Radius estimates for nearly stable H-hypersurfaces of dimension 2, 3, and 4.

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Abstract

In this paper we study the geometry of complete constant mean curvature (CMC) hypersurfaces immersed in an $(n+1)$-dimensional Riemannian manifold N ($n = 2, 3$ and 4) with sectional curvatures uniformly bounded from below. We generalise radius estimates given by Rosenberg [32] ($n = 2$) and by Elbert, Nelli and Rosenberg [13] and Cheng [2] ($n = 3, 4$) to nearly stable CMC hypersurfaces immersed in N. We also prove that certain CMC hypersurfaces effectively embedded in N must be proper.

1 Introduction

Throughout this paper, we refer to a hypersurface M immersed in a manifold N with constant mean curvature H as an H-hypersurface. Let N be an $(n+1)$-dimensional Riemannian manifold N ($n = 2, 3$ and 4) with sectional curvatures uniformly bounded from below. In their seminal papers, Rosenberg [32] ($n = 2$, see also [29, 31]) and Elbert, Nelli and Rosenberg [13] and Cheng [2] ($n = 3, 4$) prove radius estimates for stable H-hypersurfaces immersed in N. In this paper we generalise these estimates to nearly stable H-hypersurfaces.

Theorem 1.1. *Let N be an $(n+1)$-dimensional Riemannian manifold ($n = 2, 3$ and 4) with sectional curvatures uniformly bounded from below and let M be a complete, δ_n-stable, H-hypersurface with $\delta_n < \frac{27}{32}, \frac{7}{12}, \frac{19}{64}$ respectively. Then, if $|H| > 2\sqrt{|\min(0, \mathcal{K})|}$ (where $\mathcal{K} := \mathcal{K}(N)$ denotes the infimum of the sectional curvatures of N), there exists a constant $c := c(n, \delta_n, H, \mathcal{K}) > 0$ such that for any $p \in M$,*

$$\mathrm{dist}_M(p, \partial M) \le c.$$

See Section 2 for a result involving the scalar curvature of N when $n = 2$ that generalizes the main theorem in [32].

Near stability was a notion widely employed in Colding-Minicozzi Theory, that is [5, 6, 7, 8, 10], to study the geometry of embedded minimal ($H = 0$) disks. Many results about minimal hypersurfaces have employed near stability directly or extended the concept of stability to near stability (see for instance [3, 4, 14, 16, 17, 35]).

In Section 3 we use the radius estimates mentioned above together with the Stable Limit Leaf Theorem by Meeks, Perez and Ros [19] to prove the following theorem.

Theorem 1.2. *With N as in Theorem 1.1, let M be a complete H-hypersurface effectively embedded in N. Suppose that the norm of the second fundamental form of M is locally bounded (bounded in compact extrinsic balls) and $|H| > 2\sqrt{|\min(0, \mathcal{K})|}$. Then M is proper.*

See Remark 3.2 for a stronger statement when $n = 2$ and [11, 12, 30] for examples of complete H-surfaces embedded in $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ that are not proper.

Theorem 1.2 is motivated by several results in the literature. In their seminal paper [9], Colding and Minicozzi proved that a complete, minimal surface embedded in $\mathbb{R}^3$ with finite topology must be proper, see also [5, 6, 7, 8, 10]. Meeks and Rosenberg generalised this to complete minimal surfaces embedded in $\mathbb{R}^3$ with positive injectivity radius [21], see also [20]. Finally, Meeks and Tinaglia further generalised both these results to constant mean curvature (CMC) surfaces [24], see also [23, 25, 26, 27, 28].

2 Radius estimates for nearly stable H-hypersurfaces

We begin this section by reminding the reader of the notion of δ-stability.

Definition 2.1. For $\delta \in [0, 1]$, we say that a H-hypersurface M immersed in N is δ-stable if

$$\int_M \left(|\nabla f|^2 - (1-\delta)(|A|^2 + \overline{Ric}(\nu))f^2\right) \geq 0,$$

for $f \in C_0^\infty(M)$.

When $\delta = 0$, then M is stable.

In what follows, we generalize the radius estimates given in Theorem 1 in [32] and Theorem 1 in [13] for stable H-hypersurfaces to δ-stable H-hypersurfaces. We begin by generalizing Theorem 1 in [13], that is the Theorem 1.1 when $\delta = 0$.

Proof of Theorem 1.1. Our proof draws from the methods established in [13].

Since M is δ-stable, we can find a smooth function $u > 0$ on M such that the δ-stability operator satisfies

$$L^\delta u = \Delta u + (1-\delta)(|A|^2 + \overline{Ric}(\nu))u = 0,$$

see for instance Lemma 2.1 in [18]. By decomposing the symmetric shape operator into the mean curvature and the trace-less part, $A = HI + \Phi$, the square norm of Φ is

$$|\Phi|^2 = |A|^2 - n|H|^2,$$

so we can write the near-stability operator as

$$L^\delta = \Delta + (1-\delta)(|\Phi^2| + nH^2 + \overline{Ric}(\nu)).$$

We use ds^2 to denote the induced metric on M by N and conformally change the metric to $d\widetilde{s}^2 = u^{2k}ds^2$, where we will choose k later. Fix $p \in M$ and take $r > 0$ small enough such that the geodesic ball $B_M(p,r)$ centred at p and of ds-radius r is contained in the interior of M. Let γ be a $d\widetilde{s}$-geodesic which joins p to $\partial B_M(p,r)$. Let a be the ds-length of γ and $\widetilde{a}$ be the $d\widetilde{s}$-length of γ. Then we have $a \geq r$ and it suffices to prove that there exists a constant $c = c(n,H,K,\delta) > 0$ such that $a \leq c$. To this end, let R and $\widetilde{R}$ be the curvature tensors of M in the metrics ds and $d\widetilde{s}$ respectively. Choose an orthonormal basis $\{\widetilde{e}_1 = \frac{\partial\gamma}{\partial\widetilde{s}}, \widetilde{e}_2, \ldots .\widetilde{e}_n\}$ for $d\widetilde{s}$ such that $\widetilde{e}_2, \ldots .\widetilde{e}_n$ are parallel along γ and let $\widetilde{e}_{n+1} = \nu$. This yields an orthonormal basis $\{e_1 = \frac{\partial\gamma}{\partial s} = u^k\widetilde{e}_1, e_2 = u^k\widetilde{e}_2, \ldots, e_n = u^k\widetilde{e}_n\}$ for ds. Let $\overline{R}$ be the curvature tensor for the ambient manifold N. Using this notation, R_{11} (respectively $\widetilde{R}_{11}$) is the Ricci curvature in the direction of e_1 for the metric ds (respectively $d\widetilde{s}$), and $\overline{R}_{n+1,n+1}$ is $\overline{Ric}(\nu)$.

Since γ is $d\widetilde{s}$-minimising, by the second variation formula for length, we have

$$\int_0^{\widetilde{a}} \left((n-1)\left(\frac{d\phi}{d\widetilde{s}}\right) - \widetilde{R}_{11}\phi^2\right) d\widetilde{s} \geq 0,$$

for any smooth function ϕ with $\phi(0) = \phi(\widetilde{a}) = 0$. We use the formula for Ricci curvature under conformal change of metric, (see for instance the appendix in [13] for a full calculation)

$$\widetilde{R}_{11} = u^{-2k}\left(R_{11} - k(n-2)(\log u)_{ss} - k\frac{\Delta u}{u} + k\frac{|\nabla u|^2}{u^2}\right).$$

Then by δ-stability, $L^\delta u = \Delta u + (1-\delta)(|\Phi^2| + nH^2 + \overline{Ric}(\nu))u = 0$, so we can replace the Laplacian term yielding

$$\begin{aligned}\widetilde{R}_{11} = u^{-2k}(&R_{11} - k(n-2)(\log u)_{ss} \\ &+ k(1-\delta)(|\Phi|^2 + nH^2 + \overline{R}_{n+1,n+1}) + k\frac{|\nabla u|^2}{u^2}).\end{aligned}$$

Next, the Gauss equation relates the ambient curvature to the intrinsic curvature

$$R_{ijij} = \overline{R}_{ijij} + h_{ii}h_{jj} - h_{ij}^2 = \overline{R}_{ijij} + (\Phi_{ii} + H)(\Phi_{jj} + H) - (\Phi_{ij} + H\delta_{ij})^2.$$

Letting $i = 1$ and summing over $j = 2, \ldots, n$ gives

$$\begin{aligned}&R_{11} \\ &= \sum_{j=2}^n \overline{R}_{1j1j} + \sum_{j=2}^n \Phi_{11}\Phi_{jj} + (n-1)\Phi_{11}H + \sum_{j=2}^n \Phi_{jj}H + (n-1)H^2 - \sum_{j=2}^n \Phi_{1j}^2 \\ &= \sum_{j=2}^n \overline{R}_{1j1j} + \sum_{j=2}^n \Phi_{11}\Phi_{jj} + (n-2)\Phi_{11}H + \sum_{j=1}^n \Phi_{jj}H + (n-1)H^2 - \sum_{j=2}^n \Phi_{1j}^2.\end{aligned}$$

Using the traceless property $\sum_{j=1}^n \Phi_{jj} = 0$ on the second and fourth terms gives

$$R_{11} = \sum_{j=2}^n \overline{R}_{1j1j} - \Phi_{11}^2 + (n-2)\Phi_{11}H + (n-1)H^2 - \sum_{j=2}^n \Phi_{1j}^2.$$

We substitute this expression into the formula for $\widetilde{R}_{11}$ to obtain

$$\begin{aligned}&\widetilde{R}_{11} \\ &= u^{-2k}\left(\sum_{j=2}^n \overline{R}_{1j1j} - \Phi_{11}^2 + (n-2)\Phi_{11}H + (n-1)H^2 - \sum_{j=2}^n \Phi_{1j}^2\right. \\ &\left. - k(n-2)(\log u)_{ss} + k(1-\delta)(|\Phi|^2 + nH^2 + \overline{R}_{n+1,n+1}) + k\frac{|\nabla u|^2}{u^2}\right). \\ &= u^{-2k}\left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) + n - 1)H^2 + (n-2)\Phi_{11}H\right) \\ &+ u^{-2k}\left(k(1-\delta)|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 - k(n-2)(\log u)_{ss} + k\frac{|\nabla u|^2}{u^2}\right).\end{aligned}$$

Now let $\varphi = \phi \circ \widetilde{s}$ so that $\varphi(0) = \varphi(a) = 0$. We combine the above expression with the first inequality and $d\widetilde{s}^2 = u^{2k} ds^2$ to obtain

$$(n-1)\int_0^a (\varphi_s)^2 u^{-k}\, ds \geq \int_0^a \varphi^2 u^{-k} \left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} \right) ds$$
$$+ \int_0^a \varphi^2 u^{-k}$$
$$\left((kn(1-\delta) + n - 1)H^2 + (n-2)\Phi_{11}H + k(1-\delta)|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right) ds$$
$$- \int_0^a \varphi^2 u^{-k} \left(k(n-2)(\log u)_{ss} + k\frac{|\nabla u|^2}{u^2} \right) ds.$$

We replace φ by $\varphi u^{k/2}$ to eliminate the u^{-k}. Then differentiation gives $(\varphi u^{k/2})_s = \varphi_s u^{k/2} + \frac{k}{2}\varphi u^{(k-2)/2} u_s$ which yields

$$(n-1)\int_0^a (\varphi_s)^2\, ds + k(n-1)\int_0^a \varphi\varphi_s u_s u^{-1}\, ds + \frac{k^2(n-1)}{4}\int_0^a \varphi^2 u_s^2 u^{-2}\, ds$$
$$\geq \int_0^a \varphi^2 \left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} \right) ds$$
$$+ \int_0^a \varphi^2$$
$$\left((kn(1-\delta) + n - 1)H^2 + (n-2)\Phi_{11}H + k(1-\delta)|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right) ds$$
$$- \int_0^a \varphi^2 \left(k(n-2)(\log u)_{ss} + k\frac{|\nabla u|^2}{u^2} \right) ds.$$

Using the divergence theorem, we have $\int \varphi^2 (\log u)_{ss}\, ds = -2\int \varphi\varphi_s u_s u^{-1}\, ds$. Furthermore, we have $k^2 \int \varphi^2 u_s^2 u^{-2}\, ds = \int \varphi^2 (\log u^k)_s^2\, ds = k^2 \int \varphi^2 |\nabla u|^2 u^{-2}\, ds$. This allows us to combine the terms in the first and last lines as follows

$$(n-1)\int_0^a (\varphi_s)^2\, ds \geq k(n-3)\int_0^a \varphi\varphi_s u_s u^{-1}\, ds + \left(\frac{1}{k} - \frac{n-1}{4} \right) \int_0^a \varphi^2 (\log u^k)_s^2\, ds$$
$$+ \int_0^a \varphi^2 \left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} \right) ds$$
$$+ \int_0^a \varphi^2$$
$$\left((kn(1-\delta) + n - 1)H^2 + (n-2)\Phi_{11}H + k(1-\delta)|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right) ds.$$

We now use the basic inequality $a^2 + b^2 \geq -ab$ with $a = (n-2)H$ and $b = \Phi_{11}/2$ which yields

$$(n-2)^2H^2 + \frac{\Phi_{11}^2}{4} \geq -(n-2)H\Phi_{11}.$$

Replacing this in our inequality gives

$$(n-1)\int_0^a (\varphi_s)^2\, ds \geq k(n-3)\int_0^a \varphi\varphi_s u_s u^{-1}\, ds + \left(\frac{1}{k} - \frac{n-1}{4}\right)\int_0^a \varphi^2(\log u^k)_s^2\, ds$$
$$+ \int_0^a \varphi^2 \left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2\right) ds$$
$$+ \int_0^a \left(k(1-\delta)|\Phi|^2 - \frac{5}{4}\Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2\right) ds.$$

We claim that the last term is greater than zero. Using the crude estimate

$$|\Phi|^2 \geq \sum_{j=1}^n \Phi_{jj}^2 + 2\sum_{j=2}^n \Phi_{1j}^2,$$

and the traceless property $\sum_{j=1}^n \Phi_{jj} = 0$ gives us

$$|\Phi|^2 \geq \frac{n}{n-1}\Phi_{11}^2 + 2\sum_{j=2}^n \Phi_{1j}^2.$$

We now need to choose

$$k > \frac{5(n-1)}{4n(1-\delta)} \tag{1}$$

and combine it with the last inequality to estimate the last term as

$$k(1-\delta)|\Phi^2|^2 - \frac{5}{4}\Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2)$$
$$\geq \frac{5}{4}\Phi_{11}^2 + \frac{5(n-1)}{2n}\sum_{j=2}^n \Phi_{1j}^2 - \frac{5}{4}\Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 = \frac{3n-5}{2n}\sum_{j=2}^n \Phi_{1j}^2 \geq 0,$$

as required. Consequently, we now have

$$(n-1)\int_0^a (\varphi_s)^2\, ds \geq k(n-3)\int_0^a \varphi\varphi_s u_s u^{-1}\, ds + \left(\frac{1}{k} - \frac{n-1}{4}\right)\int_0^a \varphi^2(\log u^k)_s^2\, ds$$
$$+ \int_0^a \varphi^2 \left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2\right) ds.$$

After choosing k such that

$$\frac{1}{k} - \frac{n-1}{4} > 0 \quad (\text{that is } k < \frac{4}{n-1}), \tag{2}$$

we can use the inequality $a^2 + b^2 \geq -ab$ again with $a = (\frac{1}{k} - \frac{n-1}{4})^{1/2}\varphi(\log u^k)_s$ and $b = \frac{n-3}{2}(\frac{1}{k} - \frac{n-1}{4})^{-1/2}\varphi_s$ to obtain

$$\left(\frac{1}{k} - \frac{n-1}{4}\right)\varphi^2(\log u^k)_s^2 + \frac{(n-3)^2}{4}\left(\frac{1}{k} - \frac{n-1}{4}\right)^{-1}\varphi_s^2 \geq -(n-3)\varphi\varphi_s(\log u^k)_s.$$

Hence,

$$\begin{aligned}(n-1)\int_0^a (\varphi_s)^2\, ds \geq& -\frac{(n-3)^2}{4}\left(\frac{1}{k} - \frac{n-1}{4}\right)^{-1}\int_0^a (\varphi_s)^2 \\ &+ \int_0^a \varphi^2\left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2\right) ds.\end{aligned}$$

Rearranging the terms, we now have an inequality of the form

$$A\int_0^a (\varphi_s)^2\, ds \geq \int_0^a \varphi^2\left(\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2\right) ds$$

where $A := \frac{4(k(2-n)+(n-1))}{4-k(n-1)}$ is positive, thanks to the condition (2). We now want to choose $B > 0$ such that

$$B \leq \sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2$$

and therefore this would give

$$A\int_0^a (\varphi_s)^2\, ds \geq B\int_0^a \phi^2\, ds. \tag{3}$$

Recall that $\mathcal{K}$ denotes the infimum of the sectional curvatures of N. If $\mathcal{K} \geq 0$ we can set $B := (kn(1-\delta) - n^2 + 5n - 5)H^2$, which is indeed positive if $|H| > 0$ and $k > \frac{5(n-1)}{4n(1-\delta)}$ (that is assumption (1)), when $n = 2, 3$ or 4. Otherwise, note that

$$\sum_{j=2}^n \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} \geq (kn(1-\delta) + n - 1)\mathcal{K}. \tag{4}$$

Therefore setting $B := (kn(1-\delta) - n^2 + 5n - 5)H^2 + (kn(1-\delta) + n - 1)\mathcal{K}$ we have

$$B \leq \sum_{j=2}^{n} \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} + (kn(1-\delta) - n^2 + 5n - 5)H^2$$

and if

$$H^2 > \frac{kn(1-\delta) + n - 1}{kn(1-\delta) - n^2 + 5n - 5}\mathcal{K}$$

then B is also greater than zero.

Using our assumptions (1) and (2), that is $\frac{5(n-1)}{4n(1-\delta)} < k < \frac{4}{n-1}$, we can estimate the quotient

$$\begin{aligned}\frac{kn(1-\delta) + n - 1}{kn(1-\delta) - n^2 + 5n - 5} &< \frac{16n + 4(n-1)^2}{5(n-1)^2 + 4(n-1)(-n^2 + 5n - 5)} \\ &= \frac{4(n+1)^2}{(n-1)(n-5)(5-4n)} < 4.\end{aligned}$$

Therefore B is positive provided that $|H| > 2\sqrt{|\mathcal{K}|}$.

Integrating equation (3) by parts now gives

$$\int_0^a (\varphi_{ss} A + B\varphi)\varphi \, ds \leq 0.$$

Choose $\varphi = \sin(\pi s a^{-1})$, for $s \in [0, a]$ so that

$$\int_0^a \left(B - \frac{A\pi^2}{a^2}\right) \sin(\pi s a^{-1}) \, ds \leq 0,$$

which implies

$$B - \frac{A\pi^2}{a^2} \leq 0,$$

that is $a < \sqrt{A}\pi/\sqrt{B}$. Setting $c := \sqrt{A}\pi/\sqrt{B}$, this finishes the proof of the theorem, provided that we can show that we can find $\delta > 0$ and k such that

$$\frac{5(n-1)}{4n(1-\delta)} < k < \frac{4}{n-1}. \tag{5}$$

This inequality is consistent for $n = 2$ when $\delta < \frac{27}{32}$, for $n = 3$ when $\delta < \frac{7}{12}$ and for $n = 4$ when $\delta < \frac{19}{64}$. □

As in the original paper, we can prove a corollary which asserts the non-existence of certain H-hypersurfaces.

Corollary 2.1. *With N as in Theorem 1.1, let M be a complete, δ_n-stable, H-hypersurface with $\delta_n < \frac{27}{32}, \frac{7}{12}, \frac{19}{64}$ respectively. If $|H| > 2\sqrt{|\min(0, \mathcal{K})|}$, then $\partial M \neq \emptyset$.*

Proof. From the previous theorem, the radius of an intrinsic geodesic disk of M that does not meet ∂M is at most $c = c > 0$. Assuming that the boundary of M is empty, the diameter of M is at most c, so the Hopf-Rinow theorem implies that M is compact. By δ-stability, there exists a function $u > 0$ on M such that $L^\delta u = 0$. By compactness, let $p \in M$ be the minimum of the function u. Then

$$0 \leq \Delta u(p) = -(1-\delta)(|\Phi|^2(p) + nH^2 + \overline{R}_{n+1,n+1}(p))u(p).$$

The choice of H guarantees that the potential is positive, hence the right hand side is negative which yields a contradiction. □

Next we generalize Theorem 1 in [32], that is the theorem below when $\delta = 0$.

Theorem 2.2. *Let N be a 3-dimensional Riemannian manifold with scalar curvature uniformly bounded from below by $\mathcal{S}$ and let M be a complete, δ-stable, H-surface with $\delta < \frac{3}{4}$. If $3H^2 + \mathcal{S} > 0$ then for any $p \in M$,*

$$\mathrm{dist}_M(p, \partial M) \leq 2\pi\sqrt{\frac{1-\delta}{(3-4\delta)(3H^2+\mathcal{S})}}.$$

Proof. In order to prove this theorem, one can follow the proof of Theorem 1.1 with $n = 2$. Note that when $n = 2$, condition (5) becomes $\frac{5}{8(1-\delta)} < k < 4$ and thus we can take $k = \frac{1}{1-\delta}$ as long as $\delta < \frac{3}{4}$. With this choice of k, note that instead of equation (4) we have

$$\sum_{j=2}^{n} \overline{R}_{1j1j} + k(1-\delta)\overline{R}_{n+1,n+1} = \overline{R}_{1212} + \overline{R}_{3,3} \geq \mathcal{S}. \tag{6}$$

The proof then continues after defining $B := 3H^2 + \mathcal{S}$, and noting that when $n = 2$, we have $A := 4\frac{1-\delta}{3-4\delta}$. □

Just like before, one can prove a corollary which asserts the non-existence of certain H-surfaces.

Corollary 2.3. *With N as in Theorem 2.2, let M be a complete, δ-stable, H-surface with $\delta < \frac{3}{4}$. If $3H^2 + \mathcal{S} > 0$, then $\partial M \neq \emptyset$.*

3 Properness of effectively embedded H-hypersurfaces

In this section we prove Theorem 1.2. We begin by defining "effectively embedded."

Definition 3.1. Let $\phi\colon M \looparrowright N$ be an H-hypersurface. We say that M is effectively embedded if at any point $p \in \phi(M)$, there exists $\epsilon > 0$ such that either

1. $\phi^{-1}(p)$ consists of a single point $p_1 \in M$ and the connected component of $B_N(p, \epsilon) \cap \phi(M)$ containing p is an embedding of the connected component of $\phi^{-1}(B_N(p, \epsilon) \cap \phi(M))$ that contains p_1, or

2. $\phi^{-1}(p)$ consists of two points p_1 and p_2, ϕ restricted to the connected component Σ_i of $\phi^{-1}(B_N(p, \epsilon) \cap \phi(M))$ that contains p_i, $i = 1, 2$, is an embedding, the connected component of $B_N(p, \epsilon) \cap \phi(M)$ containing p is equal to $\phi(\Sigma_1 \cup \Sigma_2)$, $\phi(\Sigma_1)$ and $\phi(\Sigma_2)$ meet tangentially at p and their mean curvature vectors point in opposite directions.

Note that if M is embedded, then it is effectively embedded. This definition is natural as it includes limits of a converging sequence of embedded H-hypersurfaces, see for example [1]. Abusing the notation, when dealing with effectively embedded hypersurfaces, we will ignore the immersion ϕ and when Case 2 of Definition 3.1 occurs, we might refer to either of the $\phi(\Sigma_i)$ (that is Σ_i), $i = 1, 2$, as the connected component of $B_N(p, \epsilon) \cap \phi(M)$ (that is $B_N(p, \epsilon) \cap M$) containing p.

The proof of Theorem 1.2 is going to use the Stable Limit Leaf Theorem in [19]. To that end, we need to recall a few definitions.

Definition 3.2. Given $H > 0$, a codimension one H-lamination $\mathcal{L}$ of N is a collection of immersed (not necessarily injectively) H-hypersurfaces $\{L_\alpha\}_{\alpha \in I}$, called the leaves of $\mathcal{L}$, satisfying the following properties:

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{L_\alpha\}$ is a closed subset of N.

2. Given a leaf L_α of $\mathcal{L}$ and a small disk $\Delta \subset L_\alpha$, there exists an $\epsilon > 0$ such that, if (q, t) denote the normal coordinates for $\exp_q(t\eta_q)$ (here exp is the exponential map of N and η is the unit normal vector field to L_α pointing to the mean convex side of L_α), then:

(a) The exponential map $\exp : U(\Delta, \epsilon) = \{(q,t) \mid q \in \mathrm{Int}(\Delta), t \in (-\epsilon, \epsilon)\}$ is a submersion.

(b) The inverse image $\exp^{-1}(\mathcal{L}) \cap \{q \in \mathrm{Int}(\Delta), t \in [0, \epsilon)\}$ is a lamination of $U(\Delta, \epsilon)$.

Definition 3.3. Let $\mathcal{L}$ be an H-lamination of N and let L be a leaf of $\mathcal{L}$. We say that L is a *limit leaf* if L is contained in the closure of $\mathcal{L} - L$.

A properly effectively embedded H-hypersurface is an H-lamination with one leaf. We can now state the Stable Limit Leaf Theorem.

Theorem 3.1 (Theorem 1 in [19]). *The limit leaves of a codimension one H-lamination of a Riemannian manifold are stable.*

Finally, we are ready to begin the proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that M having locally bounded norm of the second fundamental form means that the intersection of M with any closed extrinsic ball of N has norm of the second fundamental form bounded from above by a constant that only depends on the ball.

Arguing by contradiction, suppose M is not proper. The first step in the proof is to observe that $\overline{M}$, the closure of M, has the structure of an H-lamination.

Let $p \in \overline{M}$. Since $|A|$ is locally bounded, we can apply Theorem 4.2 to give a sufficiently small harmonic chart $(U, \phi, B_N(p,r))$ such that for any $\epsilon \in (0,r)$, if Σ denotes the set of connected components of $\phi^{-1}(M \cap B_N(p,r))$ intersecting the Euclidean ball $B_{\mathbb{R}^3}(0,\epsilon)$, then there exist $\epsilon > 0$, $\rho \in (\epsilon, r)$, and $C' < \infty$ and a rotation $R \in O(\mathbb{R}^3)$ such that:

1. Every connected component of $\Sigma := R(\Sigma) \cap B(\rho) \times \mathbb{R}$ is the graph of a function u over $B(\rho)$.

2. For all such functions u, we have $\|u\|_{C^{2,\alpha}(B(\rho))} \leq C'$.

Note that $\Sigma = \Sigma_+ \cup \Sigma_-$, where Σ_+ (Σ_- respectively) is the collection of components whose mean curvature vector is pointing "upward" ("downward").

If $p \in M$ and M is proper in a neighbourhood W of p then $M \cap W = \overline{M} \cap W$ and, after possibly using a smaller chart, Σ consists of a finite number of connected components and $\overline{M}$ has the structure of an H-lamination in a neighbourhood of p. Else, the number of connected components in Σ is infinite and there exists a sequence of connected components $\sigma_n \in \Sigma$ such that $p \notin \sigma_n$ but $p \in \lim_{n\to\infty} \sigma_n$. After passing to a sub-sequence and without loss of generality, we can assume that $\sigma_n \in \Sigma_+$, for all n. Moreover, a standard compactness argument gives that σ_n converges $C^{2,\alpha}$ to a graph σ containing p, with constant mean curvature H, bounded $|A|$ and upward pointing mean curvature vector. Note that if $\phi^{-1}(\sigma) \not\subset M$ then, by definition, $\phi^{-1}(\sigma) \subset \overline{M}$. Furthermore, observe that if σ_- is a component in Σ_- that is above σ, then σ_- cannot be arbitrarily close to p, see for instance [22, Lemma 3.1]. Therefore, after possibly using an even smaller chart, no component of Σ_- is above σ. Therefore, by taking $\Delta = \sigma$ in Definition 3.2, this discussion shows that $\overline{M}$ satisfies condition 2 of Definition 3.2.

Using more or less the previous arguments, it is fairly standard to show that $\overline{M} - M$, and therefore $\overline{M}$ is a collection of H-hypersurfaces. Finally, by definition, $\overline{M}$ is a closed subset of N. This finishes to prove the observation that $\mathcal{L} := \overline{M}$ is an H-lamination.

Next, we claim that $\overline{M} \neq M$. Arguing by contradiction, suppose that $\overline{M} = M$ is the only leaf in the lamination $\mathcal{L}$. Since M is not proper and $\overline{M} = M$, M contains a limit point, namely there is a point $p \in M$ and a sequence of points $p_n \in M$ converging to p extrinsically, but not intrinsically. It's not hard to see that the set of limit points is open and closed. This implies that every point $x \in M$ is a limit point. Let U in M be a small geodesic ball centred at x which is the limit as n goes to infinity of a sequence of pairwise disjoint balls U_n in M centered at p_n. Performing this construction again for all $p_n \in U_n$, we obtain pairwise disjoint balls $U_{n,m} \subset M$ which converge to U_n in N as m goes to infinity. We can then repeat this process, for example next on $U_{n,m}$ and again on the resulting balls. This iterative process yields an uncountable number of disjoint balls on M which contradicts the second countable intrinsic property of the topology of a manifold. This shows that $\overline{M} - M$ is not empty.

Let L be a leaf $\overline{M} - M$. Then, by definition, L is a limit leaf of $\mathcal{L}$. By Theorem 3.1, L is stable and by Theorem 1 in [32], when N is a 3-dimensional manifold (see Remark 3.2), and Theorem 1 in [13], when N is a 4 or 5-dimensional manifold, L cannot exist (see Corollary 2.1 with $\delta = 0$). This final contradiction proves that M must be proper. □

Remark 3.2. *Analogously to what happened in Section 2, thanks to Theorem 1 in [32] (see Theorem 2.2 and Corollary 2.3 with $\delta = 0$) when the dimension of N is 3, one can replace the condition $|H| > 2\sqrt{|\min(0, \mathcal{K})|}$ with $3H^2 + \mathcal{S} > 0$ (where $\mathcal{S}$ is a uniform bound from* *below for the scalar curvature of N), and prove a stronger properness result.*

4 Properties of effectively embedded H-hypersurfaces with bounded $\|A\|$

To keep the paper self-contained, in this section we state a few properties of hypersurfaces effectively embedded in a manifold with bounded second fundamental form that were used in the proof of Theorem 1.2. These statements are generalizations of existing results.

Given a point x in a hypersurface M in $\mathbb{R}^n$, a neighbourhood of x is always graphical over the tangent plane of M at x. However, the size of such neighbourhood depends on x and, in general, it could be very small. However, when the norm of the second fundamental form of M is bounded, then the size of such neighbourhood is uniformly bounded from below independently of the point, see for example [36].

Analogous results are true for hypersurfaces in a manifold N. We begin by referencing a result by Hebey and Herzlich [15] that establishes that the metric respect to some harmonic coordinates is locally uniformly $C^{1,\alpha}$-controlled for any $\alpha \in (0,1)$, depending only on the bounds on the injectivity radius and sectional curvatures of N. The version stated below is presented by Rosenberg, Souam and Toubiana in the appendix of [33]. Note that the version in [33] is stated for 3-dimensional manifolds. It is not hard to see that the proof works in higher dimensions [15].

Theorem 4.1. *Let $\alpha \in (0,1)$ and $\delta > 0$. Let (N,g) be a complete Riemannian manifold with absolute sectional curvature bounds $|K| \leq \Lambda < \infty$. Let Ω be an open subset of N and define the fattening*

$$\Omega(\delta) := \{x \in N : \mathrm{dist}_N(x,\Omega) < \delta\}.$$

Suppose that there exists an $i > 0$ such that for all $x \in \Omega(\delta)$, we have $\mathrm{inj}_N(x) > i$. Then there exists a constant $Q_0 > 1$ and a radius $r_0 > 0$ which depend only on i, δ, Λ and α, but not on N, such that for any $x \in \Omega$, there exists a harmonic chart $(U, \phi, B_N(x, r_0))$ with $\phi(0) = x$. Furthermore, we have $C^{1,\alpha}$-control over the metric tensor, that is

$$Q_0^{-1}\delta_{ij} \leq g_{ij} \leq Q_0 \delta ij$$

as quadratic forms, and $\|(\phi^* g)_{ij}\|_{C^{1,\alpha}(U)} \leq Q_0$

Using the result above, by transferring the problem onto Euclidean space, in the appendix of [34], Saturnino proves the following theorem. Once again this result is stated there for surfaces in a 3-dimensional Riemannian manifold but its proof works in higher dimensions.

Theorem 4.2. *Suppose* (N, g) *is a manifold with absolute sectional curvature bounds* $|K| \leq \Lambda < \infty$ *and let* $M \subset N$ *be an effectively embedded* H*-hypersurface. Let* $\Omega \subset N$ *be an open set lying away from the boundary of* N*, and suppose the norm of the second fundamental form of* M *in* Ω *is bounded above by a constant* $C < \infty$*. Fix any* $\alpha \in (0, 1)$ *and suppose* δ*,* i*,* r_0*, and* Q_0 *are as in Theorem 4.1. Fix an* $r \in (0, r_0)$ *and let* $x \in \Omega$ *be such that* $d_M(x, \partial M) > r$*.*

Choose a harmonic chart $(U, \phi, B_N(x, r))$ *as in Theorem 4.1. For any* $\epsilon \in (0, r)$*, let* Σ *be the set of connected components of* $\phi^{-1}(M \cap B_N(x, r))$ *intersecting the Euclidean ball* $B_{\mathbb{R}^3}(0, \epsilon)$*. Then there exist* $\epsilon > 0$*,* $\rho \in (\epsilon, r)$*, and* $C' < \infty$ *depending only on* Λ*,* C*,* i*, and* α*, and a rotation* $R \in O(\mathbb{R}^3)$ *such that:*

1. *Every connected component of* $R(\Sigma) \cap B(\rho) \times \mathbb{R}$ *is the graph of a function* u *over* $B(\rho)$*.*
2. *For all such functions* u*, we have* $\|u\|_{C^{2,\alpha}(B(\rho))} \leq C'$*.*

In fact, in [34] the hypersurfaces are assumed to be properly embedded but the proof works for effectively embedded hypersurfaces. Note that in this more general case the number of connected components in Σ could be infinite and connected component must be intended in the sense described in Definition 3.1.

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