

A note on the periodic Hilbert Transform on a strip

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Abstract

In this note we prove a conjecture by Constantin–Strauss–Vărvăreucă related to the finite depth water wave problem, tightening their results. The proof uses identities related to Jacobi Theta functions. We also discuss potential implications of the improvement.

1 Introduction

In this paper we will be concerned with the periodic Hilbert transform on a strip. Let $d > 0$ and let

$$\mathcal{R}_d = \{(x, y) \in \mathbb{R}^2 : -d < y < 0\}$$

be the strip of depth d . Then, if $w \in C^{0,\alpha}(\mathbb{R})$, is 2π -periodic, has zero mean and the Fourier series expansion

$$w(x) = \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \quad x \in \mathbb{R},$$

the Hilbert transform operator \mathcal{C}_d acts as

$$(\mathcal{C}_d(w))(x) = \sum_{n=1}^{\infty} a_n \coth(nd) \sin(nx) - \sum_{n=1}^{\infty} b_n \coth(nd) \cos(nx), \quad x \in \mathbb{R}. \quad (1)$$

The Hilbert transform is of central importance in the study of the water wave problem, since it arises in the context of the Dirichlet-Neumann operator and the linearized equation around a flat wave. See [8, 9] for further references on the Hilbert Transform and [5, 11] for comprehensive surveys on the water wave problem.

In [3], the authors conjectured the following:

Conjecture 1.1 ([3] Lemma 1, Remark, p.252, abridged, see also [6] Lemma 2)). *Let us recall from Appendix A in [2] that, for any smooth 2π -periodic function $F : \mathbb{R} \rightarrow \mathbb{R}$ with mean zero over each period, we have*

$$(\mathcal{C}_{kh}(F))(x) = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \beta_{kh}(x-s) F(s) ds, \quad x \in \mathbb{R}, \quad (2)$$

where (with $d = kh$) the kernel $\beta_d : \mathbb{R} \setminus 2\pi\mathbb{Z} \rightarrow \mathbb{R}$, is given by

$$\begin{aligned} \beta_d(s) &= -\frac{s}{d} + \frac{\pi}{d} \coth\left(\frac{\pi s}{2d}\right) + \frac{\pi}{d} \sum_{n=1}^{\infty} \frac{2 \sinh\left(\frac{\pi s}{d}\right)}{\cosh\left(\frac{\pi s}{d}\right) - \cosh\left(\frac{2\pi^2 n}{d}\right)} \\ &= -\frac{s}{d} + \frac{\pi}{d} \sum_{n=-\infty}^{\infty} \left\{ \coth\left(\frac{\pi}{2d}(s - 2\pi n)\right) + \operatorname{sgn}(n) \right\}. \end{aligned} \quad (3)$$

Moreover, let us write $x = \pi^2/2d$ and consider $\beta_d(\pi/2)$ as a function of $x \in (0, \infty)$.

Since letting $d \rightarrow \infty$ is equivalent to letting $x \rightarrow 0$ in the formula

$$\pi \beta_d(\pi/2) = 2x \coth(x/2) - x - 4x \sinh(x) \sum_{n=1}^{\infty} \frac{1}{\cosh(4nx) - \cosh(x)},$$

we get

$$\pi \lim_{d \rightarrow \infty} \beta_d(\pi/2) = 4 - 8 \sum_{k=1}^{\infty} \frac{1}{16k^2 - 1} = \pi.$$

It is an interesting conjecture whether actually

$$\beta_d(\pi/2) \geq 1 \quad \text{for all } d \in (0, \infty).$$

Numerical computation in Octave/Matlab suggests that the issue is quite subtle.

Our main result in this note is the following:

Theorem 1.2. *Conjecture 1.1 is true.*

Remark 1.3. *The subtlety in this conjecture lies in the fact that the function grows extremely slowly for small values of x , as seen in Figure 1. Indeed, it is difficult to obtain a lower bound for the function via conventional means (such as for example Taylor expansions) since $\beta_d(\pi/2)$ remains so close to 1, corroborated by how $x = 0.1$ evaluates to 1 with 42 digits of accuracy according to Mathematica simulations. Hence, it is a non-trivial task to estimate the function's growth around 0. Instead we establish the lower bound by proving the equivalence between the function β and a Jacobi Theta function, and then establishing monotonicity.*

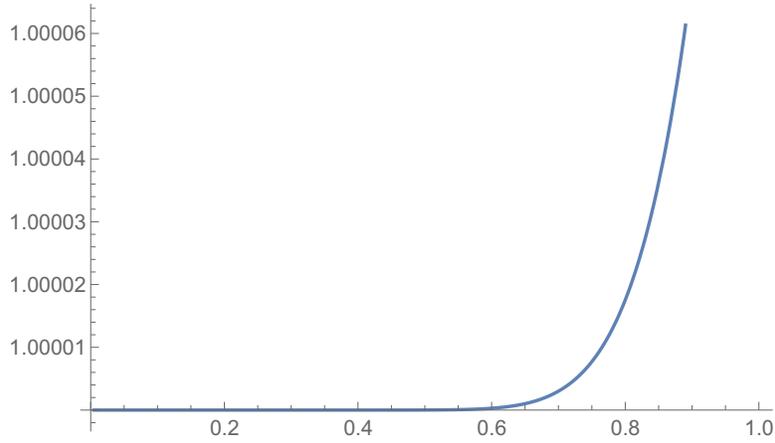


Figure 1: The function $\beta_d\left(\frac{\pi}{2}\right)$ as a function of x .

We defer the proof to Section 2. We now mention several small improvements as corollaries of the theorem. They are related to better bounds on some of the constants implied by the better bound on the function $\beta_d\left(\frac{\pi}{2}\right)$. For simplicity we refer to the corresponding papers for the definitions:

Corollary 1.4 (Strengthening of Theorem 4 of [3]). *Let $\Upsilon \geq 0$. Then, along the whole global bifurcation curve \mathcal{K}_- , we have the estimate*

$$v(0) - v(\pi) \leq \sqrt{\frac{36g^2}{\Upsilon^4} + \frac{24\pi g}{\Upsilon^2 k \beta_{hk}\left(\frac{\pi}{2}\right)}} - \frac{6g}{\Upsilon^2} \leq \sqrt{\frac{36g^2}{\Upsilon^4} + \frac{24\pi g}{\Upsilon^2 k}} - \frac{6g}{\Upsilon^2} \quad \text{if } \Upsilon > 0, \quad (4a)$$

and

$$v(0) - v(\pi) < \frac{2\pi}{k \beta_{kh}\left(\frac{\pi}{2}\right)} \leq \frac{2\pi}{k} \quad \text{if } \Upsilon = 0. \quad (4b)$$

Corollary 1.5 (Strengthening of Theorem 1.3 of [6]). *Consider a smooth water wave that belongs to the bifurcation curve \mathcal{C} in the adverse case $\gamma > 0$ and assume that either the slope $|\eta'|$ or the convexity $|\eta''|$ of the wave is bounded. Then the wave amplitude \mathcal{A} (the elevation difference between the crest and trough) is uniformly bounded by a certain constant provided γ is sufficiently small. The upper bound depends only on a certain explicit function of the constants g, Q, m and the conformal depth d .*

As an example, the upper bound can be chosen to be 12π , if we assume that each of the quantities $\gamma, Q\gamma, |m|\gamma^2$, as well as either $N\gamma^2$ or $M\gamma^4$, are less than certain explicit functions of g and d .

2 Proof of Theorem 1.2

Proof. We will show that $\beta_d\left(\frac{\pi}{2}\right) \geq 1$ for all $d \in (0, \infty)$ where $x = \pi^2/d$ and

$$\pi\beta_d\left(\frac{\pi}{2}\right) = 2x \coth\left(\frac{x}{2}\right) - x - 4x \sinh(x) \sum_{n=1}^{\infty} \frac{1}{\cosh(4nx) - \cosh(x)}. \quad (5)$$

We will also deduce that $\beta_d\left(\frac{\pi}{2}\right)$ is strictly increasing in x for $x > 0$.

We use the following intermediate lemma to begin our proof.

Lemma 2.1.

$$\sum_{n=1}^{\infty} \frac{2 \sinh(x)}{\cosh(4nx) - \cosh(x)} = \sum_{n=1}^{\infty} \frac{\sinh(x)}{\sinh\left(\frac{4n+1}{2}x\right) \sinh\left(\frac{4n-1}{2}x\right)}.$$

Proof. Use the difference to product rule for hyperbolic cosine and simplify:

$$\cosh(a) - \cosh(b) = 2 \sinh\left(\frac{a+b}{2}\right) \sinh\left(\frac{a-b}{2}\right).$$

□

Let $q = e^{-x}$. Denoting the sum by $2S$ and expanding in terms of q , we obtain

$$2S = \sum_{n=1}^{\infty} \frac{\sinh(x)}{\sinh\left(\frac{4n+1}{2}x\right) \sinh\left(\frac{4n-1}{2}x\right)} = 2 \sum_{n=1}^{\infty} \frac{(q^{-1} - q)}{(q^{-(4n+1)/2} - q^{(4n+1)/2})(q^{-(4n-1)/2} - q^{(4n-1)/2})}.$$

Simplifying by multiplying $q^{(4n+1)/2}q^{(4n-1)/2}$ in the numerator and denominator, we obtain

$$\sum_{n=1}^{\infty} \frac{(q^{4n-1} - q^{4n+1})}{(1 - q^{4n+1})(1 - q^{4n-1})} = \sum_{n=1}^{\infty} \frac{((q^{4n-1} - 1) + (1 - q^{4n+1}))}{(1 - q^{4n+1})(1 - q^{4n-1})},$$

which leads to

$$S = \sum_{n=1}^{\infty} \frac{\sinh(x)}{\cosh(4nx) - \cosh(x)} = \sum_{n=1}^{\infty} \left(\frac{1}{1 - q^{4n-1}} - \frac{1}{1 - q^{4n+1}} \right) = \sum_{n=1}^{\infty} \left(\frac{q^{4n-1}}{1 - q^{4n-1}} - \frac{q^{4n+1}}{1 - q^{4n+1}} \right).$$

By [10, Theorem 259, Chapter 12], a series of the form $\sum_k a_k z^k / (1 - z^k)$ (known as a *Lambert series*) converges if $\sum_k a_k z^k$ converges and $|z| \neq 1$. This implies that since $q \in (0, 1)$, $\sum_k q^k / (1 - q^k)$ converges since $\sum_k q^k$ converges, and for the same reason so do $\sum_n q^{4n-1} / (1 - q^{4n-1})$ and $\sum_n q^{4n+1} / (1 - q^{4n+1})$ for $x \in (0, \infty)$, implying that the whole series converges for $x \in (0, \infty)$. As a result, we may split and rearrange the terms as follows:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \left(\frac{q^{4n-1}}{1 - q^{4n-1}} - \frac{q^{4n+1}}{1 - q^{4n+1}} \right) = \left(\frac{q^3}{1 - q^3} - \frac{q^5}{1 - q^5} \right) + \left(\frac{q^7}{1 - q^7} - \frac{q^9}{1 - q^9} \right) + \dots \\ &= \frac{q}{1 - q} - \left(\left(\frac{q}{1 - q} - \frac{q^3}{1 - q^3} \right) + \left(\frac{q^5}{1 - q^5} - \frac{q^7}{1 - q^7} \right) + \dots \right) \\ &= \frac{q}{1 - q} - \sum_{n=1}^{\infty} \left(\frac{q^{4n-3}}{1 - q^{4n-3}} - \frac{q^{4n-1}}{1 - q^{4n-1}} \right). \end{aligned}$$

Plugging this back into (5) and expanding each term in q , we obtain

$$\begin{aligned} \pi\beta_d\left(\frac{\pi}{2}\right) &= x \left[2 \coth\left(\frac{x}{2}\right) - 1 - 4S \right] \\ &= x \left[2 \left(\frac{1+q}{1-q} \right) - 1 - 4 \left(\frac{q}{1-q} - \sum_{n=1}^{\infty} \left(\frac{q^{4n-3}}{1 - q^{4n-3}} - \frac{q^{4n-1}}{1 - q^{4n-1}} \right) \right) \right], \end{aligned}$$

and simplifying this expression yields

$$\pi\beta_d\left(\frac{\pi}{2}\right) = x \left[1 + 4 \sum_{n=1}^{\infty} \left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}} \right) \right]. \quad (6)$$

We now introduce the elliptic theta function $\vartheta_3(z|\tau)$, defined in [12] as

$$\vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i n z},$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, the upper half complex plane. Notice that setting $z = 0$ and writing $q = e^{\pi i \tau}$ gives the representation

$$\vartheta_3(0, q) = \vartheta_3(0|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad (7)$$

with $\tau = ix/\pi$ having positive imaginary part for $x > 0$. By [4, Theorem 312, Chapter XVII] (see [7] for the original proof), we have that

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2 = 1 + 4 \sum_{n=1}^{\infty} \left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}} \right),$$

and since the right hand side equals $x^{-1}\pi\beta_d(\pi/2)$ by (6), we have

$$\pi\beta_d\left(\frac{\pi}{2}\right) = x\vartheta_3^2(0, q).$$

Additionally, using [1, Equation (9.2)]

$$\sum_{n=-\infty}^{\infty} e^{-xn^2} = \sqrt{\frac{\pi}{x}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/x},$$

and the definition of ϑ_3 , this implies

$$x\vartheta_3^2(0, e^{-x}) = \pi\vartheta_3^2\left(0, e^{-\frac{\pi^2}{x}}\right).$$

Hence, we obtain

$$\beta_d\left(\frac{\pi}{2}\right) = \vartheta_3^2\left(0, e^{-\frac{\pi^2}{x}}\right).$$

Monotonicity then follows from (7). Recalling that $d = \pi^2/x$ and

$$\lim_{d \rightarrow \infty} \beta_d\left(\frac{\pi}{2}\right) = 1,$$

we deduce that $\beta_d(\pi/2) \geq 1$ for $d \in (0, \infty)$. □

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