Examples of Toric Scalar-flat Kähler Surfaces with Mixed-type Ends

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Abstract

Given a strictly unbounded toric symplectic 4-manifold, we explicitly construct complete toric scalar-flat Kähler metrics on the complement of a toric divisor. These symplectic 4manifolds correspond to a specific class of non-compact Kähler surfaces. We also provide an alternative construction of toric scalar-flat Kähler metrics with conical singularity along the toric divisor, following the approach of Abreu and Sena-Dias.

1 Introduction

Let X be a non-compact toric symplectic 4-manifold. In [4], Abreu and Sena-Dias construct complete toric scalar-flat Kähler metrics on strictly unbounded toric symplectic 4-manifold X. We say X is strictly unbounded if the moment polytope of X is unbounded with the unbounded edges being non-parallel. This condition is equivalent to saying there exists a finite sequence of blow-downs of X from which we obtain a minimal resolution of \mathbb{C}^2/Γ for some finite cyclic subgroup $\Gamma \subset U(2)$. The metrics constructed in [4] include the well-known examples of the LeBrun-Simanca metrics [24], the (multi-)Taub-NUT metrics [23], the gravitational instantons of Gibbons-Hawking [16] and Kronheimer [22], etc.

In this article, using *Donaldson's ansatz* [11] of toric scalar-flat Kähelr metrics, we explicitly construct complete toric scalar-flat Kähler metrics on the complement of a torus-invariant divisor in X, which exhibit *Poincaré type* singularity along the divisor. The study of Poincaré type Kähler metrics stems from the standard Poincaré cusp metric

$$\omega_{\Delta *} = \frac{\sqrt{-1}dz \wedge d\bar{z}}{(|z|\log|z|)^2} = 4\sqrt{-1}\partial\bar{\partial}\log(-\log|z|^2)$$

on the punctured unit disk. In general, for a smooth divisor D in X, we study complete Kähler metrics of Poincaré type(see Definition 2.4). Geometrically, near every point on the divisor, the Poincaré type metric is asymptotic to the model product metric given by the Poincaré cusp metric on the punctured disc and a smooth metric on the divisor. Known constructions of canonical Kähler metrics of Poincaré type include the negative Kähler-Einstein ones studied in [9], [21], [27]; the toric ones studied in [2] and [8]; and the constant scalar curvature Kähler and extremal Kähler ones studied in [25] and [13]. Most of these metrics are *not* explicit, while in this article the constructions of toric scalar-flat Kähler metrics of Poincaré type are explicit.

First, we focus on the case where X is strictly unbounded, or equivalently, as a complex surface, it arises as a finite sequence of blow-ups of minimal resolution of \mathbb{C}^2/Γ , as described in Definition-Proposition 2.1:

Theorem 1.1. (Theorem 3.1+Theorem 3.2) Given X a strictly unbounded toric symplectic 4-manifold and $D = \sum_{i=1}^{m} D_i$ a divisor on X such that each D_i is an irreducible prime divisor fixed by the torus action and $D_i \cap D_j = \emptyset$ for any $1 \le i \ne j \le m$. On $X \setminus D$, we have

- a toric scalar-flat Kähler metric with Poincaré type singularity along D, and is asymptotically locally Euclidean(ALE) on the remaining end;
- a two-parameter family of toric scalar-flat Kähler metrics with Poincaré type singularity along D, and are asymptotic to either the generalized Taub-NUT metrics or the exceptional Taub-NUT metrics on the remaining end.

Along the proof, we will see the complex structures induced from these metrics are biholomorphic to that on X away from the divisor D. Here the generalized Taub-NUT metrics live on \mathbb{C}^2 and are scalar-flat Kähler generalizations of the Ricci-flat Kähler Taub-NUT metrics. These metrics are introduced by Donaldson in [12] and later explored by Abreu and Sena-Dias in [4] and Weber in [28] and [29]. They all have quadratic curvature decay and cubic volume growth, but except for the standard Taub-NUT metric, they are not asymptotically locally flat(ALF). The exceptional Taub-NUT metrics are also scalar-flat Kähler metrics living on \mathbb{C}^2 . These metrics are studied by Weber in [29], where he showed these metrics have quadratic curvature decay and quartic volume growth but are not ALE.

A previously known example of toric scalar-flat Kähler metric of Poincaré type is first discussed by Fu-Yau-Zhou in [14], and later studied by the author in [13], which we refer to as the *Hwang-Singer metric* ω_{HS} . It lives on $\mathbb{C}^2 - \{0\}$. Near the origin, it has Poincaré type singularity, and it is asymptotically Euclidean on the other end. This S^1 -invariant metric is in fact toric and we will discuss it in detail in Example 3.1. Theorem 1.1 then gives us a two-parameter family deformation of ω_{HS} .

Intuitively, Poincaré type metrics can be viewed as the limit of a conical family of metrics when the cone angle approaches 0. Similarly, smooth metrics can be viewed as the limit when the cone angle approaches 2π . This was proved by Guenancia [17] in the Kähler-Einstein setting. For the metrics constructed in Theorem 1.1, we can explicitly write down a conical family of toric Kähler metrics connecting the Poincaré type metrics with those in [4]. This family of conical metrics is *not* necessarily scalar-flat, though.

On the other hand, Weber [30] gave a construction of toric scalar-flat Kähler metrics with conical singularity along the divisor. The cone angle along a given edge is closely related to the notion of "label" introduced there. In [30], the label is interpreted as a characterization of the growth speed of the Killing field vanishing along the edge. We formulate the problem from a different perspective, emphasizing the various *boundary conditions* specified by the cone angles. More precisely, following the method of Abreu and Sena-Dias, we give an independent construction of the conical toric scalar-flat Kähler metrics:

Theorem 1.2. (Theorem 4.1) Consider the same setting as in Theorem 1.1. Fix $\theta_i \in (0,1)$ for $i = 1, \dots, m$, on X, we have

 a conical toric scalar-flat Kähler metric with angle 2πθ_i along D_i and is asymptotically locally Euclidean(ALE) on the remaining end; a two-parameter family of them with angle 2πθ_i along D_i and are asymptotic to either the generalized Taub-NUT metrics or the exceptional Taub-NUT metrics on the remaining end.

Besides the general case where X is strictly unbounded, we construct toric scalar-flat Kähler metrics of Poincaré type when the unbounded edges of the momentum polytope are parallel:

Theorem 1.3. Consider the same setting as in Theorem 1.1 except that the unbounded edges of X are parallel. On $X \setminus D$, we have a one-parameter family of toric scalar-flat Kähler metrics with Poincaré type singularity along D and are asymptotic to the model product metric on $S^2 \times \mathbb{R}^2$ on the remaining end.

The scalar-flat metrics we constructed belong to a particular class of Poincaré type metrics, characterized by the specific behavior of their potential functions along the divisor, which we denote as the $S_{\alpha,\beta}$ type (Definition 2.5). We have the following *uniqueness* result:

Theorem 1.4. (Theorem 5.1) Given the same setting as in Theorem 1.1. Assume g is a toric scalar-flat Kähler metric on $X \setminus D$, and its symplectic potential u is of $S_{\alpha,\beta}$ type along D, then g can only be one of the metrics constructed in Theorem 1.1.

Naturally, we would ask if we still have the uniqueness result without assuming u to be of $S_{\alpha,\beta}$ type:

Question 1.1. (Strong uniqueness) Without assuming the potential function is of $S_{\alpha,\beta}$ type along the divisor in Theorem 1.4, can we still obtain the uniqueness result?

A related question is to determine, locally, whether $S_{\alpha,\beta}$ type represents the only Guillemin boundary behavior for scalar-flat Kähler metric of Poincaré type.

Outline of the article. In Section 2, we discuss the preliminaries of the construction. In Section 3, under a specific boundary condition, we give an explicit construction of toric scalar-flat Kähler metrics of Poincaré type using Donaldson's local ansatz for scalar-flat Kähler metrics, and discuss their asymptotic behavior, hence proving Theorem 1.1 and Theorem 1.3. We also include a discussion on the example of the Hwang-Singer metric. In Section 4, we use similar arguments to construct a family of conical toric scalar-flat Kähler metrics. In the Appendix 5, we show the uniqueness result with prescribed explicit boundary behavior along the divisor.

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2 Preliminaries

In this section, we recall some basics of toric Kähler metrics on toric symplectic 4-manifolds and discuss the local ansatz for finding scalar-flat Kähler metrics. First, we recall the basic definition of a toric symplectic 4-manifold:

Definition 2.1. A symplectic 4-manifold (X, ω) is said to be **toric** if it admits an effective Hamiltonian \mathbb{T}^2 -action τ of the standard torus to the diffeomorphism group of (X, ω) such that the corresponding moment map $\mu: X \to \mathbb{R}^2$ is proper onto its image P.

Here the moment map of the \mathbb{T}^2 -action is a map $\mu: X \to \mathbb{R}^2$ such that $\iota_{\xi}\omega = -d\mu$ for each infinitesimal generator ξ of \mathbb{T}^2 . For a compact symplectic 4-manifold, the moment image of μ is the convex hull of the image of fixed points of \mathbb{T}^2 in X, and the classical Atiyah-Guillemin-Sternberg tells us this image is a polytope. For a non-compact symplectic 4-manifold, we first introduce the definition of the moment polytope:

Definition 2.2. ([4], Definition 2.2) We say a convex polytope $P \subset \mathbb{R}^2$ is a moment polytope if

- (i) for each edge, we can find a primitive vector of \mathbb{Z}^2 , which is an interior normal to this edge;
- (ii) for each pair of intersecting edges, their chosen interior normals form a \mathbb{Z} -basis of \mathbb{Z}^2 .

We say two moment polytopes are equivalent if there exists a translation in \mathbb{R}^2 and a $GL(2,\mathbb{Z})$ transformation mapping one to the other, and two symplectic toric manifolds are equivalent if there is an equivariant symplectomorphism mapping one to the other. Delzant's theorem [10] tells us in the compact setting, the moment polytope determines the symplectic toric 4-manifold up to equivariant symplectomorphism. It turns out that in the non-compact setting, we also have the following correspondence:

Proposition 2.1. ([20]) There is a bijective correspondence between the equivalence class of symplectic toric 4-manifolds and the equivalence class of moment polytopes.

We are particularly interested in the following class of non-compact symplectic toric 4-manifold:

Definition-Proposition 2.1. ([4] Definition 2.4, Proposition 2.8)) A symplectic toric 4-manifold is said to be strictly unbounded if the following equivalent conditions hold:

- (i) the image of the moment map, P, is an unbounded polytope with finitely many edges, with the unbounded edges being non-parallel;
- (ii) X as a complex surface, arises as a finite sequence of blow-ups of a minimal resolution of \mathbb{C}^2/Γ for some finite cyclic group $\Gamma \subset U(2)$ such that \mathbb{C}^2/Γ has an isolated singularity point at the origin.

Let P be a moment polytope given by

$$P := \{ x \in \mathbb{R}^2 : \ell_i(x) := \langle x, \nu_i \rangle + \lambda_i \ge 0, i = 1, \cdots, d \}.$$

$$\tag{1}$$

Here $\nu_i = (\alpha_i, \beta_i) \in \mathbb{Z}^2$ are the primitive interior normals to the edges. We order its edges so that ℓ_1, ℓ_d are the unbounded edges and $\ell_i \cap \ell_{i+1} \neq \emptyset$ for $i = 1, \dots, d-1$. From [4] Remark 2.5, we know we can further assume the Delzant condition for the polytope:

$$det(\nu_i, \nu_{i+1}) = -1$$
, for $i = 1, \dots, d-1$.

Let (X_P, ω_P, τ_P) be the associated symplectic 4-manifold of P with moment map μ_P . It admits a canonical integrable, torus-invariant, compatible complex structure J_P . We denote the resulting

Kähler surface by $(X_P, \omega_P, J_P, g_P)$. Let P° be the interior of P, and consider $X_P^\circ := \mu_P^{-1}(P^\circ)$, then

$$X_P^{\circ} \cong P^{\circ} \times \mathbb{T}^2 = \{ (x, \theta) : x = (x_1, x_2) \in P^{\circ}, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 / \mathbb{Z}^2 \}.$$

Here (x, θ) are interpreted as the *action-angle coordinates* for ω_P , i.e.,

$$\omega_P = dx_1 \wedge d\theta_1 + dx_2 \wedge d\theta_2.$$

From [18], we know the symplectic potential $u_P \in C^{\infty}(P^{\circ})$ is written as

$$u_P = \frac{1}{2} \sum_{i=1}^{d} \ell_i(x) \log \ell_i(x).$$

The metric g_P is given by

$$g_P = \sum_{i,j=1}^{2} ((\operatorname{Hess} u_P)_{ij} dx_i \otimes dx_j + (\operatorname{Hess} u_P)^{ij} d\theta_i \otimes d\theta_j).$$

From [1] and [6], given any toric complex structure J which is ω_P -compatible, there exist action-angle coordinates (x, θ) on P° such that for some symmetric and positive-definite 2×2 matrix U(x), we can write J in the following form

$$J = -\sum_{i,j=1}^{2} (U(x)^{ij} \frac{\partial}{\partial x^{i}} \otimes d\theta_{j} + U(x)_{ij} \frac{\partial}{\partial \theta^{i}} \otimes dx_{j}).$$

Furthermore, the integrability of J is equivalent to the existence of $u \in C^{\infty}(P^{\circ})$ such that $U(x) = \text{Hess}_{x}(u)$. Then u is the potential corresponding to J, and the Kähler metric is written as

$$g = \sum_{i,j=1}^{2} ((\operatorname{Hess} u)_{ij} dx_i \otimes dx_j + (\operatorname{Hess} u)^{ij} dx_i \otimes dx_j).$$
⁽²⁾

For simplicity concern, we will use u_{ij} , u^{ij} to denote $(\text{Hess } u)_{ij}$, $(\text{Hess } u)^{ij}$ respectively. From [3], [2], we know when the Hessian of the symplectic potential u on P° is positive-definite and the boundary behavior of u is specified by the Guillemin's boundary condition, it determines a complex structure on X_P° which extends to (X_P, ω_P, τ_P) . We say u satisfies Guillemin's boundary condition if modulo a smooth function,

$$u(x) = \frac{1}{2} \sum_{i=1}^{d} \ell_i(x) \log \ell_i(x),$$
(3)

and its restriction to the interior of each face of P is strictly convex and smooth.

Definition 2.3. ([5], Definition 4.2) Given P, write $L := \{\ell_1(x), \dots, \ell_d(x)\}$. We say a symplectic potential $u : P^{\circ} \to \mathbb{R}$ belongs to the class S(P, L) if it is smooth, strictly convex, and satisfies the Guillemin boundary condition.

We mention that as discussed in [6] and [5] Proposition 4.3, there is an equivalent characterization of S(P, L), which we refer to as the first-order boundary conditions.

This article focuses on finding scalar-flat Kähler metrics. Direct calculations show the scalar curvature of the metric has the following expression:

$$s = -\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}.$$

Then the scalar-flat equation we aim at solving becomes $\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0$. In [11], Donaldson gave a

reformulation of Joyce's construction in [19], which allows us to write down explicit symplectic potentials of scalar-flat Kähler metrics on complex surfaces. The key is to use the *axi-symmetric harmonic function* as the local model. More precisely:

Theorem 2.1. ([11], local model) Let ξ_1, ξ_2 be two solutions to

$$\frac{\partial^2 \xi}{\partial H^2} + \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} = 0$$
(4)

on

$$\mathbb{H} \coloneqq \{ (H, r) \in \mathbb{R}^2 : r > 0 \}.$$

Then the 1-forms

$$\epsilon_1 = r \left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right), \quad \epsilon_2 = -r \left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right)$$

are closed. Let x_1, x_2 be their primitives, then the 1-form $\epsilon = \xi_1 dx_1 + \xi_2 dx_2$ is also closed. Let u be its primitive. Assume for $\xi = (\xi_1, \xi_2)$, we have

$$\det D\xi > 0. \tag{5}$$

Then u is a local symplectic potential for a scalar-flat Kähler toric metric on \mathbb{R}^4 .

Some known solutions to (4) include

$$aH + b$$
, $a\log r + b$, $\frac{1}{2}\log\left(H + \sqrt{(H+a)^2 + r^2}\right)$ (6)

where $a, b \in \mathbb{R}$. In [4], the authors used these solutions to construct scalar-flat Kähler metrics on unbounded symplectic toric 4-manifolds. These metrics belong to the class S(P, L), and are precisely those whose complex structures are equivariantly biholomorphic to J_P .

The situation is different for toric scalar-flat Kähler metrics of Poincaré type. We first recall the definition of Poincaré type Kähler metrics:

Definition 2.4. (Poincaré type Kähler metric, [7]) Given (X, ω_0) a compact complex manifold and D a smooth divisor in X with $\sigma \in H^0(X, \mathcal{O}(D))$ being a holomorphic defining section. Fix λ a sufficiently large constant such that

$$\omega_h \coloneqq \omega_0 - \sqrt{-1}\partial \overline{\partial} \log(\lambda - \log(|\sigma|^2))$$

is a positive (1,1)-form on $X \setminus D$. We say a closed, smooth (1,1)-form

$$\omega_{PT} \coloneqq \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$$

on $X \setminus D$ is a **Poincaré type Kähler metric** if

- ω_{PT} is quasi-isometric to ω_h , which means there exists some C > 0, such that $\frac{1}{C}\omega_h \leqslant \omega_{PT} \leqslant C\omega_h$ and $\forall i \ge 1$, $\sup_{X \setminus D} |\nabla^i_{\omega_h}\omega_{PT}| < \infty$;
- φ is a smooth function on $X \setminus D$ with $\varphi = O(h)$, and $\forall i \ge 1$, $\sup_{X \setminus D} |\nabla_{\omega_h}^i \varphi| < \infty$.

Let D be a torus-invariant divisor in X, and let ℓ_F be the edge corresponding to D in the moment polytope. In [5], Section 4.3, the authors introduced a special type of Guillemin boundary condition for symplectic potential u, which gives rise to a Poincaré type Kähler metric. For the setting of complex surfaces, we recall the definition as follows:

Definition 2.5. ([5], Definition 4.16) Given $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, we say a symplectic potential $u : P^{\circ} \to \mathbb{R}$ belongs to the class $S_{\alpha,\beta}(P,L,F)$ if it is strictly convex and smooth on P° , its restriction to the interior of each edge of P is strictly convex and smooth, and

$$u + (\alpha + \beta \ell_F) \log \ell_F - \frac{1}{2} \sum_{j=2}^d \ell_j \log \ell_j$$
(7)

is smooth on P.

From [5] Theorem 4.18, we know for $u \in S_{\alpha,\beta}(P,L,F)$, the induced Kähler metric (2) exhibits Poincaré type behavior along D. Let $d\lambda$ be the Lebesgue measure on \mathbb{R}^2 , we define a measure $d\lambda_i$ on ℓ_i by

$$-d\ell_i \wedge d\lambda_i = -\nu_i \wedge d\lambda_i = d\lambda.$$

On ℓ_F , the induced measure is zero for the class S(P, L, F), which is obtained by sending the corresponding label ℓ_i to infinity. Although the Guillemin boundary condition does not have a straightforward extension to describe the behavior of u near ℓ_F , the first-order boundary condition does, as pointed out in [5] Definition 4.6. Note $S_{\alpha,\beta}(P, L, F)$ is only a proper subset of S(P, L, F), which we see by comparing [5] Definition 4.6 and Proposition 4.19.

We write

$$u_{P,F,\alpha,\beta} = \frac{1}{2} \sum_{j=2}^{d} \ell_j \log \ell_j - (\alpha + \beta \ell_F) \log \ell_F$$

as the potential of the model metric for the class $S_{\alpha,\beta}(P,L,F)$. To construct scalar-flat Kähler metrics of Poincaré type in this class, we need other solutions to (4) besides the ones (6). We consider

$$\xi = \frac{1}{2} \frac{1}{\sqrt{(H+a)^2 + r^2}}, \quad a \in \mathbb{R}.$$

This solution, together with the solutions to (4) mentioned above, serve as local models for our construction in the next section.

3 Construction of scalar-flat Kähler metrics of Poincaré type

Given X a strictly unbounded symplectic toric 4-manifold and let P be its moment polytope defined by (1). Write $L = \{\ell_1(x), \dots, \ell_d(x)\}$. Let $D = \sum_{i=1}^m D_i$ be a smooth divisor on X such that each D_j is fixed by the torus action. Let ℓ_{i_j} be its image on the moment polytope. Assume

$$i_1 > 1, i_2 > i_1 + 1, \cdots, d > i_m > i_{m-1} + 1.$$
 (8)

Let $I = \{i_1, \dots, i_m\} \subset \{1, 2, \dots, d\}$ be the index set and $\ell_I = \bigcup_{j=1}^m \ell_{i_j}$ be the union of the edges corresponding to D_i , then $P \setminus \ell_I$ is the moment polytope of $X \setminus D$.



Figure 1: The moment polytope $P \setminus \ell_I$

We prove Theorem 1.1 by giving an explicit construction of the toric scalar-flat Kähler metrics of Poincaré type on $X \setminus D$. Consider $\nu = (\alpha, \beta)$ a vector in \mathbb{R}^2 s.t.

$$\det(\nu, \nu_1), \det(\nu, \nu_d) \ge 0. \tag{9}$$

As discussed in [4], since P is strictly unbounded, this set of vectors forms a cone bounded by $-\nu_1$ and ν_d .

Theorem 3.1. For X, D, P, I, ν defined as above, there exist constants $\Lambda_{i_1}, \dots, \Lambda_{i_m} > 0$ determined by the polytope P, for $(H, r) \in \mathbb{H}$, set

$$\xi_1 \coloneqq \alpha_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) \log \left(H + a_i + \sqrt{(H + a_i)^2 + r^2} \right) - \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k}{\sqrt{(H + a_{k-1})^2 + r^2}} + \alpha H,$$

$$\xi_2 \coloneqq \beta_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) \log \left(H + a_i + \sqrt{(H + a_i)^2 + r^2} \right) - \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k}{\sqrt{(H + a_{k-1})^2 + r^2}} + \beta H.$$

Here a_1, \dots, a_{d-1} are real numbers determined by P satisfying

$$a_{j-1} > a_j \text{ if } j \notin I \text{ and } a_{j-1} = a_j \text{ if } j \in I.$$

$$(10)$$

Let x_1, x_2 be the primitives of

$$\epsilon_1 = r \left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right), \quad \epsilon_2 = -r \left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right)$$

Then they define the momentum action coordinates on P° of some toric scalar-flat Kähler metric of Poincaré type on $X \setminus D$ whose symplectic potential satisfies

$$du = \xi_1 dx_1 + \xi_2 dx_2.$$

Furthermore, for each $k \in I$,

$$u \in S_{\frac{\Lambda_k}{2}, \frac{1}{2} \det(\nu_{k-1}, \nu_{k+1})}(P, L, \ell_k).$$
(11)

Proof. The first step is to show the assumption (5) in Theorem 2.1 is satisfied. We compute $D\xi$:

$$D\xi = \begin{pmatrix} \alpha + \frac{1}{2} \sum_{i=1}^{d-1} \frac{\alpha_{i+1} - \alpha_i}{\rho_i} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k H_{k-1}}{\rho_{k-1}^3} & \frac{\alpha_1}{r} + \frac{1}{2} \sum_{i=1}^{d-1} \frac{(\alpha_{i+1} - \alpha_i)r}{\rho_i(\rho_i + H_i)} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k r}{\rho_{k-1}^3} \\ \beta + \frac{1}{2} \sum_{i=1}^{d-1} \frac{\beta_{i+1} - \beta_i}{\rho_i} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k H_{k-1}}{\rho_{k-1}^3} & \frac{\beta_1}{r} + \frac{1}{2} \sum_{i=1}^{d-1} \frac{(\beta_{i+1} - \beta_i)r}{\rho_i(\rho_i + H_i)} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k r}{\rho_{k-1}^3} \end{pmatrix}.$$
(12)

Here $H_i := H + a_i, \ \rho_i := \sqrt{H_i^2 + r^2}$. Set $a_0 := -\infty, \ a_d := \infty$ and set H_0, H_d, ρ_0, ρ_d accordingly, we rewrite Equation (12) as

$$D\xi = \begin{pmatrix} \alpha + \frac{1}{2} \sum_{i=1}^{d} \alpha_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k H_{k-1}}{\rho_{k-1}^3} & \frac{1}{2r} \sum_{i=1}^{d} \alpha_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k r}{\rho_{k-1}^3} \\ \beta + \frac{1}{2} \sum_{i=1}^{d} \beta_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k H_{k-1}}{\rho_{k-1}^3} & \frac{1}{2r} \sum_{i=1}^{d} \beta_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k r}{\rho_{k-1}^3} \end{pmatrix}.$$

$$(13)$$

Note from [4] Theorem 4.1, we know the determinant of the following matrix is positive:

$$\begin{pmatrix} \alpha + \frac{1}{2} \sum_{i=1}^{d} \alpha_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) & \frac{1}{2r} \sum_{i=1}^{d} \alpha_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) \\ \beta + \frac{1}{2} \sum_{i=1}^{d} \beta_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) & \frac{1}{2r} \sum_{i=1}^{d} \beta_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) \end{pmatrix}.$$
(14)

Comparing the expression of $det(D\xi)$ with the above, it suffices to show their difference is still positive. A key step in showing the positivity is the following lemma:

Lemma 3.1.1. Given $k \in I$, $\forall i$, the term involving i and k in the expression of det $(D\xi)$ has the following expression and is non-negative:

$$\left(\beta_k \alpha_i - \alpha_k \beta_i\right) \frac{1}{4r\rho_{k-1}^3} \left(\frac{r^2 + H_{k-1}H_{i-1}}{\rho_{i-1}} - \frac{r^2 + H_{k-1}H_i}{\rho_i}\right).$$
(15)

Proof. We rewrite Equation (15) as follows:

$$\left(\beta_k \alpha_i - \alpha_k \beta_i\right) \frac{1}{4\rho_{k-1}^3} \left(r \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + H_{k-1} \left(\frac{H_{i-1}}{r\rho_{i-1}} - \frac{H_i}{r\rho_i} \right) \right)$$

Note for any $i, r, \rho_i > 0$, and from $\det(\nu_i, \nu_{i+1}) = -1$ we deduce that $\beta_k \alpha_i - \alpha_k \beta_i > 0 \iff i > k$. Thus, it suffices to show

$$\rho_i(r^2 + H_{k-1}H_{i-1}) - \rho_{i-1}(r^2 + H_{k-1}H_i) > 0 \text{ for } i > k$$
(16)

and

$$\rho_i(r^2 + H_{k-1}H_{i-1}) - \rho_{i-1}(r^2 + H_{k-1}H_i) < 0 \text{ for } i < k.$$
(17)

When i > k, we rewrite the expression as

$$\rho_i(r^2 + H_{k-1}H_{i-1}) - \rho_{i-1}(r^2 + H_{k-1}H_i) = \frac{(\rho_i - \rho_{i-1})r^2}{\rho_i H_{i-1} + \rho_{i-1}H_i} (\rho_i(H_{i-1} - H_{k-1}) + \rho_{i-1}(H_i - H_{k-1})).$$
(18)

We claim that $\rho_{i-1}H_i - \rho_iH_{i-1} > 0$. It is because $f(x) = \frac{x}{\sqrt{x^2 + y^2}}$ is an increasing function given y > 0, then $\frac{H_i}{\sqrt{H_i^2 + r^2}} > \frac{H_{i-1}}{\sqrt{H_{i-1}^2 + r^2}}$. Hence, we know

$$\rho_i H_{i-1} + \rho_{i-1} H_i > 0 \iff (\rho_{i-1} H_i - \rho_i H_{i-1}) (\rho_i H_{i-1} + \rho_{i-1} H_i) > 0 \iff r^2 (H_i^2 - H_{i-1}^2) > 0.$$
(19)

On the other hand, we have

$$\rho_i - \rho_{i-1} > 0 \iff (\rho_i + \rho_{i-1})(\rho_i - \rho_{i-1}) > 0 \iff H_i^2 - H_{i-1}^2 > 0.$$
(20)

Combining $H_{i-1} - H_{k-1} > 0$, $H_i - H_{k-1} > 0$ with (19) and (20), we obtain (16). Similarly, for i < k,

$$\rho_i(r^2 + H_{k-1}H_{i-1}) - \rho_{i-1}(r^2 + H_{k-1}H_i) = \frac{(\rho_i - \rho_{i-1})r^2}{\rho_i H_{i-1} + \rho_{i-1}H_i} (\rho_i(H_{i-1} - H_{k-1}) + \rho_{i-1}(H_i - H_{k-1})) < 0.$$
(21)

For $k, k' \in I$, the term involving k and k' in $\sum_{k \in I} \frac{\alpha_k H_{k-1}}{\rho_{k-1}^3} \sum_{k \in I} \frac{\beta_k r}{\rho_{k-1}^3} - \sum_{k \in I} \frac{\alpha_k r}{\rho_{k-1}^3} \sum_{k \in I} \frac{\beta_k H_{k-1}}{\rho_{k-1}^3}$ has the following expression and is non-negative:

$$\frac{r}{\rho_k^3 \rho_{k'}^3} (\alpha_k \beta_{k'} - \alpha_{k'} \beta_k) (H_{k-1} - H_{k'-1}).$$
(22)

Finally, for ν satisfying (9), given $k \in I$, det $D\xi$ changes by adding

$$\det(\nu,\nu_1)\left(\frac{1}{r} - \frac{r}{2\rho_1(H_1 + \rho_1)}\right) + \frac{r}{2}\sum_{i=1}^{d-1}\det(\nu,\nu_i)\left(\frac{1}{\rho_{i-1}(H_{i-1} + \rho_{i-1})} - \frac{1}{\rho_i(H_i + \rho_i)}\right) + \frac{r}{2}\det(\nu,\nu_d)\frac{1}{\rho_d(H_d + \rho_d)} + (\alpha\beta_k - \beta\alpha_k)\frac{r}{\rho_{k-1}^3}.$$
 (23)

WLOG we assume one of the unbounded edges of P is the x_1 -axis, then $\nu_1 = (0, 1)$. Consider the interior normals ν_i satisfying det $(\nu_{i-1}, \nu_i) = -1$, $i = 2, \dots, d$, we obtain $\alpha_i > 0$, $i = 2, \dots, d$ and $\frac{\beta_d}{\alpha_d} < \frac{\beta_{d-1}}{\alpha_{d-1}} < \dots < \frac{\beta_2}{\alpha_2}$. The condition (9) implies

$$\alpha\beta_k - \beta\alpha_k \ge 0 \text{ for } k \in I.$$

Thus by comparing with [4] Equation (7), we know the above additional term is non-negative. Combining (14), (15), (22), and (23), we conclude $det(D\xi) > 0$.

Next, we prove that $x = (x_1, x_2)$ define global symplectic action coordinates on $P \setminus \ell_I$ and at the same time u(x) has the desired boundary behavior on $\partial P \setminus \ell_I$. Note for

$$\xi = \frac{1}{2\sqrt{(H+a)^2 + r^2}},$$

the primitive of $\epsilon = r \left(\frac{\partial \xi}{\partial r} dH - \frac{\partial \xi}{\partial H} dr \right)$, up to constants, is given by $\frac{1}{2} \left(1 - \frac{H+a}{\sqrt{(H+a)^2 + r^2}} \right).$

Then for $\nu = 0$, up to constants, we have

$$x_{1} = \beta_{1}H + \frac{1}{2}\sum_{i=1}^{d-1} (\beta_{i+1} - \beta_{i})(H_{i} - \rho_{i}) - \frac{1}{2}\sum_{k\in I} \Lambda_{k}\beta_{k} \left(1 - \frac{H_{k-1}}{\rho_{k-1}}\right),$$

$$x_{2} = -\alpha_{1}H - \frac{1}{2}\sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_{i})(H_{i} - \rho_{i}) + \frac{1}{2}\sum_{k\in I} \Lambda_{k}\alpha_{k} \left(1 - \frac{H_{k-1}}{\rho_{k-1}}\right).$$

Then we see x extends continuously to r = 0 except at the points $(H, r) = (-a_k, 0)$ for $k \in I$. The point $(H, r) = (-a_k, 0)$ corresponds to the cusp edge ℓ_k for $k \in I$. Consider the behavior of x in the intervals on the H-axis:

(i) For
$$H > -a_1$$
, then $x_1 = \beta_1 H$, and $x_2 = -\alpha_1 H$;

(ii) for
$$1 \leq j < i_1 - 1$$
, $-a_{j+1} < H < -a_j$, then $x_1 = \beta_{j+1}H + \sum_{i=1}^j a_i(\beta_{i+1} - \beta_i)$, and
 $x_2 = -\alpha_{j+1}H - \sum_{i=1}^j a_i(\alpha_{i+1} - \alpha_i);$

(iii) for
$$j = i_1 - 1$$
, $-a_{i_1+1} < H < -a_{i_1-1} = -a_{i_1}$, then $x_1 = \beta_{i_1+1}H + \sum_{i=1}^{i_1} a_i(\beta_{i+1} - \beta_i) - \Lambda_{i_1}\beta_{i_1}$,
and $x_2 = -\alpha_{i_1+1}H - \sum_{i=1}^{i_1} a_i(\alpha_{i+1} - \alpha_i) + \Lambda_{i_1}\alpha_{i_1}$;

(iv) similar calculations show for
$$2 \le k \le m$$
, and $i_{k-1} + 1 \le j < i_k - 1$, with $-a_{j+1} < H < -a_j$,
we have $x_1 = \beta_{j+1}H + \sum_{i=1}^{j} a_i(\beta_{i+1} - \beta_i) - \sum_{\ell=1}^{k-1} \Lambda_{i_\ell}\beta_{i_\ell}$, and
 $x_2 = -\alpha_{j+1}H - \sum_{i=1}^{j} a_i(\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^{k-1} \Lambda_{i_\ell}\alpha_{i_\ell}$; for $j = i_k - 1$, with
 $-a_{i_k+1} < H < -a_{i_k-1} = -a_{i_k}$, we have $x_1 = \beta_{i_k+1}H + \sum_{i=1}^{i_k} a_i(\beta_{i+1} - \beta_i) - \sum_{\ell=1}^k \Lambda_{i_\ell}\beta_{i_\ell}$, and
 $x_2 = -\alpha_{i_k+1}H - \sum_{i=1}^{i_k} a_i(\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^k \Lambda_{i_\ell}\alpha_{i_\ell}$;
(v) for $H < -a_{d-1}$, then $x_1 = \beta_d H + \sum_{i=1}^{d-1} a_i(\beta_{i+1} - \beta_i) - \sum_{\ell=1}^m \Lambda_{i_\ell}\beta_{i_\ell}$, and
 $x_2 = -\alpha_d H - \sum_{i=1}^{d-1} a_i(\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^m \Lambda_{i_\ell}\alpha_{i_\ell}$.

We want to show the following:

1. $x = (x_1, x_2)$ gives a proper homeomorphism

$$x(H,0): \partial \mathbb{H} \setminus \bigcup_{k \in I} (a_{k-1},0) \to \partial P \setminus \ell_I.$$
(24)

2. u(x) modulo a smooth function is given by

$$\frac{1}{2} \sum_{i \notin I} \ell_i(x) \log \ell_i(x) + \frac{1}{2} \sum_{k \in I} (\det(\nu_{k+1}, \nu_{k-1})\ell_k(x) - \Lambda_k) \log \ell_k(x).$$
(25)

First, we discuss the choice of a_j . Note from [4] Theorem 1.2, there exist real numbers $a'_1 < a'_2 < \cdots < a'_{d-1}$ determined by P such that for $1 \leq j \leq d-1$,

$$\sum_{i=1}^{j} a'_{i} \det(\nu_{i+1} - \nu_{i}, \nu_{j+1}) = \lambda_{j+1}.$$
(26)

More precisely, from [26] Lemma 7.2, we know $a'_i - a'_{i-1} = \frac{L_i}{2\pi |\nu_i|^2}$. Here L_i is the length of the *i*th edge of *P*. On the other hand, from the boundary behavior of the non-cusp edges in (25), with the same proof as [26] Lemma 7.2, we have

$$a_i - a_{i-1} = \frac{L_i}{2\pi |\nu_i|^2}$$

For $k \in I$, we set $a_k = a_{k-1}$. Later we will see that this ensures the desired boundary behavior. Then we obtain the following relations between a_j and a'_j :

$$a_{j} = a'_{j} \text{ if } j < i_{1}; a_{j} = a'_{j} + \sum_{\ell=1}^{k} (a'_{i_{\ell}-1} - a'_{i_{\ell}}) \text{ if } i_{k} \leq j < i_{k+1}; a_{j} = a'_{j} + \sum_{\ell=1}^{m} (a'_{i_{\ell}-1} - a'_{i_{\ell}}) \text{ if } i_{m} \leq j < d.$$

$$(27)$$

First, we look at condition (24). For simplicity, we write $k = i_1$. Near the edge ℓ_{k+1} , as r = 0 and $-a_{k+1} < H < -a_k$, we have

$$x_1 = \beta_{k+1}H + \sum_{i=1}^k a_i(\beta_{i+1} - \beta_i) - \Lambda_k \beta_k, \quad x_2 = -\alpha_{k+1}H - \sum_{i=1}^k a_i(\alpha_{i+1} - \alpha_i) + \Lambda_k \alpha_k.$$

Then condition (24) $\ell_{k+1}(x) = 0$ translates to

$$\sum_{i=1}^{k} a_i \det(\nu_{i+1} - \nu_i, \nu_{k+1}) + \Lambda_k = \lambda_{k+1}.$$
(28)

Using (26) and the relation (27), it simplifies to

$$\Lambda_k = a'_k - a'_{k-1}.\tag{29}$$

For edges ℓ_j with i < k, the argument is the same as the standard case as in [4] Theorem 4.1; for i = k + j, with $1 \le j < i_2 - k$, the condition (24) $\ell_{k+j}(x) = 0$ on $-a_{k+j} < H < a_{k+j-1}$ becomes

$$\sum_{i=1}^{k+j-1} a_i \det(\nu_{i+1} - \nu_i, \nu_{k+j}) + \Lambda_k \det(\nu_{k+j}, \nu_k) = \lambda_{k+j}$$

Using (26) it suffices to check det $(\nu_{k+j}, \nu_k)(\Lambda_k + a'_{k-1} - a'_k) = 0$, which holds from (29). Thus we've shown condition (25) for $1 \leq j < i_2$ given the choice of Λ_k as in (29). It remains to apply the same procedure to indices i_2, \dots, i_m respectively. For simplicity, we write $t = i_2$. For $-a_{t+1} < H < -a_t$, we have

$$x_{1} = \beta_{t+1}H + \sum_{i=1}^{t} a_{i}(\beta_{i+1} - \beta_{i}) - \Lambda_{t}\beta_{t} - \Lambda_{k}\beta_{k}, \quad x_{2} = -\alpha_{t+1}H - \sum_{i=1}^{t} a_{i}(\alpha_{i+1} - \alpha_{i}) + \Lambda_{t}\alpha_{t} + \Lambda_{k}\alpha_{k}.$$

Then condition (24) $\ell_{t+1}(x) = 0$ on $-a_{t+1} < H < -a_t$ translates to

$$\sum_{i=1}^{t} a_i \det(\nu_{i+1} - \nu_i, \nu_{t+1}) + \Lambda_t + \Lambda_k \det(\nu_{t+1}, \nu_k) = \lambda_{t+1},$$
(30)

and it simplifies to

$$\Lambda_t = a'_t - a'_{t-1}.\tag{31}$$

For $i_2 < i < i_3$, condition (24) can be shown in exactly the same way. For i_3, \dots, i_m , it's now clear this procedure also works. We deduce that given the choices of $\Lambda_i = a'_i - a'_{i-1}$ for all $i \in I$, $x : (\overline{\mathbb{H}}, \partial \mathbb{H}) \to (P, \partial P)$ is a proper homeomorphism, and its restriction to \mathbb{H} is a smooth proper diffeomorphism onto P° .

Next, we look at the condition (25). Near a non-cusp edge ℓ_i , we have $\xi = \nu_i \log r + O(1)$, and the boundary condition is standard as desired. We focus on the situation near the cusp edge ℓ_k . For this, we prove the following lemmas:

Lemma 3.1.2. Assume for $k \in I$, $\Lambda_k = a'_k - a'_{k-1}$, then

$$\ell_k(x) = \rho_{k-1} + O(r^2), \tag{32}$$

and there exist smooth positive function $\delta_{k-1,k,k+1}$ such that

$$\rho_{k-1} + H_{k-1} = \rho_{k-1} \cdot \frac{2\ell_{k+1}(x) + O(r^2)}{\delta_{k-1,k,k+1}(x)}, \quad \rho_{k-1} - H_{k-1} = \rho_{k-1} \cdot \frac{2\ell_{k-1}(x) + O(r^2)}{\delta_{k-1,k,k+1}(x)}.$$
(33)

Proof. We first consider $k = i_1$. As both H_{k-1} and r go to zero, for i > k, we have $H_i - \sqrt{H_i^2 + r^2} = -\frac{r^2}{2a_i}$, and for i < k - 1, we have $H_i - \sqrt{H_i^2 + r^2} = 2a_i$. Then for the behavior of x near $\ell_k(x) = 0$, we write

$$x_1 = \beta_{k-1}H + \frac{1}{2}(\beta_{k+1} - \beta_{k-1})(H_{k-1} - \rho_{k-1}) - \frac{1}{2}\Lambda_k\beta_k\left(1 + \frac{H_{k-1}}{\rho_{k-1}}\right) + \sum_{i=1}^{k-2}a_i(\beta_{i+1} - \beta_i) + O(r^2),$$

$$x_{2} = -\alpha_{k-1}H - \frac{1}{2}(\alpha_{k+1} - \alpha_{k-1})(H_{k-1} - \rho_{k-1}) + \frac{1}{2}\Lambda_{k}\alpha_{k}\left(1 + \frac{H_{k-1}}{\rho_{k-1}}\right) - \sum_{i=1}^{k-2}a_{i}(\alpha_{i+1} - \alpha_{i}) + O(r^{2}).$$

Then

$$\nu_k \cdot x = \rho_{k-1} - a_{k-1} + \sum_{i=1}^{k-2} a_i \det(\nu_k, \nu_{i+1} - \nu_i) + O(r^2).$$

To prove Equation (32), it's equivalent to show

$$\sum_{i=1}^{k-2} a_i \det(\nu_k, \nu_{i+1} - \nu_i) - a_{k-1} + \lambda_k = 0.$$
(34)

From the relations between a_j and a'_j in Equation (27) and the expression of λ_j in Equation (26), direct computation shows this automatically holds.

Similarly, we have

$$\nu_{k-1} \cdot x = \frac{1}{2} \det(\nu_{k+1}, \nu_{k-1})(\rho_{k-1} - H_{k-1}) + \frac{\Lambda_k}{2}(1 - \frac{H_{k-1}}{\rho_{k-1}}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i) + O(r^2).$$

Under the normalization assumption $det(\nu_j, \nu_{j+1}) = -1$ for all j, straightforward calculations give

$$\det(\nu_{j+1}, \nu_{j-1}) = \frac{\alpha_{j+1} + \alpha_{j-1}}{\alpha_j} = \frac{\beta_{j+1} + \beta_{j-1}}{\beta_j}.$$
(35)

Combining (35) and (32) with the above calculations, we obtain

$$H_{k-1} = \rho_{k-1} \cdot \frac{(\nu_{k+1} - \nu_{k-1}) \cdot x + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + 2\sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i) + \Lambda_k + O(r^2)}{(\nu_{k+1} + \nu_{k-1}) \cdot x + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \Lambda_k + O(r^2)}.$$

Thus to prove Equation (33), it suffices to show

$$\lambda_{k+1} = \Lambda_k + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i),$$
(36)

and

$$\lambda_{k-1} = -\sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i).$$
(37)

We claim that

$$\lambda_{k+1} - \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \lambda_{k-1} - \Lambda_k = 0.$$
(38)

Using (26), we have

$$\lambda_{k+1} + \lambda_{k-1} = a_k - a_{k-1} + a_{k-1} \det(\nu_{k+1}, \nu_{k-1}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1} + \nu_{k+1}, \nu_{i+1} - \nu_i),$$

then by simplifying this equation with (27), we obtain (38). Then (36) is reduced to (37), which is straightforward from (26). Then we obtain Equation (33) with

$$\delta_{k-1,k,k+1} = \ell_{k+1}(x) + \ell_{k-1}(x) + O(r^2)$$

It remains to show Equations (32) and (33) for i_2, \dots, i_m . Write $t = i_2$, near $\ell_t(x) = 0$, we write

$$x_{1} = \beta_{t-1}H + \frac{1}{2}(\beta_{t+1} - \beta_{t-1})(H_{t-1} - \rho_{t-1}) - \frac{1}{2}\Lambda_{t}\beta_{t}\left(1 + \frac{H_{t-1}}{\rho_{t-1}}\right) + \sum_{i=1}^{t-2}a_{i}(\beta_{i+1} - \beta_{i}) - \Lambda_{k}\beta_{k} + O(r^{2}),$$

$$x_{2} = -\alpha_{t-1}H - \frac{1}{2}(\alpha_{t+1} - \alpha_{t-1})(H_{t-1} - \rho_{t-1}) + \frac{1}{2}\Lambda_{t}\alpha_{t}\left(1 + \frac{H_{t-1}}{\rho_{t-1}}\right) - \sum_{i=1}^{t-2}a_{i}(\alpha_{i+1} - \alpha_{i}) - \Lambda_{k}\alpha_{k} + O(r^{2}).$$

Then Equation (32) holds automatically under the assumption $\Lambda_t = a'_t - a'_{t-1}$, and the first equation in Equation (33) simplifies to

$$0 = \lambda_{t+1} - \lambda_t \det(\nu_{t+1}, \nu_{t-1}) + \lambda_{t-1} - \Lambda_t.$$
(39)

Direct computation as the previous situation shows this equation holds. For i_3, \dots, i_m , it's now clear this procedure also works.

Lemma 3.1.3. There exists a smooth and strictly positive function δ on P such that

$$\det(Hess_x(u)) = \left(\delta \prod_{i=1}^d \ell_i \prod_{k \in I} \ell_k\right)^{-1}.$$
(40)

Proof. From [12], we know $r = (\det \operatorname{Hess}_x(u))^{-1/2}$. Then it suffices to show there exists a smooth and strictly positive function δ , such that

$$\prod_{i=1}^{d} \ell_i \prod_{k \in I} \ell_k = \frac{r^2}{\delta}.$$
(41)

From the above discussions, we know

$$\frac{\partial x_1}{\partial r} = -\frac{r}{2} \sum_{i=1}^{d-1} \frac{\beta_{i+1} - \beta_i}{\rho_i} - \frac{r}{2} \sum_{k \in I} \Lambda_k \beta_k \frac{H_{k-1}}{\rho_{k-1}^3}, \quad \frac{\partial x_2}{\partial r} = \frac{r}{2} \sum_{i=1}^{d-1} \frac{\alpha_{i+1} - \alpha_i}{\rho_i} + \frac{r}{2} \sum_{k \in I} \Lambda_k \alpha_k \frac{H_{k-1}}{\rho_{k-1}^3}.$$

Then

$$\frac{\partial \ell_j}{\partial r} = \frac{\partial x_1}{\partial r} \alpha_j + \frac{\partial x_2}{\partial r} \beta_j \tag{42}$$

$$= \frac{r}{2} \sum_{i=1}^{d-1} \frac{\det(\nu_{i+1} - \nu_i, \nu_j)}{\rho_i} + \frac{r}{2} \sum_{k \in I} \frac{\Lambda_k H_{k-1}}{\rho_{k-1}^3} (\alpha_k \beta_j - \alpha_j \beta_k)$$
(43)

$$= \frac{r}{2} \left(-\frac{\det(\nu_1, \nu_j)}{\rho_1} + \sum_{i=2}^{d-1} \det(\nu_i, \nu_j) \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{\det(\nu_d, \nu_j)}{\rho_{d-1}} \right) + \frac{r}{2} \sum_{k \in I} \frac{\Lambda_k H_{k-1}}{\rho_{k-1}^3} \det(\nu_k, \nu_j).$$
(44)

We want to show for $j \notin I$, as we approach each edge E_j of P, $\frac{\partial \ell_j}{\partial r} = r\delta_j$ for some smooth and positive function δ_j ; for $j \in I$, as H approaches $-a_j$ and r approaches 0, $\ell_j^2 \ell_{j-1} \ell_{j+1} = r^2 \delta_j$ for some smooth and positive function δ_j . To show these, we have the following discussions:

- (i) When j = 1, for r = 0, $-a_1 < H$, it's immediate to see each term of (44) is positive. This implies $\frac{\partial \ell_j}{\partial r} = r\delta_j$ for some function δ_j smooth and strictly positive.
- (ii) When j > 1 and $j, j + 1 \notin I$. For $r = 0, -a_j < H < -a_{j-1}$, given $1 \leqslant i < j$, we have $\det(\nu_i, \nu_j) < 0, \frac{1}{\rho_{i-1}} \frac{1}{\rho_i} < 0$; given $j + 1 \leqslant i < d$, we have $\det(\nu_i, \nu_j) > 0, \frac{1}{\rho_{i-1}} \frac{1}{\rho_i} > 0$; given $k \in I, k < j$, we have $\det(\nu_k, \nu_j) < 0, H_{k-1} < 0$; given $k \in I, k > j$, we have $\det(\nu_k, \nu_j) < 0, H_{k-1} < 0$; given $k \in I, k > j$, we have $\det(\nu_k, \nu_j) > 0, H_{k-1} > 0$. Hence, each term of (44) is again positive.

- (iii) When j = d, for r = 0, $H < -a_d$, similar arguments show each term is positive.
- (iv) When $j + 1 \in I$, for r = 0, $-a_j = -a_{j+1} < H < -a_{j-1}$, the only difference is when $i = j + 1 \in I$, we have $\rho_i = \rho_{i-1}$ and $\det(\nu_i, \nu_j) > 0$, $H_j > 0$, thus the involved terms are still positive. All other terms are still positive from the discussion in the second case.
- (v) When $j \in I$. For $H = -a_j = -a_{j-1}$, r doesn't take the value 0, we let $r \to 0$, then $\rho_j, \rho_{j-1} \to 0$. In this case, $\frac{1}{\rho_{j-1}}$ goes to infinity, and we need a different argument. Using Equation (33), we obtain

$$r^{2} = (\rho_{j-1} + H_{j-1})(\rho_{j-1} - H_{j-1}) = \ell_{j}^{2} \cdot \frac{(2\ell_{j-1} + O(r^{2}))(2\ell_{j+1} + O(r^{2}))}{\delta_{j-1,j,j+1}^{2}},$$

here $\delta_{j-1,j,j+1}$ is a smooth and positive function. Thus $\ell_j^2 \ell_{j-1} \ell_{j+1} = r^2 \delta_j$ for some smooth and positive function δ_j .

With the preparation of these lemmas, we are ready to show the boundary condition (25) holds for the cusp edges of P. Consider $k \in I$, near $\ell_k(x) = 0$,

$$\xi_1 = \alpha_{k-1} \log r + \frac{1}{2} (\alpha_{k+1} - \alpha_{k-1}) \log(H_{k-1} + \rho_{k-1}) - \frac{\Lambda_k}{2} \frac{\alpha_k}{\rho_{k-1}} + O(1),$$

$$\xi_2 = \beta_{k-1} \log r + \frac{1}{2} (\beta_{k+1} - \beta_{k-1}) \log(H_{k-1} + \rho_{k-1}) - \frac{\Lambda_k}{2} \frac{\beta_k}{\rho_{k-1}} + O(1).$$

From Lemma 3.1.3, we know $r^2 = \delta' \ell_{k-1} \ell_k^2 \ell_{k+1}$ for some smooth and strictly positive function δ' near ℓ_k . Note from Equations (32) and (33), we obtain from (35) that

$$du = \frac{1}{2} \left(\log(\ell_{k-1})\alpha_{k-1} + \log(\ell_{k+1})\alpha_{k+1} + \left(\det(\nu_{k+1}, \nu_{k-1})\log(\ell_k) - \frac{\Lambda_k}{\ell_k} \right) \alpha_k \right) dx_1 + \frac{1}{2} \left(\log(\ell_{k-1})\beta_{k-1} + \log(\ell_{k+1})\beta_{k+1} + \left(\det(\nu_{k+1}, \nu_{k-1})\log(\ell_k) - \frac{\Lambda_k}{\ell_k} \right) \beta_k \right) dx_2 + O(1).$$
(45)

This is the desired boundary behavior. Hence proving (25). This completes the construction of the metric. The corresponding potential $u \in S_{\frac{\Lambda_k}{2}, \frac{1}{2} \det(\nu_{k-1}, \nu_{k+1})}(P, L, \ell_k)$ for $k \in I$ is immediate from the construction.

Remark 3.1. We remark here that we can modify the assumption (8) on the index set to $1 < i_1 \leq i_2 \leq \cdots \leq i_d < d$. If $k, k+1 \in I$ and $k-1, k+2 \notin I$, which means there are two adjacent cusp edges, then we have $a_{k-1} = a_k = a_{k+1}$. Now define for i < k, $\alpha'_i := \alpha_i, \beta'_i := \beta_i$; for i = k, $\alpha'_k := \alpha_k + \alpha_{k+1}, \beta'_k := \beta_k + \beta_{k+1}$; for i > k, $\alpha'_i := \alpha_{i-1}, \beta'_i := \beta_{i-1}$. Then, the situation is reduced to that discussed in the theorem. Similarly, we can extend the arguments to the case for any number of adjacent cusp edges with this argument. From this, we can relax the assumption of $D_i \cap D_j = \emptyset$ for all $i \neq j$ to allow some non-empty intersection of them.

We are interested in studying the asymptotic behavior of these Poincaré type scalar-flat Kähler metrics away from the divisor. For this, we compare them with the scalar-flat Kähler metrics on \mathbb{C}^2 discussed in [12] and [29]. From [29], we know the symplectic coordinates

$$x_1 = \frac{1}{\sqrt{2}}(-r + \sqrt{H^2 + r^2}) + \alpha H^2, \quad x_2 = \frac{1}{\sqrt{2}}(r + \sqrt{H^2 + r^2}) + \beta H^2, \text{ for } \alpha, \beta \ge 0$$

induce the Taub-NUT metric when $\alpha = \beta = 0$, the generalized Taub-NUT metrics when $\alpha > 0, \beta > 0$, and the exceptional Taub-NUT metrics when $\alpha = 0, \beta > 0$ or $\beta = 0, \alpha > 0$.

Theorem 3.2. Consider the metrics constructed in Theorem 3.1. The metric is (1) ALE if $\alpha = \beta = 0$; (2) asymptotic to a generalized Taub-NUT metric if $\alpha, \beta > 0$; (3) asymptotic to an exceptional Taub-NUT metric if either $\alpha = 0, \beta > 0$ or $\alpha > 0, \beta = 0$.

Proof. From [11], Section 3, given angular coordinates θ_1, θ_2 for x_1, x_2 , the metrics constructed in Theorem 3.1 have the following form

$$\sum u_{ij} dx_i \otimes dx_j + \sum u^{ij} d\theta_i \otimes d\theta_j.$$

Equivalently $\sum u_{ij} dx_i \otimes dx_j$ can be written as

$$r \cdot \det D\xi (dH^2 + dr^2).$$

For $H \ge 0$, and $\rho \coloneqq \sqrt{H^2 + r^2}$, as $\rho \to \infty$, we have

$$\frac{1}{\rho_i + H_i} - \frac{1}{\rho + H} = O\left(\frac{1}{\rho^2}\right), \frac{1}{\rho_i} - \frac{1}{\rho} = O\left(\frac{1}{\rho^2}\right)$$

then as $\rho \to \infty$, we rewrite $D\xi$ as

$$D\xi = \begin{pmatrix} \alpha & 0\\ \beta & 0 \end{pmatrix} + \frac{1}{r} \begin{pmatrix} r \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) & \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i)r^2\\ \frac{1}{2\rho} & \alpha_1 + \frac{1}{2\rho(\rho + H)} \\ r \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) & \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i)r^2\\ \frac{1}{2\rho} & \beta_1 + \frac{1}{2\rho(\rho + H)} \end{pmatrix} + O\left(\frac{1}{\rho^2}\right).$$
(46)

We see that compared to the Abreu and Sena-Dias case, the additional terms of the form $\frac{\alpha_k}{\rho_k}$ in ξ only create terms of the form $O\left(\frac{1}{\rho^2}\right)$. Then under similar computations as in [4] Proposition 5.1, we know

$$r \det D\xi = \frac{\det(\nu_d, \nu_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } \alpha = \beta = 0;$$

$$r \det D\xi = \det(\nu, \nu_1) \left(1 - \frac{r^2}{2\rho(H+\rho)} \right) + \det(\nu, \nu_d) \frac{r^2}{2\rho(H+\rho)} + \frac{\det(\nu_d, \nu_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } (\alpha, \beta) \neq (0, 0)$$

For $H \leq 0$, we argue the same way by considering $(H, r) \mapsto (-H, r), (\nu_1, \dots, \nu_d) \mapsto (\nu_d, \dots, \nu_1),$ $(a_1, \dots, a_d) \mapsto (-a_d, \dots, a_1)$. Hence, we deduce the desired asymptotic behavior for the first two cases. The completeness of these metrics is immediate with the above calculations, for details, see [4] Proposition 5.1.

For the last case where either $\alpha > 0, \beta = 0$ or $\alpha > 0, \beta = 0$, we compare det $D\xi$ with that for the model exceptional Taub-NUT metrics on \mathbb{C}^2 , introduced in [29]. Let ξ_{AS} be the Legendre transform of its momentum coordinate for the toric scalar-flat metric constructed in [4] for the given polytope P, and ξ_{exc} be the one for the exceptional Taub-NUT metric, then we know from [28] Section 5 that det $D\xi_{AS} = \det D\xi_{exc} + O(\rho^{-2})$. For our case, from the above expression (46), we have det $D\xi = \det D\xi_{AS} + O(\rho^{-2})$, and thus the metrics are asymptotic to the exceptional Taub-NUT metrics.

Example 3.1. (Hwang-Singer metric, [13], [14]) Consider the polytope with three edges whose normal vectors are $\nu_1 = (0, 1), \nu_2 = (1, 1), \nu_3 = (1, 0)$ and $I = \{2\}$. Let

$$L = \{x_1 = 0, x_2 = 0, x_1 + x_2 - 1 = 0\}.$$



Then, from the construction, we obtain

$$\xi_1 = \frac{1}{2} \log \left(H + \sqrt{H^2 + r^2} \right) - \frac{1}{2} \frac{1}{\sqrt{H^2 + r^2}}, \quad \xi_2 = \frac{1}{2} \log \left(-H + \sqrt{H^2 + r^2} \right) - \frac{1}{2} \frac{1}{\sqrt{H^2 + r^2}};$$
$$x_1 = \frac{1}{2} \left(H + \sqrt{H^2 + r^2} \right) + \frac{1}{2} \frac{H}{\sqrt{H^2 + r^2}}, \quad x_2 = \frac{1}{2} \left(-H + \sqrt{H^2 + r^2} \right) - \frac{1}{2} \frac{H}{\sqrt{H^2 + r^2}}.$$

Then, we obtain the symplectic potential u, which is

$$u = \frac{1}{2}x_1 \log x_1 + \frac{1}{2}x_2 \log x_2 + \frac{1}{2}(x_1 + x_2 - 1)\log(x_1 + x_2) + h$$

for some smooth function h. Let ω be the corresponding Kähler form. It lives on the complement of the zero section E of the total space of the line bundle $\mathcal{O}(-1)$ over \mathbb{CP}^1 , which we denote by Y. Then $u \in S_{\frac{1}{2},-\frac{1}{2}}(Y,L,E)$. Consider the momentum coordinate

$$\tau := 2(x_1 + x_2) = 2\sqrt{H^2 + r^2},$$

let X be the generator of the S¹-action satisfying $i_X \omega = -d\tau$, then from [30] Section 2.3, we can compute the norm $||X||^2$ as follows:



$$\|X\|^{2} = r \cdot \frac{\left(\frac{\partial \tau}{\partial H}\right)^{2} + \left(\frac{\partial \tau}{\partial r}\right)^{2}}{\frac{\partial \tau}{\partial H}\frac{\partial x_{1}}{\partial r} - \frac{\partial \tau}{\partial r}\frac{\partial x_{1}}{\partial H}} = \frac{2\tau^{2}}{2+\tau}$$

We see it is exactly the momentum profile of the Hwang-Singer metric discussed in [13] Section 2 and Section 3. If we vary the length of the edge corresponding to the divisor, then we obtain a one-parameter family of toric scalar-flat Kähler metrics of Poincaré type introduced in [14]. More precisely, consider

$$\xi_1 = \frac{1}{2}\log\left(H + \sqrt{H^2 + r^2}\right) - \frac{a}{2}\frac{1}{\sqrt{H^2 + r^2}}, \quad \xi_2 = \frac{1}{2}\log\left(-H + \sqrt{H^2 + r^2}\right) - \frac{a}{2}\frac{1}{\sqrt{H^2 + r^2}},$$

the corresponding symplectic potential is given by

$$u = \frac{1}{2}x_1 \log x_1 + \frac{1}{2}x_2 \log x_2 + \frac{1}{2}(x_1 + x_2 - a)\log(x_1 + x_2) + h_a$$

for some smooth function h_a . Then $u \in S_{\frac{a}{2},\frac{1}{2}}(Y,L,E)$.

With a similar approach, we can prove Theorem 1.3, again by explicitly constructing the toric metrics:

Proof of Theorem 1.3. Compared with the case in Theorem 3.1, all arguments of the construction work for the parallel edges case except for the choice of $\nu = (\alpha, \beta)$. Since $\nu_1 = -\nu_d$, ν needs to satisfy

$$\det(\nu, \nu_1) = 0; \quad \det(\nu, \nu_k) \ge 0, \forall k \in I,$$

then with the same arguments, we see ξ_1, ξ_2 give a one-parameter family of toric scalar-flat Kähler metrics. To understand the asymptotic behavior of these metrics, we compare them with that of the product metric g_{prod} on $S^2 \times \mathbb{R}^2$. On S^2 , we take the round metric, and on \mathbb{R}^2 , we take the hyperbolic metric. Then $S^2 \times \mathbb{R}^2$ is biholomorphic to $\mathbb{CP}^1 \times D$. The symplectic coordinates of this product metric, as discussed in [30], can be written as

$$dx_1 = \frac{1}{2} \left(-1 + \sqrt{H^2 + r^2} + \sqrt{(H-1)^2 + r^2} \right), \quad dx_2 = \frac{1}{2} \left(1 - \sqrt{H^2 + r^2} + \sqrt{(H-1)^2 + r^2} \right).$$

The toric scalar-flat metrics whose moment polytope has parallel unbounded edges are discussed by Weber in [30] and [29]. These metrics satisfy the Guillemin boundary condition. Let ξ_{Web} be the Legendre coordinate of the momentum coordinate of the metric for the given polytope, and ξ_{prod} be the one for the model metric. Then we know from [28] Section 5 that det $D\xi_{Web} = \det D\xi_{prod} + O(\rho^{-2})$. For our case, from the formula in Equation (46), we have det $D\xi = \det D\xi_{Web} + O(\rho^{-2})$, thus conclude the asymptotic behavior of the constructed metrics.

At the end of this section, we remark that for a toric scalar-flat Kähler metric whose symplectic potential lives in $S_{\alpha,\beta}(P,F)$, the α,β are **uniquely determined** by the polytope. We recall the following theorem in [5]:

Theorem 3.3. ([5], Proposition 4.20) Consider $u \in S_{\alpha,\beta}(P,L,F)$ satisfies

$$-\sum_{i,j}\frac{\partial^2 u^{ij}}{\partial x_i\partial x_j} = s_{(P,L,F)}$$

for some extremal affine linear function $s_{(P,L,F)}$ of (P,L,F). Then α,β are determined by the data (P,L,F).

Following the proof of this theorem, an explicit expression of α and β are obtained in Equations (46) and (48). For our case where $s_{(P,L,F)} = 0$, we see α, β are expressed in terms of functions on F, thus for non-compact P, they are still determined by the data (P, L, F). Hence, for a fixed polytope P, if the symplectic potential of the toric scalar-flat Kähler metric lives in $S_{\alpha,\beta}$, then the choices of α, β must agree with those in Theorem 3.1. In the Appendix 5, we will discuss a uniqueness result under this prescribed class of symplectic potential.

4 A conical family of toric metrics

Given a cone angle $2\pi\theta_0$ for $\theta_0 \in (0, 1)$, motivated by the conical family in [15] Remark 1.2, we consider the following boundary behavior of potential u of a toric metric which has conical singularity along the divisor corresponding to the edge $\ell(x) = 0$ on its moment polytope:

$$u(x) = \frac{1}{2\theta_0} \cdot \ell(x) \log \ell(x) + h_0(x)$$
(47)

for some smooth function h_0 . For the ALE scalar-flat Kähler metric of Poincaré type constructed in Theorem 3.1(i.e., $\nu = 0$) whose moment polytope is $P \setminus \ell_I$. Let u be its symplectic potential, write

$$u = \frac{1}{2} \sum_{i \notin I} \ell_i \log \ell_i + \frac{1}{2} \sum_{i \in I} (\alpha_i + \beta_i \ell_i) \log \ell_i + h$$

for some $h \in C^{\infty}(P)$. Here $\alpha_i = -\frac{1}{2}\Lambda_i, \beta_i = \frac{1}{2}\det(\nu_{i+1}, \nu_{i-1})$ for $i \in I$. Consider the smooth scalar-flat Kähler metric on P constructed by Abreu and Sena-Dias in [4], let u_{AS} be the symplectic potential, we can write

$$u_{AS} = \frac{1}{2} \sum_{i=1}^{d-1} \ell_i \log \ell_i + h_{AS}$$

for some $h_{AS} \in C^{\infty}(P)$. For any $\theta = (\theta_{i_1}, \cdots, \theta_{i_m})$, consider

$$u^{(\theta)} \coloneqq \frac{1}{2} \sum_{i \notin I} \ell_i \log \ell_i + \frac{1}{2} \sum_{i \in I} \frac{1}{\theta_i} \cdot \ell_i \log \ell_i + v_\theta + \prod_{i \in I} (1 - \theta_i) \cdot h + \prod_{i \in I} \theta_i \cdot h_{AS}.$$
(48)

Here $v_{\theta} = 0$ for $\theta = 1$ and for $\theta \in (0, 1)$,

$$v_{\theta} \coloneqq \sum_{i \in I} \left[\left(\beta_i - \frac{1}{2\theta_i} \right) \ell_i \log \left(\ell_i + \frac{\theta_i}{1 - \theta_i} \right) + \alpha_i \log \left(\ell_i + \frac{\theta_i}{1 - \theta_i} \right) + \left(\beta_i - \frac{1}{2} \right) \ell_i \log \frac{1}{1 - \theta_i} - \alpha_i \log \frac{1}{1 - \theta_i} \right]$$

Then $u^{(\theta)} \to u$ as $\theta_i \to 0$ for all i and $u^{(\theta)} \to u_{AS}$ as $\theta_i \to 1$ for all i. This family of conical metrics, however, are not necessarily scalar-flat.

In the remaining part of this section, we will follow the framework of Abreu and Sena-Dias to explicitly construct toric conical scalar-flat Kähler metrics:

Theorem 4.1. Given X satisfying the same conditions as in Theorem 3.1 and let P be the moment polytope for X. Consider an index set $J = \{j_1, \dots, j_m\} \subset \{1, 2, \dots, d\}$ with $1 \leq j_1 < \dots < j_m \leq d$. For each $j_i \in J$, fix a cone angle $2\pi\theta_{j_i}$, we write $\theta = (\theta_{j_1}, \dots, \theta_{j_m})$. For the normal vector $\nu_i = (\alpha_i, \beta_i)$, consider

$$\alpha'_{i} := \alpha_{i} \text{ if } i \notin J, \alpha'_{i} := \frac{\alpha_{i}}{\theta_{i}} \text{ if } i \in J; \quad \beta'_{i} := \beta_{i} \text{ if } i \notin J, \beta'_{i} := \frac{\beta_{i}}{\theta_{i}} \text{ if } i \in J.$$

$$(49)$$

Let $\ell_J = \bigcup_{i=1}^m \ell_{j_i}$ be the union of edges ℓ_{j_i} indexed by the elements in J, here ℓ_{j_i} corresponds to $D_i, \forall i = 1, \dots, m$. Consider $\nu = (\alpha, \beta)$ a vector in \mathbb{R}^2 satisfying (9). Set

$$\xi_1 \coloneqq \alpha_1' \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1}' - \alpha_i') \log \left(H + a_i^{(\theta)} + \sqrt{(H + a_i^{(\theta)})^2 + r^2} \right) + \alpha H,$$

$$\xi_2 \coloneqq \beta_1' \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1}' - \beta_i') \log \left(H + a_i^{(\theta)} + \sqrt{(H + a_i^{(\theta)})^2 + r^2} \right) + \beta H,$$

where $a_1^{(\theta)}, \dots, a_{d-1}^{(\theta)}$ are real numbers determined by P and θ . Let x_1, x_2 be the primitives of

$$\epsilon_1 = r \left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right), \quad \epsilon_2 = -r \left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right)$$

Then they define the momentum action coordinates on P° of some conical toric scalar-flat Kähler metric on X whose cone angle along D_i is $2\pi\theta_{j_i}$. Its symplectic potential satisfies

$$du = \xi_1 dx_1 + \xi_2 dx_2.$$

Proof. Firstly, the positivity of det $D\xi$ can be proved with the same arguments as in Theorem 3.1, it is because (α'_i, β'_i) either equals to (α_i, β_i) , or is rescaled by a positive constant $\frac{1}{\theta_i}$ from (α_i, β_i) along edges corresponding to conical divisors. Next, to check the boundary behavior of u, we note for $-a_j^{(\theta)} < H < -a_{j-1}^{(\theta)}$, we have

$$\xi_1 = \alpha_1' \log r + \sum_{i=1}^{j-2} (\alpha_{i+1}' - \alpha_i') \log r = \alpha_j' \log r + O(1), \quad \xi_2 = \beta_j' \log r + O(1),$$

which gives

$$du = \log r(\alpha'_j dx_1 + \beta'_j dx_2) + O(1).$$

We claim that we still have $r = (\delta \prod_{i=1}^{d} \ell_i)^{1/2}$ for some smooth and positive function δ . The reason is that with a similar analysis as in Lemma 3.1.3, we obtain

$$\frac{\partial x}{\partial r} = -\frac{r}{2} \sum_{i=1}^{d-1} \frac{\beta'_{i+1} - \beta'_i}{\rho_i^{(\theta)}}, \quad \frac{\partial \ell_j}{\partial r} = \frac{r}{2} \sum_{i=1}^{d-1} \frac{\det(\nu'_{i+1} - \nu'_i, \nu_j)}{\rho_i^{(\theta)}},$$

here $\nu'_i \coloneqq (\alpha'_i, \beta'_i), H_i^{(\theta)} \coloneqq H + a_i^{(\theta)}$ and $\rho_i^{(\theta)} \coloneqq \sqrt{(H_i^{(\theta)})^2 + r^2}$. Again, since α'_i, β'_i are rescaled from α_i, β_i , we still have

$$\frac{r}{2} \left(-\frac{\det(\nu'_1, \nu_j)}{\rho_1^{(\theta)}} + \sum_{i=2}^{d-1} \det(\nu'_i, \nu_j) \left(\frac{1}{\rho_{i-1}^{(\theta)}} - \frac{1}{\rho_i^{(\theta)}} \right) + \frac{\det(\nu'_d, \nu_j)}{\rho_{d-1}^{(\theta)}} \right) > 0,$$

and this proves the claim. Thus, $du = \log \ell_j (\alpha'_j dx_1 + \beta'_j dx_2) + O(1)$, giving the desired boundary behavior.

Lemma 4.1.1. Given $j \in J$,

$$a_j^{(\theta)} - a_{j-1}^{(\theta)} = \theta_j \cdot \frac{L_j}{2\pi |\nu_j|^2}.$$
(50)

Proof. Note

$$\omega|_{\ell_j} = r \frac{\partial \xi_2}{\partial r} dH \wedge d\theta_1 - r \frac{\partial \xi_1}{\partial r} dH \wedge d\theta_2,$$

then from $\xi = \frac{1}{\theta_j} \nu_j \log r + O(1)$ on ℓ_j , we get $\omega|_{\ell_j} = \frac{|\nu_j|^2}{\theta_j} dH \wedge dt$. Then
 $L_j = \int_{\ell_j} \omega = 2\pi (a_j^{(\theta)} - a_{j-1}^{(\theta)}) \cdot \frac{|\nu_j|^2}{\theta_j},$
concluding (50).

concluding (50).

Also we know for $j \notin J$, $a_j^{(\theta)} - a_{j-1}^{(\theta)} = a'_j - a'_{j-1}$. Then, we obtain the relation between $a_j^{(\theta)}$ and a'_j :

$$a_{j}^{(\theta)} = a_{j}' \text{ if } j < j_{1}; \quad a_{j}^{(\theta)} = a_{j}' + \sum_{\ell=1}^{k} (1 - \theta_{j_{\ell}})(a_{j_{\ell}-1}' - a_{j_{\ell}}') \text{ if } j_{k} \leq j < j_{k+1};$$

and $a_{j}^{(\theta)} = a_{j}' + \sum_{\ell=1}^{m} (1 - \theta_{j_{\ell}})(a_{j_{\ell}-1}' - a_{j_{\ell}}') \text{ if } j_{m} \leq j \leq d.$ (51)

From the expression of ξ_1, ξ_2 we deduce the expression of x_1, x_2 as follows:

$$x_1 = \beta_1' H + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1}' - \beta_i') (H_i^{(\theta)} - \rho_i^{(\theta)}), \quad x_2 = -\alpha_1' H - \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1}' - \alpha_i') (H_i^{(\theta)} - \rho_i^{(\theta)}).$$

When r = 0, we have

$$x_1 = \beta_1' H + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1}' - \beta_i') (H_i^{(\theta)} - |H_i^{(\theta)}|), \quad x_2 = -\alpha_1' H - \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1}' - \alpha_i') (H_i^{(\theta)} - |H_i^{(\theta)}|).$$

Equivalently, we know

$$\begin{array}{ll} \text{(i) If } -a_{1}^{(\theta)} < H, \text{ then } x_{1} = \beta_{1}'H, \quad x_{2} = -\alpha_{1}'H; \\ \text{(ii) if } -a_{j+1}^{(\theta)} < H < -a_{j}^{(\theta)}, \text{ then} \\ x_{1} = \beta_{j+1}'H + \sum_{i=1}^{j} a_{i}^{(\theta)}(\beta_{i+1}' - \beta_{i}'), \quad x_{2} = -\alpha_{i+1}'H - \sum_{i=1}^{j} a_{i}^{(\theta)}(\alpha_{i+1}' - \alpha_{i}'); \\ \text{(iii) if } H < -a_{d-1}^{(\theta)}, \text{ then } x_{1} = \beta_{d}'H - \sum_{i=1}^{d-1} a_{i}^{(\theta)}(\beta_{i+1}' - \beta_{i}'), \quad x_{2} = -\alpha_{d}'H - \sum_{i=1}^{d-1} a_{i}^{(\theta)}(\alpha_{i+1}' - \alpha_{i}'). \end{array}$$

We want to show $x' = (x'_1, x'_2)$, when restricted to (H, 0), defines a global proper homeomorphism. Again, we first focus on its behavior along ℓ_j for $j_1 \leq j < j_2$. For simplicity, we write $k = j_1$, then

$$\begin{aligned} x_1 &= \beta'_{k-1}H + \frac{1}{2}(\beta'_k - \beta'_{k-1})(H^{(\theta)}_{k-1} - \rho^{(\theta)}_{k-1}) + \sum_{i=1}^{k-2} a^{(\theta)}_i(\beta'_{i+1} - \beta'_i) + O(r^2), \\ x_2 &= -\alpha'_{k-1}H - \frac{1}{2}(\alpha'_k - \alpha'_{k-1})(H^{(\theta)}_{k-1} - \rho^{(\theta)}_{k-1}) - \sum_{i=1}^{k-2} a^{(\theta)}_i(\alpha'_{i+1} - \alpha'_i) + O(r^2). \end{aligned}$$

From (51), we know $\ell_k(x) = 0$ holds. For $k < j < j_2$, note $\ell_j(x) = 0$ is equivalent to

$$\sum_{i=2}^{j-1} \det(\nu_i, \nu_j) (a_{i-1}^{(\theta)} - a_i^{(\theta)}) - a_1^{(\theta)} \det(\nu_1, \nu_j) = \sum_{i=2}^{j-1} \det(\nu'_i, \nu_j) (a'_{i-1} - a'_i) - a'_1 \det(\nu_1, \nu_j).$$

From (51), direct computation shows the above equation holds. With essentially the same arguments applied to $j_i \leq j < j_{i+1}$ for $i \geq 2$, we see $\ell_j(x) = 0$ holds. Hence x defines a global proper homeomorphism as desired.

For the asymptotic behavior of the conical metrics, note as $\rho \to \infty$,

$$r \det D\xi = \frac{\det(\nu'_d, \nu'_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } \alpha = \beta = 0;$$

$$r \det D\xi = \det(\nu, \nu_1') \left(1 - \frac{r^2}{2\rho(H+\rho)} \right) + \det(\nu, \nu_d') \frac{r^2}{2\rho(H+\rho)} + \frac{\det(\nu_d', \nu_1')}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } (\alpha, \beta) \neq (0, 0).$$

The arguments in Theorem 3.2 still work here, and hence the asymptotic behavior of conical metrics coincide with those for the cuspidal metrics with a given choice of (α, β) .

5 Appendix: Uniqueness of toric metrics under given boundary conditions

Theorem 5.1. Consider the same setting as in Theorem 3.1. Assume g is a toric scalar-flat Kähler metric of Poincaré type and its symplectic potential u satisfies the prescribed boundary behavior given in (11), then g can only be one of the metrics constructed in Theorem 3.1.

Proof. The proof closely follows the arguments of Sena-Dias in [26], with the essential differences in Claim 5.1.3, Lemma 5.1.2 and Lemma 5.1.3.

Starting from a scalar-flat Kähler metric on a symplectic 4-manifold, Donaldson shows in [11] Theorem 1 that each solution of the scalar-flat Kähler equation locally arises from axi-symmetric harmonic functions ξ_1, ξ_2 in the way described in Theorem 2.1. Here ξ_1, ξ_2 are unique up to translation in the *H* variable and addition of constants. More precisely, given u(x) the symplectic potential of a scalar-flat Kähler metric, we consider (H, r) satisfying

$$r = (\det \operatorname{Hess} u)^{-1/2}, \quad \frac{\partial H}{\partial x_1} = -\frac{u^{2j}}{r} \frac{\partial r}{\partial x_j}, \quad \frac{\partial H}{\partial x_2} = \frac{u^{1j}}{r} \frac{\partial r}{\partial x_j}.$$
(52)

Then for the moment P° endowed with the metric $g_{poly} = u_{ij} \sum_{i,j=1}^{2} dx_i \otimes dx_j$, r is harmonic and H

is its harmonic conjugate. The metric g_{poly} induces a complex structure J_{poly} via its Hodge star. We obtain a J_{poly} -holomorphic local coordinate on P° , written as

$$z \coloneqq H + ir$$

The coordinates (H, r), as functions of x_1, x_2 , are known as the isothermal coordinates. Note the boundary behavior of u is determined on ∂P . Then r extends continuously to $\partial P \setminus \ell_I$, as does H. Thus, z extends to $\partial P \setminus \ell_I$ as a continuous function, denoted by \tilde{z} . Note r = 0 on $\partial P \setminus \ell_I$, then $\tilde{z}(\partial P \setminus \ell_I) \subset \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. From the boundary behavior of the metric we know \tilde{z} is a bijection from $\partial P \setminus \ell_I$ to $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$.

Lemma 5.1.1. The map $z: P^{\circ} \to \mathbb{H}$ is a bijection.

Proof. The proof relies on the real sub-manifold associated with the symplectic 4-manifold X. From the discussions in [20] Theorem 6.7 and [26], we know since the moment map is proper, X is symplectomorphic to the quotient of some complex plane \mathbb{C}^d by a sub-torus of the standard torus, with d being the number of edges of the moment map of X. Note complex conjugation descends to a function on X and D. We denote its fixed point set(which are real submanifolds) of X and D by $X_{\mathbb{R}}$ and $D_{\mathbb{R}}$, respectively. The moment map

$$\phi: X \backslash D \to \mathbb{R}^2 \backslash \bigcup_{k \in I} (-a_k, 0),$$

when restricted to $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ is denoted by $\phi_{\mathbb{R}}$. It is a 4 to 1 branched cover with the branched set being $\phi^{-1}(\partial P \setminus \ell_I)$ and write $\phi_{\mathbb{R}}^{-1}(P^\circ) = \bigcup_{j=0}^3 P_j$ as a disjoint union of the open sets P_j . Let $g_{\mathbb{R}}$ be the induced metric on $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ and P_0 , then $g_{poly} := \phi_{\mathbb{R}}(g_{\mathbb{R}}) = u_{ij}dx_i \otimes dx_j$. Let

$$(X_{\mathbb{R}} \backslash D_{\mathbb{R}})^{\circ} \to X_{\mathbb{R}} \backslash D_{\mathbb{R}}$$

be the orientable double cover of $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ and $\phi_{\mathbb{R}}^{\circ}$ be the lifting of $\phi_{\mathbb{R}}$ to $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^{\circ}$. For each P_j , k = 0, 1, 2, we write P_j^0 and P_j^1 as the pre-image under the double cover. Via the Hodge star operator, we obtain from the metric induced by $g_{\mathbb{R}}$ on $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^{\circ}$ a complex structure $J_{\mathbb{R}}$, whose pushforward under $\phi_{\mathbb{R}}^{\circ}$ defines a complex structure J_{poly} on $P \setminus \ell_I$.

Claim 5.1.1. Given $w \in \partial P \setminus \ell_I$, and p an element in $(\phi_{\mathbb{R}}^{\circ})^{-1}(w) \subset (X_{\mathbb{R}} \setminus D_{\mathbb{R}})^{\circ}$, then there is a neighbourhood V_p of p in $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^{\circ}$ such that $z \circ \phi_{\mathbb{R}}^{\circ}$ extends to V_p as a holomorphic function for $J_{\mathbb{R}}$.

The proof is a lifting and extension argument for the harmonic function r and the harmonic conjugate H on $X \setminus D$, and this argument is essentially the same as in [26] Lemma 6.2.

Now, we show the injectivity. For the holomorphic map z, we consider its degree; it suffices to show the degree is 1. If we consider a point $w_0 \in \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$ and let $w \in \partial P \setminus \ell_I$ be its pre-image of \tilde{z} . Fix p in the closure of $P_0^0 \cap P_1^0$. From Claim 5.1.1, there exists an extension of zto V_p . Assume V_p is small enough to admit a complex chart $z : V_p \to \mathbb{C}$. Then following the arguments of the proof of injectivity in [26] Section 6.1, we know for any $\epsilon > 0$ such that $B_{\epsilon} \cap P \subset \phi_{\mathbb{R}}^{\circ}(V_p)$, there exists δ such that $z^{-1}(B_{\delta}(w_0)) \subset B_{\epsilon}(w)$; furthermore, given a point $w'_0 \in B_{\delta}(w_0) \cap \mathbb{H}$, we can enlarge the loop γ enclosing all pre-images of z so that it also encloses w, then as w'_0 tends to w_0 . The number of pre-images of w'_0 given by the integral

$$\int_{\gamma} \frac{\frac{dz}{ds}(s)ds}{z(s) - w'_0}$$

equals to that of w_0 , which is 1.

Then we show the surjectivity. The different boundary behavior for the symplectic potential in our case compared to that in [26] doesn't cause any essential difference to the arguments of the proof. Note P is non-compact and admits non-trivial harmonic functions, from the uniformization theorem, we know there exists a holomorphic map $\kappa : P^{\circ} \to \mathbb{H}$.

Claim 5.1.2. κ extends as a homeomorphism to $P \setminus \ell_I \to \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, and it's a bijection.

Proof. First, we show the map is extendable. The different boundary behavior for the symplectic potential in our case compared to that in [26] doesn't cause any essential difference to the arguments of the proof, for details we refer the readers to Lemma 5.4.

To see the extension is bijective, we argue by contradiction. Assume it's not injective, let $w, v \in \partial P \setminus \ell_I$ such that $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Take $o \in P^\circ$ and consider the Jordan curve going through $\kappa(o)$ and $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Let C be the interior of this Jordan curve, then the segment S joining w and v satisfies $\tilde{\kappa}(S) \subset \partial A \cap \partial \mathbb{H}$, and thus $\tilde{\kappa}$ is constant on S. In particular, we see S doesn't contain ℓ_k for any $k \in I$. Then, the same arguments as in [26] give a contradiction. Similarly, we can prove that the inverse is also injective.

We use this extension map as an auxiliary to show the surjectivity of z. Let $U := z(P^{\circ})$, write

$$\partial U = (\partial U \cap \partial \mathbb{H}) \cup (\partial U \cap \mathbb{H}),$$

then the surjectivity of z is equivalent to $\partial U \cap \mathbb{H} = \emptyset$. Consider $z_{\kappa} := z \circ \kappa^{-1} : \mathbb{H} \to \mathbb{H}$, it is a holomorphic, injective map which can be extended bijectively to $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0) \to \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. Consider

$$f(w) \coloneqq \frac{1}{z_{\kappa}(-\frac{1}{w})} : \mathbb{H} \to \mathbb{H},$$

as in [26], the same arguments show that it is holomorphic and can be extended to a holomorphic function on \mathbb{C}^* with 0 being an isolated singularity using the Schwarz reflection principle. Rewrite $U = z_{\kappa}(\mathbb{H})$, we claim that

Claim 5.1.3. $\partial U \cap \mathbb{H} = \{\lim z_{\kappa}(w_{n_k}), (w_k) \text{ unbounded with } (w_{n_k}) \text{ being a subsequence}\}.$

Proof. For any point z_{∞} in the above set, we take a sequence $w_k \in \mathbb{H}$ such that $z_{\kappa}(w_k) \to z_{\infty}$. If the sequence is bounded, there is a convergent subsequence w_{n_k} in $\overline{\mathbb{H}}$ converging to $w \in \overline{\mathbb{H}}$. We have the following possibilities:

- (i) If $w \in \mathbb{H}$, then $\tilde{z}_{\kappa}(w) = z_{\kappa}(w) \in U$ but since U is open, $z_{\infty} \notin \partial U$;
- (ii) if $w \in \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, then $\tilde{z}_{\kappa} \in \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, then $z_{\infty} \notin \mathbb{H}$;
- (iii) if $w = (-a_k, 0)$ for some $k \in I$, consider $\kappa^{-1}(w_{n_k}) \coloneqq p_{n_k}$, we have $z(p_{n_k}) \to z_{\infty} \in \mathbb{H}$. From $z : P \setminus \ell_I \to \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$ is bijective we know $p_{n_k} \to z^{-1}(z_{\infty}) \in P \setminus \ell_I$ and from $\kappa : P \setminus \ell_I \to \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$ is bijective we know $\kappa(p_{n_k}) \to z_{\kappa}^{-1}(z_{\infty}) \in \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$. This implies w_{n_k} tends to an element in $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, a contradiction.

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Equivalently we know

$$\partial U \cap \mathbb{H} = \{\lim f(w_{n_k}), w_k \in \mathbb{H}, w_k \to 0, (w_{n_k}) \text{ a subsequence}\} := f(0).$$

Now, this set contains either ∞ or a single point, which is a pole. If the latter holds, we know z_{κ}^{-1} is a holomorphic function with an isolated pole, but the image can not lie in \mathbb{H} . Hence we conclude that $\partial U \cap \mathbb{H} = \emptyset$. This concludes the proof.

Now we know z is a bijection, define $\mu \coloneqq z^{-1}$, and let $\mu_{ALE} \coloneqq z_{ALE}^{-1}$ be the corresponding map for the ALE metric g_{ALE} (i.e., the case where $\nu = 0$) constructed in Theorem 3.1.

Lemma 5.1.2. Consider $\mu_0 := \mu - \mu_{ALE}$, we have $\mu_0 = r^2 f$ for some $f \in C^{\infty}(\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0))$, and f satisfies

$$f_{HH} + f_{rr} + \frac{3f_r}{r} = 0.$$

Proof. Consider

$$\eta \coloneqq (u_{x_1}, u_{x_2}), \quad \eta_{ALE} \coloneqq (u_{ALE, x_1}, u_{ALE, x_2}),$$

we write

$$\xi(H,r) = \eta \circ \mu(H,r), \quad \xi_0(H,r) = \eta \circ \mu_0(H,r).$$

To show μ_0 extends as an analytic function to $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, it's equivalent to show ξ_0 extends as an analytic function on $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$. Since ξ_0 is an axi-symmetric harmonic function on \mathbb{H} , from the mean value theorem, it is sufficient to show ξ_0 is bounded in a neighborhood of each point on $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. Write

$$\xi_0 = \eta_{ALE} \circ \mu_{ALE} - \eta \circ \mu = (\eta_{ALE} \circ \mu_{ALE} - \eta_{ALE} \circ \mu) + (\eta_{ALE} \circ \mu - \eta \circ \mu).$$

Note $\eta_{ALE} - \eta \in C^{\infty}$, and μ extends as a continuous function on $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, we know the second term is bounded. For the first term, rewrite it as $\xi_{ALE} - \xi_{ALE} \circ (\mu_{ALE}^{-1} \circ \mu)$. Near $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, there is a singularity of ξ_{ALE} with

$$\xi_{ALE} = \nu_i \log r + O(1)$$
 on each internal $-a_{i+1} < H < -a_i$ and $r = 0$ given $i \notin I$.

It's equivalent to show as $r \to 0$, $\log \frac{r}{r(\mu_{ALE}^{-1} \circ \mu)}$ is bounded. Composing with μ^{-1} it suffices to

show

- (i) For the conjugate harmonic coordinate H' for z, for $i + 1 \notin I$, $-a_{i+1} < H < -a_i \iff -a_{i+1} < H < -a_i, \text{ and for } i+1 \in I, \ H = -a_{i+1} \iff H' = -a_{i+1};$
- (ii) $\log \frac{r \circ \mu^{-1}}{r \circ \mu_{ALE}^{-1}}$ is bounded as r approaches 0.

The first claim follows from [26] Theorem 6.2 and our choice of coefficients (27). For the second claim, recall $r \circ \mu^{-1} = \det(\operatorname{Hess} u \circ \mu^{-1})^{-1/2}$, then it's equivalent to show that

$$\frac{(\det \text{ Hess } u \circ \mu^{-1})^{-1/2}}{(\det \text{ Hess } u_{ALE} \circ \mu^{-1})^{-1/2}}$$

is bounded. This follows from the assumed boundary behavior of u and u_{ALE} . Direct calculation gives $f_{HH} + f_{rr} + \frac{3f_r}{r} = 0$, for details see [26] Lemma 6.3.

Lemma 5.1.3. For the normal vector $\nu_1 = (1,0)$, consider $f_1 := f \cdot \nu_1$, then f_1 is a constant.

Proof. Since f_1 is harmonic, it suffices to show it is bounded. From $f = \frac{\mu_0}{r^2}$ and $\nu \cdot \nu_1 \ge 0$ we deduce that $f_1 \leq \frac{\mu_{ALE} \cdot \nu_1}{r^2}$. From the expression of x_1, x_2 in Theorem 3.1, we know

$$\mu_{ALE}(H,r)| \leqslant C\sqrt{H^2 + r^2 + 1}.$$

As in [26], we view f as a harmonic function in \mathbb{R}^5 , which only depends on the coordinate H and the distance to the *H*-axis, *r*. Then $\forall w \in \mathbb{R}^5$,

$$f_1(w) \leqslant \frac{C(|w|+1)}{r^2}.$$

Let \mathcal{A} denote the subset of \mathbb{R}^5 whose H coordinate does not take values $-a_k$ for any $k \in I$ when the distance r vanishes. For a fixed $z \in A$, there exists $R_z > 0$ large enough such that $\partial B(z, R_z) \subset \mathcal{A}$, and with the mean value theorem, we get $f_1(z) \lesssim \frac{1}{R^3} \int_{\partial B(z,R)} \frac{dw}{r^2}$. Then from [26] Lemma 6.4, for R large enough, direct calculations show $\int_{\partial B(z,R)} \frac{dw}{r^2} \lesssim R^2$, thus f_1 is bounded from above.

Similarly, for the normal vector ν_2 , we can define $f_2 \coloneqq f \cdot \nu_1$, then f_2 is a constant. Thus f is a constant, which implies $\xi_0 = H \cdot \nu$ for some constant vector. Hence when $\nu = 0, g$ is the ALE metric constructed in Theorem 3.1 and otherwise g is the generalized or exceptional Taub-NUT ones constructed there.

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