POTENTIALS WITH FINITE-BAND SPECTRUM AND FINITE-DIMENSIONAL REDUCTIONS OF BKM SYSTEMS

ANDREY YU. KONYAEV AND VLADIMIR S. MATVEEV

Abstract. We repeat, using methods developed for BKM systems, the famous results of S. Novikov [45], J. Moser [42, 43], and A. Veselov [50] that relate Schrödinger-Hill operators with finite-band spectra, solutions of the Neumann system, and certain solutions of the KdV equations. Our general motivation is to determine whether it is possible to apply inverse scattering methods to BKM systems, and in the conclusion, we indicate initial observations in this direction.

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1. Introduction

Jürgen Moser, in his very influential and widely cited works [41, 42, 43] attracted attention to relation of the following three a priory independent objects of interest of mathematical physics, namely the KdV equation, which is an ∞-dimensional integrable system, finite-band spectrum of Schrödinger-Hill operators, and classical finitely-dimensional integrable systems corresponding to the geodesics of ellipsoid and the Neumann system. We give necessary definitions later, in §2.2. The works [41, 42, 43], though containing new interesting results, are of rather survey nature. Certain connections discussed by Moser were published by other authors. In particular, the relation between the orbits of the Neumann system and the finite-gap solutions of the KdV equation is discussed in [1, 50]. The relation between geodesics on ellipsoid and solutions of the Neumann system is understood at least in [30]. In [45], it was shown that stationary solutions of KdV give potentials such that the corresponding Schrödinger-Hill operators have finitely many connected components of spectrum. In [24] it was shown that any quasi-periodic potential with finitely many connected components of spectrum of the corresponding Schrödinger-Hill operator is generated by this way. See also [26, 25].

J. Moser, and also many other mathematics of that time, were very excited about the relations and possibly viewed them as a wondeful coincidence, a kind of mathematical magic. The goal of the paper is to give a short proof of the relations. We will use new understanding coming from the recently found connection between finitely-dimensional integrable systems and geodesically equivalent metrics, see e.g. [36, 8] and also the very recent results of [14] on finite-dimensional reductions of BKM-systems[1]

Our note is organised as follows. First we introduce and discuss Benenti systems. All systems which we consider: the Neumann system, geodesics on the ellipsoid and the finite-dimensional reductions of the KdV equation can be described by certain

[1]BKM systems is a family of multicomponent integrable PDE-systems introduced in [9]. In the present note we mostly concentrate on KdV equation, which is a BKM system. In the Conclusion and Outlook section 5, we comment on possible generalizations for other BKM systems.

Benenti systems, which we explicitly write. Next, we prove a relatively simple Lemma 3.1 which explains when two Benenti system share the same solutions. The relation of the KdV system to the Neumann system and of the geodesic flow of the metric geodesically equivalent to the metric of ellipsoid follows directly from this Lemma. Lemma 4.1 specifies the conditions that the parameters of the solutions of the KdV equation corresponding to the Neumann system must satisfy.

Next, we consider the Sturm-Liouville-Schrödinger equation whose potential comes from a solution of the finite-dimensional reduction of the KdV equation. We show in Lemma 4.4 that one can explicitly write the solutions of this equation using solutions of the Benenti systems used in the finite-dimensional reduction of the KdV equation. Next, solutions of finite-dimensional reduction of the KdV equation that correspond to solutions of the Neumann systems can be studied using method of separation of variables. Employing this method, we show that the spectrum contains finitely many bands.

2. Preliminaries

2.1. **Benenti systems.** By *Benenti systems*[2] we understand a class of integrable finite-dimensional Hamiltonian systems on the cotangent bundle T^*M. The Hamiltonian is the sum of the kinetic energy $\frac{1}{2}g^{ij}p_ip_j$, where g is a metric of any signature, and a potential energy $V : M \to \mathbb{R}$. The metric g and the potential energy U satisfy certain differential-geometric and nondegeneracy conditions which we introduce and discuss now. We will describe these systems, and give a local classification of them in this section. We will see that, under nondegeneracy assumptions which are fulfilled in the cases we consisder in the present paper, Benenti systems are given by two functions, f and U, of one variable.

We start with a pair (metric g, g-seladjoint (1,1)-tensor L) satisfying the following condition:

$$L_{ij,k} = \lambda_j g_{ij} + \lambda_i g_{jk}. \tag{1}$$

In the equation above we use g for index manipulation and denote by comma the covariant derivative with respect to the Levi-Civita connection of g. The 1-form λ_i staying on the right hand side is necessarily $\lambda_i = \frac{1}{2} d\,\mathrm{trace}_g(L)$.

The equations (1) appeared many times independently in different branches of mathematics. In particular, in the context of integrable systems, solutions L of this equation, for a given g, are called in [22, 23] *special conformal Killing tensors*. See also [3, 34].

In differential geometry, (1) appeared at least in [46], in the context of geodesically equivalent metrics, see also [19, 39]. The equation (1) is now called *geodesic or projective compatibility* equation, see e.g. [10, 21]. Recall that two metrics g and $\bar g$ on the same manifold are *geodesically (or, which is the same, projectively) equivalent*, if every g-geodesic, after an appropriate reparameterisation, is a $\bar g$-geodesic. The relation of (8) to geodesic equivalence is as follows: if L is nondegenerate and satisfies (1), then the metric $\bar g_{ij}$ whose matrix is given by

$$\bar g = \tfrac{1}{\det(L)} g L^{-1} \tag{2}$$

[2]The terminology was suggested in [19, 29]

is geodesically equivalent to g. Moreover, if $\bar{g}$ is geodesically equivalent to g, then L reconstructed by (2) satisfies (1).

Given L^i_j satisfying (1), one can construct a family of commutative integrals for the geodesic flow of g. Namely, consider the following family of functions on the tangent bundle T^*M of the manifold M of dimension N, which is a polynomial in λ of degree $N-1$:

(3)
$$p \in T^*M \ \mapsto \ I_\lambda(p) = \tfrac{1}{2}\det(\lambda \mathrm{Id} - L) g^*((L^* - \lambda \mathrm{Id})^{-1}p, p) = I_0(p)\lambda^{N-1} + I_1(p)\lambda^{N-2} + \cdots + I_{N-1}(p).$$

In the above formula, g^* is the induced bilinear form on the cotangent bundle, and L^* is the induced operator on the cotangent bundle. In the matrix notation, the matrix of g^* is the inverse of the matrix g and the matrix of L^* is the transposed of the matrix L. In the index notation, the middle part of (3) reads

$$\tfrac{1}{2}\det(\lambda \mathrm{Id} - L) g^{ij}\left((L - \lambda \mathrm{Id})^{-1}\right)^s_i p_s p_j.$$

Theorem 2.1 ([36, 19, 23, 22, 48, 49])**.** *The functions I_i commute with respect to the standard Poisson structure on T^*M.*

Note that the function I_0 is the Hamiltonian $\frac{1}{2}g^*(p,p)$ of the geodesic flow.

The number of functionally independent integrals I_i is the degree of the minimal polynomial of L at least at one point, see [39, Lemma 5.6 and Corollary 5.7] or [48, Theorem 2 and Proposition 3]. Later we assume that L is differentially nondegenerate[3] almost everywhere. Then, by [15, §4.2], at almost every point, the degree of the minimal polynomial of L equals the dimension of the manifold[4] which implies that the geodesic flow of g is Liouville integrable. As it will be clear later, near almost every point, the integrable system can be effectively analyzed by the method of separation of variables.

Let us give two examples of geodesically compatible pairs (g, L) such that L is differentially-nondegenerate. The first is classical and is essentially due to T. Levi-Civita [32]. The metric g_{ij} and the tensor L^i_j are given by the formulas

$$g^f_{LC} = \sum_{i=1}^{N}\left(\frac{dq_i^2}{f(q_i)}\prod_{s=1, s\neq i}^{N}(q_i - q_s)\right), \tag{4}$$

$$L_{\mathrm{diag}} = \mathrm{diag}(q_1, q_2, \ldots, q_N). \tag{5}$$

Above f is an arbitrary function of one variable such that it is never zero. Note that in a more standard way to write the formula (4), the functions $f(q_i)$ in the denominators may be different functions, so the ith term in the sum (4) reads

$$\left(\frac{dq_i^2}{f_i(q_i)}\prod_{s=1, s\neq i}^{N}(q_i - q_s)\right).$$

This way is clearly equivalent to the one in (4), at least in a sufficiently small neighborhood, since the formula gives a non-degenerate metric only if $q_i \neq q_j$ for $i \neq j$, and under this assumption we can define the functions $f(q_i)$ as $f_i(q_i)$ restricted to the range of the coordinate q_i.

[3] see [15, Definition 2.10]

[4] (1,1)-tensors such that the degree of the minimal polynomial of L equals the dimension of the manifold are called gl-regular, see [16]

Remark 2.1. *In the examples interesting for the present paper, the function f is a real-analytic function. This is not a special property of our examples, but a rather general phenomenon. Indeed, the metric in the examples is real analytic. By [31], L satisfying (1) is real analytic as well. If L is differentially nondegenerate, which is the case for the Benenti systems coming from finite-dimensional reductions of BKM system, by [15, Theorem 6.2] f must be a globally defined real-analytic function. Moreover, if g is Riemannian and L is gl-regular at least in one point, which is the case for the Benenti systems coming from the Neumann systems and from the geodesic flow of the ellipsoid, f again must be a globally defined real-analytic function by [38].*

For the further use, note that the metric $\bar g^f_{LC}$ geodesically equivalent to g^f_{LC} constructed by (2) reads

$$\bar g^f_{LC} \;=\; \sum_{i=1}^{N}\left(\tfrac{(q^i)^{N-3}\mathrm{d}q_i^2}{f(q_i)}\prod_{s=1,s\neq i}^{N}\frac{(q_i-q_s)}{q_iq_s}\right). \tag{6}$$

Let us also note that, if we call g^1_{LC} the metric (4) with $f(t)\equiv 1$, then one can obtain the metric (4), i.e., the metric g^f_{LC}, by the formula

$$g^f_{LC} = g^1_{LC} f(L)^{-1}, \tag{7}$$

see e.g. [21, §1.3] for the definition and discussion of analytic functions of (1, 1)-tensors. Observe also that the metric g_1 can be invariantly characterised within g satisfying (1) with respect to a fixed differentially nondegenerate L. Indeed, the coordinates such that (g, L) are given by (4, 5) are geometrically distinguished as they are eigenvalues of L.

The second example was studied in particular in [15, Proposition 6.1], see also [12]. In the context of integrable systems, it can be found e.g. in [4, §V]. We first start with the contravariant metric g and the (1,1)-tensor L^i_j given by

$$g_1^{ij} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & w_1\\ \vdots & \iddots & \iddots & \iddots & w_2\\ 0 & 1 & w_1 & \iddots & \vdots\\ 1 & w_1 & w_2 & \cdots & w_{N-1}\end{pmatrix} \quad L_{\mathrm{comp}} = \begin{pmatrix} -w_1 & 1 & 0 & \cdots & 0\\ -w_2 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ -w_{N-1} & 0 & \ldots & 0 & 1\\ -w_N & 0 & \ldots & 0 & 0\end{pmatrix}. \tag{8}$$

The metric g_1 in this example is flat and is of splitted signature. By [15, Theorem 6.2.], in the real-analytic category, any (contravariant) metric g which is geodesically compatible with L_{comp} from (8) is given by

$$g_f = f(L_{\mathrm{comp}})g_1. \tag{9}$$

Note that the visual difference between the formulas (7) and (9) is artificial since (9) is for a contravariant metric and (7) is for a covariant metric. In fact, the formulas (9) and (7) are equivalent. Moreover, the pair (g_f, L_{comp}) given by (8), in a neighborhood of any point such that L has N real different eigenvalues, it isomorphic by a coordinate change to the pair $(g^f_{LC}, L_{\mathrm{diag}})$. The coordinate change $w(q)$ transforming (g_f, L_{comp})

to $(g^f_{LC}, L_{\rm diag})$ is given by the following relation:

$$\det(\lambda \mathrm{Id} - L_{\mathrm{diag}(q)}) = \det(\lambda \mathrm{Id} - L_{\mathrm{comp}(w)}) \tag{10}$$

Note that as $L_{\rm comp}$ is in the companion form,

$$\det(\lambda \mathrm{Id} - L_{\mathrm{comp}}) = \lambda^N + w_1\lambda^{N-1} + \dots + w_N, \tag{11}$$

so (10) gives an explicit formula $w(q)$ in terms of symmetric polynomials of q-variables:

$$w_1 = -q_1 - q_2 - \dots - q_N\ ,\ \ w_2 = q_1q_2 + q_1q_3 + \dots + q_{N-1}q_N\ ,\ \dots\ ,\ w_n = (-1)^N \det(L_{\rm diag}).$$

This observation holds, with necessary natural amendments, in the case when some different eigenvalues of L are complex-valued; of course in this case the coordinates in which (g, L) has the form (4, 5) may also be complex-valued. See [15, 20] for a discussion of complex eigenvalues of L satisfying (1) and of Nijenhuis[5] operators with complex eigenvalues.

Let us now discuss the integrals. It appears that the metric g^f_{LC}, in coordinates q_i, can be obtained by the so-called Stäckel construction, see e.g. [14]. Indeed, take the Stäckel matrix

$$S_{ij} = \begin{pmatrix} (q_1)^{N-1} & (q_1)^{N-2} & \cdots & 1 \\ (q_2)^{N-1} & (q_2)^{N-2} & \cdots & 1 \\ \vdots & & & \vdots \\ (q_N)^{N-1} & (q_N)^{N-2} & \dots & 1 \end{pmatrix} \tag{12}$$

and construct the functions $I_0, \dots, I_{N-1}$ on $\mathbb{R}^{2N}(q,p)$, quadratic in p-variables, by the formula

$$\begin{pmatrix} (q_1)^{N-1} & (q_1)^{N-2} & \cdots & 1 \\ (q_2)^{N-1} & (q_2)^{N-2} & \cdots & 1 \\ \vdots & & & \vdots \\ (q_N)^{N-1} & (q_1)^{N-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_N \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}f(q_1)(p_1)^2 \\ -\frac{1}{2}f(q_2)(p_2)^2 \\ \vdots \\ -\frac{1}{2}f(q_N)(p_N)^2 \end{pmatrix}. \tag{13}$$

It is known, see e.g. [14, Fact 1.2], that the functions Poisson commute with respect to the standard Poisson structure. Moreover, the functions $I_0, \dots, I_{N-1}$ staying in (13) coincide with those obtained via (3). In particular, the function I_0 is the Hamiltonian of the geodesic flow of the metric g^f_{LC} given by (4).

It is also known how to "introduce" the potential energy in the formula (13). Locally, in coordinates $q_1, \dots, q_N$ used in (13), the freedom is the choice of N functions of one variable. In our context, these functions are essentially the same function, so we proceed with the construction under this assumption[6]. In order to do it, we slightly modify (13)

[5](1,1)-tensor is called *Nijenhuis operator*, if its Nijenhuis torsion vanished. BKM systems were constructed in [9] within the Nijenhuis geometry project initiated in [11, 15]

[6]Actually, if the (g, L) are given by (8), in the analytic category and near the point $(w_1 = 0, \dots, w_N = 0)$, these N functions can be glued in one function; this again follows from [15, Theorem 6.2]

to obtain

$$\begin{pmatrix} (q_1)^{N-1} & (q_1)^{N-2} & \cdots & 1 \\ (q_2)^{N-1} & (q_2)^{N-2} & \cdots & 1 \\ \vdots & & & \vdots \\ (q_N)^{N-1} & (q_N)^{N-2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \tilde I_0 \\ \tilde I_1 \\ \vdots \\ \tilde I_{N-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} f(q_1)(p_1)^2 + U(q_1) \\ -\frac{1}{2} f(q_2)(p_2)^2 + U(q_2) \\ \vdots \\ -\frac{1}{2} f(q_N)(p_N)^2 + U(q_N) \end{pmatrix}. \tag{14}$$

The functions $\tilde I_i$ given by (14) Poisson commute. Each of them is the sum of the quadratic in momenta part which is I_i given by (13), and a function which depends on the position $(q_1, \dots, q_N)$ only. In particular, the function $\tilde I_0$ is the sum of the kinetic energy coming from the metric g^f_{LC} and a potential energy.

We will also need the formula for obtained system in the companion coordinates, in which g and L are given by (8). In order to do it, for a real analytic function U of one variable and for a pair (g, L) satisfying (1) and such that L is gl-regular, consider the functions $U_1, U_2, \dots, U_N$ defined by the following relation:

$$U(L) = U_0 L^{N-1} + U_1 L^{N-1} + \cdots + U_{N-1}\mathrm{Id}. \tag{15}$$

The left hand side is an analytic function of L, it is well-defined (1,1)-tensor provided U is defined at the eigenvalues of L, see e.g. the discussion in [15, 21]. The right hand side is a polynomial of order $N-1$ in L. Since L is gl-regular, (15) viewed as a system of linear equations on functions $U_0, ..., U_{N-1}$ determines the functions $U_0, \dots, U_{N-1}$ on the manifold.

Lemma 2.2. *Suppose (g, L) are geodesically compatible and L is gl-regular.*

Then, for any real-analytic function U, the functions $\tilde I_i := I_i + U_i$, where I_i is as in (3) and U_i are from (15), pairwise commute.

Lemma 2.2 was obtained within and is a natural component of the theory of conservation laws and symmetries of Nijenhuis operators developed in [17, 13, 16], where its more general versions are proved. The version staying above is easy to prove directly: the formula (15) is invariant with respect to coordinates changes, so one can prove it in any coordinate system. In the "diagonal" coordinate system, in which g and L have the form (4,5), the formula (15) is equivalent to (14).

We see that a Benenti system with potential is given by the following data: we can choose L and functions f, U of one variables. Our nondegeneracy condition on the pair is that L is differentially nondegenerate almost everywhere. This condition implies that the pair (g, L) is given by (8) almost everywhere, and by (4, 5) almost everywhere provided we allow some of the coordinates to be complex-valued.

Remark 2.2. *If $L = L_{old}$ is a solution of (1), then for any constants $const_1$ and $const_2$ the (1,1)-tensor $L_{new} = const_1 L_{old} + const_2\,\mathrm{Id}$ is also a solution. The change from L_{old} to L_{new} corresponds to the following change of f and U:*

$$f_{old}(t) = (const_1)^{-1-N} f_{new}(const_1\, t + const_2)\ , \quad U_{old} = U_{new}(const_1\, t + const_2). \tag{16}$$

This change does not affect the vector space generated by the integrals $\tilde I_0, \dots, \tilde I_{N-1}$.

2.2. **Neumann system, geodesic flow on ellipsoid and stationary solutions of the KdV equation as Benenti systems.**

2.2.1. *Neumann system, geodesic flow of the metric of the ellipsoid, and the geodesic flow of the metric geodesically equivalent to ellipsoid as Benenti systems.* *Neumann system* is the Hamiltonian system describing the movement of a particle on the sphere in a quadratic potential. That is, we consider the standard sphere $S^N \subset \mathbb{R}^{N+1}(X_1, \dots, X_{N+1})$ with the standard metric which we denote g_s, and the function $V : S^N \to \mathbb{R}$ which is the restriction of the function $\sum_{i=1}^{N+1}(X_i)^2 a_i$ to the sphere. We assume that all a_i are different, without loss of generality $a_1 < a_2 < \cdots < a_{N+1}$. Note that adding a constant to the potential energy does not affect the equations of motion and adds the constant to all a_i, so we can assume without loss of generality that $0 < a_1$. The Hamiltonian of the Neumann system is the sum of the kinetic energy coming from g_s and the potential energy V.

We will also consider the ellipsoid in $\mathbb{R}^{N+1}$ defined by the equation

$$\sum_{i=1}^{N+1} \frac{X_i^2}{a_i} = 1. \tag{17}$$

We assume that all a_i are positive and different[7] and order them by $0 < a_1 < \cdots < a_{N+1}$.

The restriction of the standard metric to the ellipsoid will be denoted by g_e. The corresponding geodesic flow is the Hamiltonian system whose Hamiltonian is the kinetic energy corresponding to g_e.

Fact 2.1. *The metric g_e of the ellipsoid and the standard metric g_s of the sphere admits L satisfying (8) which is differentially nondegenerate at almost every point.*

For the standard sphere, Fact 2.1 follows from a result of E. Beltrami[8], see e.g. [37]. For the ellipsoid, Fact 2.1 was independently and almost simultaneously obtained in [36, 47], see also [40].

Let us give a short proof of Fact 2.1, whose ingredients will be useful later. We start with the ellipsoid and consider the ellipsoidal coordinates $q_1, ..., q_N$ related to the standard coordinates $X_1, ..., X_{N+1}$ in the quadrant $\{(X_1, \dots, X_N) \in \mathbb{R}^{N+1} \mid \varepsilon_1 X_1 > 0, \dots, \varepsilon_{N+1} X_{N+1} > 0\}$, where $\varepsilon_i \in \{-1, 1\}$, by the formula

$$X_i = \varepsilon_i \sqrt{a_i \frac{\prod_{j=1}^{N}(a_i - q_j)}{\prod_{j=1, j\neq i}^{N+1}(a_i - a_j)}}. \tag{18}$$

In the ellipsoidal coordinates, the metric g_e of the ellipsoid has the form (4) with

$$f(t) = -\tfrac{4}{t} \prod_{j=1}^{N+1} (t - a_j). \tag{19}$$

Then, $L = \mathrm{diag}(q_1, \dots, q_N)$ is a solution of (8).

The calculations have shown the local existence of such L in a neighborhood of a point where elliptic coordinates are defined. Since ellipsoid is simply-connected, the metric

[7] The assumption that all a_i are different is indeed important for us; the assumption that they are positive can be omitted and instead of ellipsoid we can consider a noncompact quadric and make sense of the case when one $a_i = 0$

[8] who constructed examples of metrics geodesically equivalent to the metric of the sphere

of ellipsoid is real analytic. Next, recall that (8) viewed as a system of PDEs on L is of finite type[9]. Then, the local existence implies the global existence.

See [36, §7] and [40, §7] for the formulas for the geodesically equivalent metric and the (1,1)-tensor L in the coordinates $X_1, \dots, X_{N+1}$.

Later, we will use also the formula, in the ellipsoidal coordinates, for the metric geodesically equivalent to the metric of the ellipsoid. In view of (2), it is given by

$$\bar g_e = -\tfrac{1}{4}\sum_{i=1}^{N}\left(q_i^{N-2}\,(\mathrm{d}q_i)^2\prod_{s=1,s\neq i}^{N}\tfrac{(q_i-q_s)}{q_iq_s}\prod_{s=1}^{N+1}\tfrac{1}{(a_s-q_i)}\right) \tag{20}$$

$$= -\tfrac{1}{4}\prod_{s=1}^{N+1}a_i\sum_{i=1}^{N}q_i\left(\left(\mathrm{d}\tfrac{1}{q_i}\right)^2\prod_{s=1,s\neq i}^{N}\left(\tfrac{1}{q_s}-\tfrac{1}{q_i}\right)\prod_{s=1}^{N+1}\tfrac{1}{\frac{1}{q_i}-\frac{1}{a_s}}\right) \tag{21}$$

$$= -\tfrac{1}{4}\prod_{s=1}^{N+1}\tfrac{1}{\bar a_i}\sum_{i=1}^{N}(\mathrm{d}y_i)^2\left(\tfrac{1}{y_i}\prod_{s=1}^{N+1}\tfrac{1}{y_i-\bar a_s}\right)\left(\prod_{s=1,s\neq i}^{N}(y_s-y_i)\right), \tag{22}$$

where the new coordinates y_i are given by $y_i = \frac{1}{q_i}$ and $\bar a_i = \frac{1}{a_i}$. We see that the metric (22) is the metric (4) corresponding to the functions

$$f(t) = -4t\prod_{s=1}^{N+1}(t-\bar a_s)\prod_{s=1}^{N+1}\bar a_s. \tag{23}$$

Similarly, for the standard sphere consider the sphero-ellipsoidal coordinates[10] related to standard coordinates $X_1, ..., X_{N+1}$ in $\mathbb{R}^{N+1}$ by

$$X_i = \varepsilon_i\sqrt{\frac{\prod_{j=1}^{N}(a_i-q_j)}{\prod_{j=1,j\neq i}^{N+1}(a_i-a_j)}}. \tag{24}$$

In these coordinates, the standard metric of the sphere has the form (4) with

$$f(t) = -4\prod_{j=1}^{N+1}(t-a_j). \tag{25}$$

We again see that $L = \mathrm{diag}(q_1, \dots, q_N)$ satisfies (1).

Now, by direct calculations we see that the potential energy of the Neumann system has the form

$$\sum_{i=1}^{N+1}X_i^2a_i = \tfrac{1}{2}\left(\sum_{i=1}^{N+1}a_i-\sum_{i=1}^{N}q_i\right). \tag{26}$$

[9]It closes after 2 prolongations, see e.g. [27]

[10]Also called sphero-conical coordinates.

By direct calculations we see that it corresponds to the function[11]

$$U(t) = -t^N + t^{N-1}\sum_{i=1}^{N+1} a_i. \tag{27}$$

Let us summarise the content of this section. We recalled that two finitely-dimensional systems of interest, the geodesic flow of the metric geodesically equivalent to the metric of the ellipsoid, and the Neumann system, belong to the class of Benenti systems. They correspond to the following functions f, U used in (14): The geodesic flow of the metric geodesically equivalent to the metric of the ellipsoid corresponds to f given by (20) and to $U = 0$, and the Neumann system corresponds to f given by (25) and to U given by (27).

Remark 2.3. *Let us observe from (26) that the potential energy corresponding to the Neumann system is equal to const − trace(L). In coordinates such that L is given by (8), trace of L clearly equals $-w_1$.*

Remark 2.4. *The range of the coordinates $(q_1, \dots, q_N)$, both in the case of ellipsoidal coordinates and sphero-ellipsoidal coordinates, is given by $a_1 < q_1 < a_2 < q_2 < \cdots < q_N < a_{N+1}$.*

2.2.2. *KdV equation as BKM system with n=1 and its finite-dimensional reductions.* The KdV equation is the following partial differential equation on the unknown function u of two variables x and t:

$$u_t = -\tfrac{1}{2}u_{xxx} + \tfrac{3}{2}uu_x. \tag{28}$$

There are different almost equivalent forms of the equation. For example, by re-scaling of x and t one can change the coefficients on the right hand side of the equation, and by multiplying u and t by -1 one can change the signs on the right hand side. In particular, the version of the KdV equation staying in [41] and in Wikipedia is

$$u_t = -u_{xxx} + 6uu_x. \tag{29}$$

Note also that the transformation $u_{new} = u_{old} + \text{const}$ makes from (28) the equation

$$u_t = -\tfrac{1}{2}u_{xxx} + \tfrac{3}{2}uu_x + \text{const}\,\tfrac{3}{2}u_x. \tag{30}$$

Similarly the transformation $u_{new}(x,t) = u_{old}(x + \text{const}\, t, t)$ makes from (28) the equation

$$u_t = -\tfrac{1}{2}u_{xxx} + \tfrac{3}{2}uu_x - \text{const}u_x. \tag{31}$$

Combining (30) with (31), we see that by the appropriate choice of constants the transformations compensate one another.

One may view the KdV equation, and actually all BKM systems, as dynamical systems on the space of real-analytic curves. Indeed, for a real-analytic function $x \mapsto u(x)$, the solution $u(x,t)$ of the KdV equation such that $u(x,0) = u(x)$ exists and is unique by the Kovalevskaya Theorem and can be viewed as a family of functions $x \mapsto u(x,t)$ depending on the parameter t.

[11]The second term $t^{N-1}\sum_{i=1}^{N+1} a_i$ in (27) can be ignored as it corresponds to the addition of the constant $\sum_{i=1}^{N+1} a_i$ in (26) and does not change the equations of motion

By *a finite-dimensional reduction* of the KdV equation one understands a finite-dimensional family of functions $x \mapsto u(x)$ such that the family is invariant with respect to the dynamical system above[12]. This notion naturally generalises to the BKM systems.

The finite-dimensional reduction of the KdV equation related to the topic of the paper was suggested in [45], where S. Novikov considered the so called *stationary solutions*; we will call it *stationary reduction*. The corresponding family of functions is defined as follows.

It is known, that the KdV equation admits symmetries, that are partial differential equations of the form

$$u_t = B[u], \tag{32}$$

where $B[u]$ is a differential polynomial, that is, a polynomial in $u_x := \frac{\partial u}{\partial x}$, $u_{xx} := \frac{\partial u}{\partial x}$ etc. satisfying the following property: The system of equations

$$u_t = -\tfrac{1}{2}u_{xxx} + \tfrac{3}{2}uu_x \quad \text{and} \quad u_\tau = B[u], \tag{33}$$

viewed as a system of equations on the unknown function $u(x, t, \tau)$ is compatible.

It is known and easy to see that the KdV equation has a trivial symmetry of differential degree 1

$$u_t = u_x. \tag{34}$$

The right hand side of this symmetry will be denoted by B_0, so the trivial symmetry reads $u_t = B_0[u]$.

The KdV equation itself is also a symmetry of differential degree 3, we denote its right hand side by B_1. The next symmetry has differential degree 5, we denote the corresponding right hand side by B_2 and so on, so the Nth nontrivial symmetry, whose right hand side is denoted by B_N, has differential degree $2N + 1$.

It is known that for any i, j the differential symmetry constructed by B_i is symmetry for the differential symmetry constructed by B_j.

As a finite-dimensional family of curves, S. Novikov [45] considered the family of solutions of the ordinary differential equation on u given by

$$B_N[u] + \lambda_1 B_1[u] + ... + \lambda_{N-1}B_{N-1}[u] = 0 \tag{35}$$

with constant coefficients $\lambda_1, \dots, \lambda_{N-1}$. Because for every i, j the differential symmetry constructed by B_j is a symmetry of that constructed by B_i, the family of curves is invariant with respect to the flow of all differential symmetries. Indeed, if we take a solution $u(x, t)$ of the PDE $u_t = B_i[u]$ such that the curve $x \mapsto u(x, 0)$ is a solution of $B_N[u] = 0$, then for any t the curve $x \mapsto u(x, t)$ is a solution of $B_N[u] = 0$, so the family of curves is invariant with respect to the flow of (28). For the fixed choice of $\lambda_1, \dots, \lambda_{N-1}$, the dimension of this family of the curves is $2N+1$, as $2N+1$ initial data determine the solution of ODE of degree $2N + 1$ on one unknown function $u(x)$. If we identify curves of the form $u(x)$ and $u(x + \text{const})$, the dimension of the family is $2N$, it coincides with the dimension of the cotangent space to the N-dimensional manifold.

[12]In other words, if $u(x)$ belongs to this family, then for the solution $u(x, t)$ of the KdV equation such that $u(x, 0) = u(x)$, the curves $x \mapsto u(x, t)$ lies in this family for any sufficiently small t

The solutions of the KdV equations such that for any t the curve $x \mapsto u(x, 0)$ solves the ODE (35) are called *stationary solutions*[13].

As explained in [9, Example 3.1], KdV equation in the form (28) is the BKM IV system corresponding to a special choice of parameters. In [14], a finite-dimensional reduction of BKM systems is studied. Though visually the finite-dimensional reduction procedure in [14] is different from that of in e.g. [45], it appears that for the KdV systems the finite-dimensional reduction constructed in [14] coincides with the one considered in [45].

The following results of [14] are relevant to the present paper. The parameters of the finite-dimensional reduction from [14] are the number N (which has nothing to do with the natural number N from the construction of BKM systems) and a monic polynomial

$$C(\mu) = \mu^{2N+1} + 0\mu^{2N} + c_2\mu^{2N-1} + \cdots + c_{2N+1} \tag{36}$$

of degree $2N+1$, whose second highest coefficient is zero. The reduction is constructed as follows: consider the following polynomial in μ whose coefficients depend on x:

$$w(x; \mu) = \mu^N + \mu^{N-1}w_1(x) + \mu^{N-2}w_2(x) + \cdots + w_N(x) \tag{37}$$

Next, consider the system of ODEs on the functions $w_1, \ldots, w_n$ given by

$$C(\mu) = m_0\left(w_{xx}w - \tfrac{1}{2}w_x^2\right) + (\mu - \tfrac{1}{2}w_1)w^2. \tag{38}$$

We observe that both sides of (38) are polynomial in μ of degree $2N+1$ such that the free terms and linear terms on the right and left side coincide, so (38) is a system of $2N$ ordinary differential equations of the second order on n unknown functions $w_1, ..., w_N$. The system depends on $2N$ parameters $c_2, c_3, \ldots, c_{2N+1}$ and on the parameter m_0. [14]

Next, for any coefficients $c_2, c_3, \ldots, c_{2N+1}$ consider the following family of 1-dimensional curves given by $x \mapsto u(x) = 2w_1(x)$, where $w_1(x)$ comes from a solution of (38).

Theorem 2.3 ([14]). *For every coefficients $c_2, c_3, \ldots, c_{2N+1}$, the family of the curves above is invariant with respect to the flow of KdV.*

This gives us a finite-dimensional reduction of the KdV equation. Actually, the result of [14] is valid for all BKM systems, not only for the KdV. For the KdV systems the reduction is essentially the stationary reduction to the one constructed in [45]. Namely, for every function $x \mapsto u(x)$ from the family corresponding to the finite-dimensional reduction from [45], there exist constants $c_2, \ldots, c_{2N+1}$ such that the function lies in the family described above.

The next result of [14] which will be used in the present paper, and which also explains why the visually overdetermined system of equations (38) has solutions, is as follows:

[13] There exist different nonequivalent definitions of stationary solutions of KdV in the literature. In particular, in [7] a much more restrictive stationarity condition $B_N[u] = 0$ is considered

[14] Though a generic system of $2N$ ordinary differential equation on N functions does not have a solutions, the system (38) can be solved for any values of the parameters $c_2, c_3, \ldots, c_{2N+1}$ and the solution depends on the choice of initial values $w_1(x_0), ..., w_N(x_0)$, so for fixed initial constants $c_2, \ldots, c_{2N+1}$ the family of functions is, locally, N-dimensional. In fact, solutions of the system are trajectories of a Lagrangian system whose Lagrangian is the sum of kinetic energy coming from a flat metric and the potential energy of a special form.

We consider the Benenti system constructed by (g, L_{comp}) from (8), so the function $f(t) = 1$, and by the function[15]

(39)
$$U(t) = \frac{1}{m_0}\left(t^{2N+1} + \sum_{i=2}^{N+1} c_i t^{2N-i+1}\right) = \frac{1}{m_0}\left(C(t) - c_{N+2}t^{N-1} - c_{N+3}t^{N-2} - \cdots - c_{2N+1}\right).$$

Theorem 2.4 ([14])**.** *For every parameters $c_2, \dots, c_{N+1}$, the trajectories of this Benenti systems viewed as functions $x \mapsto w(x)$ on $\mathbb{R}^N$ are solutions of (38) corresponding to the polynomial C. Moreover, any solution of (38) is a trajectory of this Benenti system.*

Let us give additional explanations on the relations between parameters $c_2, \dots, c_{N+1}$ of the potential energy in the Benenti system and the parameters $c_2, \dots, c_{2N+1}$ of (38). The parameters of the Benenti system are the first N parameters of (38). The additional N parameters $c_{N+2}, \dots, c_{2N+1}$ can be thought as the values of the integrals $\tilde I_0, ..., \tilde I_{N-1}$ corresponding to the Benenti system, so a solution of (38) corresponding to the parameters $c_2, \dots, c_{2N+1}$ corresponds to a trajectory of the Benenti system corresponding to the parameters $c_2, \dots, c_{N+1}$ such that the values of the integrals $\tilde I_0, \dots, \tilde I_{N-1}$ on this trajectory are $\tilde I_0 = -c_{N+2}, \tilde I_1 = -c_{N+1}, \dots, \tilde I_{N-1} = -c_{2N+1}$.

Combining Theorems 2.3 and 2.4, we see that, for a fixed t, all stationary solutions of KdV, viewed as the curves $x \mapsto u(x)$, can be described as follows: we take the Benenti system above, and for any its solution $(w_1(x), ..., w_N(x))$ set $u(x) = 2w_1(x)$. The curves $u(x)$ constructed by this method are stationary solutions of KdV and for any stationary solution $u(x,t)$ of KdV the curve $x \mapsto u(x,t)$ can be obtained by this procedure.

Remark 2.5. *Though the unknown function $u(x,t)$ in the KdV equation depends on two variables, x and t, the dependence on the variable t was not used in Theorems 2.3, 2.4 and plays no role in our paper. The dependence on t, for general BKM systems, was understood as a part of the finite-dimensional-reduction approach of [14], and corresponds to the flow of one of the integrals. The KdV case was understood ways before.*

Though we do not use the dependence of t at all in the present paper, let us note that for a solution $u(x,t)$ the curves $x \mapsto u(x,t_1)$ and $x \mapsto u(x,t_2)$ will correspond to the same spectrum of the corresponding Schrödinger operator.

3. When solutions of one Benenti system are solutions of another

Lemma 3.1. *Consider two families of Benenti systems sharing the same L: one constructed by $U_1(t)$ and $f_1(t)$, another by U_2 and f_2. Suppose for certain values of the integrals $\tilde I^1_0 = H^1_0, ..., \tilde I^1_{N-1} = H^1_{N-1}$ of the first system and for certain values of the integrals $\tilde I^2_0 = H^2_0, ..., \tilde I^2_{N-1} = H^2_{N-1}$ of the second system the products we have*

(40)
$$(-U_1(t)+H^1_0 t^{N-1}+\cdots+H^1_{N-2}t+H^1_{N-1})f_1(t) = (-U_2(t)+H^2_0 t^{N-1}+\cdots+H^2_{N-2}t+H^2_{N-1})f_2(t).$$

Then, every solution of the first Benenti system, viewed as a curve on $\mathbb{R}^N$, corresponding to the values of the integrals $H^1_0, \dots, H^1_{N-1}$, is a solution of the second Benenti system corresponding to the values of the integrals $H^2_0, \dots, H^2_{N-1}$.

[15]The function $U(t)$ depends on the choice of parameters $c_2, \dots, c_{N+1}$

Proof. Without loss of generality we may assume that (g, L) are given by (4, 5). First we view solutions of both system as curves $(q(\tau), p(\tau))$ on $\mathbb{R}^{2N} = T^*\mathbb{R}^N$. In view of (14), we have for any $i = 1, \ldots, N$

$$\tfrac{1}{2} f(q_i) p_i^2 = -U(q_i) + H_0 q_i^{N-1} + \cdots + H_{N-2} q_i + H_{N-1}. \tag{41}$$

Applying Legendre transformation, we see that (41) is equivalent to the system of ODEs

$$\frac{1}{2f(q_i)} \left(\frac{\dot q_i}{\prod_{s=1; s\neq i}^{N} (q_i - q_s)} \right)^2 = -U(q_i) + H_0 q_i^{N-1} + \cdots + H_{N-2} q_i + H_{N-1}. \tag{42}$$

We see that the condition (40) implies that the solutions of both systems viewed now as curves on $\mathbb{R}^N(q)$ satisfy the same system of ODEs and therefore every solution of one system is a solution of the other. □

Let us now compare, with the help of Lemma 3.1 and Remark 2.2, the systems in question: the Neumann system, the geodesic flow of the metric geodesically equivalent to the metric of ellipsoid, and the Benenti system coming from the finite-dimensional reduction of the KdV system via Theorem 2.3.

For all of them, the product $f(t)(-U(t) + H_0 t^{N-1} + \cdots + H_{N-1})$ is a polynomial of degree $2N + 1$ with positive leading coefficient. Clearly, the value of the leading coefficient is not important, once it is positive, as one can scale it by a positive constant by rescaling the time. The polynomial $f(t)(-U(t) + H_0 t^{N-1} + \cdots + H_{N-1})$ has certain restrictions which we will discuss now.

The restriction in the KdV systems is that the second highest coefficient of $f(t)(-U(t) + H_0 t^{N-1} + \cdots + H_{N-1})$ is zero. This restriction does not affect anything as the transformation $L_{new} = L_{new} + \mathrm{const}\mathrm{Id}$ can make the second highest coefficient arbitrary.

Next, in the Benenti systems describing the Neumann system and the geodesic flow of the metric geodesically equivalent to the ellipsoid, the product $f(t)(-U(t) + H_0 t^{N-1} + \cdots + H_{N-1})$ has at least $N + 1$ real different zeros $a_1, \ldots, a_{N+1}$. This restriction is essential.

Note also that not all values of the integrals allow solutions. Indeed, the system of ODEs (42) has no real solution if $f_\alpha(t)(-U(t) + H_0 q t^{N-1} + \cdots + H_{N-1}) < 0$.

As it is clear from the introduction, for our paper the relation of the stationary solutions of KdV and the solutions of the Neumann systems is most important. Lemma 3.1 implies that stationary solutions of KdV, viewed as curves $x \mapsto u(x)$, correspond to the evolution of $2\mathrm{trace}(L)$ along the solution of the Neumann system. By Remark 2.3, up to addition of a constant, $-\mathrm{trace}(L)$ is the potential energy of the Neumann system. This relates the potential energy of the Neumann system evaluated along a solution of the Neumann system to stationary solutions of KdV. The relation is of course known [42, 43, 50].

Concerning the relation of the geodesics on ellipsoid to stationary solutions of the KdV system, Lemma 3.1 and the discussion above established such a relation between the metric geodesically equivalent to the metric of ellipsoid and the KdV system. The solutions of the Hamiltonian systems corresponding to the geodesic flow of the ellipsoid and to the geodesic flow of the metric geodesically equivalent to the metric of the ellipsoid give the same curves on the manifold, but differently parameterised. And indeed, in [42, §3.5] the reparameterisation is given.

Remark 3.1. *Lemma 3.1 and discussion above links the solution of Neumann system, geodesics of the metric geodesically equivalent to the metric of the ellipsoid and the finite-dimensional reductions of the KdV system. The arguments used in the discussion are essentially local calculations in a special coordinate chart, which, e.g. in the case of Neumann system, covers almost whole but not the whole sphere. A natural question is therefore whether the links between systems survives when the solution of the Neumann system leaves the coordinate chart, or possibly never passes through the coordinate chart.*

The answer is "yes". Indeed, all considered systems are real-analytic, so if two solutions coincide locally they coincide globally. Moreover, the limit of a sequence of solutions is a solution, so even if the solution never passes through the special coordinate chart, the link remains.

4. Stationary solutions corresponding to the Neumann system and finite-bandness of the corresponding Sturm-Liouville-Hill problem

4.1. **The property of the polynomial C(t) for solutions of KdV coming from the solution of Neumann system.** By Lemma 3.1, see also discussion short after, the solutions of the Neumann system, viewed as curves on the N-dimensional sphere, are closely related to finite-dimensional reductions of the KdV system corresponding to the polynomial

$$C(t) = \tfrac{1}{2}(t^N + H_0 t^{N-1} + \cdots + H_{N-2}t + H_{N-1}) \prod_{s=1}^{N+1}(t - a_i). \tag{43}$$

From the formula (43) we immediately see that the polynomial $C(t)$ has $N+1$ real roots $a_1, \dots, a_{N+1}$. The condition that the sphero-ellipsoidal coordinates q_i used for the description of the Neumann system are necessarily real-valued and satisfy, see Remark 2.4,

$$a_1 < q_1 < a_2 < q_2 < \cdots < q_N < a_{N+1}, \tag{44}$$

give further assumptions on the roots of $C(t)$, which we discuss now.

Lemma 4.1. *Consider a solution of the Neumann system and the corresponding polynomial $C(t)$ given by (43), where $H_0, H_1, \dots, H_{N-1}$ are the values of the integrals $\tilde I_0, \dots, \tilde I_{N-1}$ corresponding to this solution.*

Then, all roots of polynomial $C(t)$ are real numbers. Moreover, if we denote roots, counted with their multiplicities, by $r_1 \le r_2 \le \cdots \le r_{2N+1}$, and by $q_1 \le \cdots \le q_N$ the eigenvalues of the (1,1)-tensor L, again counted with their multiplicities, then

$$r_1 \le r_2 \le q_1 \le r_3 \le r_4 \le q_2 \le r_5 \le r_6 \le q_3 \le \cdots \le r_{2N-2} \le r_{2N-1} \le q_{2N} \le r_{2N+1}, \tag{45}$$

that is, q_i lies on the interval $[r_{2i}, r_{2i+1}]$).

We recall that in the sphero-ellipsoidal coordinate system $q_1, \dots, q_N$ the operator L has the form $\mathrm{diag}(q_1, \dots, q_N)$, so the notation q_i used for eigenvalues in Lemma 4.1 is compatible with that in §2.2. Note though that for the Neumann system the operator L

is defined also at the points where the sphero-ellisoidal coordinate system is not defined. These points are precisely the points where L has multiple eigenvalues.

Proof. As the property of a polynomial to have a complex root is an open property, we may work at a point $(q_1, \ldots, q_N)$ of our sphero-ellipsoidal coordinate system and assume that all component of the velocity vector $(\dot{q}_1, \ldots, \dot{q}_N)$ are different from 0. From (43), we see that the leading coefficient of the polynomial is positive so for $t << -1$ we have $C(t) < 0$. Next, from (42) we see that the value of $C(t)$ are negative at each $t = q_i$. Since the polynomial $\prod_{s=1}^{N+1}(t - a_i)$ has different signs at $t = x_i$ and at $t = q_{i+1}$, the polynomial $(t^N + H_0 t^{N-1} + \cdots + H_{N-2}t + H_{N-1})$ should also have different signs at q_i and at q_{i+1} implying the existence, in view of (44), of $N-1$ real roots on the interval $[a_1, a_{N+1}]$. Thus, $2N$ roots, counted with multiplicities, of our polynomial C of degree $2N+1$, are real, so the remaining root is real as well. □

By Theorems 2.3, 2.4, solutions of the KdV system coming from finite-dimensional reduction correspond to the solutions of the Benenti system with $f(t) = 1$ and $U(t) = -(t^{2N+1} + c_2 t^{2N-1} + \cdots + c_{2N+1})$. By Lemmas 3.1, every solution of the Neumann system corresponds to a solution of the Benenti system above. Lemma 4.1 tells which solutions of the Benenti system above correspond to the solutions of some Neumann system: necessary conditions are that all roots of the polynomial $C(t)$ are real and that the eigenvalues of the corresponding L are real and satisfy (45). The next Lemma shows, in particular, that these are also sufficient conditions. It also implies that if all eigenvalues of L are real and satisfy (45) at one point of the solution of the Benenti system above, then it is true at every point.

Lemma 4.2. *Suppose $(w_1(x), ..., w_n(x))$ is a solution of (38) such that the polynomial $C(t)$ has only real roots which we list with their multiplicities and denote by $r_1 \leq \cdots \leq r_{2N+1}$. Denote by $q_1(x), \ldots, q_N(x)$ the zeros of the polynomial (37), listed with their multiplicities. Assume that $(w_1(0), \cdots, w_N(0))$ are such that all $q_i(0)$ are real valued and satisfy $r_{2i} \leq q_i(0) \leq r_{2i+1}$. Then, the solution $(w_1(x), ..., w_n(x))$, with the initial values $(w_1(0), \cdots, w_N(0))$, can be extended for all $x \in \mathbb{R}$.*

Moreover, suppose for a certain i and for certain $\tilde{x}$ we have $r_{2i} < q_i(\tilde{x}) < r_{2i+1}$ and denote by I_i the connected component of the set $\{x \in \mathbb{R} \mid r_{2i} < q_i(x) < r_{2i+1}\}$ containing 0.

Then the function $q_i(x)$ smoothly depends on x for $x \in I_i$, moreover $\frac{d}{dx}q_i \neq 0$ for all $x \in I_i$ and satisfies (42).

Proof. First assume that for each i we have $r_{2i} < q_i(0) < r_{2i+1}$. Then, we take the Neumann system such that $a_i = r_{2i}$ for $i < N+1$ and $a_N = r_{2N+1}$. Next, take a point on S^N such that sphero-ellipsoidal coordinates of this points are $(q_1(0), \ldots, q_N(0))$. The corresponding function $f(t)$ in the equation (42) is then equal to $-4\prod_{s=1}^{N+1}(q_i - a_s)$, see (25). Next, take $H_0, \ldots, H_{N-1}$ such that the right hand side of the equation (42) has roots $r_1, r_3, \ldots, r_{2i-1}, \ldots, r_{2N-1}$. This is clearly possible since for the Neumann system $U(t)$ is given by (27), so the right hand side of (42) is polynomial of degree N in which we freely choose all nonleading coefficients. Then, at out point $(q_1(0), \ldots, q_N(0))$ of the sphere the products $f(q_i(0))U(q_i(0))$ are positive and we can choose $\left(\frac{\mathrm{d}}{\mathrm{d}x}q_1(0), \ldots, \frac{d}{dt}q_N(0)\right)$ such that the equation (42) is satisfied. The solution of the Neumann system with this initial data will correspond to the solution of (38) with the initial values $(w_1(0), \cdots, w_N(0))$. Since the solution of Neumann system is defined on a closed manifold and therefore can be extended for the whole $\mathbb{R}$, the solution of (38) can be extended to the whole $\mathbb{R}$ as well. At the points of the interval I_i, the function $C(q_i(x))$ is not zero, so (38) implies that $\frac{d}{dt}q_i(x) \neq 0$. Lemma 4.2 is proved under the assumption that all roots of the polynomial C are different.

The remaining case when for a certain i we have $r_{2i} = q_i(0)$ or $r_{2i+1} = q_i(0)$ can be proved by passage to the limit. We take the sequence of the polynomials $C_n(t)$ satisfying the standard assumptions such that it converge to our polynomial $C(t)$, and the sequence of initial data satisfying assumptions used in the proof above such that it converges to our initial data. The limit of the corresponding solutions of the Neumann system is a solution of the Neumann system. As it lives on a compact manifold, it exists for all $x \in \mathbb{R}$.

Finally, for $x \in I_i$, $q_i(x)$ is an isolated root so it depends smoothly on x. Since the equation (42) are fulfilled for the sequence of the solutions considered above, it is fulfilled for the limit solution. But then (42) implies $\frac{d}{dx}q_i \neq 0$. □

Corollary 4.2.1. *Any solution of the KdV system corresponding to a solution of the Neumann system is bounded.*

Proof. Indeed, the solution is (up to a constant) the trace of L, which is the sum of x_i, which is clearly bounded from above by Na_N and from below by Na_1. □

An alternative proof: up to a constant, the solution is essentially the potential energy of the Neumann system evaluated along a solution of the Neumann system. The potential energy is clearly bounded as it is a continuous function on the sphere S^N which is compact. □

4.2. **Proof that the Sturm-Liouville-Hill problem corresponding to the Neumann system has finite-band spectrum.** In the previous section we discussed the conditions on the polynomial $C(t)$ for the stationary solutions of the KdV equations coming from the Neumann system, in particular, they should have only real roots. The goal of this section is to prove the following Theorem:

Theorem 4.3. *Denote by $r_1 \leq r_2 \leq \cdots \leq r_{2N+1}$ the roots of the polynomial C, counted with their multiplicities.*

Then, $\frac{1}{2}\lambda$ lies in the spectrum of $-\frac{\partial}{\partial x^2} + \frac{1}{2}u$ if and only if

$$\lambda \in [r_1, r_2] \cup [r_3, r_4] \cup \cdots \cup [r_{2N-1}, r_{2N}] \cup [r_{2N+1}, +\infty). \tag{46}$$

We see that the spectrum consists of finitely many closed intervals and one half-line $[2r_{2N+1}, +\infty)$. In jargon, the intervals are called *bands* and the half-line $[r_{2N+1}, +\infty)$ is called *the infinite band*. Note that the number of (finite) bands is at most N. Since r_i may be equal to r_{i+1}, the number of bands can be smaller than N and some of the bands could be actually points.

Let us also observe that the set (46) coincides with

$$\{\lambda \in \mathbb{R} \mid C(\lambda) \geq 0\}. \tag{47}$$

Lemma 4.4 and the discussion afterwards relates the KdV equation, via (38), to the spectral problem for Schrödinger-Hill equation.

Lemma 4.4. *Let $w(x), \sigma(x)$ be smooth functions defined on an interval such that $w > 0$ and $m \neq 0$, C be constants. Consider the primitive function $\phi(x) = \int \frac{\mathrm{d}s}{2w(s)}$ and the functions*

$$\psi_1(x) = \sqrt{w(x)}\, e^{\sqrt{-\frac{2C}{m}}\phi(x)} \quad \text{and} \quad \mathrm{sign}(w(x))\,\psi_2(x) = \sqrt{w(x)}\, e^{-\sqrt{-\frac{2C}{m}}\phi(x)}. \tag{48}$$

Then, equation

$$\psi_{i,xx} + \frac{1}{2}\frac{\sigma(x)}{m}\psi_i = 0, \quad i = 1, 2 \tag{49}$$

holds if ans only if equation

$$w_{xx}(x)w(x) - \frac{1}{2}w_x^2(x) + \frac{\sigma(x)w^2(x) - C}{m} = 0. \tag{50}$$

holds.

Remark 4.1.

(1) The function $\phi(x)$ is defined up to an addition of a constant, which corresponds to multiplication of the function ψ_1 by a constant and division of ψ_2 by the same constant.
(2) The change of the sign of the function w does not affect the equation (50). If $w < 0$ on the interval we work, the functions

$$\sqrt{-w(x)}\, e^{-\sqrt{-\frac{2C}{m}}\phi(x)} \quad \text{and} \quad -\sqrt{-w(x)}\, e^{\sqrt{-\frac{2C}{m}}\phi(x)}$$

are solution of (49).
(3) The functions ψ_1, ψ_2 clearly have the property $\psi_1\psi_2 = w$. One easily shows that, within the 2-dimensional solution space of (49), the property $\psi_1\psi_2 = w$ defines (up to swapping, multiplication one by a constant and division of the second by the same constant) two solutions of (49) which form a basis in the space of all solutions.

Proof of Lemma 4.4. Note that the equations (49) and (50) are both second order ODEs so substituting ψ_i given by (48) in (49) will give a second order differential equation on w. We will show that the formula for ψ_i is chosen such that the obtained equation is equivalent to (50).

Let us do the corresponding calculations. To simplify the formulas, we assume $w > 0$, denote $k = \sqrt{-2\frac{C}{m}}$ and consider $\psi := \psi_1$. We get

$$\psi_x = \frac{1}{2}\frac{w_x}{\sqrt{w}}e^{k\phi} + \frac{k}{2\sqrt{w}}e^{k\phi},$$

$$\psi_{xx} = \frac{1}{2}\frac{w_{xx}}{\sqrt{w}}e^{k\phi} - \frac{1}{4}\frac{w_x^2}{w^{3/2}}e^{k\phi} + \frac{k}{2}\frac{w_x}{w^{3/2}}e^{k\phi} - \frac{k}{2}\frac{w_x}{w^{3/2}}e^{k\phi} + \frac{k^2}{4w^{3/2}}e^{k\phi} - \frac{1}{2}\sigma(\lambda)\sqrt{w}e^{k\phi} =$$

$$= \frac{1}{2}\left(\frac{w_{xx}}{w} - \frac{1}{2}\frac{w_x^2}{w^2} - \frac{C}{mw^2}\right)\sqrt{w}e^{k\phi} = -\frac{1}{2}\frac{\sigma}{m}\psi_1.$$

Thus, the equation (49) holds for the function ψ_1. The proof for ψ_2 is analogous.

Let us now prove the Lemma in the other direction. Assume that (49) holds. The same computation give us

$$0 = \left(\frac{w_{xx}}{w} - \frac{1}{2}\frac{w_x^2}{w^2} - \frac{C}{mw^2} + \frac{\sigma}{m} \right) \psi_1.$$

As w is not zero by assumptions, (48) implies that ψ_1 is not zero so (50) holds. □

The equation (49) is just the Sturm-Liouville equation corresponding to the potential $\frac{1}{2}\frac{\sigma(x)}{m}$; its operator is

$$\mathscr{H} = \tfrac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\sigma(x)}{m}\,\mathrm{Id}\,.$$

In this notation, the equation (49) reads

$$\mathscr{H}\psi = 0. \tag{51}$$

Next, note that (50) is just the equation (38) with $m = m_0$ and $\sigma(x, \lambda) = \lambda - \frac{1}{2}w_1$. We assume $m_0 = 1$ for simplicity, see the discussion around (28). In this case, (50) has the form

$$w_{xx}(x, \lambda)w(x, \lambda) - \frac{1}{2}w_x^2(x, \lambda) + \underbrace{\sigma(x, \lambda)}_{\lambda - \frac{1}{2}w_1} w^2(x, \lambda) = C(\lambda). \tag{52}$$

In view of Theorem 2.3, the corresponding Sturm-Liouville equation from Lemma 4.4 reads then

$$\psi_{xx} + \tfrac{1}{2}\left(\lambda - u\right)\psi = 0. \tag{53}$$

It is the so-called Hill-Schrödinger equation, i.e., the equation on the eigenfunctions with eigenvalue $\frac{1}{2}\lambda$ for the operator

$$\left(-\tfrac{\partial}{\partial x^2} + \tfrac{1}{2}u\right)\psi = \tfrac{1}{2}\lambda\psi. \tag{54}$$

Note that (54) is an ordinary linear differential equation of the second order, so it has precisely two linearly independent solutions for each λ, on any connected open interval where u is defined. We say that $\frac{1}{2}\lambda \in \mathbb{R}$ lies in *spectrum*, if u is defined on the whole $\mathbb{R}$ and there exists a <u>bounded</u> non-zero solution ψ of (54).

Proof of Theorem 4.3. First, let us show, that the bands, indeed, lie in the spectrum. First, observe that the roots $r_1, ..., r_{2N+1}$ lie in the spectrum. Under the assumption $\lambda = r_i$, both solutions from (48) coincide and give us a solution

$$\psi(x, \lambda) = \sqrt{|w(x, \lambda)|} \tag{55}$$

which is bounded in view of Corollary 4.2.1.

Next, consider λ from the open interval (r_{2i-1}, r_{2i}). For each $\lambda \in (r_{2i-1}, r_{2i})$ we have $C(\lambda) > 0$, as no root of the characteristic polynomial of L lies in such an interval. Therefore, $w(x, \lambda)$ does not change the sign and we may assume $w(\lambda) > 0$. Since $C(\lambda)$ is positive, $\sqrt{-2C(\lambda)}$ is purely imaginary so $e^{\sqrt{-2C(\lambda)}\phi(x)}$ is bounded. Then, the solution $\psi_1 = \sqrt{w(x)}e^{\sqrt{-2C(\lambda)}\phi(x)}$ from (48) is bounded as well. Thus, $\frac{1}{2}\lambda$ lies in the spectrum.

The infinite band can be handled using a similar argument: Take $\lambda \in [r_{2N+1}, +\infty)$. We again have that λ cannot be a root of the characteristic polynomial of L. Therefore, $w(x;\lambda)$ does not change the sign and we may think that it is positive. Since $C(\lambda)$ is positive, $\sqrt{-2C(\lambda)}$ is purely imaginary so $e^{\sqrt{-2C(\lambda)}\phi(x)}$ is bounded. Finally, the solution $\psi_1 = \sqrt{|w(x)|}e^{\sqrt{-2C(\lambda)}\phi(x)}$ from (48) is bounded as well. Thus, $\frac{1}{2}\lambda$ lies in the spectrum.

Next, let us show that for $\lambda \in \mathbb{R}$ such that $C(\lambda) < 0$ the ODE (54) does not have bounded solutions. We start with the case when $\lambda < r_1$. In this case, $w(x)$ is separated from zero, we may assume that it is positive. The function $\phi(x) = \int \frac{\mathrm{d}s}{2w(s)}$ is therefore unboundend. Since $\sqrt{-2C(\lambda)}$ is a real positive number, the function ψ_1 from (48) is unbounded for $x \to \infty$ and goes to zero for $x \to -\infty$. The function ψ_1 from (48) is unbounded for $x \to -\infty$ and goes to zero for $x \to \infty$. Therefore, no nontrivial linear combination of ψ_1 and ψ_2 is bounded.

Finally, let us consider the most complicated case $\lambda \in (r_{2i}, r_{2i+1})$. Without loss of generality, we may assume that $i = 1$ and $\lambda = 0$, so $r_2 < 0$ and $r_3 > 0$. As above, we denote by $q_1, \dots, q_N$ the eigenvalues of L and consider their dependence on the "time" x, that is, the functions $q_i(x)$. We reserve the notation $\tilde{x}$ for values such that $q_1(\tilde{x}) = \lambda = 0$. Note also that the assumption $\lambda = 0$ implies $w(x) = q_1(x)q_2(x)\cdots q_N(x)$ with all $q_i \geq r_3 > 0$ for $i \geq 2$. The functions q_i are well-defined at least continuous functions. We know from Lemma 4.2 that the derivative $\frac{\mathrm{d}q_1}{\mathrm{d}x}$ is not zero at $\tilde{x}$ such that $q_1(\tilde{x}) = 0$. Actually, by (42), it is given at any point x by

$$\frac{\mathrm{d}q_1(x)}{\mathrm{d}x} = \pm\frac{\sqrt{-2C(0)}}{(q_2(x) - q_1(x))\,(q_3(x) - q_1(x))\cdots(q_N(x) - q_1(x))}, \tag{56}$$

so at $x = \tilde{x}$ it is given by

$$\frac{\mathrm{d}q_1(\tilde{x})}{\mathrm{d}x} = \pm\frac{\sqrt{-2C(0)}}{q_2(\tilde{x})q_3(\tilde{x})\cdots q_N(\tilde{x})}. \tag{57}$$

This in particular implies that, at the point $\tilde{x}$, the derivative of w is given by

$$\frac{\mathrm{d}w(\tilde{x})}{\mathrm{d}x} = \frac{\mathrm{d}q_1(\tilde{x})q_2(\tilde{x})\cdots q_N(\tilde{x})}{\mathrm{d}x} = q_2(\tilde{x})\cdots q_N(\tilde{x})\frac{\mathrm{d}q_1(\tilde{x})}{\mathrm{d}x} = \pm\sqrt{-2C(0)}. \tag{58}$$

We see that as $C(0) \neq 0$, the derivative of q_1 and of w is not zero at $\tilde{x}$. The sign $\pm$ depends on whether the functions q_1, w are locally increasing or decreasing at a small neighborhood of $\tilde{x}$. Note also that the derivative of w at the points such that $q_1 = 0$ has the same absolute value and its sign depends on whether the sign of w changes from "+" to "−" or from "−" to "+" when x passes this point.

This in particular implies that the points $\tilde{x}$ such that $q_1(\tilde{x}) = 0$ exist, as the derivative of the function $q_1(x)$ is nonzero for x such that $r_1 < q_1(x) < r_2\,,\ q_1(x) \neq 0$ by (42).

Lemma 4.4 gives us two eigenfunctions ψ_1 and ψ_2 of the equation (54) at the interval where w does not change sign. Note that as ψ_1, ψ_2 are solutions of a linear ODE with smooth coefficients, they can be extended to solutions defined and smooth on the whole $\mathbb{R}$. The solutions ψ_1, ψ_2 are linearly independend and therefore form a basis in the two-dimensional linear space of the solutions of (54).

Let us understand how the solutions ψ_1, ψ_2 behave near zeros of w. At a zero $\tilde{x}$ of w, using (57), we have that the primitive function $\phi(x) = \int^x \sqrt{-2C(0)}\frac{1}{2w(s)}ds$ behaves asymptotically, for $x \to \tilde{x}$, as const $\pm \frac{1}{2}\ln|w(x)|$. This implies that one of the functions $\psi_1(x)$ and $\psi_2(x)$ behaves asymptotically, for $x \to \tilde{x}$, as

$$\text{const} \cdot w \quad \text{and another as} \qquad 1/\text{const}$$

(for a certain nonzero constant const which of course may depend on the choice of the point $\tilde{x}$ at which $q_1 = 0$).

Next, observe that as objects we considered were real-analytic, we can think that our parameter x is complex-valued, $x \in \mathbb{C}$. The solutions $w(x)$ and $q_1(x)$ are then holomorphic functions of $x \in \mathbb{C}$. Since the derivatives of q_1 and of w are not zero at the points $\tilde{x}$ such that $q_1 = 0$, such points are isolated in a small neighborhood of the real line. We denote this neighborhood by W.

We denote by $\dots < \tilde{x}_{-1} < \tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \dots$ the points of $\mathbb{R}$ such that $q_1(\tilde{x}_i) = 0$, the sequence of such $\tilde{x}_i$ could be unbounded in both directions, in one direction, or can simply be finite. Assume without loss of generality that $w > 0$ on the interval $(\tilde{x}_0, \tilde{x}_1) \subset \mathbb{R}$. Then it changes sign from "+" to "−" at the points x_0, x_1, so it is negative on the intervals (x_{-1}, x_0) and (x_1, x_2), positive on the intervals $(\tilde{x}_2, \tilde{x}_3)$, $(\tilde{x}_{-2}, \tilde{x}_{-1})$, negative on the intervals $(\tilde{x}_{-3}, \tilde{x}_{-2})$, $(\tilde{x}_3, \tilde{x}_4)$ and so on. The intervals such that w is positive will be called *positive intervals,* and such that w is negative will be called *negative intervals.*

Next, denote by $\widetilde{W}$ the neighborhood W without negative intervals. As at the endpoints of the negative intervals the the derivatives of w at $\tilde{x}_i$ and $\tilde{x}_{i+1}$ coincide in absolute values but have different signs by (58), the corresponding residues cancels. Then, the primitive function $\phi = \int \frac{1}{2w(s)}\mathrm{d}s$ from Lemma 4.4 is well-defined on the whole $\widetilde{W}$. This implies that the formula (48) globally defined the restriction of global solutions of (49) to $\widetilde{W}$.

Recall now that the product $\psi_1\psi_2$ equals ω, and at the points $x \in \mathbb{R}$ with $|q_1(r)| > \varepsilon > 0$ the function w is separated from zero. Our next goal is to show now that for $x \in \mathbb{R}_{>0}$ the function ψ_1 achieves arbitrary big values. Recall that the equation (49) is a linear second order ODE and its solution space is 2-dimensional, so any solution ψ on $\widetilde{W}$ is given by

$$\text{const}_1\psi_1 + \text{const}_2\psi_2. \tag{59}$$

At the points where ψ_1 is big in its absolute value, the function ψ_2 is small so a bounded linear combination $\text{const}_1\psi_1 + \text{const}_2\psi_2$ for unbounded ψ_1 implies $\text{const}_1 = 0$.

In order to do it, let us decompose the function $w(x)$ in the product $q_1(x)\bar{w}(x)$, where $\bar{w}(x) = q_2 q_3 \cdots q_N$. In our setup, $\bar{w}(x)$ is positive for all x. Consider a pair $(a(x), b)$, where a is a function and b is a sufficiently small positive constant such that

$$a(x)\bar{w}(x) + bq_1(x) = 1. \tag{60}$$

The existence of such a pair $(a(x), b)$ is clear. Indeed, if we take $b > 0$ sufficiently small, then $1 - bq_1(x) > \varepsilon > 0$ for all x. Then the function $a(x)$ is given by

$$a(x) = \frac{1 - bq_1(x)}{\bar{w}(x)}.$$

If b is small enough, $a(x)$ is separated from zero.

The condition (60) implies that

$$\frac{1}{q_1(x)\bar{w}(x)} = \frac{a(x)}{q_1(x)} + \frac{b}{\bar{w}(x)}.$$

Combining this formula with (48), we obtain

$$\psi_1 = \left(\sqrt{q_1} e^{\sqrt{-2C(0)} \int \frac{a(s)}{2q_1(s)} \mathrm{d}s}\right) \cdot \left(\sqrt{\bar{w}} e^{\sqrt{-2C(0)} \int \frac{b}{2\bar{w}(s)} \mathrm{d}s}\right). \tag{61}$$

The second factor on the right hand side of (61) is unbounded for $x \to +\infty$ as the function $\frac{b}{2q_1(s)}$ is positive, $\sqrt{-2C(0)}$ is positive and $\bar{w}$ is separated from zero. Note that the second factor

$$\left(\sqrt{\bar{w}} e^{\sqrt{-2C(0)} \int \frac{b}{2\bar{w}(s)} \mathrm{d}s}\right)$$

can be extended to the whole real line $\mathbb{R}$, as the integrand $\frac{b}{2\bar{w}(s)}$ has no poles.

The function $q_1(s)$ in the first factor must, for certain arbitrary large s, be different from zero, as its derivative is different from zero by Lemma 4.2 at the points where $q_1 = 0$. Its derivative $\frac{\mathrm{d}}{\mathrm{d}x} q_1(x)$ satisfies (42) and therefore for every fixed $\tilde{q}_1 \in (r_1, r_2)$ the derivative $\frac{\mathrm{d}}{\mathrm{d}x} q_1(x)$ is bounded from zero on the set $\{s \in \mathbb{R} \mid q_1(s) = \tilde{q}_1\}$. Then, the first bracket (61) cannot be arbitrarily close to zero implying that (61) unbounded for $x \to +\infty$. As explained above, this means that const_1 in (59) is zero,

Similarly, the function ψ_2 is bounded for $x \to -\infty$ implying $\mathrm{const}_2 = 0$. Theorem is proved. □

5. Conclusion and outlook

The goal of this note was to understand the famous results of [45, 42, 43] using new approach and new methods coming from recent investigation [9, 14] of BKM systems. We have shown that it is possible, at least for a part of results, and the proofs in the present paper are shorter than that of [42, 43]. Of course the proofs use the preliminary work done in [9, 14]. But still, at least in our eyes, we replaced the "mathematics magic" by "mathematical methods"; in particular the wonderful observations of H. Knörrer [30] relating the geodesics of ellipsoid to the solutions of the Neumann systems, and of Veselov [50] relating finite-gap solutions of the KDV to the solutions of the Neumann system are direct corollaries of Lemma 3.1, which can be applied to a much wider class of systems.

But in fact the main motivation behind this investigation is as follows: the de-facto main approach of tackling KdV equations is the so-called inverse scattering method. Inverse scattering method is based on the understanding of the eigenfunctions of the Schrödinger operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(t, x)$, where $u(t, x)$ is essentially a solution of the KdV equation, and studying their evolution when we change t. It is clearly related to the problem we consider, and actually the results and methods of [45, 42, 43] came from this approach, see also [5, 33, 6] for recent developments. A natural question on which we will concentrate in our next investigation is how to adapt the inverse scattering method for studying general BKM systems, with the goal to generalise famous results on the KdV equation for all BKM systems.

Let us indicate first observation in this direction. First let us recall that in the finite-dimensional reduction of BKM systems studied in [14] the analog of (38) is

$$w_{xx}(x,\lambda)w(x,\lambda) - \frac{1}{2}w_x^2(x,\lambda) + \frac{\sigma(w,\lambda)w^2(x,\lambda) - C(\lambda)}{m(\lambda)} = 0. \tag{62}$$

Above $m(\lambda)$ is a polynomial of degree $\leq n$ with constant coefficients used for the construction of the BKM system, C is a polynomial of degree $2N+1$, $w = w(x)$ is the monic polynomial (37) of degree N. The function $\sigma(w,\lambda)$ is the monic polynomial of degree n in λ, whose coefficients are uniquely defined by the condition that $\frac{\sigma(w,\lambda)w^2(x,\lambda)-C(\lambda)}{m(\lambda)}$ is a monic polynomial of degree $2N-1$. They are functions of $w_1, w_2, \dots, w_n$.

The system (62) is polynomial in λ of degree $2N-1$, so it is equivalent to a system of $2N$ ODEs of the second order on N functions $w_1, \dots, w_N$. This system is equivalent to the system of Euler-Lagrange equation, whose Lagrangian is the sum of the kinetic energy coming from the flat metric (8) and a certain potential energy depending on parameters. As in the KdV case, the system is a Benenti system. The potential is slightly more complicated compared to that in the KdV case and the corresponding function U is a polynomial of a possibly higher degree, if $m(\lambda) = \text{const}$, or a rational function if $m(\lambda) \neq \text{const}$.

Having a solution of this system of ODEs, one constructs a solution of the corresponding BKM system as follows: recall that in order to choose a BKM system we should choose a differentially nondegenerate Nijenhuis operator $\tilde{L}$ on $\mathbb{R}^n$ with coordinates $u_1, \dots, u_n$. The solution $(u_1, \dots, u_n)$ corresponding to the BKM system is constructed by the solution w of the (62) by the formula

$$\sigma(w,\lambda) = \det(\lambda \mathrm{Id} - \tilde{L}). \tag{63}$$

For a given $\tilde{L}$, the right hand side of (63) is an explicit expression in u from which we may reconstruct u. In particular, if in the coordinates u the operator $\tilde{L}$ has the companion form (8) (with w replaced by u and N replaced by n), then $\det(\lambda \mathrm{Id} - \tilde{L}) = \lambda^n + u_1\lambda^{n-1} + \cdots + u_n$, so the formula (63) gives us an explicit formula for $u_1, \dots, u_n$,

Note that the equation (62) has the form (38). The dependence on λ is though different from that of in the KdV case. The analog of the equation (53) in this case is the equation

$$\psi_{xx} + \tfrac{1}{2}\frac{\det(\lambda \mathrm{Id} - \tilde{L}(u(x))}{m(\lambda)}\psi = 0. \tag{64}$$

We see that for $n \geq 2$ the dependence on the λ is not linear, differently from the Hill-Schrödinger equation. Indeed, the function $\frac{1}{2}\frac{\det(\lambda \mathrm{Id} - \tilde{L}(u(x))}{m(\lambda)}$ is a rational function in λ whose coefficients depend on x.

The special case $m(\lambda) = \text{const} = m_0$ of this equation appeared in the literature, see e.g. [35, 2], under the name "energy-dependent potentials", in the relation to multicomponent integrable systems. The scattering problem for such systems were studied quite recently, see e.g. [44, 28].

We say that λ lies in the spectrum of (64), if there exists a bounded nonzero solution ψ. For certain u coming from finite-dimensional reductions of the BKM systems, the spectrum is the finite-band one, in the sense it contains finitely many connected components. It will be interesting to understand whether in the class of quasi-periodic coefficients of the equation (64) the finite-bandness of the spectrum is equivalent to the property that the coefficients came from a BKM system. Moreover, as mentioned above, it will be generally interesting to develop the inverse scattering approach to BKM systems; the corresponding Sturm-Liouville equation is expected to be (64).

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Faculty of Mechanics and Mathematics, Moscow State University, 119992, Moscow, Russia

Email address: maodzund@yandex.ru

Institut für Mathematik, Friedrich Schiller Universität Jena, 07737 Jena, Germany

Email address: vladimir.matveev@uni-jena.de