

# THE LINEARIZED ISRAEL-STEWART EQUATIONS WITH A PHYSICAL VACUUM BOUNDARY

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ABSTRACT. In this article, we consider the Israel-Stewart equations of relativistic viscous fluid dynamics with bulk viscosity. We investigate the evolution of the equations linearized about solutions that satisfy the physical vacuum boundary condition and establish local well-posedness of the corresponding Cauchy problem.

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## 1. INTRODUCTION

The study of relativistic fluids comprises a large and active field of study in mathematics and physics. Many important physical systems in astrophysics, high-energy physics, and cosmology are described with the formalism of relativistic hydrodynamics. See, e.g., [75] for an overview of the topic and its applications. Relativistic hydrodynamics is also a fertile source of mathematical problems, see e.g. [12, Chapter IX] and the reviews [3, 17].

On the mathematical side, most of the community's focus has been on studies of relativistic ideal fluids, i.e., fluids without viscosity, which are described by the well-known relativistic Euler equations. While many important problems remain regarding the mathematical properties of the relativistic Euler equations (see the two recent reviews [3, 17] for a discussion

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of recent progress and open problems), there are also compelling reasons to investigate relativistic fluids *with viscosity*. Indeed, there are two important situations where viscosity is needed for an accurate description:

- The quark-gluon plasma. The quark-gluon plasma is an exotic state of matter that forms at extreme temperatures and densities achieved in modern-day particle colliders such as the Large Hadron Collider and the Relativistic Heavy Ion Collider. During a brief period of time after the collision, matter deconfines and quarks and gluons enter into a plasma-like state. Study of this plasma-like state provides researchers with a unique opportunity to understand properties of matter at extreme conditions. In fact, the discovery of the quark-gluon plasma was named by the American Physical Society as one of the ten most important discoveries in physics in the first decade of this century [1]. Careful analysis of data from these experiments reveals that the quark-gluon plasma behaves like a relativistic fluid with viscosity [76].
- Neutron star mergers. The recent detection of gravitational waves produced in the merger of binary neutron stars [2] opened the door to a whole range of previously inaccessible phenomena and has the potential to revolutionize our understanding of nature [16, 50]. While for a long time theoretical studies of mergers of neutron stars assumed that they could be reasonably modeled by the Einstein-Euler system, recent state-of-the-art numerical simulations convincingly show that viscous effects can be relevant for such systems [4]. In particular, effects of viscosity on the gravitational-wave signal emitted by mergers of neutron stars are expected to be within the sensitivity range of the next-generation of gravitational wave detectors [63, 64].

Despite their importance in physics, the mathematical study of theories of relativistic fluids with viscosity has traditionally been slow. The last decade, however, has seen increasing interest from the mathematical community in theories of relativistic viscous fluids, with several rigorous results finally established, including local [5–8, 25, 31–33] and global [28–30, 79, 80] well-posedness, formation of singularities [19, 27], existence of viscous shocks [26, 71, 72], and the study of the Einstein constraint equations [21] for viscous matter.

In this paper, we initiate the study of relativistic *viscous* hydrodynamics in a free-boundary setting, thus providing a further step on the mathematical development of relativistic fluids with viscosity. In the absence of viscosity, the study of free-boundary relativistic fluids goes back to the '30s with the seminal works of Tolman, Oppenheimer, and Volkoff [70, 82, 83], wherein they derived the now-famous TOV equations that describe a general relativistic static star, and Oppenheimer and Snyder [69], who introduced the first model of gravitational collapse and subsequent formation of a black hole in general relativity, in what is now known as the Oppenheimer-Snyder model. Subsequent works on relativistic free-boundary ideal fluids include [9, 10, 18, 34, 36–41, 47, 52–62, 66–68, 73, 74, 84]. See section 1.6 of [20] for a review of previous works on free-boundary relativistic fluids without viscosity.

Inclusion of viscosity is important in relativistic free-boundary models in order to describe dissipative effects in star evolution, especially in the case of neutron stars (see [22, 23], references therein, and the above discussion), and, more recently, accretion disks near black holes [11, 14, 24], where viscous effects can be important to understand some measurements made by the Event Horizon Telescope [13]. This work is the first one to mathematically address a model of relativistic viscous hydrodynamics in a free-boundary context. More precisely, we investigate equations of Israel-Stewart type originally introduced in [15] and that are widely used in applications devoted to viscous effects in the quark-gluon plasma

[76] and neutron star mergers [77, 78]. We focus on the case when the fluid satisfies the well-known physical vacuum boundary condition, which in the absence of viscosity is needed to ensure that the fluid-vacuum interface can accelerate. We remark that when viscosity is present, further conditions, in comparison to the non-viscous case, are needed to ensure that the free boundary moves with bounded non-zero acceleration. In other words, our first task below is to determine what the physical vacuum boundary condition means in the viscous context, i.e., what assumptions are needed in order to guarantee that the free boundary can accelerate.

As said, the model we will investigate was originally introduced in [15]. This model is sometimes referred to as the DNMR model after its authors, although for historical reasons it is more often called a Israel-Stewart, Israel-Stewart-like, or Israel-Stewart-Mueller model due to its similarities with a model of relativistic viscous fluids introduced by Israel and Stewart in the '70s [43–46, 81], which in turn was inspired by a model of non-relativistic viscous fluids introduced earlier by Mueller [65]. The DNMR or Israel-Stewart model contains a wide range of fluid variables, including three types of viscosity, namely, bulk and shear viscosity and heat conductivity. The local well-posedness of these equations has been obtained only in the special case when bulk viscosity is the only viscous contribution. Thus, it would be premature to study it in a free boundary context with all viscous contributions, and we restrict ourselves to the situation where only bulk viscosity is present.

In comparison with previous works on ideal fluids with a physical vacuum boundary, the main new difficulty we encounter is that viscosity in Israel-Stewart models is governed by a new variable satisfying an additional equation of motion, see the third equation in (1.3). This is a transport equation sourced by the divergence of the fluid's velocity, and a simple derivative counting shows that direct application of transport estimates leads to a derivative loss. In the case of a physical vacuum boundary, this problem is compounded by the need to control the fluid variables in weighted spaces that have to be introduced due to the fluid's degeneracy on the free boundary. We overcome this difficulty by uncovering some special structures at the level of the linearized equations.

It turns out that the linearized equations by themselves already present a rich and complex structure, and their local well-posedness cannot be derived from standard techniques. Thus, here we focus on establishing local well-posedness for the linearized Israel-Stewart equations with a physical vacuum boundary and bulk viscosity. The solution to the linearized equations will be the basis for our solution to the full nonlinear problem that we will present in an upcoming work [85]. In order to shorten the presentation, we will focus on aspects of the problem that are genuinely new compared to the case of an ideal fluid. Thus, we will be brief when arguments are a simple modification of those employed in the case of an ideal fluid with a physical vacuum boundary, mostly notably to those found in [20, 42, 48].

**1.1. The Equations of Motion.** In this article, we consider the equations of Israel-Stewart type with bulk viscosity proposed in [15], which is the most widely used model of relativistic fluids with viscosity [76]. We consider the situation when the only viscous contribution comes from bulk viscosity, thus ignoring the dissipation due to shear and heat flux. The fluid is characterized by an energy-momentum tensor

$$T_{\mu\nu} = \varrho u_\mu u_\nu + (p + \Pi)\Delta_{\mu\nu}, \tag{1.1}$$

and a conserved quantity called rest-mass current

$$J^\mu = nu^\mu,$$

where  $\varrho$  is the energy density,  $\Pi$  is the bulk viscosity,  $n$  is the baryon density,  $p$  is the pressure subject to the equation of state

$$p = p(\varrho, n),$$

$u$  is the 4-velocity of the fluid subject to the normalization condition

$$u^\mu u_\mu = -1, \tag{1.2}$$

and the 2-tensor  $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$  is the projection onto the space orthogonal to  $u$ . Additionally,  $g$  is the space-time metric, which we will take to be the Minkowski metric in that this already contains many difficulties in the study of a free-boundary fluids which we introduce below.

The evolution of the fluid is given by the system

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0, \\ \partial_\mu J^\mu = 0, \\ \tau_\Pi u^\mu \partial_\mu \Pi + \Pi + \lambda \Pi^2 + \zeta \partial_\mu u^\mu = 0, \end{cases} \tag{1.3}$$

where the first equation is the conservation of energy and momentum, the second equation is the conservation of mass and the third equation is the additional relaxation-type equation satisfied by the bulk scalar  $\Pi$ , in which  $\tau_\Pi = \tau_\Pi(\varrho)$  is the bulk relaxation time,  $\lambda = \lambda(\varrho)$  is a transport coefficient and  $\zeta = \zeta(\varrho)$  is the coefficient of bulk viscosity.

Throughout and including the above, we adopt the rectangular spacetime coordinate  $\{x^\alpha\}_{\alpha=0,1,2,3}$  of  $\mathbb{R}^{1+3}$ , where we identify  $x_0$  with the time coordinate  $t$ . We use  $x = (x^1, x^2, x^3)$  to denote the spatial coordinates. Einstein sum convention is employed. The vector field  $\partial_\alpha$  is a short-hand expression for  $\frac{\partial}{\partial x^\alpha}$ . Greek indices will vary from 0 to 3, and Latin indices will vary from 1 to 3. Indices are raised and lowered with the Minkowski metric  $g$ .

**1.2. Physical Vacuum Boundary.** It was shown in [6] that the system is strongly hyperbolic if the following causality condition is satisfied

$$\left[ \frac{\zeta}{\tau_\Pi} + n \left( \frac{\partial p}{\partial n} \right)_\varrho \right] \frac{1}{\varrho + p + \Pi} \leq 1 - \left( \frac{\partial p}{\partial \varrho} \right)_n, \tag{1.4}$$

where  $\rho + p + \Pi > 0$  and  $\tau_\Pi > 0$ . One can use the standard theory of FOSH developed by Friedrichs-Lax-Kato [35, 49, 51] to prove the local well-posedness.

In this article, we consider the physical situation where vacuum states are allowed, i.e., the density  $\varrho$  is allowed to vanish, and the boundary has finite non-zero acceleration. Then, the gas is located in the region where the density is non-vanishing, i.e., the following moving domain at time  $t$ ,

$$\Omega_t = \{x \in \mathbb{R}^d \mid \rho(t, x) > 0\},$$

whose boundary denoted by  $\Gamma_t := \partial\Omega_t$  is called the vacuum boundary at time  $t$ , which is advected by the fluid velocity  $u$ . So, the dynamics of the fluid is confined within

$$\mathcal{D}_T = \bigcup_{t \in [0, T]} \Omega_t$$

for the time existence  $T > 0$  of the solutions.

We will consider the gas where the pressure  $p$  is determined by the density  $\varrho$  through the following equation of state,

$$p(\varrho) = \varrho^{\kappa+1}, \quad (1.5)$$

where  $\kappa > 0$ . Such gas is called a barotropic gas. With the equation of state (1.5), the variable  $n$  gets decoupled from the system (1.3), which, with the energy-momentum tensor  $T$  given by (1.1), can be written as the following Cauchy problem

$$\begin{cases} u^\mu \partial_\mu \varrho + (\varrho^{\kappa+1} + \varrho + \Pi) \partial_\mu u^\mu = 0, \\ (\varrho^{\kappa+1} + \varrho + \Pi) u^\mu \partial_\mu u^\alpha + (\kappa + 1) \varrho^\kappa \Delta^{\alpha\mu} \partial_\mu \varrho + \Delta^{\alpha\mu} \partial_\mu \Pi = 0, \\ \tau_\Pi u^\mu \partial_\mu \Pi + \Pi + \lambda \Pi^2 + \zeta \partial_\mu u^\mu = 0, \\ u^\mu u_\mu = -1, \\ (\varrho, u, \Pi)|_{t=0} = (\rho_0, u_0, \Pi_0), \end{cases} \quad (1.6)$$

where the first two equations in the system (1.6) is derived from the projections of the first equation in the system (1.3) onto the space parallel and orthogonal to  $u$  and  $\rho_0, u_0, \Pi_0$  are initial data for the system. Due to the absence of matter ( $\varrho = 0$ ) on the free boundary  $\Gamma_t$ , it is natural to assume that the bulk viscosity  $\Pi$  also vanishes on  $\Gamma_t$ . Then, the system (1.6) becomes degenerate due to the vanishing of both  $\varrho$  and  $\Pi$ . In this case, the classical theories no longer apply.

It turns out that the decay rates of  $\varrho$  and  $\Pi$  play a crucial role in the problem. In view of finite speed of propagation of hyperbolic system, the behavior of the fluid away from  $\Gamma_t$  is essentially the same as the non-free-boundary case and thus can be solved with classical theory. Near the free boundary  $\Gamma_t$ , obtained from the second equation of (1.6), the fluid evolves with the 4-acceleration

$$a^\alpha = u^\mu \partial_\mu u^\alpha = -\frac{(\kappa + 1) \varrho^\kappa \Delta^{\alpha\mu} \partial_\mu \varrho + \Delta^{\alpha\mu} \partial_\mu \Pi}{\varrho^{\kappa+1} + \varrho + \Pi}. \quad (1.7)$$

We will consider appropriate decay rates that allow  $\Gamma_t$  to move with a bounded non-zero acceleration. We denote by  $d(t, x) := \text{dist}(x, \Gamma_t)$  the distance from the free boundary  $\Gamma_t$  at time  $t$ . Assume that at the leading order  $\varrho$  and  $\Pi$  decays like a power of  $d$ , i.e.,

$$\varrho \simeq d^\sigma, \Pi \simeq d^\eta \quad (1.8)$$

for some  $\sigma, \eta > 0$ , where by  $f \simeq g$  we mean there exists a constant  $C > 0$  such that

$$\frac{1}{C}g \leq f \leq Cg.$$

Under this assumption, from (1.7), the 4-acceleration behaves like

$$a \simeq \begin{cases} d^{\sigma\kappa-1} + d^{\eta-\sigma-1} & \sigma \leq \eta \\ \infty & \sigma > \eta \end{cases}.$$

To obtain a bounded nonzero acceleration of  $\Gamma$ , we either have

$$\begin{cases} \sigma = \frac{1}{\kappa}, \\ \eta \geq \sigma + 1, \end{cases} \quad (1.9)$$

or

$$\begin{cases} \sigma \geq \frac{1}{\kappa}, \\ \eta = \sigma + 1. \end{cases}$$

We will define the physical vacuum boundary condition by (1.9) since this reduces to the case without viscosity in [20]. **In particular, we will solve the case  $\eta = \frac{1}{\kappa} + 2$ .** We make this choice because any choice  $\eta < \frac{1}{\kappa} + 2$  leads to singular behavior of the coefficients of the linearized equation and the energy estimate cannot be closed with our method, and any choice  $n > \frac{1}{\kappa} + 2$  can be solved similarly as  $\eta = \frac{1}{\kappa} + 2$  under similar assumptions. We also make some technical assumptions on the coefficients of the relaxation-type equation  $\Pi$  satisfies. We assume that  $\tau_{\Pi} = \frac{1}{2 + \frac{1}{\kappa}}$ ,  $\zeta(\varrho) \simeq \varrho^{2\kappa+1}$  is a smooth function and  $\lambda = \lambda(\varrho)$  is a smooth function with bounded derivatives. We remark that the above assumption is compatible with the causality condition (1.4). Under these assumptions, we introduce new variables

$$r = \varrho^{\kappa}, \pi = \Pi^{\frac{\kappa}{2\kappa+1}}.$$

As a result, the decay rates of the new variable near the free boundary are  $r, \pi \simeq d$ . Using  $(r, u, \pi)$  as our new variables, we write (1.6) as follows,

$$\begin{cases} u^{\mu} \partial_{\mu} r + \kappa r a_1 \partial_{\mu} u^{\mu} = 0, & (1.10a) \\ u^{\mu} \partial_{\mu} u^{\alpha} + \left(1 + \frac{1}{\kappa}\right) a_2 \Delta^{\alpha\mu} \partial_{\mu} r + \left(2 + \frac{1}{\kappa}\right) r a_3 \Delta^{\alpha\mu} \partial_{\mu} \pi = 0, & (1.10b) \\ u^{\mu} \partial_{\mu} \pi + \pi + \lambda \left(r^{\frac{1}{\kappa}}\right) \pi^{3 + \frac{1}{\kappa}} + r a_4 \partial_{\mu} u^{\mu} = 0, & (1.10c) \\ u^{\mu} u_{\mu} = -1, & (1.10d) \\ r|_{\Gamma_t} = 0, \pi|_{\Gamma_t} = 0, & (1.10e) \\ (\varrho, u, \pi)|_{t=0} = (\rho_0, u_0, \pi_0), & (1.10f) \end{cases}$$

where

$$a_0 = \frac{\pi}{r}, a_1 = r + 1 + r^2 a_0^{2 + \frac{1}{\kappa}}, a_2 = \frac{1}{a_1}, a_3 = a_0^{1 + \frac{1}{\kappa}} a_2, a_4 = \frac{\zeta\left(r^{\frac{1}{\kappa}}\right)}{r \pi^{1 + \frac{1}{\kappa}}}.$$

## 2. BASIC ENERGY ESTIMATE FOR THE LINEARIZED EQUATIONS

In this section, we formulate the linearized problem of system (1.10) and establish the basic energy estimate for the linearized equations, assuming suitable conditions on the background solution about which we linearize. This energy estimate is a crucial component in proving the local well-posedness of the linearized system.

**2.1. The Linearized Equations.** We consider a one-parameter family of solutions  $(r_{\lambda}, u_{\lambda}, \pi_{\lambda})$  to the system (1.10) such that  $(r_{\lambda}, u_{\lambda}, \pi_{\lambda})|_{\lambda=0} = (r, u, \pi)$ . Then, the variables  $(\tilde{r}, \tilde{u}, \tilde{\pi}) :=$

$\frac{d}{d\lambda}|_{\lambda=0}(r_\lambda, u_\lambda, \pi_\lambda)$  defined on the moving domain  $\Omega_t$  solve the following linearized problem,

$$\begin{cases} u^\mu \partial_\mu \tilde{r} + \kappa r a_1 \partial_\mu \tilde{u}^\mu + \partial_\mu r \tilde{u}^\mu + V_1 \tilde{r} + r^2 Z_1 \tilde{\pi} = f, & (2.1a) \\ u^\mu \partial_\mu \tilde{u}^\alpha + \left(1 + \frac{1}{\kappa}\right) a_2 \Delta^{\alpha\mu} \partial_\mu \tilde{r} + \left(2 + \frac{1}{\kappa}\right) r a_3 \Delta^{\alpha\mu} \partial_\mu \tilde{\pi} \\ \quad + (V_2)^\alpha \tilde{r} + (W_2)_\beta^\alpha \tilde{u}^\beta + (Z_2)^\alpha \tilde{\pi} = g^\alpha, & (2.1b) \\ u^\mu \partial_\mu \tilde{\pi} + r a_4 \partial_\mu \tilde{u}^\mu + V_3 \tilde{r} + (W_3)_\alpha \tilde{u}^\alpha + Z_3 \tilde{\pi} = h, & (2.1c) \\ u^\mu \tilde{u}_\mu = 0, & (2.1d) \\ (\tilde{r}, \tilde{u}, \tilde{\pi})|_{t=0} = (\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0), & (2.1e) \end{cases}$$

where  $V, W, Z$  are coefficients involving the background solution  $(r, u, \pi)$  given as the following,

$$\begin{aligned} V_1 &= \kappa \left[ 2r + 1 + \left(1 - \frac{1}{\kappa}\right) r^2 a_0^{2+\frac{1}{\kappa}} \right] \partial_\mu u^\mu, \\ (V_2)^\alpha &= \left\{ \left( \frac{1}{\kappa} r a_0^{2+\frac{1}{\kappa}} - 1 \right) a_2 \left[ \left(1 + \frac{1}{\kappa}\right) \partial_\mu r + \left(2 + \frac{1}{\kappa}\right) r a_0^{1+\frac{1}{\kappa}} \partial_\mu \pi \right] \right. \\ &\quad \left. - \frac{1}{\kappa} \left(2 + \frac{1}{\kappa}\right) a_0^{1+\frac{1}{\kappa}} \partial_\mu \pi \right\} a_2 \Delta^{\alpha\mu} \\ V_3 &= \frac{1}{\kappa} \left( r^{\frac{1}{\kappa}-1} \pi^{3+\frac{1}{\kappa}} \lambda'(r^{\frac{1}{\kappa}}) + \frac{\zeta'(r^{\frac{1}{\kappa}}) r^{\frac{1}{\kappa}-1}}{\pi^{1+\frac{1}{\kappa}}} \right), \\ W_1 &= 0, \\ (W_2)^{\alpha\beta} &= \left\{ \partial_\mu u^\alpha g^{\mu\beta} + a_2 (u^\mu g^{\alpha\beta} + u^\alpha g^{\mu\beta}) \left[ \left(1 + \frac{1}{\kappa}\right) \partial_\mu r + \left(2 + \frac{1}{\kappa}\right) r a_0^{1+\frac{1}{\kappa}} \partial_\mu \pi \right] \right\}, \\ (W_3)^\beta &= g^{\mu\beta} \partial_\mu \pi, \\ Z_1 &= \kappa \left(2 + \frac{1}{\kappa}\right) a_0^{1+\frac{1}{\kappa}} \partial_\mu u^\mu, \\ (Z_2)^\alpha &= - \left(2 + \frac{1}{\kappa}\right) a_0^{\frac{1}{\kappa}} a_2 \Delta^{\alpha\mu} \left\{ \left(1 + \frac{1}{\kappa}\right) r a_0 a_2 \partial_\mu r + \left(2 + \frac{1}{\kappa}\right) r^2 a_0^{2+\frac{1}{\kappa}} a_2 \partial_\mu \pi \right. \\ &\quad \left. - \left(1 + \frac{1}{\kappa}\right) \partial_\mu \pi \right\}, \\ Z_3 &= 1 + \left(3 + \frac{1}{\kappa}\right) \pi^{2+\frac{1}{\kappa}} \lambda(r^{\frac{1}{\kappa}}) - \left(1 + \frac{1}{\kappa}\right) \frac{\zeta(r^{\frac{1}{\kappa}})}{\pi^{2+\frac{1}{\kappa}}} \partial_\mu u^\mu. \end{aligned}$$

Here, we introduce the source terms  $(f, g, h)$  to make the equations inhomogeneous, as the proof remains the same whether or not these source terms are included. Since each member of the one-parameter family  $(r_\lambda, u_\lambda, \pi_\lambda)$  does not necessarily share the same domain—unlike when working in a Lagrangian frame—there are no boundary conditions imposed on the linearized variables  $(\tilde{r}, \tilde{u}, \tilde{\pi})$ .

**2.2. Function Spaces.** We define the functional framework of weighted Sobolev spaces for our analysis here. This functional framework is dictated by the degenerate hyperbolic nature of the problem and corresponding estimates.

**Definition 2.1.** The weighted  $L^2$  space with weight  $\omega \geq 0$ , denoted by  $L^2(\omega)$ , is the space of all distributions  $f$  such that

$$\|f\|_{L^2(\omega)}^2 = \int \omega |f|^2 dx < \infty.$$

The weight  $\omega$  will be played out by  $r$  which is a solution to (1.10) that satisfies the physical vacuum condition (1.9). With the above definition, the base space of the solutions  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  to the linearized system (2.1) is

$$\mathcal{H} := \mathcal{H}(\Omega_t) = L^2(r^{\frac{1-\kappa}{\kappa}}(t, \bullet)) \times L^2(r^{\frac{1}{\kappa}}(t, \bullet)) \times L^2(r^{\frac{1}{\kappa}}(t, \bullet)).$$

The energy admitted by the linearized system is the following,

$$\begin{aligned} E(t) &= \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 G^{\alpha\beta} \tilde{u}_\alpha \tilde{u}_\beta + \frac{(2\kappa+1)\kappa}{\kappa+1} r \frac{r\pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \right] dx \\ &= \frac{1}{2} \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 \tilde{u}_\alpha \tilde{u}^\alpha + \frac{(2\kappa+1)\kappa}{\kappa+1} r \frac{r\pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \right] dx. \end{aligned} \quad (2.2)$$

It can be shown the energy  $E(t)$  can serve as an equivalent norm of the base space  $\mathcal{H}$ . (See Lemma 2.8) Now, we define high-order weighted Sobolev spaces.

**Definition 2.2.** We define  $H^{j,\sigma} := H^{j,\sigma}(\Omega_t)$ , where  $j \geq 0$  is an integer and  $\sigma > -\frac{1}{2}$ , to be the space of all distributions in  $\Omega_t$  such that

$$\|f\|_{H^{j,\sigma}}^2 := \sum_{|\alpha| \leq j} \|r^\sigma \partial^\alpha f\|_{L^2(\Omega_t)} < \infty.$$

The weighted spaces  $H^{j,\sigma}$  satisfy the following hardy-type embedding. We refer readers to [42] for the proofs.

**Lemma 2.3.** Assume that  $j_1 > j_2 \geq 0$  and  $\sigma_1 > \sigma_2 > -\frac{1}{2}$  with  $j_1 - j_2 = \sigma_1 - \sigma_2$ . Then we have

$$H^{j_1, \sigma_1} \subset H^{j_2, \sigma_2}.$$

As a corollary of the above lemma we have embeddings of the weighted Sobolev spaces into standard Sobolev spaces:

**Corollary 2.4.** Assume that  $\sigma > 0$  and  $\sigma \leq j$ . Then we have

$$H^{j,\sigma} \subset H^{j-\sigma}.$$

Then, by the standard Sobolev embedding theorem, we have

**Corollary 2.5.**

$$H^{j,\sigma} \subset C^s, \quad 0 \leq s \leq j - \sigma - \frac{d}{2}.$$

We define the high-order space  $\mathcal{H}^{2l}$  for the regular solutions as

$$\mathcal{H}^{2l} = H^{2l, \frac{1}{2\kappa} + l - \frac{1}{2}} \times H^{2l, \frac{1}{2\kappa} + l} \times H^{2l, \frac{1}{2\kappa} + l}.$$



**2.3. Basic Energy Estimate.** Here, we establish the basic energy estimate of the solutions  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  to the linearized equations (2.1) under suitable assumptions on the background solutions  $(r, u, \pi)$ . Throughout the proof, we will use the following differential formula for the moving domain,

$$\frac{d}{dt} \int_{\Omega_t} f dx = \int_{\Omega_t} \frac{1}{u^0} D_t f dx + \int_{\Omega_t} \partial_i \left( \frac{u^i}{u^0} \right) dx. \quad (2.3)$$

**Notation 2.6.** We denote the material derivative parallel to the 4-velocity by

$$D_t = u^\mu \partial_\mu.$$

**Notation 2.7.** We let  $y = (y^1, y^2, y^3)$  be the coordinate on the initial surface  $\Omega_0$  induced by the rectangular spacetime coordinate  $\{x^\alpha\}_{\alpha=0,1,2,3}$ . We define the map  $\eta : I \times \Omega_0 \rightarrow \mathbb{R}^{1+3}$  to be the flow map of the 4-velocity, where  $I$  is the interval of proper time containing  $t = 0$ , i.e.,  $\eta = \eta(\tau, y)$  solve the following equation

$$\begin{cases} \partial_\tau \eta = u \circ \eta \\ n(0, y) = (0, y) \end{cases}. \quad (2.4)$$

Similar as before, we identify the proper time  $\tau$  with  $y^0$ . We use Capital Latin indices  $K, L, M, \dots$ , etc. ranging from 0 to 3 to denote the indices of Lagrangian spacetime coordinates. We use  $\partial_K$  to denote the derivative  $\frac{\partial}{\partial y^K}$  with respect to the Lagrangian coordinate.

We use the following lemma to control the coefficient  $a_0 = \frac{\pi}{r}$  of the linearized equation (2.1).

**Lemma 2.8.** Let  $(r, u, \pi)$  be the solutions to the nonlinear system (1.10) such that

$$\sup_{t \in [0, T]} \|r\|_{W^{2l+1, \infty}(\Omega_t)}, \sup_{t \in [0, T]} \|\pi\|_{W^{2l+1, \infty}(\Omega_t)}, \sup_{t \in [0, T]} \|u\|_{W^{2l+1, \infty}(\Omega_t)} \leq K,$$

where  $K > 0$  is a constant. Then, there exists  $T^* = T^* \left( K, \left\| \frac{\pi_0}{r_0} \right\|_{W^{2l, \infty}(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{W^{2l, \infty}(\Omega_0)} \right) > 0$  such that

$$\begin{aligned} \sup_{t \in [0, T^*]} \left\| \frac{\pi}{r} \right\|_{W^{2l, \infty}(\Omega_t)} &\leq 2 \left\| \frac{\pi_0}{r_0} \right\|_{W^{2l, \infty}(\Omega_0)}, \\ \sup_{t \in [0, T^*]} \left\| \frac{r}{\pi} \right\|_{W^{2l, \infty}(\Omega_t)} &\leq 2 \left\| \frac{r_0}{\pi_0} \right\|_{W^{2l, \infty}(\Omega_0)}, \\ \sup_{t \in [0, T^*]} \left\| \frac{\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}} \right\|_{W^{2l, \infty}(\Omega_t)} &\leq 2 \left\| \frac{\zeta(r_0^{\frac{1}{\kappa}})}{r_0^{2+\frac{1}{\kappa}}} \right\|_{W^{2l, \infty}(\Omega_0)}. \end{aligned}$$

*Proof.* We prove the lemma by induction and continuity argument. We first prove the base case  $l = 0$ . Assume

$$\begin{aligned} \sup_{t \in [0, T^*]} \left\| \frac{\pi}{r} \right\|_{L^\infty(\Omega_t)} &\leq 2 \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \\ \sup_{t \in [0, T^*]} \left\| \frac{r}{\pi} \right\|_{L^\infty(\Omega_t)} &\leq 2 \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)}, \\ \sup_{t \in [0, T^*]} \left\| \frac{\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}} \right\|_{L^\infty(\Omega_t)} &\leq 2 \left\| \frac{\zeta(r_0^{\frac{1}{\kappa}})}{r_0^{2+\frac{1}{\kappa}}} \right\|_{L^\infty(\Omega_0)}, \end{aligned}$$

for  $T^* > 0$ . We compute the evolution for  $a_0 = \frac{\pi}{r}$ ,  $\frac{1}{a_0} = \frac{r}{\pi}$  and  $\frac{\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}}$  and obtain the following

$$\begin{aligned} D_t(a_0) &= [\kappa(r+1) - 1] a_0 + \left( \kappa r^2 - \lambda(r^{\frac{1}{\kappa}}) r^{2+\frac{1}{\kappa}} \right) a_0^{3+\frac{1}{\kappa}} - \frac{\zeta(r^{\frac{1}{\kappa}})}{r \pi^{1+\frac{1}{\kappa}}} \partial_\mu u^\mu, \\ D_t\left(\frac{1}{a_0}\right) &= [1 - \kappa(r+1)] \frac{1}{a_0} + \left( \lambda(r^{\frac{1}{\kappa}}) r^{2+\frac{1}{\kappa}} - \kappa r^2 \right) a_0^{1+\frac{1}{\kappa}} + \frac{\zeta(r^{\frac{1}{\kappa}}) r}{\pi^{3+\frac{1}{\kappa}}} \partial_\mu u^\mu, \\ D_t\left(\frac{\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}}\right) &= \partial_\mu u^\mu (r+1 + r^2 a_0^{2+\frac{1}{\kappa}}) \left( \frac{(2\kappa+1)\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}} - \frac{\zeta'(r^{\frac{1}{\kappa}})}{r^2} \right). \end{aligned} \quad (2.5)$$

We integrate above along the flow line and by a direct estimate, we have for any  $(\tau, y) \in \eta^{-1}(\mathcal{D}_{T^*})$ ,

$$\begin{aligned} |a_0 \circ \eta(\tau, y)| &\leq \left| \frac{\pi_0}{r_0}(y) \right| + T^* \mathcal{C} \left( K, \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)} \right), \\ \left| \left( \frac{1}{a_0} \right) \circ \eta(\tau, y) \right| &\leq \left| \frac{r_0}{\pi_0}(y) \right| + T^* \mathcal{C} \left( K, \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)} \right), \\ \left| \left( \frac{\zeta(r^{\frac{1}{\kappa}})}{r^{2+\frac{1}{\kappa}}} \right) \circ \eta(\tau, y) \right| &\leq \left| \frac{\zeta(r_0^{\frac{1}{\kappa}})}{r_0^{2+\frac{1}{\kappa}}}(y) \right| + T^* \mathcal{C} \left( K, \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)} \right). \end{aligned}$$

We take  $T^* > 0$  such that

$$T^* \mathcal{C} \left( K, \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)} \right) < \frac{1}{2} \min \left\{ \left\| \frac{\pi_0}{r_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{r_0}{\pi_0} \right\|_{L^\infty(\Omega_0)}, \left\| \frac{\zeta(r_0^{\frac{1}{\kappa}})}{r_0^{2+\frac{1}{\kappa}}} \right\|_{L^\infty(\Omega_0)} \right\}$$

to close the bootstrap. The high order case can be proven by differentiating the evolution (2.5), induction and a similar continuity argument.  $\square$

We also need the following Gronwall-type inequality for the energy estimate.

**Lemma 2.9.** *Let  $d(t)$  be a non-negative function that satisfies the following integral inequality,*

$$d(t) \leq c(t) + \int_0^t a(s) d(s) + b(s) d^\alpha(s) ds, \quad (2.6)$$

where  $a, b, c$  are continuous non-negative functions for  $t \geq 0$  and  $0 < \alpha < 1$ . Then,

$$\begin{aligned} d(t) &\leq \left( c^{1-\alpha}(t) \exp \left( (1-\alpha) \int_0^t a(s) ds \right) \right. \\ &\quad \left. + (1-\alpha) \int_0^t b(s) \exp \left( (1-\alpha) \int_s^t a(r) dr \right) ds \right)^{\frac{1}{1-\alpha}} \end{aligned} \quad (2.7)$$

*Proof.* Given  $\tau > 0$ , let  $c_\tau = c(\tau)$ . For  $t \in [0, \tau]$ , we define

$$\phi(t) = c_\tau + \int_0^t a(s) d(s) + b(s) d^\alpha(s) ds.$$

From (2.6), we have  $d(t) \leq \phi$  for  $t \in [0, \tau]$ . We differentiate  $\phi$ ,

$$\begin{aligned}\phi'(t) &= a(t)d(t) + b(t)d^\alpha(t) \\ &\leq a(t)\phi(t) + b(t)\phi^\alpha(t).\end{aligned}$$

Dividing both sides by  $\phi^\alpha$ , this gives us a linear differential inequality in  $\phi^{1-\alpha}$ ,

$$(\phi^{1-\alpha}(t))' \leq (1-\alpha)a(t)\phi^{1-\alpha}(t) + (1-\alpha)b(t).$$

By an elementary method of integrating factor,

$$(\psi(t)\phi^{1-\alpha}(t))' \leq (1-\alpha)b(t)\psi(t),$$

where

$$\psi(t) = \exp\left(\int_0^t (\alpha-1)a(s) ds\right).$$

Taking integral with respect to  $t$ , and considering that  $\tau$  is arbitrary, we have (2.7).  $\square$

Now, we prove the basic estimate for the linearized equation.

**Remark 2.10.** *Due to the finite speed of propagation, we can divide the domain into two regions: the bulk region away from the free boundary and the region close to the free boundary. In the bulk region, we can apply the classical theory of hyperbolic equations. Our primary focus is on the degenerate quasilinear hyperbolic system near the free boundary, where  $r \ll 1$  is small. In this case, we interpret each integral over  $\Omega_t$  as being restricted to the region near the free boundary  $\Gamma_t$ . By using a partition of unity argument, we can extend the solution to the entire domain  $\Omega_t$ .*

**Theorem 2.11.** *Let  $(r, u, \pi)$  be sufficiently smooth solutions to (1.10) satisfying the physical vacuum boundary condition (1.9) such that*

$$\sup_{t \in [0, T]} \|(r, u, \pi)\|_{W^{1, \infty}(\Omega_t)} \leq K, \quad (2.8)$$

where  $K > 0$  is a constant. Then, there exists  $T = T(K) > 0$  such that if the functions  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  are sufficiently smooth solutions to the linearized Cauchy problem (2.1) with  $(f, g, h) \in L^1([0, T], \mathcal{H})$ , then the solutions obey the following estimate,

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}} \leq \mathcal{C}(K) \left( \|(\tilde{r}(0), \tilde{u}(0), \tilde{\pi}(0))\|_{\mathcal{H}} + \int_0^t \|(f, g, h)\|_{\mathcal{H}} ds \right), \quad (2.9)$$

for any  $t \in [0, T]$ , where  $\mathcal{C}(K) > 0$  is a constant continuously dependent on and increasing with respect to  $K$ .

*Proof.* We multiply the equation (2.1a) by  $r^{\frac{1}{\kappa}-1}$ , the equation (2.1b) by  $\frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2G_\alpha^\beta$  and the equation (2.1c) by  $\frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}$ , we have the system

$$\left\{ \begin{array}{l} r^{\frac{1}{\kappa}-1}u^\mu\partial_\mu\tilde{r} + \kappa r^{\frac{1}{\kappa}}a_1\partial_\mu\tilde{u}^\mu + r^{\frac{1}{\kappa}-1}\partial_\mu r\tilde{u}^\mu + r^{\frac{1}{\kappa}-1}V_1\tilde{r} + r^{\frac{1}{\kappa}+1}Z_1\tilde{\pi} = r^{\frac{1}{\kappa}-1}f, \\ \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2G_\alpha^\beta u^\mu\partial_\mu\tilde{u}^\alpha + \kappa r^{\frac{1}{\kappa}}a_1\Delta^{\beta\mu}\partial_\mu\tilde{r} + \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}+1}a_0^{1+\frac{1}{\kappa}}a_1\Delta^{\beta\mu}\partial_\mu\tilde{\pi} \\ \quad + \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2(V_2)^\beta\tilde{r} + \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2G_\alpha^\beta(W_2)^\alpha_\gamma\tilde{u}^\gamma \\ \quad \quad \quad + \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2(Z_2)^\beta\tilde{\pi} = \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2G_\alpha^\beta g^\alpha, \\ \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}u^\mu\partial_\mu\tilde{\pi} + \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}+1}a_0^{1+\frac{1}{\kappa}}a_1\partial_\mu\tilde{u}^\mu \\ \quad + \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}V_3\tilde{r} \\ \quad + \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}(W_3)_\alpha\tilde{u}^\alpha \\ \quad + \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}Z_3\tilde{\pi} = \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}h. \end{array} \right. \quad (2.10)$$

We let  $\mathcal{L}$  be the operator of the above linear system, then the equation can be denoted as

$$\mathcal{L} \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} = \begin{pmatrix} r^{\frac{1}{\kappa}-1}f \\ \frac{\kappa^2}{\kappa+1}r^{\frac{1}{\kappa}}a_1^2G_\alpha^\bullet g^\alpha \\ \frac{(2\kappa+1)\kappa}{\kappa+1}r^{\frac{1}{\kappa}}\frac{r\pi^{1+\frac{1}{\kappa}}a_0^{1+\frac{1}{\kappa}}a_1}{\zeta(r^{\frac{1}{\kappa}})}h. \end{pmatrix} \quad (2.11)$$

Now, we multiply the first equation of (2.10) by  $\tilde{r}$ , the second by  $\tilde{u}$  and the third by  $\tilde{\pi}$ . This yields the following identities,

$$\begin{aligned}
& \frac{1}{2} r^{\frac{1}{\kappa}-1} D_t \tilde{r}^2 + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa}-1} \partial_\mu r \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa}-1} V_1 \tilde{r}^2 + r^{\frac{1}{\kappa}+1} Z_1 \tilde{\pi} \tilde{r} = r^{\frac{1}{\kappa}-1} f \tilde{r}, \\
& \frac{1}{2} \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 D_t (\tilde{u}^\alpha \tilde{u}_\alpha) + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{r} \tilde{u}^\mu + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{\pi} \tilde{u}^\mu \\
& \quad + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (V_2)^\beta \tilde{r} \tilde{u}_\beta + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (W_2)_\gamma \tilde{u}^\gamma \tilde{u}_\alpha + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (Z_2)^\beta \tilde{\pi} \tilde{u}_\beta = \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 g^\alpha \tilde{u}_\alpha, \\
& \frac{1}{2} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} D_t \tilde{\pi}^2 + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{\pi} \\
& \quad + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} V_3 \tilde{r} \tilde{\pi} + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} (W_3)_\alpha \tilde{u}^\alpha \tilde{\pi} \\
& \quad + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} Z_3 \tilde{\pi}^2 = \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} h \tilde{\pi}.
\end{aligned}$$

We sum over these identities and obtain

$$\begin{aligned}
& \frac{1}{2} r^{\frac{1}{\kappa}-1} D_t \tilde{r}^2 + \frac{1}{2} \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 D_t (\tilde{u}^\alpha \tilde{u}_\alpha) + \frac{1}{2} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} D_t \tilde{\pi}^2 = r^{\frac{1}{\kappa}-1} f \tilde{r} \\
& \quad + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 g^\alpha \tilde{u}_\alpha + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} h \tilde{\pi} \\
& \quad - \left( \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa}-1} \partial_\mu r \tilde{u}^\mu \tilde{r} + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{r} \tilde{u}^\mu \right) \\
& \quad - \left( \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{\pi} \tilde{u}^\mu + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{\pi} \right) \\
& \quad - \left( r^{\frac{1}{\kappa}-1} V_1 \tilde{r}^2 + r^{\frac{1}{\kappa}+1} Z_1 \tilde{\pi} \tilde{r} \right) \\
& \quad - \left( \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (V_2)^\beta \tilde{r} \tilde{u}_\beta + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (W_2)_\gamma \tilde{u}^\gamma \tilde{u}_\alpha + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (Z_2)^\beta \tilde{\pi} \tilde{u}_\beta \right) \\
& \quad - \left( \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} V_3 \tilde{r} \tilde{\pi} + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} (W_3)_\alpha \tilde{u}^\alpha \tilde{\pi} \right. \\
& \quad \left. + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} Z_3 \tilde{\pi}^2 \right). \quad (2.12)
\end{aligned}$$

The blue terms form a perfect derivative modulo some terms that can be controlled by the energy, i.e.,

$$\begin{aligned}
& \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa}-1} \partial_\mu r \tilde{u}^\mu \tilde{r} + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{r} \tilde{u}^\mu = \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) \\
& \quad - r^{\frac{1}{\kappa}} \left[ (1+\kappa) \partial_\mu r + (2\kappa+1) r a_0^{1+\frac{1}{\kappa}} \partial_\mu \hat{\Pi} \right] \tilde{r} \tilde{u}^\mu. \quad (2.13)
\end{aligned}$$

We also group the purple terms into one term. We will show later that this term can be controlled.

$$\begin{aligned} & \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu \tilde{\pi} \tilde{u}^\mu + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu \tilde{u}^\mu \tilde{\pi} \\ &= \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu (\tilde{\pi} \tilde{u}^\mu). \end{aligned} \quad (2.14)$$

Differentiating the energy with (2.3), we obtain

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 \tilde{u}_\alpha \tilde{u}^\alpha + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \tilde{\pi}^2 \right] dx \\ &= \frac{1}{2} \int_{\Omega_t} \frac{1}{u^0} D_t \left\{ r^{\frac{1}{\kappa}-1} \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 \tilde{u}_\alpha \tilde{u}^\alpha + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \tilde{\pi}^2 \right] \right\} dx \\ &\quad + \frac{1}{2} \int_0^t \left\{ r^{\frac{1}{\kappa}-1} \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 \tilde{u}_\alpha \tilde{u}^\alpha + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \tilde{\pi}^2 \right] \right\} \partial_i \left( \frac{u^i}{u^0} \right) dx \\ &= (i) + (ii). \end{aligned}$$

The latter term can be controlled readily, i.e.,

$$(ii) \lesssim \left\| \partial_i \left( \frac{u^i}{u^0} \right) \right\|_{L^\infty(\Omega_t)} E(t) \lesssim \mathcal{C}(K) E(t) \cong \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2,$$

where  $\mathcal{C}(K) > 0$  is a constant depending on  $K > 0$  subject to change line by line. Now, we control the former term,

$$\begin{aligned} (i) &= \frac{1}{2} \int_{\Omega_t} \left( \frac{1}{\kappa} - 1 \right) \frac{1}{u^0} r^{\frac{1}{\kappa}-2} D_t r \left[ \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 (\tilde{u}_\alpha \tilde{u}^\alpha) + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \tilde{\pi}^2 \right] dx \\ &\quad + \frac{1}{2} \int_{\Omega_t} \frac{1}{u^0} r^{\frac{1}{\kappa}-1} \left( D_t \tilde{r}^2 + \frac{\kappa^2}{\kappa+1} r a_1^2 D_t (\tilde{u}_\alpha \tilde{u}^\alpha) + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} D_t \tilde{\pi}^2 \right) dx \\ &\quad + \frac{1}{2} \int_{\Omega_t} \frac{\kappa^2}{\kappa+1} \frac{1}{u^0} r^{\frac{1}{\kappa}-1} D_t (r a_1^2) \tilde{u}_\alpha \tilde{u}^\alpha dx + \frac{1}{2} \int_{\Omega_t} \frac{(2\kappa+1)\kappa}{\kappa+1} \frac{1}{u^0} r^{\frac{1}{\kappa}-1} D_t \left[ \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} \right] \tilde{\pi}^2 dx \\ &= (iii) + (iv) + (v) + (vi). \end{aligned}$$

We must be cautious when a derivative falls on the powers of  $r$ , as this could potentially produce terms with insufficient weight. However, this issue does not arise when the derivative is the material derivative  $D_t$  due to (1.10a)

$$D_t r = -\kappa r a_1 \partial_\mu u^\mu = r O(K),$$

By a direct estimate (also using Remark 2.12 to control the time derivatives  $\partial_t(r, u, \pi)$ ),

$$(iii), (v), (vi) \lesssim \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2.$$

For (iv), we will use the identity (2.12)

$$\begin{aligned}
(iv) &= \int_{\Omega_t} \frac{1}{u^0} r^{\frac{1}{\kappa}-1} f \tilde{r} + \frac{1}{u^0} \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 g^\alpha \tilde{u}_\alpha + \frac{1}{u^0} \frac{(2\kappa+1)^2}{\kappa+1} r^{\frac{1}{\kappa}} \frac{a_0^{\frac{1}{\kappa}}(r+1)}{1+(1+\kappa)r} h \tilde{\pi} \, dx \\
&\quad - \int_{\Omega_t} \frac{1}{u^0} \left( \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa-1}} \partial_\mu r \tilde{u}^\mu \tilde{r} + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{r} \tilde{u}^\mu \right) \, dx \\
&\quad - \int_{\Omega_t} \frac{1}{u^0} \left( \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{\pi} \tilde{u}^\mu + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{\pi} \right) \, dx \\
&\quad - \int_{\Omega_t} \frac{1}{u^0} \left( r^{\frac{1}{\kappa}-1} V_1 \tilde{r}^2 + r^{\frac{1}{\kappa}+1} Z_1 \tilde{\pi} \tilde{r} \right) \, dx \\
&\quad - \int_{\Omega_t} \frac{1}{u^0} \left( \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (V_2)^\beta \tilde{r} \tilde{u}_\beta + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (W_2)_\gamma^\alpha \tilde{u}^\gamma \tilde{u}_\alpha + \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 (Z_2)^\beta \tilde{\pi} \tilde{u}_\beta \right) \, dx \\
&\quad - \int_{\Omega_t} \frac{1}{u^0} \left( \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} V_3 \tilde{r} \tilde{\pi} + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} (W_3)_\alpha \tilde{u}^\alpha \tilde{\pi} \right. \\
&\quad \quad \left. + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} Z_3 \tilde{\pi}^2 \right) \, dx \\
&= (vii) + (viii) + (ix) + (x).
\end{aligned}$$

By Cauchy-Schwartz, we have

$$\begin{aligned}
(vii) &= \int_{\Omega_t} \frac{1}{u^0} r^{\frac{1}{\kappa}-1} f \tilde{r} + \frac{1}{u^0} \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 g^\alpha \tilde{u}_\alpha + \frac{1}{u^0} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}} \frac{r \pi^{1+\frac{1}{\kappa}} a_0^{1+\frac{1}{\kappa}} a_1}{\zeta(r^{\frac{1}{\kappa}})} h \tilde{\pi} \, dx \\
&\lesssim \mathcal{C}(K) \|(f, g, h)\|_{\mathcal{H}} \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}.
\end{aligned}$$

Similarly, (x), which is the sum of last 3 terms of (iv) can be estimated directly,

$$(x) \lesssim \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2.$$

From the blue terms (2.13),

$$\begin{aligned}
(viii) &= - \int_{\Omega_t} \frac{1}{u^0} \left( \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{r} + r^{\frac{1}{\kappa-1}} \partial_\mu r \tilde{u}^\mu \tilde{r} + \kappa r^{\frac{1}{\kappa}} a_1 \partial_\mu \tilde{r} \tilde{u}^\mu \right) \, dx \\
&= - \int_{\Omega_t} \frac{1}{u^0} \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) - \frac{1}{u^0} r^{\frac{1}{\kappa}} \left[ (1+\kappa) \partial_\mu r + (2\kappa+1) r a_0^{1+\frac{1}{\kappa}} \partial_\mu \hat{\Pi} \right] \tilde{r} \tilde{u}^\mu \, dx \\
&\lesssim - \int_{\Omega_t} \frac{1}{u^0} \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) \, dx + \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2.
\end{aligned}$$

From the purple terms (2.14)

$$\begin{aligned}
(ix) &= - \int_{\Omega_t} \frac{1}{u^0} \left( \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{\pi} \tilde{u}^\mu + \frac{(2\kappa+1)\kappa}{\kappa+1} r^{\frac{1}{\kappa}+1} a_0^{1+\frac{1}{\kappa}} a_1 \partial_\mu \tilde{u}^\mu \tilde{\pi} \right) \, dx \\
&= - \int_{\Omega_t} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu (\tilde{\pi} \tilde{u}^\mu) \, dx.
\end{aligned}$$

Combining all the inequalities above, we have

$$\begin{aligned} \frac{dE}{dt} \lesssim & - \int_{\Omega_t} \frac{1}{u^0} \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx - \int_{\Omega_t} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu (\tilde{\pi} \tilde{u}^\mu) dx \\ & + \mathcal{C}(K) \|(f, g, h)\|_{\mathcal{H}} \|\tilde{r}, \tilde{u}, \tilde{\pi}\|_{\mathcal{H}} + \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2. \end{aligned}$$

Now, we take the integral with respect to time,

$$\begin{aligned} E(t) \lesssim & E(0) - \int_0^t \int_{\Omega_s} \frac{1}{u^0} \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx ds \\ & - \int_0^t \int_{\Omega_s} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu (\tilde{\pi} \tilde{u}^\mu) dx ds + \int_0^t \mathcal{C}(K) \|(f, g, h)\|_{\mathcal{H}} \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}} ds \\ & + \int_0^t \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2 ds \end{aligned}$$

We integrate the blue term by part,

$$\begin{aligned} - \int_0^t \int_{\Omega_s} \frac{1}{u^0} \kappa \partial_\mu \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx ds &= \int_0^t \int_{\Omega_s} \partial_\mu \left( \frac{1}{u^0} \right) \kappa \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx ds \\ &\quad - \int_{\Omega_0} \frac{1}{u^0} \kappa \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx - \int_{\Omega_t} \frac{1}{u^0} \kappa \left( r^{\frac{1}{\kappa}} a_1 \tilde{r} \tilde{u}^\mu \right) dx \\ &\lesssim \epsilon (E(t) + E(0)) \cong \epsilon \left( \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2 + \|(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0)\|_{\mathcal{H}}^2 \right), \end{aligned}$$

where  $\epsilon > 0$  is a sufficiently small number due to an extra half power of  $r$ , which is assumed to be small (see Remark 2.10). A similar argument is applied to the purple term, yielding the following,

$$- \int_0^t \int_{\Omega_s} \frac{(2\kappa+1)\kappa}{\kappa+1} r^{1+\frac{1}{\kappa}} a_0^{\frac{1}{\kappa}+1} a_1 \partial_\mu (\tilde{\pi} \tilde{u}^\mu) dx ds \lesssim \epsilon \left( \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2 + \|(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0)\|_{\mathcal{H}}^2 \right).$$

Absorbing  $\epsilon \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2$  to the LHS, we have

$$\begin{aligned} \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2 &\lesssim \|(\tilde{r}(0), \tilde{u}(0), \tilde{\pi}(0))\|_{\mathcal{H}}^2 + \int_0^t \mathcal{C}(K) \|(f, g, h)\|_{\mathcal{H}} \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}} ds \\ &\quad + \int_0^t \mathcal{C}(K) \|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}}^2 ds. \end{aligned}$$

By the Grönwall-type inequality (2.7), we have our basic energy estimate (2.9).  $\square$

**Remark 2.12.** *In this proof, we treat the time derivatives of the background solutions as  $L^\infty$  coefficients, i.e., we assume  $\|\partial_t(r, u, \pi)\|_{L^\infty} \leq \mathcal{C}(K)$ . This differs slightly from (2.8), where the Sobolev norms are considered only in space and not in time. However, we can express the time derivatives in terms of spatial derivatives using (1.10). Let  $\mathbb{L}(r, u, \pi)$  denote the operator associated with the nonlinear system (1.10). This operator can be expressed as*

$$\mathbb{L}(r, u, \pi) = A^0 \partial_t \begin{pmatrix} r \\ u \\ \pi \end{pmatrix} + A^i \partial_i \begin{pmatrix} r \\ u \\ \pi \end{pmatrix} + B \begin{pmatrix} r \\ u \\ \pi \end{pmatrix},$$



where  $A^0 = A^0[r, u, \pi]$ ,  $A^i = A^i[r, u, \pi]$ , and  $B = B[r, u, \pi]$  are given by

$$\begin{aligned}
A^0 &= \begin{pmatrix} u^0 & \kappa r a_1 & 0 & 0 & 0 & 0 \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_0^0 & u^0 & 0 & 0 & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_0^0 \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_1^0 & 0 & u^0 & 0 & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_1^0 \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_2^0 & 0 & 0 & u^0 & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_2^0 \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_3^0 & 0 & 0 & 0 & u^0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_3^0 \\ 0 & r a_4 & 0 & 0 & 0 & u^0 \end{pmatrix}, \\
A^i &= \begin{pmatrix} u^i & 0 & \kappa r a_1 \delta_1^i & \kappa r a_1 \delta_2^i & \kappa r a_1 \delta_3^i & 0 \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_0^i & u^i & 0 & 0 & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_0^i \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_1^i & 0 & u^i & 0 & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_1^i \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_2^i & 0 & 0 & u^i & 0 & \left(2 + \frac{1}{k}\right) r a_3 \Delta_2^i \\ \left(1 + \frac{1}{k}\right) a_2 \Delta_3^i & 0 & 0 & 0 & u^i & \left(2 + \frac{1}{k}\right) r a_3 \Delta_3^i \\ 0 & 0 & r a_4 \delta_1^i & r a_4 \delta_2^i & r a_4 \delta_3^i & u^i \end{pmatrix}, \\
B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 + \lambda \left(r \frac{1}{\kappa}\right) \pi^{2 + \frac{1}{\kappa}} \end{pmatrix}.
\end{aligned}$$

The determinant  $\det A^0 \neq 0$  when  $r$  is small, i.e., near the boundary. Furthermore,

$$(A^0)^{-1} = \begin{pmatrix} \mathcal{O}(1) & r\mathcal{O}(1) & 0 & 0 & 0 & 0 \\ \mathcal{O}(1) & \mathcal{O}(1) & 0 & 0 & 0 & r\mathcal{O}(1) \\ \mathcal{O}(1) & r\mathcal{O}(1) & \mathcal{O}(1) & 0 & 0 & r\mathcal{O}(1) \\ \mathcal{O}(1) & r\mathcal{O}(1) & 0 & \mathcal{O}(1) & 0 & r\mathcal{O}(1) \\ \mathcal{O}(1) & r\mathcal{O}(1) & 0 & 0 & \mathcal{O}(1) & r\mathcal{O}(1) \\ r\mathcal{O}(1) & r\mathcal{O}(1) & 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix},$$

where  $\mathcal{O}(1)$  represents some  $L^\infty$  coefficients in the energy estimate. Therefore (see Notation 3.2 for the notation  $\simeq$ ),

$$\begin{aligned}
\partial_t r &\simeq \partial r + r \partial u + r^2 \partial \pi + r^2 \pi, \\
\partial_t u &\simeq \partial r + \partial u + r \partial \pi + r \pi, \\
\partial_t \pi &\simeq r \partial r + r \partial u + \partial \pi + \pi.
\end{aligned}$$

### 3. LOCAL WELLPOSEDNESS OF LINEARIZED EQUATIONS AND ITS HIGH-ORDER ESTIMATE

In this section, we build on the basic energy estimate established in the previous section for the linearized equations to prove their local well-posedness. The basic energy estimate yields only a weak solution to the system. To construct a more regular solution, we introduce weighted elliptic operators derived from the second-order evolution equations of the main system. The relevant properties of this operator, which are crucial for proving the existence of a regular solution, will also be demonstrated in this section.

**3.1. Bookkeeping Scheme.** Here, we introduce a bookkeeping scheme to track the crucial terms and omit the less important terms during the energy estimate of the linearized equations. This allows us to greatly simplify the analysis. In view of the weighted version of the Sobolev embedding (2.3), we make the following definition,

**Definition 3.1.** *The order of terms of the form  $r^a \partial^b \tilde{r}$ ,  $r^a \partial^b \tilde{u}$  and  $r^a \partial^b \tilde{\pi}$  are defined to be  $b - a$ ,  $b - a + \frac{1}{2}$  and  $b - a + \frac{1}{2}$ , respectively.*

In an estimate of order  $2l$ , if  $b - a = l$ , by the embedding theorem (2.3),

$$\|r^a \partial^b \tilde{r}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \|\tilde{r}\|_{H^{2l, l + \frac{1}{2\kappa} - \frac{1}{2}}}.$$

Similarly, if  $b - a + \frac{1}{2} = l$ , then

$$\begin{aligned} \|r^a \partial^b \tilde{u}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} &\lesssim \|\tilde{u}\|_{H^{2l, l + \frac{1}{2\kappa}}}, \\ \|r^a \partial^b \tilde{\pi}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} &\lesssim \|\tilde{\pi}\|_{H^{2l, l + \frac{1}{2\kappa}}}. \end{aligned}$$

We refer to such terms as exact.

If  $b - a < l$  for  $\tilde{r}$ ,  $b - a + \frac{1}{2} < l$  for  $\tilde{u}$  and  $\tilde{\pi}$ , then we have a similar control as above,

$$\|r^a \partial^b \tilde{r}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \|\tilde{r}\|_{H^{2l, l + \frac{1}{2\kappa} - \frac{1}{2}}}, \quad (3.1)$$

$$\|r^a \partial^b \tilde{u}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \|\tilde{u}\|_{H^{2l, l + \frac{1}{2\kappa}}}, \quad (3.2)$$

$$\|r^a \partial^b \tilde{\pi}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \|\tilde{\pi}\|_{H^{2l, l + \frac{1}{2\kappa}}}, \quad (3.3)$$

These terms are referred to as lower-order.

**Notation 3.2.** *We will use the notation*

$$\text{Expression 1} \simeq \text{Expression 2}$$

*if both Expression 1 and Expression 2 are of the same order and*

$$\text{Expression 1} = \text{Expression 2} + L.O.T.,$$

*where L.O.T. contains all the terms lower-order than Expression 1, 2.*

For each term in an expression, we will retain the powers of undifferentiated  $r$  of the coefficients, as these powers of  $r$  are essential for the terms to be controlled. We will omit the parts of the coefficients involving the derivatives  $\partial^{\geq 1} r$ ,  $\partial^{\geq 1} \pi$  of  $r$ ,  $\pi$  of order higher than 1, as well as terms involving  $u, a_0, a_1, \dots, a_4$  and their derivatives. The reasons for this are twofolds. First, these parts of the coefficients can be treated as  $L^\infty$  coefficients in the estimate. Second, when derivatives are applied to these terms, they will generate lower-order terms.

As an example of the bookkeeping scheme, from the first equation (2.1a) of the linearized system in the homogeneous case (i.e.,  $f = 0$ ):

$$D_t \tilde{r} = -\kappa r a_1 \partial_\mu \tilde{u}^\mu - \partial_\mu r \tilde{u}^\mu - V_1 \tilde{r} - r^2 Z_1 \tilde{\pi} \simeq r \partial \tilde{u} + \tilde{u}. \quad (3.4)$$

**Remark 3.3.** *Since our weighted Sobolev norms only involve spatial derivatives, in the estimate, we need to write all time derivatives in terms of the spatial derivatives. Linearizing the expressions in Remark 2.12, we can show that*

$$\partial_t \tilde{r} \simeq \partial \tilde{r} + L.O.T.,$$

$$\partial_t \tilde{u} \simeq \partial \tilde{u} + L.O.T.,$$

$$\partial_t \tilde{\pi} \simeq \partial \tilde{\pi} + L.O.T..$$

From now on, we use  $\partial$  denotes only spatial derivatives.

**Remark 3.4.** Similar to (3.4), we have

$$\begin{aligned} D_t \tilde{u} &\simeq \partial \tilde{r}, \\ D_t \tilde{\pi} &\simeq r \partial \tilde{u} + \tilde{u}. \end{aligned}$$

Counting the order of RHS,  $D_t \tilde{r}, D_t \tilde{u}, D_t \tilde{\pi}$  have an order of  $\frac{1}{2}, 1, \frac{1}{2}$ . So, we assign order  $\frac{1}{2}$  to the material derivative  $D_t$ .

**3.2. Weighted Elliptic Operators.** The linearized equations (2.1) in its homogeneous form ( $f, g, h = 0$ ) can be rewritten as a second-order evolution that has a wave-like structure. Applying  $D_t$  to the equation (2.1a) and using the equation (2.1b), with the bookkeeping scheme to capture the principal part, we have

$$\begin{aligned} D_t^2 \tilde{r} &\simeq \tilde{L}_1 \tilde{r}, \\ D_t^2 \tilde{u}^\alpha &\simeq \left( \tilde{L}_2 \tilde{u} \right)_\alpha, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \tilde{L}_1 \tilde{r} &= (\kappa + 1) H^{ij} \left( r \partial_i \partial_j \tilde{r} + \frac{1}{\kappa} \partial_i r \partial_j \tilde{r} \right), \\ \left( \tilde{L}_2 \tilde{u} \right)_\alpha &= (\kappa + 1) A_\alpha^i H^{jk} \left( \partial_i (r \partial_j \tilde{u}_k) + \frac{1}{\kappa} \partial_j r \partial_i \tilde{u}_k \right), \end{aligned}$$

with

$$\begin{aligned} A_\alpha^i &= g_\alpha^i - \frac{u^i}{u^0} g_\alpha^0, \\ H^{ij} &= \delta^{ij} - \frac{u^i u^j}{(u^0)^2}. \end{aligned}$$

We say the second-order evolution (3.5) has an wave-like structure because later we will show  $\tilde{L}_1$  and  $\tilde{L}_2$  satisfy elliptic estimates. More precisely, the operator  $\tilde{L}_2$  is only associated to the divergence of  $\tilde{u}$ . To gain the full coercivity, we need to pair it with an operator  $\tilde{L}_3$  associated with the curl of  $\tilde{u}$ , which is defined as follows,

$$\left( \tilde{L}_3 \tilde{u} \right)_\alpha = (\kappa + 1) A_\alpha^i H^{ij} r^{-\frac{1}{\kappa}} \partial_j \left( r^{\frac{1}{\kappa}+1} (\partial_k \tilde{u}_i - \partial_i \tilde{u}_k) \right).$$

We can check that  $\tilde{L}_2 \tilde{L}_3 \simeq \tilde{L}_3 \tilde{L}_2 \simeq 0$ . Although we will not use these operators directly in connection to the wave equations, they play an important role in the high-order energy estimates and in constructing more regular solutions of the linearized equation. We sometimes employ a slightly modified versions  $\hat{L}_1, \hat{L}_2 + \hat{L}_3$  of these operators  $\tilde{L}_1, \tilde{L}_2 + \tilde{L}_3$  for a different purpose, but they share the same principal parts,

$$\begin{aligned} \hat{L}_1 \tilde{r} &= (\kappa + 1) r^{1-\frac{1}{\kappa}} \partial_i \left( r^{\frac{1}{\kappa}} H^{ij} \partial_j \tilde{r} \right), \\ \left( \hat{L}_2 \tilde{u} \right)_\alpha &= (\kappa + 1) A_\alpha^i \partial_i \left( r^{1-\frac{1}{\kappa}} \partial_j \left( r^{\frac{1}{\kappa}} H^{ij} \tilde{u}_k \right) \right), \\ \left( \hat{L}_3 \tilde{u} \right)_\alpha &= (\kappa + 1) A_\alpha^l \Delta_{ml} r^{-\frac{1}{\kappa}} \partial_j \left( r^{\frac{1}{\kappa}+1} H^{jk} H^{im} (\partial_k \tilde{u}_i - \partial_i \tilde{u}_k) \right). \end{aligned} \tag{3.6}$$

We have following properties for these operators,

**Lemma 3.5.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). Then, the operator  $\hat{L}_1$  defined as an unbounded operator of the Hilbert space  $H^{0, \frac{1}{2\kappa} - \frac{1}{2}}$  with the domain*

$$\mathcal{D}(\hat{L}_1) := \{f \in H^{0, \frac{1}{2\kappa} - \frac{1}{2}} \mid \hat{L}_1 f \in H^{0, \frac{1}{2\kappa} - \frac{1}{2}} \text{ in the distributional sense}\}$$

*is a non-negative, self-adjoint operator.*

*Proof.* Similar statements and proofs can be found in [20, 42]. We first show that  $\mathcal{D}(\hat{L}_1^*) = \mathcal{D}(\hat{L}_1)$ . It is sufficient to show  $\mathcal{D}(\hat{L}_1^*) \subset \mathcal{D}(\hat{L}_1)$ . Let  $\tilde{s} \in \mathcal{D}(\hat{L}_1^*)$ , then

$$(\hat{L}_1 \tilde{r}, \tilde{s})_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} = (\tilde{r}, \hat{L}_1^* \tilde{s})_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}, \forall \tilde{r} \in \mathcal{D}(\hat{L}_1).$$

We want to show  $\tilde{s} \in \mathcal{D}(\hat{L}_1)$ , i.e.,  $\hat{L}_1 \tilde{s} \in H^{0, \frac{1}{2\kappa} - \frac{1}{2}}$ . For any  $\phi \in C_c^\infty$ ,

$$\begin{aligned} (\hat{L}_1 \tilde{s})(\phi) &= \int \tilde{s}(\kappa + 1) \partial_j (r^{\frac{1}{\kappa}} H^{ij} \partial_i (r^{\frac{1}{\kappa} - 1} \phi)) dx \\ &= (\tilde{s}, \hat{L}_1 (r^{\frac{1}{\kappa} - 1} \phi))_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \\ &= (\hat{L}_1^* \tilde{s}, r^{\frac{1}{\kappa} - 1} \phi)_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \\ &= \int \hat{L}_1^* \tilde{s} \phi dx. \end{aligned}$$

Since  $\hat{L}_1^* \tilde{s} \in H^{0, \frac{1}{2\kappa} - \frac{1}{2}}$ ,  $\hat{L}_1 \tilde{s} \in H^{0, \frac{1}{2\kappa} - \frac{1}{2}}$  in the distributional sense and thus  $\tilde{s} \in \mathcal{D}(\hat{L}_1)$ . In view of Lemma 3.11, the non-negativity and self-adjointness can be proven by a simple application of integration by parts.  $\square$

With a similar argument as the above lemma, we obtain

**Lemma 3.6.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). Then, the operator  $\hat{L}_2 + \hat{L}_3$  defined as an unbounded operator of the Hilbert space  $H^{0, \frac{1}{2\kappa}}$  with the domain*

$$\mathcal{D}(\hat{L}_2 + \hat{L}_3) := \{f \in H^{0, \frac{1}{2\kappa}} \mid (\hat{L}_2 + \hat{L}_3)f \in H^{0, \frac{1}{2\kappa}} \text{ in the distributional sense}\}$$

*is a non-negative, self-adjoint operator.*

**Remark 3.7.** *In Lemma 3.6, the operator  $\hat{L}_2 + \hat{L}_3$  is an unbounded operator on a subspace of  $H^{0, \frac{1}{2\kappa}}$  consisting of vectors orthogonal to  $u$  with respect to the Minkowski metric  $g$ . For simplicity, we slightly abuse notation by using  $H^{0, \frac{1}{2\kappa}}$  to denote this subspace, to which the linearized 4-velocity belongs.*

**3.3. Elliptic Estimates.** Here, we show the elliptic estimate of the operator  $\tilde{L}_1$  and  $\tilde{L}_2 + \tilde{L}_3$

**Lemma 3.8.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). For  $\tilde{r}$  sufficiently smooth, we have*

$$\|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} + \|\tilde{r}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}. \quad (3.7)$$

*For a sufficiently smooth  $\tilde{u}$  orthogonal to  $u$ , we have*

$$\|\tilde{u}\|_{H^{2, \frac{1}{2\kappa}}} \lesssim \left\| (\tilde{L}_2 + \tilde{L}_3) \tilde{u} \right\|_{H^{0, \frac{1}{2\kappa}}} + \|\tilde{u}\|_{H^{0, \frac{1}{2\kappa}}}. \quad (3.8)$$

*Proof.* By a partition of unity, we can localize the estimates to a small ball. We consider the interesting case where the ball is centered at  $x_0$  on the free boundary. The elliptic estimate is standard otherwise. We pick a coordinate  $e_3 = \frac{(\nabla r)(x_0)}{|(\nabla r)(x_0)|}$  and  $e_1, \dots, e_2$  an orthonormal basis of the tangent space of the free boundary at  $x_0$ . We have

$$|\partial_3 - 1|, |\partial_{i'} r| < \epsilon,$$

where  $\epsilon > 0$  is sufficiently small and  $i' = 1, 2$ . We assume the ball is small enough such that  $H^{ij}$  is a constant modulo perturbative error that can be absorbed onto the other side. So, we simply assume  $H^{ij}$  is a constant in the estimates. To prove (3.7), we first prove

$$\|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} + \frac{1}{2}}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} + \|\tilde{r}\|_{H^{1, \frac{1}{2\kappa} - \frac{1}{2}}}. \quad (3.9)$$

Compute  $\left\| \tilde{L}_1 \tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}$ ,

$$\begin{aligned} \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}^2 &= \int_{\Omega_t} (\kappa + 1)^2 H^{ij} H^{kl} \left( r \partial_i \partial_j \tilde{r} + \frac{1}{\kappa} \partial_i r \partial_j \tilde{r} \right) \left( r \partial_k \partial_l \tilde{r} + \frac{1}{\kappa} \partial_k r \partial_l \tilde{r} \right) dx \\ &\simeq (i) + (ii) + (iii) + (iv), \end{aligned}$$

where (i) is the product of the 2-derivatives, (ii), (iii) are the products of the 1-derivatives and 2-derivatives, and (iv) is the product of the 1-derivatives. For (i) we integrate by parts with respect to  $\partial_i$ ,

$$\begin{aligned} (i) &= \int_{\Omega_t} r^{\frac{1}{\kappa} + 1} H^{ij} H^{kl} \partial_i \partial_j \tilde{r} \partial_k \partial_l \tilde{r} dx \\ &= - \int_{\Omega_t} \left( \frac{1}{\kappa} + 1 \right) r^{\frac{1}{\kappa}} H^{ij} H^{kl} \partial_j \tilde{r} dx - \int_{\Omega_t} r^{\frac{1}{\kappa} + 1} H^{ij} H^{kl} \partial_j \tilde{r} \partial_i \partial_k \partial_l \tilde{r} dx \\ &= (v) + (vi). \end{aligned}$$

By Cauchy-Schwartz with  $\epsilon > 0$ ,

$$\begin{aligned} |(v)| &\lesssim \int_{\Omega_t} |H^{ij} H^{kl} \partial_i r| \left( r^{\frac{1}{2\kappa} - \frac{1}{2}} |\partial_j \tilde{r}| \right) \left( r^{\frac{1}{2\kappa} + \frac{1}{2}} |\partial_k \partial_l \tilde{r}| \right) dx \\ &\lesssim \mathcal{C}(K) \|\tilde{r}\|_{H^{1, \frac{1}{2\kappa} - \frac{1}{2}}} + \epsilon \|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} + \frac{1}{2}}}^2, \end{aligned}$$

where  $\mathcal{C}(K)$  is a constant continuously depending on  $K > 0$  and

$$\|(r, u, \pi)\|_{W^{1, \infty}} \lesssim K.$$

For (vi), we integrate by parts with respect to  $\partial_k$ ,

$$\begin{aligned} (vi) &= \int_{\Omega_t} \left( \frac{1}{\kappa} + 1 \right) r^{\frac{1}{\kappa}} \partial_k r H^{ij} H^{kl} \partial_j \tilde{r} \partial_i \partial_l \tilde{r} dx + \int_{\Omega_t} r^{\frac{1}{\kappa} + 1} H^{ij} H^{kl} \partial_j \partial_k \tilde{r} \partial_i \partial_l \tilde{r} dx \\ &= (vii) + (viii). \end{aligned}$$

We notice that (ii), (iii) and (vii) can be estimated similarly to (v),

$$|(ii)|, |(iii)|, |(vii)| \lesssim \mathcal{C}(K) \|\tilde{r}\|_{H^{1, \frac{1}{2\kappa} - \frac{1}{2}}}^2 + \epsilon \|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} + \frac{1}{2}}}^2.$$

By the ellipticity of the metric  $H$ ,

$$(viii) \gtrsim \left\| r^{\frac{1}{2\kappa} + \frac{1}{2}} \partial^2 \tilde{r} \right\|_{L^2}^2.$$

For (iv),

$$|(iv)| \lesssim \mathcal{C}(K) \|\tilde{r}\|_{H^1, \frac{1}{2\kappa} - \frac{1}{2}}^2.$$

Combining all the above inequalities, we have

$$\|\tilde{r}\|_{H^2, \frac{1}{2\kappa} + \frac{1}{2}} \lesssim_{\mathcal{C}(K)} \left\| \tilde{L}_1 \tilde{r} \right\|_{H^0, \frac{1}{2\kappa} - \frac{1}{2}} + \|\tilde{r}\|_{H^1, \frac{1}{2\kappa} - \frac{1}{2}}.$$

Now, we prove

$$\|\tilde{r}\|_{H^1, \frac{1}{2\kappa} - \frac{1}{2}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^0, \frac{1}{2\kappa} - \frac{1}{2}} + \|\tilde{r}\|_{H^0, \frac{1}{2\kappa} - \frac{1}{2}}. \quad (3.10)$$

Consider the integral and integrate by parts with respect to  $\partial_n$ ,

$$\begin{aligned} \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \tilde{L}_1 \tilde{r} \partial_n \tilde{r} &= \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \partial_n \tilde{r} H^{ij} \left( r \partial_i \partial_j \tilde{r} + \frac{1}{\kappa} \partial_i r \partial_j \tilde{r} \right) dx \\ &= -(\kappa+1) \frac{1}{\kappa} \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \partial_i r H^{ij} \partial_n \tilde{r} \partial_j \tilde{r} dx \\ &\quad - \frac{\kappa+1}{2} \int_{\Omega_t} r^{\frac{1}{\kappa}} \partial_n (H^{ij} \partial_i \tilde{r} \partial_j \tilde{r}) dx \\ &\quad + (\kappa+1) \frac{1}{\kappa} \int_{\Omega_t} r^{\frac{1}{\kappa}-1} \partial_i r H^{ij} \partial_n \tilde{r} \partial_j \tilde{r} dx \\ &= \frac{\kappa+1}{2\kappa} r^{\frac{1}{\kappa}-1} \partial_n H^{ij} \partial_i \tilde{r} \partial_j \tilde{r} dx \\ &\gtrsim \left\| r^{\frac{1}{2\kappa} - \frac{1}{2}} \partial \tilde{r} \right\|_{L^2}^2. \end{aligned}$$

Upon applying Cauchy-Schwartz with  $\epsilon > 0$  on the LHS, we have (3.10). For the basic elliptic estimate (3.8), we again first prove the inequality

$$\left\| \vec{u} \right\|_{H^2, \frac{1}{2\kappa} + 1} \lesssim \left\| \left( \tilde{L}_2 + \tilde{L}_3 \right) \tilde{u} \right\|_{H^0, \frac{1}{2\kappa}} + \left\| \vec{u} \right\|_{H^1, \frac{1}{2\kappa}}.$$

We use  $\vec{u}$  to represent the spatial part of the full linearized 4-velocity  $\tilde{u}$ . This can be proved similarly to the inequality (3.9). Then, we prove the following with a similar semi-geodesic coordinate change in [20] to reduce the estimate into the Euclidean case in [42],

$$\left\| \vec{u} \right\|_{H^1, \frac{1}{2\kappa}} \lesssim \left\| \left( \tilde{L}_2 + \tilde{L}_3 \right) \tilde{u} \right\|_{H^0, \frac{1}{2\kappa}} + \left\| \vec{u} \right\|_{H^0, \frac{1}{2\kappa}}.$$

Combining the 2 inequalities, we have

$$\left\| \vec{u} \right\|_{H^2, \frac{1}{2\kappa} + 1} \lesssim \left\| \left( \tilde{L}_2 + \tilde{L}_3 \right) \tilde{u} \right\|_{H^0, \frac{1}{2\kappa}} + \left\| \vec{u} \right\|_{H^0, \frac{1}{2\kappa}}.$$

To bound the full linearized 4-velocity, we use the orthogonality to write  $\tilde{u}^0 = \frac{u^i}{u^0} \tilde{u}_i$ , so

$$\left\| \tilde{u}^0 \right\|_{H^2, \frac{1}{2\kappa} + 1} = \left\| \frac{u^i}{u^0} \tilde{u}_i \right\|_{H^2, \frac{1}{2\kappa} + 1} \lesssim \left\| \left( \tilde{L}_2 + \tilde{L}_3 \right) \tilde{u} \right\|_{H^0, \frac{1}{2\kappa}} + \left\| \tilde{u} \right\|_{H^0, \frac{1}{2\kappa}},$$

which finishes the proof.  $\square$

**Remark 3.9.** *The operator  $\hat{L}_1$  and  $\hat{L}_2 + \hat{L}_3$  satisfy a similar basic elliptic estimate as in 3.7 and 3.8 because they differ from  $\tilde{L}_1$  and  $\tilde{L}_2 + \tilde{L}_3$  only by lower-order terms.*

**Corollary 3.10.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). For  $\tilde{r}$  sufficiently smooth, we have*

$$\|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} - \frac{1}{2}}} \lesssim \left\| (\hat{L}_1 + I)\tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}.$$

For a sufficiently smooth  $\tilde{u}$  orthogonal to  $u$ , we have

$$\|\tilde{u}\|_{H^{2, \frac{1}{2\kappa}}} \lesssim \left\| (\hat{L}_2 + \hat{L}_3 + I)\tilde{u} \right\|_{H^{0, \frac{1}{2\kappa}}}.$$

*Proof.* By the elliptic estimate (3.7), we have

$$\begin{aligned} \left( (\hat{L}_1 + I)\tilde{r}, (\hat{L}_1 + I)\tilde{r} \right)_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} &= \left\| \hat{L}_1\tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}^2 + 2 \left( \hat{L}_1\tilde{r}, \tilde{r} \right)_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} + \|\tilde{r}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}^2 \\ &\gtrsim \left\| \hat{L}_1\tilde{r} \right\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}^2 + \|\tilde{r}\|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}}^2 \\ &\gtrsim \|\tilde{r}\|_{H^{2, \frac{1}{2\kappa} + \frac{1}{2}}}^2, \end{aligned}$$

where  $\left( \hat{L}_1\tilde{r}, \tilde{r} \right)_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} \geq 0$  follows from integration by parts. A similar proof works for the velocity case.  $\square$

We can then identify the domain of  $\hat{L}_1$  and  $\hat{L}_2 + \hat{L}_3$

**Corollary 3.11.**

$$\begin{aligned} \mathcal{D}(\hat{L}_1) &= H^{2, \frac{1}{2\kappa} + \frac{1}{2}} \\ \mathcal{D}(\hat{L}_2 + \hat{L}_3) &= H^{2, \frac{1}{2\kappa} + 1}, \end{aligned}$$

*Proof.* To show:  $\mathcal{D}(\hat{L}_1) = H^{2, \frac{1}{2\kappa} + \frac{1}{2}}$ , it is sufficient to show  $\mathcal{D}(\hat{L}_1) \subset H^{2, \frac{1}{2\kappa} + \frac{1}{2}}$ . Take  $\phi \in \mathcal{D}(\hat{L}_1) \subset L^2(r^{\frac{1}{\kappa} - 1})$ . Let  $\sigma = \frac{1}{\kappa} - 1$ . Then,  $r^{\frac{\sigma}{2}}\hat{L}_1\phi \in L^2$ , we take  $\phi_n \in C_c^\infty(\Omega)$  such that  $\phi_n \rightarrow r^{\frac{\sigma}{2}}\hat{L}_1\phi$  in  $L^2$ . We define  $\psi_n \in C_c^\infty(\Omega)$  by

$$\begin{cases} \hat{L}_1\psi_n = \frac{\phi_n}{r^{\frac{\sigma}{2}}} \\ \psi_n|_{\partial\text{supp}(\phi_n)} = 0 \end{cases}$$

By the elliptic estimates, we have

$$\|\psi_n - \psi_m\|_{H^{2, \frac{\sigma}{2} + 1}} \lesssim \left\| \hat{L}_1(\psi_n - \psi_m) \right\|_{H^{0, \frac{\sigma}{2}}} = \left\| \frac{\phi_n - \phi_m}{r^{\frac{\sigma}{2}}} \right\|_{H^{0, \frac{\sigma}{2}}} = \|\phi_n - \phi_m\|_{L^2} \rightarrow 0$$

as  $n, m \rightarrow \infty$ . Thus,  $\psi_n \rightarrow \psi \in H^{2, \frac{\sigma}{2} + 1}$ . So,

$$r^{\frac{\sigma}{2}}\hat{L}_1\psi = \lim_{n \rightarrow \infty} r^{\frac{\sigma}{2}}\hat{L}_1\psi_n = \lim_n \phi_n = r^{\frac{\sigma}{2}}\hat{L}_1\phi,$$

which implies  $\hat{L}_1(\psi - \phi) = 0$ , where  $\psi - \phi \in H^{0, \frac{\sigma}{2}}$ . Consider this under the a subdomain  $\Omega' \subset \Omega$ , where  $r$  is strictly positive on  $\partial\Omega'$ . Since 0 is  $C^\infty$ ,  $\psi - \phi$  is  $C^\infty$  on  $\Omega'$  by the standard elliptic theory. By the weighted elliptic estimate,

$$\|\psi - \phi\|_{H^{2, \frac{\sigma}{2} + 1}} \lesssim \left\| \hat{L}_1(\psi - \phi) \right\|_{H^{0, \frac{\sigma}{2}}} = 0.$$

So,  $\phi = \psi \in H^{2, \frac{1}{2\kappa} + \frac{1}{2}}$ .  $\mathcal{D}(\hat{L}_2 + \hat{L}_3) = H^{2, \frac{1}{2\kappa} + 1}$  can be proven similarly.  $\square$

We have the following high-order elliptic estimates.

**Lemma 3.12.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). For  $\tilde{r}$  sufficiently smooth, we have*

$$\|\tilde{r}\|_{H^{2l, \frac{1}{2\kappa} + l - \frac{1}{2}}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{2l-2, \frac{1}{2\kappa} + l - \frac{3}{2}}} + \|\tilde{r}\|_{H^{0, \frac{1}{2\kappa}}}$$

For a sufficiently smooth  $\tilde{u}$  orthogonal to  $u$ , we have

$$\|\tilde{u}\|_{H^{2l, l + \frac{1}{2\kappa}}} \lesssim \left\| \left( \tilde{L}_2 + \tilde{L}_3 \right) \tilde{u} \right\|_{H^{2l-2, l-1 + \frac{1}{2\kappa}}} + \|\tilde{u}\|_{H^{0, \frac{1}{2\kappa}}}.$$

*Proof.* We will prove the case for  $\tilde{r}$  only, as the proof for  $\tilde{u}$  follows similarly. We pick a coordinate  $e_3 = \frac{(\nabla r)(x_0)}{|(\nabla r)(x_0)|}$  and  $e_1, e_2$  an orthonormal basis of the tangent space of the free boundary at  $x_0$ . We have

$$|\partial_3 - 1|, |\partial_{i'} r| < \epsilon,$$

where  $\epsilon > 0$  is sufficiently small and  $i' = 1, 2$ . Let  $\sigma = \frac{1}{2\kappa} - \frac{1}{2}$ . It is sufficient to show that

$$\|\tilde{r}\|_{H^{2l, \frac{1}{2\kappa} + l - \frac{1}{2}}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{2l-2, \frac{1}{2\kappa} + l - \frac{3}{2}}} + \|\tilde{r}\|_{H^{2l-2, l-1 + \frac{1}{2\kappa}}}.$$

To show this, we show that for any  $l \in \mathbb{N}$  and any integer  $0 \leq a \leq l$ ,

$$\|r^a \partial^{l+a} \tilde{r}\|_{H^{0, \sigma}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{2l-2, l-1 + \sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l + \sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1 + \sigma}}. \quad (3.11)$$

We prove this statement by induction. Commuting  $r^a \partial^{l-1+a}$  and  $\tilde{L}_1$ , we obtain

$$r^a \partial^{l-1+a} \tilde{L}_1 \tilde{r} = \tilde{L}_1 (r^a \partial^{l-1+a}) + \left[ r^a \partial^{l-1+a}, \tilde{L}_1 \right] \tilde{r},$$

where

$$\begin{aligned} \left[ r^a \partial^{l-1+a}, \tilde{L}_1 \right] \tilde{r} &= (\kappa + 1)(l - 1 + a) r^a H^{ij} \partial_r \partial_i \partial_j \partial^{l-2+a} \tilde{r} \\ &\quad + (\kappa + 1) r^a H^{ij} \sum_{\gamma=2}^{l-1+a} \binom{l-1+a}{\gamma} \partial^\gamma r \partial^{l-1+a-\gamma} \partial_i \partial_j \tilde{r} \\ &\quad + \frac{\kappa + 1}{\kappa} r^a H^{ij} \sum_{\gamma=1}^{l-1+a} \binom{l-1+a}{\gamma} \partial^\gamma \partial_{i'r} \partial^{l-1+a-\gamma} \partial_j \tilde{r} \\ &\quad + (\kappa + 1) r^a \sum_{\gamma=1}^{l-1+a} \binom{l-1+a}{\gamma} \partial^\gamma H^{ij} \partial^{l-1+a-\gamma} \left( r \partial_i \partial_j \tilde{r} + \frac{1}{\kappa} \partial_{i'} r \partial_j \tilde{r} \right) \\ &\quad - 2a(\kappa + 1) r^a H^{ij} \partial_{i'r} \partial_j \partial^{l-1+a} \tilde{r} - (\kappa + 1)a(a-1) r^{a-1} H^{ij} \partial_{i'r} \partial_j r \partial^{l-1+a} \tilde{r} \\ &\quad - (\kappa + 1) a r^a H^{ij} \partial_i \partial_j r \partial^{l-1+a} \tilde{r} - \frac{1}{\kappa} (\kappa + 1) a r^{a-1} H^{ij} \partial_{i'r} \partial_j r \partial^{l-1+a} \tilde{r} \\ &\simeq r^a \partial^{l+a} \tilde{r} + r^{a-1} \partial^{l-1+a} \tilde{r}. \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} \left\| r^a \partial^{l-1+a} \tilde{L}_1 \tilde{r} \right\|_{H^{0, \sigma}} &= \left\| \tilde{L}_1 (r^a \partial^{l-1+a}) + \left[ r^a \partial^{l-1+a}, \tilde{L}_1 \right] \tilde{r} \right\|_{H^{0, \sigma}} \\ &\gtrsim \left\| \tilde{L}_1 (r^a \partial^{l-1+a}) \right\|_{H^{0, \sigma}} - \left\| r^a \partial^{l+a} \tilde{r} \right\|_{H^{0, \sigma}} - \left\| r^{a-1} \partial^{l-1+a} \tilde{r} \right\|_{H^{0, \sigma}} \end{aligned}$$

By the basic elliptic estimate (3.7),

$$\left\| r^a \partial^{l-1+a} \tilde{r} \right\|_{H^{2, 1 + \sigma}} \lesssim \left\| \tilde{L}_1 \tilde{r} \right\|_{H^{2l-2, l-1 + \sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l + \sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1 + \sigma}}$$



Considering

$$\|r^a \partial^{l-1+a} \tilde{r}\|_{H^{2,1+\sigma}} \gtrsim \|r \partial^2 (r^a \partial^{l-1+a} \tilde{r})\|_{H^{0,\sigma}} - \|r^a \partial^{l+a} \tilde{r}\|_{H^{0,\sigma}} - \|r^{a-1} \partial^{l-1+a} \tilde{r}\|_{H^{0,\sigma}}$$

We use the induction again and obtain the  $(a+1)$ -th case of the induction. It remains to show the base case  $a=0$ . To show this, we instead show the following equivalent statement: for  $b=0, 1, \dots, l$ ,

$$\|\partial_n^b \partial^{l-b} \tilde{r}\|_{H^{0,\sigma}} \lesssim \|\tilde{L}_1 \tilde{r}\|_{H^{2l-2, l-1+\sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l+\sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1+\sigma}}, \quad (3.12)$$

where  $\partial'$  represents the tangential derivatives. We commute  $\partial_n^b \partial^{l-1-b}$  and  $\tilde{L}_1$ ,

$$\begin{aligned} \partial_n^b \partial^{l-1-b} \tilde{L}_1 \tilde{r} &= \tilde{L}_1 (\partial_n^b \partial^{l-1-b} \tilde{r}) + [\partial_n^b \partial^{l-1-b}, \tilde{L}_1] \tilde{r} \\ &\simeq \tilde{L}_1^b (\partial_n^b \partial^{l-1-b} \tilde{r}) + (\kappa+1)(l-b-1) \partial' r \partial^i \partial_j \partial_n^b \partial^{l-b-2} \tilde{r} \\ &\quad + (\kappa+1)b \partial_n r H^{i'j} \partial_{i'} \partial_j \partial_n^{b-1} \partial^{l-b-1} \tilde{r} - (\kappa+1)b H^{i'j} \partial_{i'} r \partial_j \partial_n^b \partial^{l-b-1} \tilde{r}. \end{aligned}$$

Here,  $\tilde{L}_1^b \tilde{r} = (\kappa+1)H^{ij}(r \partial_i \partial_j \tilde{r} + (\frac{1}{\kappa} + b) \partial_i r \partial_j \tilde{r})$  has a similar structure to  $\tilde{L}_1^b$  and a review of the proof of Lemma 3.8 shows that  $\tilde{L}_1^b$  satisfies a similar elliptic bound. So, by induction and the smallness of  $\partial' r$ , we have

$$\|\partial_n^b \partial^{l-b-1} \tilde{r}\|_{H^{1,\sigma}} \lesssim \|\partial_n^b \partial^{l-b-1} \tilde{r}\|_{H^{2,1+\sigma}} \lesssim \|\tilde{L}_1 \tilde{r}\|_{H^{2l-2, l-1+\sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l+\sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1+\sigma}}.$$

Upon using the induction hypothesis on  $b$  again,

$$\|\partial_n^{b+1} \partial^{l-b-1} \tilde{r}\|_{H^{0,\sigma}} \lesssim \|\tilde{L}_1 \tilde{r}\|_{H^{2l-2, l-1+\sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l+\sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1+\sigma}} \quad (3.13)$$

For the base case  $b=0$ , we commute  $\partial^{l-1}$  and  $\tilde{L}_1$

$$\begin{aligned} \partial^{l-1} \tilde{L}_1 \tilde{r} &= \tilde{L}_1 \partial^{l-1} \tilde{r} + [\partial^{l-1}, \tilde{L}_1] \tilde{r} \\ &\simeq \tilde{L}_1 \partial^{l-1} \tilde{r} + (\kappa+1)(l-1) H^{ij} \partial' r \partial_i \partial_j \partial^{l-2} \tilde{r}. \end{aligned}$$

By another application of Lemma 3.8, we have the base case  $b=0$ . Set  $a=l$  in (3.11), we have

$$\|\tilde{r}\|_{H^{2l, l+\sigma}} \lesssim \|\tilde{L}_1 \tilde{r}\|_{H^{2l-2, l-1+\sigma}} + \epsilon \|\tilde{r}\|_{H^{2l, l+\sigma}} + \|\tilde{r}\|_{H^{2l-2, l-1+\sigma}}.$$

We soak the term  $\epsilon \|\tilde{r}\|_{H^{2l, l+\sigma}}$  to the LHS, which finishes the proof.  $\square$

Adding the identity to the operators, we have the following estimates

**Corollary 3.13.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9). For  $\tilde{r}$  sufficiently smooth, we have*

$$\|\tilde{r}\|_{H^{2l, \frac{1}{2\kappa} + l - \frac{1}{2}}} \lesssim \|(\hat{L}_1 + I) \tilde{r}\|_{H^{2l-2, \frac{1}{2\kappa} + l - \frac{3}{2}}}.$$

For a sufficiently smooth  $\tilde{u}$  orthogonal to  $u$ , we have

$$\|\tilde{u}\|_{H^{2l, l + \frac{1}{2\kappa}}} \lesssim \|(\hat{L}_2 + \hat{L}_3 + I) \tilde{u}\|_{H^{2l-2, l-1 + \frac{1}{2\kappa}}}.$$

This corollary implies that  $\hat{L}_1 + I$  and  $\hat{L}_2 + \hat{L}_3 + I$  are injective mapping from  $H^{2l, \frac{1}{2\kappa} + l - \frac{1}{2}}$  to  $H^{2l-2, \frac{1}{2\kappa} + l - \frac{3}{2}}$  and mapping from  $H^{2l, \frac{1}{2\kappa} + l}$  to  $H^{2l-2, \frac{1}{2\kappa} + l - 1}$ . Since injective self-adjoint operators have dense ranges, Lemma 3.5 and Lemma 3.6 imply  $\hat{L}_1 + I$  and  $\hat{L}_2 + \hat{L}_3 + I$  are isomorphisms.

**3.4. Local Wellposedness of the Linearized Equations.** Here we establish the local well-posedness of the linearized equation. The idea of the proof is similar to the proof of Proposition 5.6 in [48].

**Theorem 3.14.** *Let  $(r, u, \pi)$  be a sufficiently smooth solution to (1.10) satisfying the physical vacuum boundary condition (1.9) such that*

$$\sup_{t \in [0, T]} \|(r, u, \pi)\|_{W^{2l+1, \infty}(\Omega_t)} \leq K, \quad (3.14)$$

where  $K > 0$  is a constant and  $l$  is an integer. There exists  $T = T(K) > 0$  such that for any  $(f, g, h) \in L^1([0, T], \mathcal{H}^{2l})$ , there exist unique solutions  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  to the Cauchy problem (2.1), satisfying the following estimate,

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}^{2l}} \leq \mathcal{C}(K) \left( \|(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0)\|_{\mathcal{H}^{2l}} + \int_0^t \|(f, g, h)\|_{\mathcal{H}^{2l}} ds \right), \forall t \in [0, T], \quad (3.15)$$

where  $\mathcal{C}(K) > 0$  is a constant continuously dependent on  $K$ .

*Proof.* We will construct the solution by using a duality argument. We first consider the zero initial data case, i.e.,  $(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0) = (0, 0, 0)$ . Define the set of all test functions as

$$\mathcal{A} := \left\{ \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \in C_c^\infty(\mathcal{D}_T) \left| \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \Big|_{t=T} = 0 \right. \right\}.$$

Let  $\mathcal{L}$  denote the linear operator as in (2.11). We consider the weak formulation of the Cauchy problem, i.e., we say  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  solves the Cauchy problem if

$$\int_0^T \int_{\Omega_t} \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \cdot \mathcal{L}^* \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} dxdt = \int_0^T \int_{\Omega_t} \begin{pmatrix} r^{\frac{1}{\kappa}-1} f \\ \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 G^{\alpha\beta} g_\alpha \\ \frac{(2\kappa+1)^2}{\kappa+1} r^{\frac{1}{\kappa}} \frac{a_0^\kappa (r+1)}{1+(\kappa+1)r} h \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} dxdt,$$

where

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} = \tilde{\phi}\bar{\phi} + \tilde{\psi}^\alpha \bar{\psi}_\alpha + \tilde{\chi}\bar{\chi},$$

and  $\mathcal{L}^*$  is the adjoint operator of  $\mathcal{L}$ . It can be checked that  $\mathcal{L}^*$  has a similar structure to  $\mathcal{L}$ . We consider the adjoint problem

$$\mathcal{L}^* \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} \tilde{\Phi} \\ \tilde{\Psi} \\ \tilde{X} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} (0) = 0.$$

A similar method as in the proof of the basic energy estimate (2.9) gives us

$$\left\| \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \right\|_{L_t^\infty \mathcal{H}} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} \tilde{\Phi} \\ \tilde{\Psi} \\ \tilde{X} \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^*}.$$

Define a map  $J_0 : \mathcal{L}(A) \rightarrow \mathbb{R}$  by

$$J_0 \left( \mathcal{L}^* \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \right) = \int_0^T \int_{\Omega_t} \begin{pmatrix} r^{\frac{1}{\kappa}-1} f \\ \frac{\kappa^2}{\kappa+1} r^{\frac{1}{\kappa}} a_1^2 G^{\alpha\beta} g_\alpha \\ \frac{(2\kappa+1)^2}{\kappa+1} r^{\frac{1}{\kappa}} \frac{a_0^\kappa (r+1)}{1+(\kappa+1)r} h \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} dx dt.$$

We show that  $J_0$  is bounded by the following,

$$\begin{aligned} \left| J_0 \left( \mathcal{L}^* \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \right) \right| &\lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}} \left\| \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \right\|_{L^\infty \mathcal{H}} \\ &\lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}} \left\| \mathcal{L}^* \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\chi} \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^*}. \end{aligned}$$

By Hahn-Banach theorem, we can extend  $J_0$  to  $J \in (L_t^1(\mathcal{H}^*))^*$ . By Riesz representation theorem, there exists a unique  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L^\infty(\mathcal{H})$  to represent  $J$ , which gives us a weak solution. We will prove the high-order case by induction. First, we prove the case where  $l = 1$ , i.e., we establish a solution  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L^\infty([0, T], \mathcal{H}^2)$  given a source  $(f, g, h) \in L^1([0, T], \mathcal{H}^2)$  satisfying the estimate

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{L_t^\infty \mathcal{H}^2} \leq \mathcal{C}(K) \|(f, g, h)\|_{L_t^1 \mathcal{H}^2},$$

where  $(r, u, \pi)$  satisfies

$$\sup_{t \in [0, T]} \|(r, u, \pi)\|_{W^{3, \infty}(\Omega_t)} \leq K.$$

Let us first argue formally. We apply  $\hat{L}_1$ ,  $\hat{L}_2 + \hat{L}_3$  and  $\hat{L}_1$  to the first, second and third equation of the linearized system (2.1). Commuting with  $\mathbb{L}'(r, u, \pi)$ , we get

$$\mathbb{L}'(r, u, \pi) \begin{pmatrix} \hat{L}_1 \tilde{r} \\ (\hat{L}_2 + \hat{L}_3) \tilde{u} \\ \hat{L}_1 \tilde{\pi} \end{pmatrix} = \begin{pmatrix} \hat{L}_1 f + \mathcal{f} \\ (\hat{L}_2 + \hat{L}_3) g + \mathcal{g} \\ \hat{L}_1 h + \mathcal{h} \end{pmatrix},$$

where  $\mathcal{F} = \mathcal{F}[\tilde{r}, \tilde{u}, \tilde{\pi}]$ ,  $\mathcal{G} = \mathcal{G}[\tilde{r}, \tilde{u}, \tilde{\pi}]$  and  $\mathcal{H} = \mathcal{H}[\tilde{r}, \tilde{u}, \tilde{\pi}]$  are all the commutators given by the following

$$\begin{aligned}
\mathcal{F} &= \left[ D_t, \hat{L}_1 \right] \tilde{r} + \kappa r a_1 \partial_\mu \left[ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \right]^\mu - \hat{L}_1 \left( \kappa r a_1 \partial_\mu \tilde{u}^\mu \right) \\
&\quad + \partial_\mu r \left[ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \right]^\mu - \hat{L}_1 \left( \partial_\mu r \tilde{u}^\mu \right) - \left( \hat{L}_1 V_1 \right) \tilde{r} - \hat{L}_1 \left( r^2 Z_1 \right) \tilde{\pi}, \\
\mathcal{G} &= \left[ D_t, \left( \hat{L}_2 + \hat{L}_3 \right) \right] \tilde{u} \\
&\quad + \left( 1 + \frac{1}{\kappa} \right) a_2 \Delta^{\bullet\mu} \partial_\mu \left( \hat{L}_1 \tilde{r} \right) - \left( \hat{L}_2 + \hat{L}_3 \right) \left[ \left( 1 + \frac{1}{\kappa} \right) a_2 \Delta^{\bullet\mu} \partial_\mu \tilde{r} \right] \\
&\quad + \left( 2 + \frac{1}{\kappa} \right) r a_3 \Delta^{\bullet\mu} \partial_\mu \left( \hat{L}_1 \tilde{\pi} \right) - \left( \hat{L}_2 + \hat{L}_3 \right) \left[ \left( 2 + \frac{1}{\kappa} \right) r a_3 \Delta^{\bullet\mu} \partial_\mu \tilde{\pi} \right] \\
&\quad + V_2 \hat{L}_1 \tilde{r} - \left( \hat{L}_2 + \hat{L}_3 \right) \left( V_2 \tilde{r} \right) \\
&\quad + \left( W_2 \right)_\beta \left[ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \right]^\beta - \left( \hat{L}_2 + \hat{L}_3 \right) \left( \left( W_2 \right)_\beta \tilde{u}^\beta \right) \\
&\quad + Z_2 \left( \hat{L}_1 \tilde{\pi} \right) - \left( \hat{L}_2 + \hat{L}_3 \right) \left( Z_2 \tilde{\pi} \right), \\
\mathcal{H} &= \left[ D_t, \hat{L}_1 \right] \tilde{\pi} + r a_4 \partial_\mu \left[ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \right]^\mu - \hat{L}_1 \left( r a_4 \partial_\mu \tilde{u}^\mu \right) \\
&\quad - \hat{L}_1 \left( r V_3 \right) \tilde{r} + \left( W_3 \right)_\alpha \left[ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \right]^\alpha - \hat{L}_1 \left( \left( W_3 \right)_\alpha \tilde{u}^\alpha \right) - \hat{L}_1 Z_3 \tilde{\pi}.
\end{aligned} \tag{3.16}$$

and satisfying

$$\begin{aligned}
\| \mathcal{F}[\tilde{r}, \tilde{u}, \tilde{\pi}] \|_{H^{0, \frac{1}{2\kappa} - \frac{1}{2}}} &\lesssim \mathcal{C}(K) \| (\tilde{r}, \tilde{u}, \tilde{\pi}) \|_{\mathcal{H}^2}, \\
\| \mathcal{G}[\tilde{r}, \tilde{u}, \tilde{\pi}] \|_{H^{0, \frac{1}{2\kappa}}} &\lesssim \mathcal{C}(K) \| (\tilde{r}, \tilde{u}, \tilde{\pi}) \|_{\mathcal{H}^2}, \\
\| \mathcal{H}[\tilde{r}, \tilde{u}, \tilde{\pi}] \|_{H^{0, \frac{1}{2\kappa}}} &\lesssim \mathcal{C}(K) \| (\tilde{r}, \tilde{u}, \tilde{\pi}) \|_{\mathcal{H}^2}.
\end{aligned} \tag{3.17}$$

It might appear that above inequality is not true because of **the red terms** in (3.16). However, we remark here that these **red terms** cancel with each other at the top order and reduce to terms that can be controlled. Formally, by the basic energy estimate (2.9), we will have

$$\begin{aligned}
\left\| \begin{pmatrix} \hat{L}_1 \tilde{r} \\ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u} \\ \hat{L}_1 \tilde{\pi} \end{pmatrix} \right\|_{\mathcal{H}} &\lesssim \left\| \begin{pmatrix} \hat{L}_1 \tilde{r}_0 \\ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u}_0 \\ \hat{L}_1 \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}} + \int_0^t \left\| \begin{pmatrix} \hat{L}_1 f \\ \left( \hat{L}_2 + \hat{L}_3 \right) g \\ \hat{L}_1 h \end{pmatrix} + \mathcal{C}(K) \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{pmatrix} \right\|_{\mathcal{H}} dt \\
&\lesssim \left\| \begin{pmatrix} \hat{L}_1 \tilde{r}_0 \\ \left( \hat{L}_2 + \hat{L}_3 \right) \tilde{u}_0 \\ \hat{L}_1 \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}} \\
&\quad + \mathcal{C}(K) \left( \int_0^t \left\| \begin{pmatrix} \hat{L}_1 f \\ \left( \hat{L}_2 + \hat{L}_3 \right) g \\ \hat{L}_1 h \end{pmatrix} \right\|_{\mathcal{H}} dt + \int_0^t \left\| \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{pmatrix} \right\|_{\mathcal{H}} dt \right) \\
&\lesssim \left\| \begin{pmatrix} \tilde{r}_0 \\ \tilde{u}_0 \\ \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}^2} + \mathcal{C}(K) \left( \int_0^t \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{\mathcal{H}^2} dt + \int_0^t \left\| \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \right\|_{\mathcal{H}^2} dt \right)
\end{aligned}$$

By the elliptic estimates (3.10), we have

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}^2} \lesssim \left\| \begin{pmatrix} \tilde{r}_0 \\ \tilde{u}_0 \\ \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}^2} + \mathcal{C}(K) \left( \int_0^t \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{\mathcal{H}^2} dt + \int_0^t \left\| \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \right\|_{\mathcal{H}^2} dt \right).$$

By the standard grönwall's inequality, we have

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}^2} \lesssim \mathcal{C}(K) \left( \left\| \begin{pmatrix} \tilde{r}_0 \\ \tilde{u}_0 \\ \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}^2} + \int_0^t \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{\mathcal{H}^2} dt \right).$$

In the same manner, we can prove the high-order case,

$$\|(\tilde{r}, \tilde{u}, \tilde{\pi})\|_{\mathcal{H}^{2l}} \lesssim \mathcal{C}(K) \left( \left\| \begin{pmatrix} \tilde{r}_0 \\ \tilde{u}_0 \\ \tilde{\pi}_0 \end{pmatrix} \right\|_{\mathcal{H}^{2l}} + \int_0^t \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{\mathcal{H}^{2l}} dt \right). \quad (3.18)$$

To make the argument more rigorous, we define

$$\begin{pmatrix} \tilde{r}^0 \\ \tilde{u}^0 \\ \tilde{\pi}^0 \end{pmatrix} \in L_t^\infty(\mathcal{H})$$

by solving the Cauchy problem (2.1) with the source term replaced by

$$\begin{pmatrix} \hat{L}_1 f \\ (\hat{L}_2 + \hat{L}_3)g \\ \hat{L}_1 h \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \left\| \begin{pmatrix} \tilde{r}^0 \\ \tilde{u}^0 \\ \tilde{\pi}^0 \end{pmatrix} \right\|_{L_t^\infty \mathcal{H}} &\lesssim \mathcal{C}(K) \left\| \begin{pmatrix} \hat{L}_1 f \\ (\hat{L}_2 + \hat{L}_3)g \\ \hat{L}_1 h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}} \\ &\lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^2}. \end{aligned}$$

Next, for each  $i$ , we define

$$\begin{pmatrix} \tilde{r}^i \\ \tilde{u}^i \\ \tilde{\pi}^i \end{pmatrix} \in L_t^\infty(\mathcal{H})$$

by solving the Cauchy problem (2.1) with zero data with the source term replaced by

$$\begin{pmatrix} \not{L}[\hat{L}_1^{-1} \tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1} \tilde{u}^{i-1}, \hat{L}_1^{-1} \tilde{\pi}^{i-1}] \\ \not{Q}[\hat{L}_1^{-1} \tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1} \tilde{u}^{i-1}, \hat{L}_1^{-1} \tilde{\pi}^{i-1}] \\ \not{R}[\hat{L}_1^{-1} \tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1} \tilde{u}^{i-1}, \hat{L}_1^{-1} \tilde{\pi}^{i-1}] \end{pmatrix}.$$

By the commutator estimate (3.17) and the basic energy estimate (2.9), we have

$$\begin{aligned}
\left\| \begin{pmatrix} \tilde{r}^i \\ \tilde{u}^i \\ \tilde{\pi}^i \end{pmatrix} \right\|_{L_t^\infty(\mathcal{H})} &\lesssim \mathcal{C}(K) \left\| \begin{pmatrix} \mathcal{F}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \\ \mathcal{G}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \\ \mathcal{H}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \end{pmatrix} \right\|_{L_t^1\mathcal{H}} \\
&\lesssim \left\| \begin{pmatrix} \hat{L}_1^{-1}\tilde{r}^{i-1} \\ (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1} \\ \hat{L}_1^{-1}\tilde{\pi}^{i-1} \end{pmatrix} \right\|_{L_t^1\mathcal{H}^2} \\
&\lesssim \left\| \begin{pmatrix} \tilde{r}^{i-1} \\ \tilde{u}^{i-1} \\ \tilde{\pi}^{i-1} \end{pmatrix} \right\|_{L_t^1\mathcal{H}} \\
&\lesssim \mathcal{C}(K)T \left\| \begin{pmatrix} \tilde{r}^{i-1} \\ \tilde{u}^{i-1} \\ \tilde{\pi}^{i-1} \end{pmatrix} \right\|_{L_t^\infty\mathcal{H}} \\
&\lesssim \dots \lesssim (\mathcal{C}(K)T)^{i-1} \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1\mathcal{H}^2}.
\end{aligned}$$

If  $T$  is chosen small enough, define

$$\begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} := \sum_{i=0}^{\infty} \begin{pmatrix} \tilde{r}^i \\ \tilde{u}^i \\ \tilde{\pi}^i \end{pmatrix}$$

and it satisfies

$$\left\| \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \right\|_{L_t^\infty\mathcal{H}} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1\mathcal{H}^2}.$$

By continuity, we have

$$\begin{aligned}
\mathbb{L}'(r, u, \pi) \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} &= \sum_{i=0}^{\infty} \mathbb{L}'(r, u, \pi) \begin{pmatrix} \tilde{r}^i \\ \tilde{u}^i \\ \tilde{\pi}^i \end{pmatrix} \\
&= \begin{pmatrix} \hat{L}_1 f \\ (\hat{L}_2 + \hat{L}_3)g \\ \hat{L}_1 h \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} \mathcal{F}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \\ \mathcal{G}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \\ \mathcal{H}[\hat{L}_1^{-1}\tilde{r}^{i-1}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}^{i-1}, \hat{L}_1^{-1}\tilde{\pi}^{i-1}] \end{pmatrix} \quad (3.19) \\
&= \begin{pmatrix} \hat{L}_1 f \\ (\hat{L}_2 + \hat{L}_3)g \\ \hat{L}_1 h \end{pmatrix} + \begin{pmatrix} \mathcal{F}[\hat{L}_1^{-1}\tilde{r}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}, \hat{L}_1^{-1}\tilde{\pi}] \\ \mathcal{G}[\hat{L}_1^{-1}\tilde{r}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}, \hat{L}_1^{-1}\tilde{\pi}] \\ \mathcal{H}[\hat{L}_1^{-1}\tilde{r}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}, \hat{L}_1^{-1}\tilde{\pi}] \end{pmatrix}.
\end{aligned}$$

Now, let

$$(\hat{r}, \hat{u}, \hat{\pi}) := (\hat{L}_1^{-1}\tilde{r}, (\hat{L}_2 + \hat{L}_3)^{-1}\tilde{u}, \hat{L}_1^{-1}\tilde{\pi}) \in L_t^\infty\mathcal{H}^2.$$

Then, we have

$$\mathbb{L}'(r, u, \pi) \begin{pmatrix} \hat{L}_1 \hat{r} \\ (\hat{L}_2 + \hat{L}_3) \hat{u} \\ \hat{L}_1 \hat{\pi} \end{pmatrix} = \begin{pmatrix} \hat{L}_1 f \\ (\hat{L}_2 + \hat{L}_3) g \\ \hat{L}_1 h \end{pmatrix} + \begin{pmatrix} \mathcal{F}[\hat{r}, \hat{u}, \hat{\pi}] \\ \mathcal{G}[\hat{r}, \hat{u}, \hat{\pi}] \\ \mathcal{H}[\hat{r}, \hat{u}, \hat{\pi}] \end{pmatrix},$$

commuting the operators  $\begin{pmatrix} \hat{L}_1 \\ \hat{L}_2 + \hat{L}_3 \\ \hat{L}_1 \end{pmatrix}$  and  $\mathbb{L}'(r, u, \pi)$ , we have

$$\begin{pmatrix} \hat{L}_1 \\ \hat{L}_2 + \hat{L}_3 \\ \hat{L}_1 \end{pmatrix} \left( \mathbb{L}'(r, u, \pi) \begin{pmatrix} \hat{r} \\ \hat{u} \\ \hat{\pi} \end{pmatrix} - \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right) = 0.$$

By the invertibility of  $\hat{L}_1$  and  $\hat{L}_2 + \hat{L}_3$ , we know that  $(\hat{r}, \hat{u}, \hat{\pi})$  solves the Cauchy problem and we have the estimate,

$$\|(\hat{r}, \hat{u}, \hat{\pi})\|_{L_t^\infty \mathcal{H}^2} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^2}.$$

To establish the result for general  $l$ , we assume the existence of the solution

$$(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L_t^\infty \mathcal{H}^{2l}$$

to the Cauchy problem (2.1) satisfying the estimate

$$\left\| \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \right\|_{L_t^\infty \mathcal{H}^{2l}} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^{2l}}$$

given the source terms

$$(f, g, h) \in L_t^1 \mathcal{H}^{2l}$$

and then show that  $l + 1$  case holds. If given

$$(f, g, h) \in L_t^1 \mathcal{H}^{2l+2}$$

The inductive assumption and the same argument for the case  $l = 1$  give us a solution  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L_t^\infty \mathcal{H}^{2l}$  to (3.19) satisfying the estimate,

$$\left\| \begin{pmatrix} \tilde{r} \\ \tilde{u} \\ \tilde{\pi} \end{pmatrix} \right\|_{L_t^\infty \mathcal{H}^{2l}} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^{2l+2}}.$$

As in the case  $l = 2$ , with the high-order elliptic estimates Lemma 3.10, we define

$$(\hat{r}, \hat{u}, \hat{\pi}) := (\hat{L}_1^{-1} \tilde{r}, (\hat{L}_2 + \hat{L}_3)^{-1} \tilde{u}, \hat{L}_1 \tilde{\pi}) \in L_t^\infty \mathcal{H}^{2l+2}.$$

$(\hat{r}, \hat{u}, \hat{\pi})$  solves the Cauchy problem with the following estimates by the elliptic bounds Lemma 3.10,

$$\|(\hat{r}, \hat{u}, \hat{\pi})\|_{L_t^\infty \mathcal{H}^{2l+2}} \lesssim \mathcal{C}(K) \left\| \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\|_{L_t^1 \mathcal{H}^{2l+2}}.$$

Now, if  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L^\infty([0, T], \mathcal{H}^{2l})$  is the solution when given a source  $(f, g, h) \in C_c^\infty(\mathcal{D}_T)$ . By solving time derivatives in terms of spatial derivatives and omitting the lower-order terms as in 3.3, we have

$$\begin{aligned}\partial_t \tilde{r} &\simeq \partial \tilde{r} + f + rg + r^2 h \\ \partial_t \tilde{u} &\simeq \partial \tilde{u} + f + g + rh \\ \partial_t \tilde{\pi} &\simeq \partial \tilde{\pi} + rf + rg + h.\end{aligned}\tag{3.20}$$

Then, by the embedding theorem (2.3), we have

$$\partial_t(\tilde{r}, \tilde{u}, \tilde{\pi}) \in L_t^\infty \mathcal{H}^{2l-2}$$

Consequently,  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in C([0, T], \mathcal{H}^{2l})$ . Since here  $l$  can be arbitrary, re-using the equation (3.20), we can improve the regularity of the solution,

$$(\tilde{r}, \tilde{u}, \tilde{\pi}) \in C^\infty(\mathcal{D}_T)$$

By a density argument, we can obtain  $(\tilde{r}, \tilde{u}, \tilde{\pi}) \in C([0, T], \mathcal{H}^{2l})$  for  $(f, g, h) \in L_t^1 \mathcal{H}^{2l}$ . Now we consider the non-zero initial data case. If the initial data  $(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0) \in \mathcal{H}$ , we consider the system for  $\hat{r} := \tilde{r} - \tilde{r}_0$ ,  $\hat{u} := \tilde{u} - \tilde{u}_0$  and  $\hat{\pi} := \tilde{\pi} - \tilde{\pi}_0$ , where  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  solves the system (2.1). The system for  $(\hat{r}, \hat{u}, \hat{\pi})$  is the system (2.1) with zero initial data, where the source terms  $(f, g, h)$  are replaced by  $(f, g, h)$  minus some derivatives of  $(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0)$ . For now, we assume  $(f, g, h), (\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0) \in C_c^\infty(\mathcal{D}_T)$ . Then, we have solutions  $(\hat{r}, \hat{u}, \hat{\pi}) \in C_c^\infty(\mathcal{D}_T)$  and  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  solve the system (2.1) with nonzero initial data  $(\tilde{r}_0, \tilde{u}_0, \tilde{\pi}_0)$ , so  $(\tilde{r}, \tilde{u}, \tilde{\pi})$  satisfies the high-order estimate (3.18). The proof finishes with another density argument.  $\square$

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