An Algebraic Generalization of the Ramanujan Sum

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In honor of George Andrews and Bruce Berndt on the occasion of their 85th birthday.

Abstract

Ramanujan sums have attracted significant attention in both mathematical and engineering disciplines due to their diverse applications. In this paper, we introduce an algebraic generalization of Ramanujan sums, derived through polynomial remaindering. This generalization is motivated by its applications in Restricted Partition Theory and Coding Theory. Our investigation focuses on the properties of these sums and expresses them as finite trigonometric sums subject to a coprime condition. Interestingly, these finite trigonometric sums with a coprime condition, which arise naturally in our context, were recently introduced as an analogue of Ramanujan sums by Berndt, Kim, and Zahaescu. Furthermore, we provide an explicit formula for the size of Levisthesin codes with an additional parity condition (also known as Shifted Varshamov-Tenengolts deletion correction codes), which have found many interesting applications in studying the Little-Offord problem, DNA-based data storage and distributed synchronization. Specifically, we present an explicit formula for a particularly important open case $\mathrm{SVT}_{t,b}(s \pm \delta, 2s+1)$ for s or $s+1$ are divisible by 4 and for small values of δ.

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1 Introduction

Using the notation in [7], we denote a polynomial

$$R_k(q) = \sum_{t=0}^{k-1} c_k(t) q^t,$$

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where $c_k(t)$ is the Ramanujan (trigonometric) sum for t an integer. In this paper, our motivation for a generalization of the Ramanujan sum is the elegant congruence relation [8]:

$$R_k(q) \equiv (q)_{k-1} \mod 1 - q^k, \tag{1}$$

where $(q)_k = (1-q)\cdots(1-q^k)$ denotes the restricted q-product for a positive integer k. As $R_k(q)$ is a polynomial of degree $(k-1)$, it is indeed the remainder when $(q)_{k-1}$ is divided by $1-q^k$. The equation (1) also provides a combinatorial interpretation:

$$c_k(t) = \begin{pmatrix} \text{number of even-sized} \\ \text{subsets of } \mathcal{A}_k \text{ whose} \\ \text{elements sum to } t \bmod k \end{pmatrix} - \begin{pmatrix} \text{number of odd-sized} \\ \text{subsets of } \mathcal{A}_k \text{ whose} \\ \text{elements sum to } t \bmod k \end{pmatrix}, \tag{2}$$

where $\mathcal{A}_k = \{1, 2, \ldots, k-1\}$, and by convention, we consider the empty set to be even-sized.

Surprisingly, although several generalizations of the Ramanujan sum have been considered in the literature, none of them are based on the above combinatorial interpretation. In this paper, we present a combinatorial-based generalization of the Ramanujan sum using polynomial remaindering. Our approach centers on the base set $\mathcal{A}_s$ for various values of s within the range $1 \le s \le k$. We denote this generalization as $\sigma_k^{(b)}(t; s)$, introducing two additional parameters: s and b. A notable feature of our generalization is its linear recurrence property with respect to both t and s.

The generalization $\sigma_k^{(b)}(t; s)$ essentially consists of two types of finite trigonometric sums: one is a direct sum, while the other is subject to a coprime condition (similar to the one given in [4] and the references therein). For $1 \le s \le k$,

$$\sigma_k^{(0)}(t; s) = \frac{1}{k} \sum_{j=0}^{k-1} \alpha^{-jt} (\alpha^j)_{s-1} \qquad \text{and} \qquad \sigma_k^{(1)}(t; s) = \frac{1}{k} \sum_{\substack{j=0 \\ (j,k)=1}}^{k-1} \alpha^{-jt} (\alpha^j)_{s-1},$$

where $\alpha = e^{2\pi i/k}$. Thus, $\sigma_k^{(1)}$ has an additional coprime condition compared to $\sigma_k^{(0)}$. Additionally, from the observation that $(\alpha^j)_{k-1} = k$ if $\gcd(j, k) = 1$ and 0 otherwise, we have

$$\sigma_k^{(b)}(t; k) = c_k(t).$$

Recently, in [8], we observed that the values $\sigma_k^{(1)}(t; s)$ serve as coefficients within the framework of Sylvester Wave Theory as applied to restricted partitions. In this work, we explore this generalization further and present a relevant formula in coding theory.

It is well-known that Ramanujan sums play a crucial role in determining the size of a specific class of deletion codes known as Levisthesin codes [1]. These codes are quite versatile, capable of correcting various types of errors, including deletions, insertions, bit reversals, and transpositions of adjacent bits (see [1] and the references therein). Despite their usefulness in numerous applications such as the Little-Offord problem, DNA-based data storage and distributed synchronization, many problems related to these codes remain open.

For positive integers k, s with $1 \le s \le k$, Levisthesin codes $L_t(s, k)$ is a set of binary codes $(b_{s-1}, \cdots, b_1)$ that correct single-bit deletions, defined by

$$\sum_{j=1}^{s-1} j b_j \equiv t \bmod k$$

Clearly, this congruence relation is associated with the distinct partitions over the set $\mathcal{A}_s$ given above. Levenshtein introduced the concept of edit distance, showing that $L_t(s, 2s+1)$ can correct single deletion, insertion, transposition, and substitution errors. Bibak et al. [2] provided an explicit formula for $|L_t(s, 2s+1)|$, the cardinality of $L_t(s, 2s+1)$, in terms of the Ramanujan sums.

Levisthesin codes with the parity condition (also known as Shifted Varshamov-Tenengolts codes) $\mathrm{SVT}_{t,b}(s; k)$ are a subset of $L_t(s, k)$ that include an additional parity condition, i.e.,

$$\sum_{j=1}^{s-1} jb_j \equiv t \bmod k \quad \text{and} \quad \sum_{j=1}^{s-1} b_j \equiv b \bmod 2.$$

From this definition, we have $|L_t(s, 2s+1)| = |\mathrm{SVT}_{t,0}(s, k)| + |\mathrm{SVT}_{t,1}(s, k)|$. The reason these codes are called "shifted" is that they can correct a single deletion when the location of the deleted bit is known to be within certain consecutive positions. Bibak et al. [1] provided a formula using trigonometric functions. In this work, we present an explicit formula for the significant case $\mathrm{SVT}_{t,b}(s \pm \delta, 2s+1)$ for small values of δ such as $0, 1, 2, 3$, with $s \equiv 0$ or 3 mod 4. The results for this case are known in the literature, and determining the scenario when $\delta > 0$ is particularly challenging. The elegance of our approach lies in deriving explicit formulas for these difficult cases using only linear recurrence and the Gauss sum, which is a novel contribution to this literature.

The remainder of the paper is organized into two sections. In Section 2, we discuss the definition, general properties and a combinatorial interpretation of our generalization, along with the associated trigonometric sums. In Section 3, we derive the size of $\mathrm{SVT}_{t,b}(s, 2s+1)$ for cases where $4 \mid s$ or $4 \mid (s+1)$.

2 Our Generalization and its Properties

Let $(q)_k = (1-q)\cdots(1-q^k)$ and $(q)_0 = 1$ be the restricted q-products. Our parametric generalization of the Ramanujan sum is based on polynomial remaindering.

Definition 2.0.1. *[Generalization of the Ramanujan Sum] Let b and t be two non-negative integers, s and k be a positive integers such that $1 \le s \le k$. The generalized Ramanujan sum $\sigma_k^{(b)}(t; s)$ is given as the coefficients of the polynomial*

$$R_{k,s}^{(b)}(q) = \sum_{t=0}^{k-1} \sigma_k^{(b)}(t; s) q^t = \frac{1}{k^b} (q)_{k-1}^b (q)_{s-1} \pmod{1 - q^k}.$$

As $R_{k,s}^{(b)}(q)$ is a polynomial of degree $(k-1)$, it is in fact the remainder when divided by $1 - q^k$.

As discussed in the Introduction, the parameter s extends the combinatorial meaning of the Ramanujan sum. In fact, $s = k$ corresponds to the Ramanujan sum case, i.e., $\sigma_k^{(b)}(t; k) = c_k(t)$. Furthermore, the parameter b is crucial for generating two distinct types of finite trigonometric sums, particularly for small values of s.

Similar to the Ramanujan sum, the sequence of values $\sigma_k^{(b)}(t;s)$ with respect to t is a periodic sequence with period k. Specifically, for $t \geq 1$, we have

$$\sigma_k^{(b)}(t;s) = \sigma_k^{(b)}(t\%k;s),$$

where % denotes the remainder operator. A key feature of these sums is that they satisfy a generalized Pascal-like linear recurrence relation.

Theorem 2.0.1 (Linear Recurrence)**.** *For $k \geq 1$ and $1 \leq k < s$ we have*

$$\sigma_k^{(b)}(t;s+1) = \sigma_k^{(b)}(t;s) - \sigma_k^{(b)}(t-s;s).$$

Proof. Follows from observing that $(q)_s = (q)_{s-1} - q^s(q)_{s-1}$. □

Using the above recurrence relation one can perform efficient computations of the values with the initial data (Proposition 2.0.4):

$$\sigma_k^{(0)}(t;1) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere.} \end{cases}$$

For $b \geq 1$,

$$\sigma_k^{(b)}(t;1) = \frac{1}{k} c_k(t).$$

The time required to compute all the values $\sigma_k^{(b)}(t;s)$ will be linear with respect to the total number of values. Using the linear recurrence, we can easily obtain values of $\sigma_k^{(b)}(t;s)$ for specific cases. For instance, we have the following result.

Proposition 2.0.2. *For p prime we have*

$$\sigma_p^{(b)}(t;p-1) = t - \frac{p+1}{2} \text{ for } 1 \leq t \leq p.$$

Proof. By linear recurrence and

$$\sigma_p^{(b)}(p;p) = c_p(p) = p-1 \text{ and } \sigma_p^{(b)}(t;p) = c_p(t) = -1 \text{ for } 1 \leq t \leq p-1.$$

□

Ramanujan sum has the following beautiful arithmetical structure:

$$c_k(t) = \sum_{d|(k,t)} \mu\left(\frac{k}{d}\right) d.$$

Using the above linear recurrence one can understand the arithmetical structure of $\sigma_k^{(1)}(t;s)$. Indeed, we have

$$\sigma_k^{(1)}(t;2) = \frac{1}{k}\left(c_k(t) - c_k(t-1)\right)$$

and

$$\sigma_k^{(1)}(t;3) = \frac{1}{k}\left(c_k(t) - c_k(t-1) - c_k(t-2) + c_k(t-3)\right).$$

$s \backslash t$	0	1	2	3	4	5
1	1	0	0	0	0	0
2	1	-1	0	0	0	0
3	1	-1	-1	1	0	0
4	0	-1	-1	0	1	1
5	1	-1	-2	-1	1	2
6	2	1	-1	-2	-1	1

$s \backslash t$	0	1	2	3	4	5
1	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
3	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
4	0	-1	-1	0	1	1
5	1	-1	-2	-1	1	2
6	2	1	-1	-2	-1	1

Table 1: The table on the left displays $\sigma_6^{(0)}(t;s)$, while the table on the right displays $\sigma_6^{(b)}(t;s)$ for $b \geq 1$.

For $k > 2$, we can consolidate the above equations for $s \in \{1,2,3\}$ and $s \leq t \leq k+s-1$ as follows:

$$\sigma_k^{(1)}(t;s) = \frac{1}{k}\sum_{\delta=0}^{s-1}\sum_{d|(k,t-\delta)}(-1)^{\mathrm{wt}(\delta)}\mu\left(\frac{k}{d}\right)d,$$

where $\mathrm{wt}(\delta)$ denotes the number of 1s in the binary representation of δ. This expression shows that the value of $\sigma_k^{(1)}(t;s)$ depends on the greatest common divisor of k and the translates of t.

Table 1 provides values for the case $k = 6$. From the table, it can be observed that the values $\sigma_k^{(b)}(t;s)$ are either integers or rational numbers of the form $\frac{a}{k}$, where a is an integer and k is the given base. Moreover, after a certain s, the rows in both tables are equal. The last row corresponds to the Ramanujan sum. These observations can be made more precise for the general case using the following lemma.

Lemma 2.0.3. *For $b \geq 1$, we have*

$$\sigma_k^{(b)}(t;s) = \frac{1}{k}\sum_{\xi\in\Delta_k}(\xi)_{s-1}\xi^{-t}, \tag{3}$$

where $\Delta_k = \{\alpha^h : \alpha = e^{2\pi i/k}, (h,k) = 1 \text{ and } 1 \leq h \leq k\}$ is the collection all primitive k^{th} roots of unity.

Proof. Since $\left\{\sigma_k^{(b)}(t;s)\right\}_{t=0}^{\infty}$ is a periodic sequence, the generating function

$$F(q) = \sum_{t\geq 0}\sigma_k^{(b)}(t;s)q^t = R_{k,s}^{(b)}(q)\left(\sum_{t=0}^{\infty}q^{kt}\right),$$

has a finite Fourier series expansion given by

$$F(q) = \sum_{t\geq 0}\sigma_k^{(b)}(t;s)q^t = \sum_{t\geq 0}\left(\sum_{j=0}^{k-1}a(j)\xi^{jt}\right)q^t, \tag{4}$$

where $a(j) = \frac{1}{k} R_{k,s}^{(b)}(\xi^{-j})$ and $\xi = e^{2\pi i/k}$. Comparing the coefficients of any fixed q^t gives us the finite Fourier series for $\sigma_k^{(b)}(t;s)$, namely

$$\sigma_k^{(b)}(t;s) = \frac{1}{k} \sum_{j=0}^{k-1} R_{k,s}^{(b)}(\xi^{-j}) \xi^{jt}.$$

As ξ^{-j}, for $0 \le j < k$, is a k^{th} root of unity, using

$$(\xi^{-j})_{k-1} = (1-\xi^{-j})(1-\xi^{-2j}) \cdots (1-\xi^{-j(k-1)}) = \begin{cases} k & \text{for } (j,k)=1 \\ 0 & \text{otherwise} \end{cases},$$

we obtain

$$a(j) = \frac{1}{k} R_{k,s}^{(b)}(\xi^{j}) = \begin{cases} \frac{1}{k}(\xi^{-j})_{s-1} & \text{for } (j,k)=1 \\ 0 & \text{otherwise} \end{cases}.$$

Since Δ_k is defined only for $(j,k)=1$ and by (4) we have the result. □

Remark 2.0.1. *Our generalization clearly extends the Ramanujan sum in a direct and insightful manner. For $s = k$ and $b \ge 0$, we have*

$$c_k(t) = \sigma_k^{(b)}(t;k) = \frac{1}{k} \sum_{\xi \in \Delta_k} (\xi)_{k-1} \xi^{-t} = \sum_{\xi \in \Delta_k} \xi^{-t},$$

which precisely matches the definition originally provided by Ramanujan [6].

We can compute the initial values with the above formula.

Corollary 2.0.4. *For $k, b \ge 1$ and $0 \le t < k$ we have*

$$\sigma_k^{(b)}(t;1) = \frac{1}{k} \sigma_k^{(b)}(t;k) = \frac{1}{k} c_k(t).$$

Proof. Using $(\xi)_{k-1} = k$ for $\xi \in \Delta_k$,

$$\sigma_k^{(b)}(t;k) = \frac{1}{k} \sum_{\xi \in \Delta_k} (\xi)_{k-1} \xi^{-t} = c_k(t) = k \left(\frac{1}{k} \sum_{\xi \in \Delta_k} \xi^{-t} \right) = k \sigma_k^{(b)}(t;1).$$

□

Notice that the right hand side of (3) is independent of b. Therefore, it suffices to consider the values of $\sigma_k^{(b)}(t;s)$ only for the two cases $b = 0, 1$.

Corollary 2.0.5. *For $b \ge 1$, $\sigma_k^{(b+1)}(t;s) = \sigma_k^{(b)}(t;s)$.*

Following similar steps of the proof of Lemma 2.0.3 and by the finite Fourier series, $\sigma_k^{(0)}(t;s)$ is given by

$$\sigma_k^{(0)}(t;s) = \frac{1}{k} \sum_{j=0}^{k-1} (\alpha^j)_{s-1} \alpha^{-jt}, \tag{5}$$

where $\alpha = e^{2\pi i/k}$. Notice that there is not coprime condition. Nevertheless, it is easy to show that the sum $\sigma_k^{(0)}(t;s)$ could have a coprime condition for larger values of s.

Lemma 2.0.6. *Let $s \geq k/p$, where p is the smallest prime divisor of k ($p = 1$ if k is a prime). Then,*

$$\sigma_k^{(0)}(t;s) = \frac{1}{k} \sum_{\substack{j=0 \\ (j,k)=1}}^{k-1} \alpha^j \alpha^{-jt} = \frac{1}{k} \sum_{\xi \in \Delta_k} (\xi)_{s-1} \xi^{-t}, \tag{6}$$

where $\alpha = e^{2\pi i/k}$ and Δ_k is the collection of primitive k-th roots of unity. Consequently,

$$\sigma_k^{(1)}(t;s) = \sigma_k^{(0)}(t;s) \quad \textit{for} \quad s \geq k/p, \tag{7}$$

and, thus, $\sigma_k^{(1)}(t;s)$ is an integer.

Proof. From the case $s \geq k/p$ we can see that the set $\{k-s, \cdots, k-1\}$ does not contain any divisor of k. Thus, the factors $(1 - x^{k-s-1}), \cdots, (1 - x^{k-1})$ do not vanish for any $x = \alpha^j, j = 1, 2, \cdots, (k-1)$. Therefore, we have $R_{k,s}^{(0)}(\alpha^j) = (\alpha^{-j})_{s-1}$ for $(j,k) = 1$ and $R_{k,s}^{(0)}(\alpha^j) = 0$ for $(j,k) \neq 1$. The result follows. □

Corollary 2.0.7. *For $b \geq 0$ and $s \geq k/p$ where p is the smallest prime dividing k ($p = 1$ if k is a prime) we have*

$$R_{k,s}^{(b)}(q) = (q)_{s-1} \mod 1 - q^k.$$

Consequently, for this case the value $\sigma_k^{(b)}(t;s)$ is an integer.

Theorem 2.0.8 (Backward Linear Recurrence)**.** *For $S > 0$,*

$$\sigma_k^{(b)}(t;s) = -\frac{1}{k} \sum_{j=1}^{k-1} j \sigma_k^{(b)}(t - js; s+1)$$

Proof. Let ξ be a k^{th} root of unity. Then, we have

$$(1-\xi)(\xi + 2\xi^2 + \cdots + (k-1)\xi^{k-1}) = -k.$$

Consider

$$\begin{aligned} \sigma_k^{(1)}(t;s) &= \frac{1}{k} \sum_{\xi \in \Delta_k} (\xi)_{s-1} \xi^{-t} \frac{1-\xi^s}{1-\xi^s} = -\frac{1}{k^2} \sum_{\xi \in \Delta_k} (\xi)_s \xi^{-t} \sum_{j=1}^{k-1} j \xi^{js} \\ &= -\frac{1}{k} \sum_{j=1}^{k-1} j \sigma_k^{(1)}(t - js; s+1). \end{aligned}$$

Similarly, we can obtain an expression for $\sigma_k^{(0)}(t;s)$. □

2.1 Finite Trigonometric Sums

Our generalization introduces finite trigonometric sums subject to a coprime condition, which were recently proposed as analogues to the Ramanujan sum in [4]. This extension not only enriches the theoretical framework but also opens new avenues for practical applications in related fields.

We have two types of sums: when $b = 0$, the sums are direct sums, while for $b = 1$, the sums are analogues to the Ramanujan sum under a coprime condition. The two sums are

$$\sigma_k^{(0)}(t;s) = \frac{1}{k}\sum_{j=0}^{k-1} \alpha^{-jt}(\alpha^j)_{s-1} \qquad \text{and} \qquad \sigma_k^{(1)}(t;s) = \frac{1}{k}\sum_{\substack{j=0\\(j,k)=1}}^{k-1} \alpha^{-jt}(\alpha^j)_{s-1},$$

where $\alpha = e^{2\pi i/k}$. By Lemma 2.0.3 we can write

$$\sum_{d|k}\frac{d}{k}\sigma_d^{(1)}(t;s) = \frac{1}{k}\sum_{d|k}\sum_{\xi\in\Delta_d}\xi^{-t}(\xi)_{s-1} = \frac{1}{k}\sum_{j=0}^{k-1}\alpha^{-jt}(\alpha^j)_{s-1} = \sigma_k^{(0)}(t;s),$$

where α is a primitive k^{th} root of unity. Thus, the two types of sums are connected with the Dirichlet convolution.

Proposition 2.1.1.

$$k\sigma_k^{(0)}(t;s) = \sum_{d|k} d\sigma_d^{(1)}(t;s)$$

where we assume that $\sigma_d^{(1)}(t;s) = 0$ if $s > d$. That is, $k\sigma_k^{(0)} = I * (k\sigma_k^{(1)})$, where $*$ is the Dirichlet convolution and $I(n) = 1$ constant function. By Möbius inversion we have

$$k\sigma_k^{(1)}(t;s) = \sum_{d|k}\mu\left(\frac{k}{d}\right)d\sigma_d^{(0)}(t;s).$$

As one would expect the analogues of the Ramanujan sums considered in [4] are indeed related to the Ramanujan sum. We will explain the idea through the simple case $\sigma_k^{(b)}(0;2)$. Consider

$$\sigma_k^{(0)}(0;2) = \frac{1}{k}\sum_{j=0}^{k-1}(1-\xi^j),$$

where $\xi = e^{2\pi i/k}$. Taking the factor $\xi^{l/2}$ out and mutliplying and dividing by $2i$ we get

$$\sigma_k^{(0)}(0;2) = -2i\frac{1}{k}\sum_{j=0}^{k-1}\left(\frac{\xi^{j/2}-\xi^{-j/2}}{2i}\right)\xi^{j/2}.$$

Using the formulae $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $e^{i\theta} - \cos\theta + i\sin\theta$, we have

$$\sigma_k^{(0)}(0;2) = -2i\frac{1}{k}\sum_{j=0}^{k-1}\sin\frac{\pi j}{k}(\cos\frac{\pi j}{k} + i\sin\frac{\pi j}{k}).$$

Comparing the real parts we have the identity

$$\sigma_k^{(0)}(0;2) = \frac{2}{k}\sum_{j=0}^{k-1} \sin^2\left(\frac{\pi j}{k}\right).$$

Similarly, we have

$$\sigma_k^{(1)}(0;2) = \frac{2}{k}\sum_{\substack{j=0\\(j,k)=1}}^{k-1} \sin^2\left(\frac{\pi j}{k}\right).$$

Further, the values of $\sigma_k^{(b)}(0;2)$ can be easily determined:

$$\sigma_k^{(1)}(0;2) = \frac{1}{k}\sum_{\substack{j=1\\(j,k)=1}}^{k}(1-\xi^j) = \frac{1}{k}\sum_{\substack{j=1\\(j,k)=1}}^{k}\xi^0 - \frac{1}{k}\sum_{\substack{j=1\\(j,k)=1}}^{k}\xi^j = \frac{c_k(k)-c_k(k-1)}{k},$$

where $\xi = e^{2\pi i/k}$.

Theorem 2.1.2. *For $k \geq 1$ we have the following identities:*

$$\sum_{j=0}^{k-1}\sin^2\left(\frac{\pi j}{k}\right) = \frac{k}{2}$$

and

$$\sum_{\substack{j=0\\(j,k)=1}}^{k-1}\sin^2\left(\frac{\pi j}{k}\right) = \frac{c_k(k)-c_k(k-1)}{2}.$$

Now, let us consider $\sigma_k^{(b)}(0;3)$. Taking a similar approach as before we have

$$\sigma_k^{(0)}(0;3) = \frac{1}{k}\sum_{j=0}^{k-1}(1-\xi^j)(1-\xi^{2j}),$$

where $\xi = e^{2\pi i/k}$. By pulling out the factors $\xi^{j/2}$ and $\xi^{2j/2}$ and multiplying and diving by $2i$ we have

$$\sigma_k^{(0)}(0;3) = -\frac{4}{k}\sum_{j=0}^{k-1}\frac{(\xi^{j/2}-\xi^{-j/2})}{2i}\frac{(\xi^{j}-\xi^{-j})}{2i}\xi^{3j/2}.$$

Again, using the trigonometric formulae and comparing the real and imaginary parts we have:

Theorem 2.1.3. *For $k > 2$ we have the following results:*

1.

$$\sum_{l=0}^{k-1}\sin\left(\frac{\pi l}{k}\right)\sin\left(\frac{2\pi l}{k}\right)\cos\left(\frac{3\pi l}{k}\right) = \begin{cases}-\frac{k}{4} & k \geq 4\\ -\frac{3}{2} & k = 3\end{cases}$$

2.

$$\sum_{\substack{1\le l\le k\\(l,k)=1}} \sin\left(\frac{\pi l}{k}\right)\sin\left(\frac{2\pi l}{k}\right)\cos\left(\frac{3\pi l}{k}\right) = \frac{c_k(k-1)+c_k(k-2)-c_k(k)-c_k(k-3)}{4}$$

3.

$$\sum_{l=0}^{k-1}\sin\left(\frac{\pi l}{k}\right)\sin\left(\frac{2\pi l}{k}\right)\sin\left(\frac{3\pi l}{k}\right) = 0$$

4.

$$\sum_{\substack{1\le l\le k\\(l,k)=1}} \sin\left(\frac{\pi l}{k}\right)\sin\left(\frac{2\pi l}{k}\right)\sin\left(\frac{3\pi l}{k}\right) = 0.$$

We now consider the case s divisible by 4 and $k \geq s(s+1)/4$.

$$\sigma_k^{(1)}(k - s(s+1)/4; s) = \frac{1}{k}\sum_{\xi\in\Delta_k}(1-\xi)(1-\xi^2)\cdots(1-\xi^s)\xi^{-(k-s(s+1)/4)}.$$

Taking a factor $\xi^{j/2}$ from each of the j^{th} factor, multiplying and dividing by $2i$ we get

$$\sigma_k^{(0)}\left(k - \frac{s(s+1)}{4}; s+1\right) = 2^s\sum_{h=1}^{k}\sin\frac{h\pi}{k}\cdots\sin\frac{sh\pi}{k}$$

$$\sigma_k^{(1)}\left(k - \frac{s(s+1)}{4}; s+1\right) = 2^s\sum_{\substack{h=1\\(h,k)=1}}^{k}\sin\frac{h\pi}{k}\cdots\sin\frac{sh\pi}{k}$$

Theorem 2.1.4. *Let k and s be two positive integers such that s is divisible by 4 and $k/p \leq s \leq k$ for p the smallest prime divisor of k. Then, the following equation holds:*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k}\sin\frac{h\pi}{k}\cdots\sin\frac{sh\pi}{k} = \sum_{h=1}^{k}\sin\frac{h\pi}{k}\cdots\sin\frac{sh\pi}{k} = \frac{1}{2^s}\sigma_k^{(b)}\left(k-\frac{s(s+1)}{4}; s+1\right).$$

Suppose $s = 4$ then from the above equation we have

$$\frac{1}{2^4}\sigma_k^{(b)}(k-5;5) = \sum_{\substack{h=1\\(h,k)=1}}^{k}\sin\frac{h\pi}{k}\sin\frac{2h\pi}{k}\sin\frac{3h\pi}{k}\sin\frac{4h\pi}{k}.$$

For sufficiently large k, that is k larger than 30 we have the values to be zero.

2.2 Combinatorial Interpretation

As stated in the Introduction, our algebraic generalization of the Ramanujan sum in Definition 2.0.1 directly reflects combinatorial significance through the coefficients of the restricted q-product. Suppose we write the q-product in the form

$$(q)_{s-1} = (1-q)\cdots(1-q^{s-1}) = \sum_{m=0}^{s(s-1)/2} a_m q^m.$$

Then, the coefficient a_m can be interpreted as difference of number of even-size partition ($E_s(m)$) and odd-size partitions ($O_s(m)$) of m considering distinct part from the set $\{1,2,\cdots,s-1\}$, namely $a_m = E_s(m) - O_s(m)$.

The polynomial remainder with $1-q^k$ can also be treated as sieving of the coefficients of a polynomial. In particular, we have

$$R_{k,s}^{(0)}(q) = \sum_{t=0}^{k-1} \sigma_k^{(0)}(t;s) q^t = (q)_{s-1} \mod 1-q^k.$$

Therefore,

$$\sigma_k^{(0)}(t;s) = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} a_{jk+t} = \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \left(E_s(jk+t) - O_s(jk+t)\right).$$

The case $s=k$ corresponding to the Ramanujan sum is well known [5].

Corollary 2.2.1. *For $k>0$ and $1 \le s \le k$, we can write*

$$\sigma_k^{(0)}(t;s) = \sum_{\mathrm{sum}_k(A)=t} (-1)^{|A|}, \qquad \text{for } A \subseteq \{1,2,\cdots,s-1\}, \tag{8}$$

where $|A|$ is the cardinality of A and $\mathrm{sum}_k(A)$ stands for the sum modulo k of the elements of A, with the convention $\mathrm{sum}_k(\emptyset)=0$.

Proof. The largest sum possible with distinct parts from $\{1,2,\cdots,s-1\}$ is $s(s-1)/2$. By the combinatorial interpretation given above and iterating over various sums considering the number of even-size and odd-size sums we have:

$$\sigma_k^{(0)}(t;s) = \sum_{\substack{l=0 \\ l\%k=t}}^{s(s-1)/2} (E_s(l) - O_s(l)) = \sum_{\substack{\mathrm{sum}_k(A)=t \\ |A| \text{ even}}} 1 \; - \sum_{\substack{\mathrm{sum}_k(B)=t \\ |B| \text{ odd}}} 1,$$

where $A, B \subseteq \{1,2,\cdots,s-1\}$. □

The combinatorial interpretation of $\sigma_k^{(1)}(t;s)$ is a bit cumbersome but it is also straightforward. By the sieving of coefficients, for an arbitrary $s \in \{0,1,\cdots,k-1\}$, we have the following interpretation from the defintion:

$$\sigma_k^{(1)}(t;s) = \frac{1}{k} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \left(\tilde{E}_{k,s}(jk+t) - \tilde{O}_{k,s}(jk+t)\right).$$

Here $\tilde{E}_{k,s}(l)$ is the number of partitions of l into an even number of parts each less than k and permitting parts of size $< s$ at most twice and others at most once. Similarly, $\tilde{O}_{k,s}(l)$ is the number of partitions of l into an odd number of parts each less than k and permitting parts of size $< s$ at most twice and others at most once.

From the integrality of $\sigma_k^{(1)}(t;s)$, for $s \geq \frac{k}{p}$, where p is the smallest prime divisor of k (with $p = 1$ when k is prime), we have demonstrated that the difference in parity is a multiple of k for sufficiently large s.

From a combinatorial perspective, our generalization accurately captures the parity difference among distinct partitions. This is particularly evident when s divides k. If s divides k, then $1 - q^s$ divides $1 - q^k$. Consequently,

$$R_s(q) = R_{k,s}^{(0)}(q) \mod 1 - q^s.$$

By sieving the indices of the coefficients of $R_{k,s}(q)$ with s and comparing terms from the above equation, we obtain the following result:

Proposition 2.2.2. *For positive integers k and s with $s \mid k$, we have*

$$c_s(t) = \sum_{j=0}^{k/s-1} \sigma_k^{(0)}(js + t; s).$$

This equation demonstrates that the generalized Ramanujan sum effectively captures the parity difference among distinct partitions. Specifically, the parity difference for $t \mod s$ is equivalent to the sum of the parity differences for $t + js \mod k$, where $0 \leq j < k/s$.

Proposition 2.2.3. *For positive integers k and s with $s \mid k$, we have*

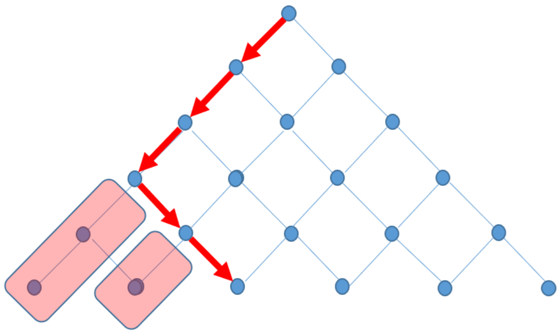
$$\sum_{j=0}^{k/s-1} \sigma_k^{(1)}(js + t; s) = 0.$$

Example 2.2.1. *Let $k = 12$ and $s = 6$. Then, we have*

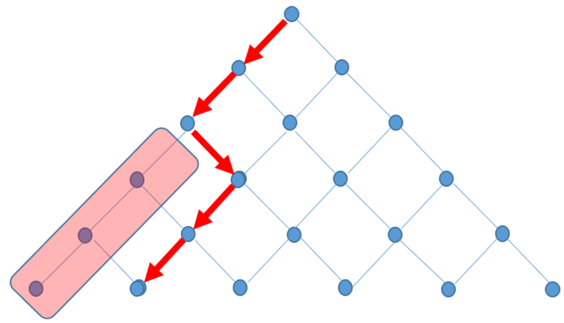
$$\begin{aligned} R_{12,6}(x) &= (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \mod (1-x^{12}) \\ &= -x^{10} - x^9 - x^8 + x^7 + x^6 + x^5 - x^3 + 1 \\ &= \sum_{t=0}^{11} \sigma_{12}(t;6) x^t. \end{aligned} \tag{9}$$

Consider the case $t = 3$. Then, we have the sets $\{1,2\}, \{3\}, \{4,5\}, \{2,3,4\}, \{1,3,5\}$ and $\{1,2,3,4,5\}$ satisfying the property $\mathrm{sum}_{12}(A) = t$. These subsets can also be visualized as paths in a Pascal-Lattice (see Figure 1). One can also represent these paths as binary vectors [9]. From (Corollary 2.2.1) we have

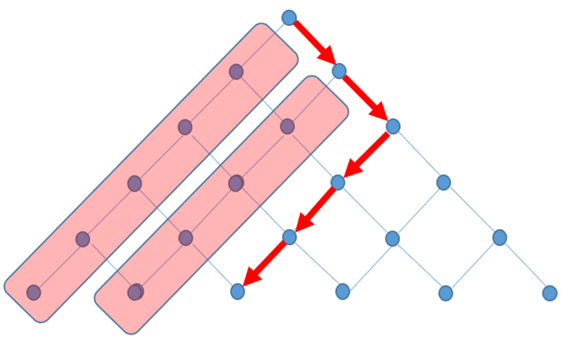
$$\sigma_{12}(3;6) = (-1)^{|\{1,2\}|} + (-1)^{|\{3\}|} + (-1)^{|\{1,2,3,4,5\}|} = 1 - 2 = -1, \tag{10}$$
$$\sigma_{12}(9;6) = (-1)^{|\{4,5\}|} + (-1)^{|\{2,3,4\}|} + (-1)^{|\{1,3,5\}|} = 1 - 2 = -1. \tag{11}$$



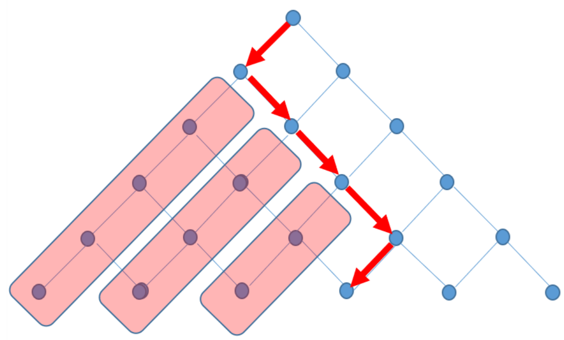
Even number of parts: $2 + 1$



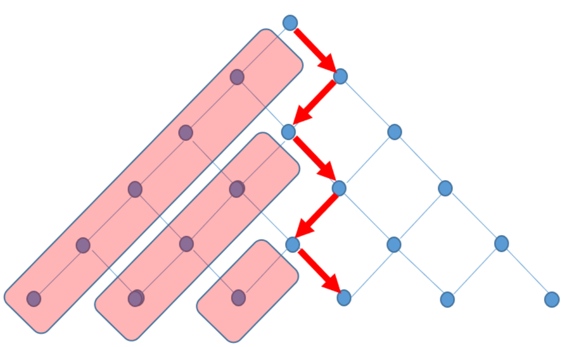
Odd number of parts: 3



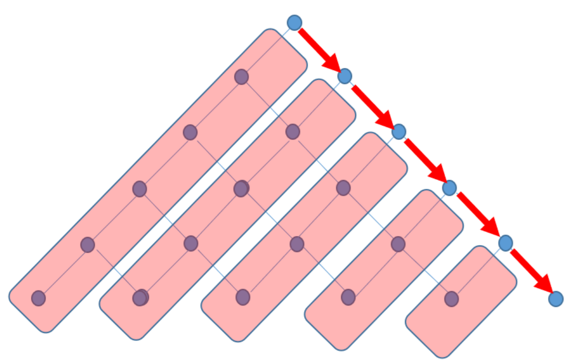
Even number of parts: $5 + 4$



Odd number of parts: $4 + 3 + 2$



Odd number of parts: $5 + 3 + 1$



Odd number of parts: $5 + 4 + 3 + 2 + 1$

Figure 1: Case $k = 6$ and $s = 6$: $c_6(3) = \#\text{even partitions} - \#\text{odd partitions} = 2 - 4 = -2$. Alternatively, consider $k = 12$ and $s = 6$ to compute $\sigma_{12}(3; 6)$ and $\sigma_{12}(9; 6)$ from equations (10) and (11) to obtain $c_6(t) = \sigma_{12}(3; 6) + \sigma_{12}(9; 6)$.

We could also the case as $k = 6$ and $s = 6$ in order to compute $c_6(t)$. In this case,

$$R_{6,6}(x) = x^5 - x^4 - 2x^3 - x^2 + x + 2 = \sum_{t=0}^{5} c_6(t)x^t.$$

We have

$$c_6(3) \;\; = \;\; \sigma_{12}(3;6) + \sigma_{12}(9;6) = -1 - 1 = -2.$$

The above decomposition of Ramanujan sum can be extended to obtain natural decomposition of arithmetic functions based on their expansion. Suppose an arithmetic function is expanded as

$$f(n) = \sum_{s=1}^{\infty} a_s c_s(t),$$

where the series converges absolutely. Let $k = sm$. Then, we can write

$$f(n) = \sum_{s=1}^{\infty} a_s c_s(n) = \sum_{s=0}^{\infty} a_s \left(\sum_{j=0}^{m-1} \sigma_{ms}^{(0)}(js + n; s) \right) = \sum_{j=0}^{m-1} \sum_{s=0}^{\infty} a_s \sigma_{ms}^{(0)}(js + n; s).$$

Defining the functions

$$f_j(n) = \sum_{s=1}^{\infty} a_s \sigma_{ms}^{(0)}(js + n; s),$$

for $j = 0, \cdots, m-1$. We have the decomposition $f(n) = \sum_{j=0}^{m-1} f_j(n)$. A question that naturally arises in this context is: What is the arithmetic function defined by

$$g_{r,j}(n) = \sum_{s=1}^{\infty} \frac{1}{s^r} \sigma_{ms}^{(0)}(js + n; s)$$

for $r > 0$? By the well-known Dirichlet series expansion we have

$$\sum_{j=0}^{m-1} g_{r,j}(n) = \sum_{s=1}^{\infty} \frac{1}{s^r} c_s(n) = \frac{\sigma_{r-1}(n)}{n^{r-1}\zeta(r)},$$

where σ_r is the divisor function.

We also have an explicit expression for the following function:

$$f_{\alpha,s}(t) = \sum_{k=1}^{\infty} \frac{\sigma_k^{(1)}(t;s)}{k^{\alpha}},$$

for $\alpha > 0$ a positive integer. Suppose $(1-x)_{s-1} = \sum_{j=0}^{s(s-1)/2} a_j x^j$. Then, we have

$$\sigma_k^{(1)}(t;s) = \frac{1}{k} \sum_{j=0}^{s(s-1)/2} a_j c_k(t-j),$$

where a_j is the difference of even parity distinct partitions of j and odd parity distinct partitions of j with parts from $\{1, 2, \cdots, s-1\}$. Substituting for $\sigma_k^{(1)}(t, s)$ we have

$$\begin{aligned} f_{\alpha,s}(t) &= \sum_{k=1}^{\infty} \frac{\sigma_k^{(1)}(t; s)}{k^\alpha} = \sum_{j=0}^{s(s-1)/2} a_j \sum_{k=1}^{\infty} \frac{c_k(t-j)}{k^{\alpha+1}} \\ &= \left(\frac{1}{\zeta(\alpha+1)} \sum_{\substack{j=0 \\ t \neq j}}^{s(s-1)/2} \frac{a_j \sigma_\alpha(|t-j|)}{|t-j|^\alpha} \right) + \frac{6}{\pi^2} \zeta(\alpha+1). \end{aligned}$$

3 Size of $\mathbf{SVT}_{t,b}(s, 2s+1)$, for $4|s$ or $4|(s+1)$

In this section we show that our algebraic generalization of the Ramanujan sum is helpful in the context of deletion correction codes. We will look at an application associated with the Levishtein codes with a parity condition (or the Shifted Varshamov-Tenengolts codes) used for deletion correction (see [1, 2] and references therein).

A coefficient of $(q)_{s-1} = (1-q) \cdots (1-q^{s-1})$ is determined by which terms in each factor are considered. This can be represented in a binary code form based on the notion of position sum. Let $\mathbf{x} = (b_1, \cdots, b_{s-1}) \in \mathbb{F}_2^{s-1}$. Then, define the position sum as

$$\mathrm{PS}(\mathbf{x}) = \sum_{j=1}^{s-1} j b_j.$$

The Hamming weight of a code is defined in the usual manner, $\mathrm{wt}(\mathbf{x}) = \sum_{j=1}^{s-1} b_j$.

Definition 3.0.1. *Let k, s be positive integers, $t \in \mathbb{Z}_k$, and $r \in \{0, 1\}$. The Shifted Varshamov-Tenengolts code $\mathrm{SVT}_{t,r}(s, k)$ is the set of all binary s-tuples $\mathbf{x} = (b_1, \cdots, b_{s-1})$ such that*

$$\mathrm{PS}(\mathbf{x}) = \sum_{j=1}^{s-1} j b_j \equiv t \mod k, \quad \mathrm{wt}(\mathbf{x}) = \sum_{j=1}^{s-1} b_j \equiv r \mod 2.$$

Corollary 3.0.1. *For $k > 0$ and $1 < s \leq k$ we have*

$$\sigma_k^{(0)}(t; s) = |\mathrm{SVT}_{t,0}(s, k)| - |\mathrm{SVT}_{t,1}(s, k)| = \sum_{\mathrm{PS}(\mathbf{x}) \% k \equiv t} (-1)^{\mathrm{wt}(\mathbf{x})}, \quad \mathbf{x} \in \mathbb{F}_2^{s-1}.$$

We will determine the size of $\mathrm{SVT}_{t,r}(s; 2s+1)$, where s or $s+1$ is divisible by 4.

Theorem 3.0.2 ([2], Theorem 4.5)**.**

$$\begin{aligned} |SVT_{t,1}(s, 2s+1)| \;&+\; |SVT_{t,0}(s, 2s+1)| = \\ &\frac{1}{2s+1} \sum_{d|2s+1} (-1)^{\frac{d - \boldsymbol{I}_{4|d-1}}{4}} 2^{\frac{2s+1-d}{2d}} c_d \left(\frac{1}{16} (d - \boldsymbol{I}_{4|d-1})(3d+1) - t \right), \end{aligned}$$

where $\boldsymbol{I}_{4|d-1}$ is 1 if 4 divides $d-1$ and -1 if 4 doesn't divide $d-1$.

In view of the above theorems, if one wishes to determine the size of $\mathrm{SVT}_{t,r}(s; 2s+1)$ we just need to determine $\sigma^{(0)}_{2s+1}(t; s+1)$.

Theorem 3.0.3. *For a positive number s such that $4|s$ or $4|(s+1)$, and $k = 2s+1$ we have*

$$|\mathrm{SVT}_{t,0}(s, 2s+1)| - |\mathrm{SVT}_{t,1}(s, 2s+1)| = \sigma^{(0)}_k(t; s+1) = \frac{1}{k}\sum_{j=0}^{k-1} c_k\left(t + j^2 + \frac{s(s+1)}{4}\right).$$

If k is a square then

$$|\mathrm{SVT}_{t,0}(s, 2s+1)| - |\mathrm{SVT}_{t,1}(s, 2s+1)| = \sigma_k(t; s+1) = \frac{1}{\sqrt{k}} c_k\left(t + \frac{s(s+1)}{4}\right).$$

Proof. For $\xi \in \Delta_k$ we have $(\xi)_{k-1} = k$. So, from $k = 2s+1$ we can write

$$\begin{aligned}(1-\xi)\cdots(1-\xi^{k-1}) &= (1-\xi)\cdots(1-\xi^s)(1-\xi^{s+1})\cdots(1-\xi^{2s})\\ &= (1-\xi)\cdots(1-\xi^s)(1-\xi^{-s})\cdots(1-\xi^{-1})\\ &= (-1)^s\xi^{-s(s+1)/2}(1-\xi)^2\cdots(1-\xi^s)^2 = k.\end{aligned}$$

By the quadratic Guass sum we have $\sqrt{k} = d_\xi \sum_{j=0}^{k-1} \xi^{j^2}$, d_ξ is a constant coming from the Jacobi symbol. Further, from [3, pg. no. 91], we have $G(\xi)^2 = (-1)^s k$ where $G(\xi) = \sum_{j=0}^{k-1} \xi^{j^2}$ is the Gauss sum. Therefore, we have the right hand side of the above expression

$$\xi^{-s(s+1)/2}(1-\xi)^2\cdots(1-\xi^s)^2 = (-1)^s k = G(\xi)^2.$$

Taking square root both sides we have

$$\xi^{-s(s+1)/4}(1-\xi)\cdots(1-\xi^s) = G(\xi) = \sum_{j=0}^{k-1}\xi^{j^2}. \tag{12}$$

Thus

$$(1-\xi)\cdots(1-\xi^s) = \sum_{j=0}^{k-1}\xi^{j^2+s(s+1)/4}.$$

Now, consider

$$\begin{aligned}\sigma^{(0)}_k(t; s+1) &= \frac{1}{k}\sum_{\xi\in\Delta_k}\xi^{-t}(1-\xi)\cdots(1-\xi^s)\\ &= \frac{1}{k}\sum_{\xi\in\Delta}\sum_{j=0}^{k-1}\xi^{-t}\xi^{j^2+s(s+1)/4} = \frac{1}{k}\sum_{j=0}^{k-1} c_k(j^2 + s(s+1)/4 - t).\end{aligned}$$

For the case when k is a perfect square, we have:

$$\begin{aligned}\sigma_k(t; s) &= \frac{1}{k}\sum_{\xi\in\Delta_k}\xi^{-t}(1-\xi)\cdots(1-\xi^s)\\ &= \frac{1}{k}\sum_{\xi\in\Delta_k}\sqrt{k}\xi^{-t+s(s+1)/4} = \frac{1}{\sqrt{k}}c_k(s(s+1)/4 - t).\end{aligned}$$

□

Remark 3.0.1. *It is interesting to note that, in the above calculation, without knowledge of the Gauss sum, one would obtain erroneous results. Consider the case $k = 17$ and $s = 8$. Using the derived formula, one would have*

$$\sigma_{17}^{(0)}(t; 9) = \frac{1}{17} \sum_{j=0}^{16} c_{17}(j^2 + 18 - t). \tag{13}$$

Instead of using (12), if we had substitutued $\sqrt{k}$ on the right-hand side, we would have

$$\sigma_{17}^{(0)}(t; 9) = \frac{1}{\sqrt{17}} c_{17}(18 - t), \tag{14}$$

which is clearly absurd as the problem at hand is a counting problem. This leads us to an important lesson: in calculations involving square roots, it is crucial to be mindful of the correct choice of $\sqrt{k}$.

From Theorem 3.0.2 and 3.0.3, we can obtain both an explicit and an efficiently computable formula for the size of $\mathrm{SVT}_{t,r}(s, 2s+1)$ when s or $s+1$ is divisible by 4. Furthermore, using the recurrence relation Theorem 2.0.1 and 2.0.8, we can also provide direct formulas for $\sigma_k^{(0)}(t; s \pm \delta) = \mathrm{SVT}_{t,0}(s \pm \delta, 2s + 1) - \mathrm{SVT}_{t,1}(s \pm \delta, 2s + 1)$ for $\delta = 1, 2, 3$ which are otherwise very difficult to determine. In order to compute $|\mathrm{SVT}_{t,b}(s \pm \delta, 2s + 1)|$ we define

$$\eta_k(t; s) = \frac{1}{k} \sum_{j=0}^{k-1} (1 + \alpha^j) \cdots (1 + \alpha^{(s-1)j}) \alpha^{-jt}$$

where $\alpha = e^{\frac{2\pi i}{k}}$. It can be easily shown that

$$\eta_k(t; s + 1) = |\mathrm{SVT}_{t,1}(s, k)| + |\mathrm{SVT}_{t,0}(s, k)|.$$

Thus, given by the Theorem 3.0.2 for the case of our interest. Moreover, similar to $\sigma_k^{(0)}(t; s)$, we have the linear recurrence relation

$$\eta_k(t; s + 1) = \eta_k(t; s) + \eta_k(t - s; s).$$

and the following backward linear recurrence.

Theorem 3.0.4. *For $s > 0$ and k is an odd positive integer,*

$$\eta_k(t; s) = \frac{1}{2} \sum_{j=0}^{k-1} (-1)^j \eta_k(t - js; s + 1).$$

Proof. The result follows from the observation

$$(1 + \xi)(1 - \xi + \xi^2 - \cdots + \xi^{k-1}) = 2.$$

□

In conclusion, we can write

$$|\mathrm{SVT}_{t,b}(s \pm \delta, 2s + 1)| = \frac{1}{2} \Big(\eta_{2s+1}(t; s + 1 \pm \delta) + (-1)^b \sigma_{2s+1}^{(0)}(t; s + 1 \pm \delta) \Big).$$

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