}.
$$

This is the lift c_0 of b_0 in the notation of the proof. Therefore, associated to this cycle is the vector

$$
w_{c_{X_1}} = v_{0_1} + v_{b_2} + v_{0_2} + v_{b_1}.
$$

Let $c_{X_1}^1$ be the projection of $w_{c_{X_1}}$ to W^1 , and similarly for $c_{X_1}^{\tau}$. Then

$$
c_{X_1}^1 = e_{0_1}e_{b_2}
$$
 and $c_{X_1}^7 = e_{0_2}e_{b_1}$.

If $\pi: Y \to X$ is the natural projection, then $\pi(c_{X_1}^1)$ and $\pi(c_{X_1}^T)$ are both cycles in X and equal to c_{0b} .

Since $|G| = 2$ and there are two special cycles in X, it follows that there are *four* tuples of special cycles in $C(X) \times C(X)$. These are in one-to-one correspondence between special cycles of Y. Indeed,

$$
(c_{0b}, c_{0b}) \longleftrightarrow c_{0b0b} = e_{0_1}e_{b_2}e_{0_2}e_{b_1}
$$

\n
$$
(c_{0b}, c_{0r}) \longleftrightarrow c_{0b0r} = e_{0_1}e_{b_2}e_{0_2}e_{r_1}
$$

\n
$$
(c_{0r}, c_{0b}) \longleftrightarrow c_{0r0b} = e_{0_1}e_{r_2}e_{0_2}e_{b_1}
$$

\n
$$
(c_{0r}, c_{0r}) \longleftrightarrow c_{0r0r} = e_{0_1}e_{r_2}e_{0_2}e_{r_1}.
$$

Consider the tuple of special cycle \vec{c} in $C(X) \times C(X)$. We index the components of \vec{c} by elements in G. When $\vec{\mathbf{c}} = (c_{0b}, c_{0b})$, we see that the special cycles

$$
c_{1}=c_{\tau}=c_{0b}=b_{0}.
$$

This choice of \vec{c} corresponds to $c_0 = c_{0b0b}$. (This is already clear from the correspondence above). When $\vec{\mathbf{c}} = (c_{0b}, c_{0r})$, we see that the special cycles

$$
c_1 = c_{0b} = b_0
$$

$$
c_\tau = c_{0r} = b_0 + w_{br}.
$$

Therefore, for this choice of \vec{c} , we see that $b_{\tau} = w_{br}$. It follows that \vec{c} corresponds to $c_0 + e_{b_1}e_{r_1}$. We note that $e_{b_1}e_{r_1}$ is a non-special cycle. Computations are similar for $\vec{c} = (c_{0r}, c_{0b})$; this yields $b_1 = w_{br}$. One can also work with $\vec{\mathbf{c}} = (c_{0r}, c_{0r})$ in which case $b_1 = b_\tau = w_{br}$.

Let $C'_g \subset W^g$ be the subspace generated by non-special cycles $e_{b_1}e_{r_1}$ and $e_{b_2}e_{r_2}$. Note that C'_g has a two basis and that none of its basis vectors have a non-zero entry at (e_0, g) . It follows that

$$
C(Y) = \mathbb{F}_2 c_0 \oplus C'_1 \oplus C'_g,
$$

which clearly has a 2 basis.

The next result no longer requires the voltage assignment to be unramified. In other words, the derived graph we obtain is a (ramified) branched cover.

Corollary 3.7. Let X be a finite planar graph and let Y be the derived graph of a single voltage assignment. Then Y is planar.

Proof. Let Y^{unr} be the corresponding unramified derived graph. Note that the immersion $\iota: Y^{\text{unr}} \to Y$ is a contraction and that contractions of planar graphs are planar.

Example 3.8. We once again start with the same graph X as in Example [3.5.](#page-7-0)

We no longer require that the vertices are unramified in this extension. In fact, we assume that the vertices B, C are totally ramified, i.e., the corresponding inertia groups are isomorphic to G at each layer. The derived graph $Y = X_1 = X(\mathbb{Z}/2\mathbb{Z}, \mathcal{I}, \tau)$ is the following

This is easily seen to be a contraction of the hexagon obtained in Example [3.5.](#page-7-0) The vertices B_1 and B_2 (resp. C_1 and C_2) have been identified.

Now, when $G \simeq (\mathbb{Z}/2^2 \mathbb{Z}) = \mathbb{Z}/4 \mathbb{Z}$, the derived graph $Y = X(\mathbb{Z}/4 \mathbb{Z}, \mathcal{I}, \tau)$ we obtain is the following

This is a contraction of the cyclic 12-gon graph above in Example [3.5.](#page-7-0)

4. Towers of Derived Graphs

Throughout this section, we assume that $\mathcal{G} \cong \mathbb{Z}_p^d$. By [\[DV23,](#page-22-2) p. 1336], we know that all unramified \mathbb{Z}_p^d -towers arise as derived graphs.

Lemma 4.1. Let $I \subset \mathcal{G}$ be a closed subgroup. Let $s_n(\sigma)$ denote the order of the image of σ in $\mathcal{G}/\mathcal{G}^{p^n}$. If $s_n(\sigma) \neq 1$, then

$$
s_{n+1}(\sigma) = ps_n(\sigma).
$$

Proof. Let $\gamma_1, \ldots, \gamma_d$ be topological generators of G. For every $1 \leq i \leq d$ and $m \geq 0$, there exist integers $1 \leq a_{i,m} \leq p^m$ such that

$$
\sigma \equiv \prod_{i=1}^d \gamma_i^{a_{i,m}} \pmod{\mathcal{G}^{p^m}}
$$
 and $a_{i,m} \equiv a_{i,m+1} \pmod{p^m}$.

Set $b_m = \min_i (v_p(a_{i,m}))$. By definition $s_m(\sigma) = p^{m-b_m}$.

Let n be a fixed integer satisfying $s_n(\sigma) > 1$. This is equivalent to $b_n < n$. This inequality, combined with the congruence condition mod p^n ensures that $b_{n+1} = b_n$; hence $s_{n+1}(\sigma) = ps_n(\sigma)$.

Lemma [4.1](#page-10-1) implies that given a group $I_v \subset \mathcal{G}$ of rank k there exist topological generators $\sigma_{v,1}, \ldots, \sigma_{v,k}$ such that for all $n \geq n_0$

$$
I_v \mathcal{G}^{p^n} / \mathcal{G}^{p^n} = \bigoplus_{t=1}^k C_{n,t},
$$

where $C_{n,t}$ is the cyclic subgroup of $\mathcal{G}/\mathcal{G}^{p^n}$ generated by the image of $\sigma_{v,t}$. For each vertex v, let $n_{0,v}$ be an integer that is large enough such that $\text{rank}_p(I_v \mathcal{G}/\mathcal{G}^{p^{n_{0,v}}}) = \text{rank}_p(I_v)^4$ $\text{rank}_p(I_v \mathcal{G}/\mathcal{G}^{p^{n_{0,v}}}) = \text{rank}_p(I_v)^4$.

Define $\mathbf{\vec{j}} = (j_1, \ldots, j_k)$ where for each $1 \le t \le k$, the entry j_t can take all (non-negative) integral values between 0 and $s_n(\sigma_{v,t}) - 1$ (both included). For $n \geq n_{0,v}$, define

(4.1)
$$
\boldsymbol{\omega}_{v,n} = \sum_{\vec{\mathbf{j}}} \prod_{t=1}^k \sigma_{v,t}^{j_t} \in \mathbb{Z}_p[\![T_1,\ldots,T_d]\!].
$$

Definition 4.2. Let X be a finite graph with vertices v_1, \ldots, v_s and voltage assignment α . Let X_{∞} be a branched cover of X. Label the unramified vertices in X_{∞}/X as v_1, \ldots, v_r and the ramified ones as v_{r+1}, \ldots, v_s . For each vertex v, define

$$
P_v = \deg(v)(v, 1) - \sum_{e \in \mathbb{E}_v^o(X)} (t(e), \alpha(e)I_{t(e)}) \in \text{Div}(X_\infty)
$$

• Define

$$
\Pr_{\Lambda}^{\text{unr}}(X_{\infty}) := \Lambda\text{-submodule of Div}_{\Lambda}(X_{\infty})\ generated\ by\ \{P_{v_l} \mid 1 \leq l \leq r\}.
$$

 \bullet Set

$$
n_0 = \max_{r+1 \leq l \leq s} n_{0,v_l}
$$
 where n_{0,v_l} is defined as in the previous paragraph.

For each $n \geq n_0$, define

$$
\Pr_n^{\text{ram}}(X_\infty) = \Lambda\text{-submodule of Div}_\Lambda(X_\infty) generated by $\omega_{v_l,n} P_{v_l}$ for $r+1 \leq l \leq s$.
$$

⁴For an abelian group G, we define its p-rank to be the $\mathbb{Z}/p\mathbb{Z}$ -dimension of G[p].

Remark 4.3. For the unramified vertices these are the generators one would expect for the principal divisors. The factors $\omega_{v_l,n}$ occur for the ramified vertices due to the immersion map $X_\infty^{\text{unr}} \to X_\infty$ as we will see in the following proof.

Notation: We henceforth set $P_{v_l} = P_l$ and $\boldsymbol{\omega}_{v_l} = \boldsymbol{\omega}_l$.

4.1. Picard Group of Ramified Derived Graphs. In this section, we prove the d-dimensional analogue of [\[GV24,](#page-22-9) Theorem 5.4]. Let G be any group, let $Y = X(G, \mathcal{I}, \alpha)$ denote the derived graph of X and write $Y^{\text{unr}} = X(G, \alpha)$. Let $\iota: Y^{\text{unr}} \to Y$ be the immersion. There is a natural surjective group morphism

$$
\iota_*\colon \operatorname{Div}(Y^{\mathrm{unr}}) \longrightarrow \operatorname{Div}(Y).
$$

Theorem 4.4. Let $X_\infty = X(\mathbb{Z}_p^d, \mathcal{I}, \alpha)$ denote the (ramified) derived graph of X and $X_\infty^{\text{unr}} = X(\mathbb{Z}_p^d, \alpha)$ be the unramified derived graph. Denote the respective layers of the derived graph as X_n and X_n^{unr} . Define

$$
N_n = \Pr_{\Lambda}^{\text{unr}}(X_{\infty}) + (\omega_n(T_1), \dots, \omega_n(T_d)) \operatorname{Div}_{\Lambda}(X_{\infty}) + \Pr_n^{\text{ram}}(X_{\infty}),
$$

where $\omega_n(T_i) = \gamma_i^{p^n} - 1$ for $1 \leq i \leq d$. For $n \gg 0$,

$$
Div_{\Lambda}(X_{\infty})/N_n \cong Pic(X_n).
$$

Proof. Fix a layer X_n and consider the projection map

$$
\pi_n\colon X_\infty\longrightarrow X_n.
$$

This induces a surjective map on the divisors, namely

$$
\pi_n\colon \operatorname{Div}_\Lambda(X_\infty)\longrightarrow \operatorname{Div}(X_n).
$$

To complete the proof of the theorem, it is sufficient to show that $\pi_n^{-1}(\Pr(X_n)) = N_n$.

For the layer X_n , consider a vertex $v \in V(X_n)$ and let P_v^n be the principal divisor associated to v as a vertex^{[5](#page-11-0)} in X_n^{unr} . Define

$$
P_{v,n}=\iota_*(P_v^n).
$$

A similar definition can be made over X_{∞} . For each vertex v in X_n , we have

$$
\pi_n(P_{v,\infty})=P_{v,n}
$$

Recall that associated to each ramified vertex v, we can associate a group $I_v \subseteq \mathbb{Z}_p^d$ of rank $k = k(v)$. Let $n \geq n_{0,v}$ where $n_{0,v}$ is a large enough integer such that $\text{rank}_p(I_v\mathcal{G}/\mathcal{G}^{p^{n_{0,v}}}) = \text{rank}_p(I_v)$. Then

$$
\pi_n(\boldsymbol{\omega}_{v,n} P_{v,\infty}) = \sum_{\vec{\mathbf{j}}} \prod_{t=1}^k \sigma_{v,t}^{j_t} P_{v,n} = \sum_{\sigma \in I_v \mathcal{G}^{p^n}/\mathcal{G}^{p^n}} \sigma P_{v,n}
$$

and $[GV24, Corollary 4.7]$ implies that $\pi_n(N_n) = Pr(X_n)$. Therefore

$$
N_n \subset \pi_n^{-1}(\Pr(X_n))
$$
 and $\pi_n^{-1}(\Pr(X_n)) = N_n + \ker(\pi_n)$.

Write Aug (I_{v_l}) to denote the augmentation ideal of I_{v_l} . The arguments in [\[GV24\]](#page-22-9) now show

$$
\mathrm{Div}_{\Lambda}(X_{\infty}) = \Lambda^r \oplus \bigoplus_{l=r+1}^s \Lambda/\mathrm{Aug}(I_{v_l}).
$$

Taking appropriate projections,

Div
$$
(X_n) \cong \mathbb{Z}_p[\mathcal{G}/\mathcal{G}^{p^n}]^r \oplus \bigoplus_{l=r+1}^s \mathbb{Z}_p[\mathcal{G}/I_{v_l}\mathcal{G}^{p^n}].
$$

Therefore for $n \gg 0$,

$$
\ker(\pi_n)=(\omega_n(T_1),\ldots,\omega_n(T_d))\operatorname{Div}_\Lambda(X_\infty).
$$

In particular, $\ker(\pi_n) \subset N_n$ and $\pi_n^{-1}(\Pr(X_n)) = N_n$.

Even though the module $Pr_n^{\text{ram}}(X_\infty)$ is necessary to describe $Pic(X_n)$ as a quotient of $Div_\Lambda(X_\infty)$ it does not play a role when we take protective limits as the following corollary points out.

 \Box

⁵Note that there are many choices for this unramified counterpart of v and we make a choice.

Corollary 4.5. With notation as above,

$$
\operatorname{Pic}\nolimits_{\Lambda}(X_{\infty}) \cong \operatorname{Div}\nolimits_{\Lambda}(X_{\infty})/\operatorname{Pr}\nolimits^{\operatorname{unr}}_{\Lambda}(X_{\infty}).
$$

Proof. This is analogous to [\[GV24,](#page-22-9) Corollary 5.5] using Theorem [4.4](#page-11-1) instead of [GV24, Theorem 5.4]. \square

Let $w_{v,\infty}^{\text{unr}}$ denote the vertex $(v,1_G)$ in $X_{\infty}^{\text{unr}} = X(\mathcal{G},\alpha)$. Let Δ be the operator on $\text{Div}_{\Lambda}(X_{\infty}^{\text{unr}})$ given by

$$
\Delta w_{v,\infty}^{\text{unr}} = \begin{cases} P_v^{\text{unr}} & \text{if } v \text{ is an unramified vertex in } X_\infty / X \\ (\sigma_{v,1} - 1) w_{v,\infty}^{\text{unr}} & \text{if } v \text{ is a ramified vertex in } X_\infty / X \text{ and } \text{rank}(I_v) = 1 \\ w_{v,\infty}^{\text{unr}} & \text{if } v \text{ is a ramified vertex in } X_\infty / X \text{ and } \text{rank}(I_v) > 1, \end{cases}
$$

where

$$
P_v^{\text{unr}} = \deg(v)(v, 1) - \sum_{e \in \mathbb{E}_v^o(X)} (t(e), \alpha(e)) \in \text{Div}(X_\infty^{\text{unr}}).
$$

In previous references, the operator P_v^{unr} is often written as $\mathcal{L}_{\Lambda}^{\text{unr}}(-)$ and is called the Laplacian operator. This notation is consistent with how P_v was expressed in Definition [4.2;](#page-10-3) only now $I_{t(e)}$ is trivial.

Remark 4.6. If X is a finite connected graph and $X_{\infty} = X_{\infty}^{\text{unr}}$ is an unramified \mathbb{Z}_p^d -tower of X, this operator Δ was already studied in [\[KM24\]](#page-22-8) and was denoted by Δ_{∞} .

Theorem 4.7. Let $X_{\infty} = X(\mathbb{Z}_p^d, \mathcal{I}, \alpha)$ denote the (ramified) derived graph of X. With this notation

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = (\det(\Delta)).
$$

Proof. Write $X_{\infty}^{\text{unr}} = X(\mathbb{Z}_p^d, \alpha)$ to denote the unramified derived graph. Define^{[6](#page-12-1)}

$$
\mathrm{Pr}'_\Lambda(X_\infty^{\mathrm{unr}}) = \Lambda\text{-submodule of } \mathrm{Div}_\Lambda(X_\infty^{\mathrm{unr}}) \text{ generated by } \{P_{v_l}^{\mathrm{unr}} \mid 1 \leq l \leq r\}.
$$

Note that the quotient $\text{Div}_{\Lambda}(X_\infty^{\text{unr}})/\text{Pr}'_{\Lambda}(X_\infty^{\text{unr}})$ has Λ -rank $s-r$. Also, recall the surjection

$$
\iota_*: \mathrm{Div}_{\Lambda}(X_\infty^{\mathrm{unr}}) \longrightarrow \mathrm{Div}_{\Lambda}(X_\infty)
$$

Under this map, the pre-image of $Pr_{\Lambda}^{unr}(X_{\infty})$ is precisely $ker(\iota_*) + Pr'_{\Lambda}(X_{\infty}^{unr})$. Now, it follows immediately from Corollary [4.5](#page-12-2) that

$$
\mathrm{Pic}_{\Lambda}(X_{\infty})\cong \mathrm{Div}_{\Lambda}(X_{\infty})/\mathrm{Pr}^{\mathrm{unr}}_{\Lambda}(X_{\infty})=\mathrm{Div}_{\Lambda}(X^{\mathrm{unr}}_{\infty})/\ker(\iota_*)+\mathrm{Pr}'_{\Lambda}(X^{\mathrm{unr}}_{\infty}).
$$

Thus it suffices to compute the characteristic ideal of $Div_{\Lambda}(X_{\infty}^{unr})/ker(\iota_*) + Pr'_{\Lambda}(X_{\infty}^{unr})$. Consider the exact sequence

$$
0 \to \text{Im}(\Delta)/\ker(\iota_*) + \text{Pr}'_{\Lambda}(X^{\text{unr}}_{\infty}) \to \text{Div}_{\Lambda}(X^{\text{unr}}_{\infty})/\ker(\iota_*) + \text{Pr}'_{\Lambda}(X^{\text{unr}}_{\infty}) \to \text{Div}_{\Lambda}(X^{\text{unr}}_{\infty})/\text{Im}(\Delta) \to 0.
$$

To prove the theorem we need to show that the first term is pseudo-null. This module is generated by

$$
\{w_{v,\infty}^{\text{unr}} \mid v \in \{v_{r+1}, \ldots, v_s\}, \text{ rank}(I_v) \ge 2\}
$$

Each $w_{v,\infty}^{\text{unr}}$ is annihilated by Aug (I_v) . As rank $(I_v) \geq 2$, the module $\Lambda/\text{Aug}(I_v)$ is pseudo-null giving the desired result.

Remark 4.8. If $X_{\infty} = X_{\infty}^{\text{unr}}$, the above theorem asserts that

$$
char_{\Lambda}(\mathrm{Pic}_{\Lambda}(X_{\infty}^{\mathrm{unr}})) = (\det(\Delta_{\infty})).
$$

On the other hand, it was proven in [\[KM24,](#page-22-8) Theorem 5.2] that if X_{∞}/X is an unramified \mathbb{Z}_p^d -extension of X (obtained as a derived graph of X) then

$$
(\det(\Delta_{\infty})) = \begin{cases} \text{char}_{\Lambda}(\text{Jac}_{\Lambda}(X_{\infty})) & \text{if } d \geq 2 \\ (T) \operatorname{char}_{\Lambda}(\text{Jac}_{\Lambda}(X_{\infty})) & \text{if } d = 1. \end{cases}
$$

Therefore, we conclude

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = \begin{cases} char_{\Lambda}(Jac_{\Lambda}(X_{\infty})) & \text{if } d \geq 2 \\ (T) char_{\Lambda}(Jac_{\Lambda}(X_{\infty})) & \text{if } d = 1. \end{cases}
$$

Compare also with [\[KM24,](#page-22-8) Remark 5.4].

⁶We want to emphasize that $Pr_{\Lambda}^{unr}(X_{\infty})$ was defined as a Λ -submodule of $Div_{\Lambda}(X_{\infty})$ generated by $\{P_{v_l} \mid 1 \leq l \leq r\}$ in Definition [4.2](#page-10-3) which is different from $\Pr'_{\Lambda}(X_\infty^{\text{unr}})$. The two definitions coincide if $X_\infty = X_\infty^{\text{unr}}$.

Example 4.9. We give an example of an unramified cover of X that can also be found in $[KM24]$. Here the dashed line between A and B in the graph at the bottom layer has non-trivial voltage assignment, say τ .

For convenience of the reader, we draw another way to interpret the 18-gon in the third diagram above:

The characteristic polynomial of $Pic_\Lambda(X_\infty)$ is T^2 for this unramified tower.

For ramified branched covers X_{∞} , it has been pointed out [\[GV24,](#page-22-9) Section 6] that $\det(\Delta) = \det(D - B(T))$ where $D, B(T)$ were matrices introduced in Definition [2.10.](#page-4-1) We now provide examples of the same.

Example 4.10. Reconsider the branched covering from Example [3.8.](#page-9-0) We calculate the matrices

$$
D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ T+1 & -T & 0 \\ 1 & 0 & -T \end{bmatrix}.
$$

It follows that

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = det(D - B(T)) = det\begin{bmatrix} 2 & 0 & 0 \\ -(1+T) & T & 0 \\ -1 & 0 & T \end{bmatrix} = 2T^{2}.
$$

Example 4.11. In this example $G = (\mathbb{Z}/3\mathbb{Z}, +)$ for each subsequent layer and the vertex $R \in V(X)$ is totally ramified whereas the vertex $U \in V(X)$ is unramified. The dashed edge has non-trivial voltage assignment.

We calculate the two matrices D and $B(T)$ using Definition [2.10,](#page-4-1)

$$
D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B(T) = \begin{bmatrix} 0 & 0 \\ \tau^{-1} + 1 & 1 - (1+T)^{p^0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tau^{-1} + 1 & -T \end{bmatrix}
$$

It follows that the characteristic ideal

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = det(D - B(T)) = det\begin{bmatrix} 2 & 0 \\ -\tau^{-1} - 1 & T \end{bmatrix} = 2T.
$$

This example can be generalized in the following manner:

Example 4.12. Let X be a cycle graph with m vertices. Let α be a single voltage assignment with image in \mathbb{Z}_p . Let $I_v = \mathbb{Z}_p$ for exactly one vertex and set $I_{v'} = 1$ for all $v' \neq v$. Let $X_\infty = X(\mathbb{Z}_p, \alpha, I)$ and let X_n be the intermediate covers. Assume that the image of α generates \mathbb{Z}_p topologically. Then

$$
v_p(|\mathrm{Jac}(X_n)|) = v_p(m)p^n.
$$

Indeed, X_n looks similar to the flower graph above. Instead of leaves consisting of one vertex and two edges, the leaves have $(m - 1)$ vertices and m edges. To count the spanning trees we need to choose a spanning tree for each leaf, i.e. we have to choose one edge that we "delete". That leaves m choices per leaf. As there are p^n leaves the claim follow.

In the following picture we choose $p = 3$ and draw the base graph and the first layer.

We calculate the two matrices D and $B(T)$ using Definition [2.10,](#page-4-1)

$$
D = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times m} \quad \text{and } B(T) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \tau^{-1} & 0 & 0 & -T \end{bmatrix}_{m \times m}
$$

Therefore,

$$
char_{\Lambda}(\text{Pic}_{\Lambda}(X_{\infty})) = det(D - B(T)) = mT
$$

Example 4.13. In the subsequent example we consider the case when $\mathcal{G} = \mathbb{Z}_3^2$, i.e. the first layer is a $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ covering. We assume that the vertex U is unramified and that the vertex R is totally ramified.

In the next step we will get a flower with 27 leaves. Thus, for X_n we obtain

$$
v_p(|\operatorname{Jac}(X_n)|) = v_p(2)p^{2n}.
$$

If we compute the characteristic ideal using the matices

$$
D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}
$$
 and
$$
B(T) = \begin{bmatrix} 0 & 0 \\ \tau^{-1} + \sigma^{-1} & -1 \end{bmatrix}
$$
,

we obtain that the characteristic ideal is given by

$$
\det(D - B(T)) = 2.
$$

If we let R only ramifiy at the group generated by τ , i.e. $\langle \tau \rangle$, we obtain

$$
B(T) = \begin{bmatrix} 0 & 0 \\ \tau^{-1} + \sigma^{-1} & -(\tau - 1) \end{bmatrix}.
$$

In this case the characteristic ideal is given by 2T .

4.2. Growth Formula. In this section, the aim is to prove an analogue of Iwasawa's growth number formula for the number of spanning trees at the *n*-th layer of the \mathbb{Z}_p^d -tower.

Definition 4.14. Let M be a Λ -module. A structure on M consists of an integer z and tuples (I_i, M_i) $1 \leq j \leq z$, where the M_j are submodules of M and the I_j are subgroups of G. For all n, set

$$
A_n = (\omega_n(T_1), \ldots, \omega_n(T_d))M + \sum_{j=1}^z \omega_{j,n} M_j,
$$

where $\omega_{j,n}$ is defined with respect to the subgroup I_j . With this notation, define $\mathcal{M}_n = M/A_n$.

Lemma 4.15. There is a constant c such that $p^{2dn+c}\mathcal{M}_n[p^{\infty}]=0$.

Proof. Even though our definition of structures differs from the one given in $\lfloor CM81 \rfloor$, the proof of Theorem 4.5 in loc. cit. carries over almost verbatim. We repeat it for the convenience of the reader.

Let N be the submodule generated by all the M_i . Let $W = \Omega^d$, where Ω is the group of p-power roots of unity. For each $\zeta \in W$ there is a a natural map $M/N \to (M/N) \otimes \mathbb{Z}_p[\zeta]$ induces by the valuation $\Lambda \to \mathbb{Z}_p[\zeta]$. By [\[CM81,](#page-21-0) Lemma 2.6] there exists a constant c such that $p^c x$ annihilates $(M/N)_{\zeta}[p^{\infty}]$ and $M_{\zeta}[p^{\infty}]$ for all choices of ζ . Let n_0 be the index such that the rank of I_i in $\mathcal{G}/\mathcal{G}^{p^n}$ is stable for all $n \geq n_0$. Let $\zeta \in W$ be an element with ord $(\zeta) > p^{n_0}$. The evaluation $\Lambda \to \mathbb{Z}_p[\zeta]$ sends $\omega_{j,n}$ to zero in this case. Thus,

$$
(\mathcal{M}_n)_{\zeta}[p^{\infty}] = M_{\zeta}[p^{\infty}]
$$

is annihilated by p^c . It remains to estimate the contribution of the $\zeta \in W$ with $\text{ord}(\zeta) \leq p^{n_0}$.

For all $n \geq n_0$ we define

$$
\omega_n = \prod_{i=1}^d \frac{(T_i+1)^{p^n}-1}{(T_i+1)^{p^{n_0}}-1}.
$$

Note that ω_n describes the trace from $\mathcal{G}/\mathcal{G}^{p^n}$ to $\mathcal{G}/\mathcal{G}^{p^{n_0}}$. Let N' be the image of N in M_ζ . Then the image of $\omega_n N$ in M_ζ is $p^{(n-n_0)d}N'$. It follows that $(M/\omega_n N)_\zeta$ and $(M/N)_\zeta$ have the same \mathbb{Z}_p -rank. Furthermore $(M/\omega_n N)_{\zeta} [p^{\infty}]$ is annihilated by p^{nd+c} . Furthermore we have a natural surjection

$$
(M/\omega N)_{\zeta} \to (M/A_n)_{\zeta} \to (M/N)_{\zeta}.
$$

All three terms have the same \mathbb{Z}_p -rank. Therefore, $(M/A_n)_{\zeta} [p^{\infty}]$ is annihilated by p^{nd+2c} .

Summing over all ζ of order at most p^n we obtain that

$$
\oplus_{\zeta \in W, \mathrm{ord}(\zeta) \leq p^n} (M/A_n)_{\zeta} [p^{\infty}]
$$

is annihilated by p^{nd+2c} . It now follows from [\[CM81,](#page-21-0) Lemma 2.7] that $\mathcal{M}_n[p^{\infty}]$ is annihilated by p^{2dn+2c}

Write

$$
M=\mathrm{Div}_\Lambda(X_\infty)\simeq\mathrm{Div}_\Lambda(X_\infty)/\mathrm{Pr}^{\mathrm{unr}}_\Lambda(X_\infty).
$$

Set $z = s - r$ and write $(I_j, M_j) = (I_{v_{j+r}}, \Lambda P_{v_{j+r}})$ for $1 \leq j \leq s - r$. Note that as $1 \leq j \leq s - r$, the vertices v_{j+r} are the *ramified* ones in X_{∞}/X . This defines a structure in the above sense; by Theorem [4.4](#page-11-1) we have

$$
Pic(X_n) = Div_{\Lambda}(X_{\infty})/Pr_{\Lambda}^{unr}(X_{\infty}) + (\omega_n(T_1), \ldots, \omega_n(T_d)) Div_{\Lambda}(X_{\infty}) + Pr_n^{ram}(X_{\infty})
$$

= M/A_n .

Theorem 4.16. Fix $d \geq 2$. Let X_{∞} denote the derived graph $X(\mathbb{Z}_p^d, \mathcal{I}, \alpha)$ and X_{∞}^{unr} denote the derived graph $X(\mathbb{Z}_p^d, \alpha)$. Assume that all intermediate X_n are connected. Let f be a generator of the characteristic ideal of $\text{Jac}_{\Lambda}(X_{\infty})$. Then

$$
\text{ord}_p(|\operatorname{Jac}(X_n)|) = \mu(f)p^{nd} + \lambda(f)np^{(d-1)n} + O(p^{(d-1)n}).
$$

Remark 4.17. Recall that the number of spanning trees of a graph X is equal to the size of the Jacobian of X. The above statement can hence be rewritten in terms of $\text{ord}_p(\kappa(X_n)).$

Proof. We begin by arranging the ramified vertices in increasing order of the rank of the corresponding subgroups I_v . In other words, assume that $rank(I_{v_j}) \leq rank(I_{v_{j+1}})$ for $r + 1 \leq j \leq s$. Let $s' \geq r + 1$ be the index such that $rank(I_{v_i}) \geq 2$ for all $j \geq s'$. For $r + 1 \leq i \leq s' - 1$ we define $\overline{T}_i = \sigma_{v_i,1} - 1$. Decompose $Div_{\Lambda}(X_{\infty})$ as follows

$$
\mathrm{Div}_{\Lambda}(X_{\infty})=\mathfrak{M}_1\oplus\mathfrak{M}_2\oplus\mathfrak{M}_3,
$$

where \mathfrak{M}_1 is Λ -free, $\mathfrak{M}_2 = \bigoplus_{i=1}^{s'-1} \Lambda/\widetilde{T}_i$ and \mathfrak{M}_3 is pseudo-null and \mathbb{Z}_p -free; the contribution to \mathfrak{M}_1 comes from the unramified vertices and to \mathfrak{M}_2 from those ramified vertices for which the rank of I_v is 1. For $1 \leq i \leq 3$ we denote by $\overline{\mathfrak{M}_i}$ the image of \mathfrak{M}_i in Pic_A(X_∞). Clearly, $\overline{\mathfrak{M}_3}$ is pseudo-null. By [\[CM81,](#page-21-0) Lemma 3.1],

$$
rank_p(\pi_n(\overline{\mathfrak{M}_3})) \leq rank_p(\overline{\mathfrak{M}_3}/(\omega_n(T_1), \ldots, \omega_n(T_d))\overline{\mathfrak{M}_3}) = O(p^{n(d-2)}).
$$

Thus, $\text{ord}_p(|\pi_n(\overline{\mathfrak{M}_3})[p^\infty]|) = O(2ndp^{n(d-2)})$ by Lemma [4.15.](#page-15-0) Set $M' = M/\overline{\mathfrak{M}_3}$ where M is defined as before.

Define M'_j as the image of M_j in M' . The tuples $(I_{v_{j+r}}, M'_j)$ for $1 \leq j \leq z'$ define a structure on M' , here $z' = s' - r - 1$. Let \mathcal{M}'_n be the module associated to this structure. The structure defined on M' is an admissible structure in the sense of $\lfloor CM81$, Definitions 4.1 and 4.6. By $\lfloor CM81$, Theorem 4.13 and using the estimate for the contribution of $\overline{\mathfrak{M}_3}$ we obtain

$$
\mathrm{ord}_p(|\mathcal{M}_n[p^{\infty}])|) = \mathrm{ord}_p(|\mathcal{M}'_n[p^{\infty}]|) + O(p^{n(d-1)}) = \mu p^{nd} + \lambda np^{(d-1)n} + O(p^{(d-1)n}).
$$

Upon noting that M and M' have the same characteristic ideal, the result follows. \square

5. Results on Dual Graphs

Definition 5.1. Let X be a planar graph. The dual graph of X, denoted by X^{\vee} , is a graph that has a vertex for each face of X and an edge for each pair of faces in X that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge.

Each edge e of X has a corresponding dual edge, with endpoints the dual vertices corresponding to the faces on either side of e.

5.1. Jacobian of Dual Graphs. Let $\mathcal G$ be any uniform pro-p p-adic Lie extension. We first observe that the dual graphs of each layer of a $\mathcal{G}\text{-}$ tower of planar graphs need not be a $\mathcal{G}\text{-}$ tower.

Non-example 5.2. Recall the tower of graphs in Example [3.8.](#page-9-0)

A

 $B \longleftrightarrow$ C

τ

 A_1

 A_1

 A_2 A3 A_4

 \overline{B} \overline{C}

 A_2

 $B \leq C$

It is obvious that the second and third graph are not (branched) coverings of the first one.

Later on we will see examples where the dual graphs are actually coverings again. The following theorem describes the Galois structure in this case.

Theorem 5.3. Let X_{∞}/X be a G-tower of planar connected graphs with intermediate layers denoted by X_n . Assume that X_n^{\vee}/X^{\vee} are branched coverings. Then X_∞^{\vee}/X^{\vee} is also a G-tower of planar graphs with intermediate layers X_n^{\vee} and the Galois action can be chosen to be compatible with the process of taking duals.

Remark 5.4. For some cases where the hypothesis is satisfied we refer the reader to Corollary [5.11.](#page-20-1)

Proof. We fix a planar representation of X and let $\pi_n: X_n \to X$ be the natural projection. Given a directed edge e of X_n there are exactly two directed edges e_1 and e_2 in X_n^{\vee} that intersect e. We wish to choose one of them (as the dual edge) depending on the orientation of e. We fix once and for all a choice e^{\vee} for X and define $e_i = e^{\vee}$ if $\pi_n(e)^{\vee} = \pi_n(e_i)$.

Define the action of \mathcal{G}_n on the set of directed edges $\mathbb{E}(X_n^{\vee})$ by

$$
g \cdot e^{\vee} = (g \cdot e)^{\vee}.
$$

We extend the action to the vertices of X_n^{\vee} by defining $g(o(e^{\vee})) = o(ge^{\vee})$ and $gt(e^{\vee}) = t(ge^{\vee})$. Thus, g extends naturally to a morphism of graphs. This defines a free action without inversion, because the action of \mathcal{G}_n on $\mathbb{E}(X_n)$ is a free action without inversion.

Claim: This action is compatible with the tower.

Justification: Let $e \in \mathbb{E}(X_n^{\vee})$ and let $\pi_{n,n-1} : X_n^{\vee} \to X_{n-1}^{\vee}$ be the natural projection. By abuse of notation we also denote the projection $X_n \to X_{n-1}$ by $\pi_{n,n-1}$. Let $\psi_{n,n-1}$: $G_n \to G_{n-1}$ be the corresponding projection on the groups. We need to show

$$
\pi_{n,n-1}(g \cdot e^{\vee}) = \psi_{n,n-1}(g) \cdot \pi_{n,n-1}(e^{\vee}).
$$

By construction,

$$
\pi_{n,n-1}(g \cdot e^{\vee}) = \pi_{n,n-1}((g \cdot e)^{\vee}) = (\pi_{n,n-1}(g \cdot e))^{\vee} = (\psi_{n,n-1}(g) \cdot \pi_{n,n-1}(e))^{\vee} = \psi_{n,n-1}(g) \cdot \pi_{n,n-1}(e^{\vee}).
$$

This completes the proof of the claim and the theorem.

We elucidate this via an example.

Example 5.5. Let X be the following graph with 4 vertices.

Let τ be a topological generator of \mathbb{Z}_p . Let X_∞ be the derived graph of this voltage assignment and let X_n be the intermediate coverings. By Theorem [3.4](#page-6-3) all X_n are planar. The graph X_n has the following form (picture for $p = 2$ and $n = 1$).

The dual graphs of the base graph has the following shape:

If we look at the dual of the first layer for $p = 2$ we obtain

The second graph is a branched covering of the first one, where the voltage assignment is trivial on all black edges and τ on the red one. For higher n the graph X_n will consist of p^n squares connected by red edges as drawn in the example $p = 2$ and $n = 1$ as above. The dual graph has one vertex for each square denoted by O_1, \ldots, O_{p^n} and two additional vertices created by the red edges of X_n . We will denote these vertices by I_1 and I_2 . Each O_i is connected to each I_j by two black edges. Furthermore there are p^n red edges connecting I₁ to I₂. Thus, X_n^{\vee} is the derived graph of the voltage assignment that is totally ramified in I₁ and I₂ and assigns τ to the red edge in X_0^{\vee} and 1 to all other edges.

Definition 5.6. Let $X = (V, \mathbb{E})$ be a directed graph. A dart assignment is a function

$$
\omega\colon \mathbb{E}\longrightarrow \mathbb{Z},
$$

such that $\omega(e) + \omega(\overline{e}) = 0$, where \overline{e} denotes the inverse of e. Set $F(X)$ to denote the faces of X, and $f^{\circ}(X)$ denotes the boundary of $F(X)$ in a counterclockwise orientation.

$$
\partial \omega = \sum_{v \in V(X)} \sum_{e \in \mathbb{E}_v^o(X)} \omega(e)v
$$

$$
\partial^* \omega = \sum_{f \in F(X)} \sum_{e \in f^o(X)} \omega(e)f.
$$

Theorem 5.7. Keep the notation introduced above and assume that (X_n^{\vee}) is a tower as well. Then as Galois modules

$$
Jac(X_n)\simeq Jac(X_n^{\vee}).
$$

Proof. As abstract groups, the isomorphism of the Jacobian groups is proven in $CR00$, Theorem 2. Let u be an element in Div⁰(X). By [\[CR00,](#page-21-1) Proposition 3.2] there exists a dart assignment ω such that $\partial \omega = u$. It follows that $\partial^*(\omega) \in Div^0(X^{\vee})$. By [\[CR00,](#page-21-1) proof of Theorem 2] this assignment induces a well-defined bijective homomorphism

$$
\theta
$$
: Jac (X) \longrightarrow Jac (X^{\vee}) .

It remains to check that θ is a G-homomorphism. Let $x \in \text{Jac}(X)$ and let u be a lift in $\text{Div}^0(X)$. Then

$$
gu = g \sum_{v \in V(X)} u_v v = \sum_{v \in V(X)} u_v g(v) = \sum_{v \in V(X)} u_{g^{-1}v} v = \sum_{v \in V(X)} \sum_{e \in \mathbb{E}_{g^{-1}v}^{\circ}} \omega(e) v
$$

$$
= \sum_{v \in V(X)} \sum_{e \in \mathbb{E}_v^{\circ}} \omega(g^{-1}e) v
$$

$$
= \sum_{v \in V(X)} \sum_{e \in \mathbb{E}_v^{\circ}} (g\omega)(e) v
$$

$$
= \partial(g\omega).
$$

Therefore,

$$
\theta(gx) = \partial^*(g\omega) + \Pr(X^{\vee}) \in \text{Jac}(X^{\vee}).
$$

Furthermore,

$$
\partial^*(g\omega)=\sum_{f\in F(X)}\sum_{e\in f^\circ(X)}\omega(g^{-1}e)f=\sum_{f\in F(X)}\sum_{e\in f^\circ(X)}\omega(e)gf=g\partial^*(\omega).
$$

Therefore

$$
g\theta(x) = \theta(gx).
$$

 \Box

Remark 5.8. It is crucial that we work with the Jacobian group here and not with the whole Picard group. For Picard groups the above statement is in general not true. Indeed, consider a graph X with 1 vertex and one loop. Let X_{∞}/X be an unramified \mathbb{Z}_p -tower of X with intermediate layers X_n . In this case $Pic(X_n) \cong \Lambda/(T^2, \omega_n)$ where ω_n was defined previously in Theorem [4.4.](#page-11-1) For $n > 0$ the dual graph X_n^{\vee} has 2 vertices and p^n edges between these vertices. In this case

$$
Pic(X_n^{\vee}) \cong ((\Lambda/T)e_1 \oplus (\Lambda/T)e_2) / (\nu_{n,0}(e_1 - e_2)).
$$

These two modules are certainly not isomorphic as Λ-modules. But

$$
Jac(X_n^{\vee}) = ((\Lambda/T)(e_1 - e_2)) / \nu_{n,0}(e_1 - e_2) \cong ((\Lambda/T)(e_1 - e_2)) / p^n(e_1 - e_2) \cong T\Lambda/(T^2, \omega_n) \cong Jac(X_n)
$$

Recall that if X and X^{\vee} are dual planar graphs and T is a spanning tree for X, then the complement of the edges dual to $\mathcal T$ is a spanning tree for X^{\vee} , see [\[Big71\]](#page-21-2). It now follows that

$$
\mathrm{ord}_p(\kappa(X_n)) = \mathrm{ord}_p(\kappa(X_n^{\vee})).
$$

In other words, the generalized Iwasawa invariants associated with the finite layers above the graph X and X^{\vee} must coincide. Another way to see the above equality is to recall the fact that $\kappa(X_n) = \# \operatorname{Jac}(X_n)$, the equality is immediate from Theorem [5.7.](#page-18-0)

By Theorem [5.7,](#page-18-0) we can conclude that $\text{Jac}_{\Lambda}(X_{\infty}) \simeq \text{Jac}_{\Lambda}(X_{\infty}^{\vee})$ as Λ -modules. Therefore,

$$
char_{\Lambda}(Jac_{\Lambda}(X_{\infty}))=char_{\Lambda}(Jac_{\Lambda}(X_{\infty}^{\vee})).
$$

Example 5.9. Revisiting Example [5.5](#page-17-1) we can now compute the characteristic ideal of $Jac(X_{\infty})$ by computing the characteristic ideal of Jac(X_{∞}^{\vee}). While the computation of Jac(X_{∞}) involves the computation of a 5×5 matrix, the computation of the characteristic ideal of Jac (X_{∞}^{\vee}) only involves the computation of the determinant of the following 3×3 matrix

$$
\frac{1}{T} \det \left(\begin{bmatrix} 4 & 0 & 0 \\ -2 & T & 0 \\ -2 & 0 & T \end{bmatrix} \right) = 4T.
$$

5.2. Applications.

Proposition 5.10. Let X be a planar graph with two faces. The dual graph exists and is described as follows

- X^{\vee} has two vertices
- the number of edges between these vertices is the length of the cycle in X .
- one of the vertices has $|\{$ edges in $X\}\rangle \{$ edges in the cycle of $X\}$ loops

Proof. Let D be a drawing of X. Then D has two faces an inner and an outer one. The edges separating the two faces are precisely the ones in the cycle contained in X. In the dual graph each such edge corresponds to an edge between the two vertices. Each further edge in X , only touches the outer face. Thus, it corresponds to a loop. \Box

Corollary 5.11. Let (X_n) be an \mathbb{Z}_p -tower of connected planar graphs. Assume that all X_n have two faces. Then

$$
char_{\Lambda}(Jac_{\Lambda}(X_{\infty}))=T
$$

Remark 5.12. The condition that each X_n is planar and has two faces implies that X_n/X_0 is unramified. Indeed, let e_n be the number of edges of X_n and let v_n be the number of vertices. As X_n has two faces we have

$$
0 = e_n - v_n
$$

by the Euler characteristic formula. By definition $e_n = p^n e_0$. Thus, $v_n = p^n v_0$ and X_n/X is unramified by Lemma [2.9.](#page-4-2) Furthermore in X_0 there is exactly one cycle path. Thus, if we assume that all X_n are connected we need to have $d = 1$ by [\[KM24,](#page-22-8) Lemma 2.5].

Proof. By assumption, each X_n^{\vee} has two vertices. Define $\pi_n^{\vee}: X_n^{\vee} \to X_0^{\vee}$ as the identity map on the vertices and $\pi_n(e^{\vee}) = \pi_n(e)^{\vee}$.

Claim: X_n^{\vee}/X_0^{\vee} is a branched covering.

Justification: Let e_1, \ldots, e_s be the edges of X_0 that form the unique cycle of X_0 . Each of these edges has p^n preimages in X_n and as X_n/X_0 is unramified all these preimages together form a cycle of length p^ns without tails and backtracks in X_n . As X_n has only two faces, this is the unique cycle in X_n . Each of these p^ns edges corresponds to an edge connecting the two vertices of X_n^{\vee} .

As X_0 has only two faces each vertex on the cycle of X_0 can be seen as the root of a finite tree. Let T be such a tree with root v. Let $v' \in X_n$ be a preimage of v. As X_n/X_0 is unramified, v' is the root of a tree in X_n that is isomorphic to T, i.e. all preimages of edges in T lie completely in the outer face of X_n . This implies that each loop of X_0^{\vee} has exactly p^n preimages in X_n^{\vee} that are all loops.

We have thus verified that there is a p^{n} -to-1 correspondence between the loops of X_n^{\vee} and the loops of X_0^{\vee} and one between the edges connecting the two vertices in X_n^{\vee} and in X_0^{\vee} . This shows that X_n^{\vee}/X_0^{\vee} is a totally ramified p^n -cover.

To complete the proof of the corollary, it suffices to work with the dual tower (X_n^{\vee}) . Instead of computing the characteristic ideal of the Jacobian, it suffices to determine the characteristic ideal of the Picard group. The tower (X_n^{\vee}) is totally ramified. In particular $I_v = \mathcal{G}$ for all $v \in V(X_0^{\vee})$. By Corollary [4.5](#page-12-2) when $d = 1$, we obtain

$$
\mathrm{Div}_{\Lambda}(X_{\infty}^{\vee})/\mathrm{Pr}_{\Lambda}(X_{\infty}^{\vee}) \cong \Lambda/(T) \oplus \Lambda/(T).
$$

The characteristic ideal of the Picard group is T^2 and the claim follows from [\[GV24,](#page-22-9) Proposition 5.8]. \Box

Let us elucidate this via an example where the base graph X is not self-dual.

Example 5.13. We start with a graph X that has two faces, six edges, and one cycle. Let the dashed edge between A and B be assigned a non-trivial voltage assignment and every other edge be assigned the trivial voltage assignment. We draw the first two layers of a \mathbb{Z}_2 -tower. Each subsequent layer will have a cycle with $2ⁿ$ vertices and each such vertex will have two additional edges. This is indeed an unramified cover.

Note that for this unramified tower

char_Λ(Pic_Λ(X_∞)) = T^2 and char_Λ(Jac_Λ(X_∞)) = T.

We now draw the respective dual graphs. For the dual graphs, we write I to denote the vertex that corresponds to the 'inner face' and O to denote the one that corresponds to the 'outer face'. It is easy to observe that the subsequent layers are indeed forming a branched \mathbb{Z}_2 -cover of X^{\vee} .

We now independently calculate the characteristic polynomial(s) of $Pic_\Lambda(X_\infty^\vee)$ and $Jac_\Lambda(X_\infty^\vee)$. Both the vertices are ramified in this tower; i.e., there are no unramified vertices. Thus D is the 0-matrix and $B(T)$ is the 2 \times 2 diagonal matrix diag($-T$, $-T$). It now follows easily that

char_Λ(Pic_Λ(X_{∞}^{\vee})) = T^2 and char_Λ(Jac_Λ(X_{∞}^{\vee})) = T.

Corollary 5.14. Assume that $(X_n)_n$ is an unramified \mathbb{Z}_p -tower of planar graphs. Assume that for each n X_n^{\vee}/X is a branched covering. Then $T | \text{char}_{\Lambda}(\text{Jac}_{\Lambda}(X_{\infty}))$.

Proof. Let e_n be the number of undirected edges, v_n be the number of vertices, and f_n be the number of faces of X_n . As all graphs are planar, the Euler characteristic formula asserts that for all n

> $2 = v_n - e_n + f_n$ $= p^{n}(v_0 - e_0) + f_n$ since X_n/X_0 is unramified.

Rearranging the terms,

$$
f_n = 2 - p^n(v_0 - e_0) = 2 + p^n(f_0 - 2).
$$

By Lemma [2.9,](#page-4-2) we know that X_n^{\vee} has exactly two totally ramified vertices and that all other vertices are unramified. By construction, the matrix $D - B(T)$ has the form

The determinant of this matrix is clearly divisible by T^2 . The claim follows from $[GV24,$ Theorem 5.9 and the fact that $char_{\Lambda}(Jac_{\Lambda}(X_{\infty})) = char_{\Lambda}(Jac_{\Lambda}(X_{\infty}^{\vee}))$. $\mathbb{Z}^{(1)}_{\infty}$)).

Example 5.15. The above corollary can be applied to the unramified cover given in Example [5.5.](#page-17-1) Indeed, if one computes the characteristic ideal of $Jac(X_{\infty})$ one obtains 4T which is clearly divisible by T.

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