From λ-connections to $PSL_2(\mathbb{C})$-opers with apparent singularities

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Abstract

On a Riemann surface of genus > 1, we discuss how to construct opers with apparent singularities from $SL_2(\mathbb{C})$ λ-connections (E, ∇_λ) and sub-line bundles L of E. This construction defines a rational map from a space which captures important data of triples (E, L, ∇_λ) to a space which parametrises the positions and residue parameters of the induced apparent singularities. We show that this is a Poisson map with respect to natural Poisson structures. The relations to wobbly bundles and Lagrangians in the moduli spaces of Higgs bundles and λ-connections are discussed.

Contents

1 Introduction

Let X be a compact connected Riemann surface of genus $g \geq 2$, and G a simple complex Lie group. The de Rham moduli space $\mathcal{M}_{dR}$ of irreducible holomorphic G-connections (E, ∇) on

X carries a hyperkähler structure, and in particular has an associated twistor space $\mathcal{M}_{dR} \times \mathbb{P}^1$. The fiber $\mathcal{M}_\lambda$ over a fixed $\lambda \in \mathbb{C}$ is the moduli space of irreducible λ-connections (E, ∇_λ). In particular, $\mathcal{M}_1$ can be identified with $\mathcal{M}_{dR}$ and $\mathcal{M}_0$ with the Hitchin moduli space $\mathcal{M}_H$ of G-Higgs bundles (E, ϕ) on X. The Hodge moduli space $\mathcal{M}_{Hod}$, which is the restriction of the twistor space to $\mathbb{C} \subset \mathbb{P}^1$, provides an interpolation between these moduli spaces and hence a framework for the heurestic that a holomorphic connection is a deformation of a Higgs bundle.

In this paper, for $G = SL_2(\mathbb{C})$, we will describe λ-connections (E, ∇_λ) and the associated moduli spaces using sub-line bundles L of the underlying rank-2 bundles E. Central to our consideration will be the moduli space $\mathcal{N}_d$ of pairs (E, L), where the rank-2 bundles E have trivial determinant and $\deg(L) = d$ is fixed. There is a natural forgetful map from $\mathcal{N}_d$ to the moduli space $\mathcal{N}$ of rank-2 bundles with trivial determinant. In the range $0 < -2d \leq g - 1$, the generic fibers of the forgetful map from $\mathcal{N}_d$ to the moduli space $\mathcal{N}$ of stable bundles are finite, making $\mathcal{N}_d$ a very useful auxiliary space to explicitly investigate $\mathcal{N}$ and more generally the Segre strata on $\mathcal{N}$.

We will consider bundles on $\mathcal{N}_d$ that capture important data of triples (E, L, ∇_λ) in this paper. For $\lambda = 0$, i.e. the case of Higgs bundles, it is the cotangent space $T^*\mathcal{N}_d$ that captures important data of triples (E, L, ϕ). For example, for $-2d = g-1$, away from the loci where E is unstable or has nilpotent Higgs fields with kernel L, we have $T^*\mathcal{N}_d$ as the moduli space of triples (E, L, ϕ) with E stable. In [7], the author with collaborator have constructed a rational map

$$\mathrm{SoV} : T^*\mathcal{N}_d \dashrightarrow (T^*X)^{[m]}, \qquad m = 2g-2-2d. \tag{1.1}$$

THEOREM 1.1. *[7]* SoV *is a dominant Poisson map with respect to the natural Poisson structure of cotangent bundles.*

The map SoV generalises Sklyanin's Separation of Variables approach for the Gaudin model, which can be regarded as a variant of the Hitchin moduli space – this explains our notation for the map. This map can also be regarded as the classical limit of Drinfeld's approach to the geometric Langlands correspondence [9, 12].

Main results The main goal of this paper is to construct the generalisation of SoV and Theorem 1.1 for λ-connections. We will consider an affine bundle $\mathcal{M}_\lambda^d$ on $\mathcal{N}_d$ modeled over $T^*\mathcal{N}_d$; it is the analogue of the restriction to $\mathcal{N}$ of the moduli space $\mathcal{M}_\lambda$, which is an affine bundle modeled over $T^*\mathcal{N}$. The main result of this paper is an affine analogue of Theorem 1.1.

THEOREM 1.2. *There exists a rational dominant Poisson map with respect to natural Poisson structures*

$$\mathrm{SoV}_\lambda : \mathcal{M}_\lambda^d \dashrightarrow \mathcal{M}_{\mathrm{op},\lambda}^m, \qquad m = 2g-2-2d, \tag{1.2}$$

where $\mathcal{M}_{\mathrm{op},\lambda}^m$ is the symmetric product of an affine bundle modeled over T^*X.

The construction of SoV_λ and the proof that it is a Poisson map are analogous to those of SoV. For the latter, the key ingredients are constructions of certain effective divisors on smooth spectral curves of Higgs bundles (E, ϕ) from triples (E, ϕ, L), which induce points in $T^*\mathcal{N}_d$. The authors called these effective divisors Baker-Akhiezer divisors in [7]. The analogues of these divisors are the positions and residue parameters of simple apparent singularities of *branched projective structures*, or equivalently *opers* with gauge group $PSL_2(\mathbb{C})$ on X. The map SoV_λ encodes the positions and residue parameters of apparent singularities of opers that are induced from triples (E, L, ∇_λ). In fact, for $-2d = g - 1$, an open dense subset of $\mathcal{M}_{\mathrm{op},\lambda}^m$ parametrises opers on X with $m = 3g - 3$ simple apparent singularities at which no quadratic differential vanishes. This explains the notation we have used for this affine bundle. The parametrisation is done through the positions and residue parameters of the apparent singularities. For the case $g = 0$ and meromorphic connections with regular singularities, Oblezin has studied a similar map [19].

We note that in this paper, the Poisson structures in the main theorem 1.2 are natural to the constructions of the domain and target spaces as affine bundles. One can also consider the symplectic structures defined by pulling-back the natural symplectic structure of the restriction of $\mathcal{M}_{dR}$ to $\mathcal{N}$ along $\mathcal{N}_d \dashrightarrow \mathcal{N}$ on one hand, and the natural symplectic structure on the monodromy representation variety $\mathrm{Hom}(\pi_1, PSL_2(\mathbb{C})) / \sim$ along the monodromy map on the other hand. It is natural to expect that the Poisson structures in our paper are indeed symplectic structures and coincide with those defined by pull-back. In the punctured Riemann surface setting, these identities were proved by Pinchbeck [21] and Iwasaki [17] respectively. We also note that Pinchbeck has discussed an idea similar to Theorem 1.2.

Structure of the paper We start by, in Section 2, discussing the characterisation of λ-connections using sub-line bundles. In particular, we construct the affine bundle $\mathcal{M}_\lambda^d$ by explicit affine coordinate transformations, and discuss how it is essentially a symplectic reduction of an affine bundle $\widetilde{\mathcal{M}_\lambda^d}$ modeled over $T^*\mathcal{M}_d$, where $\mathcal{M}_d$ is the moduli space of embeddings $(L \hookrightarrow E)$. This is the analogue of the fact that $T^*\mathcal{N}_d$ is essentially a symplectic reduction of $T^*\mathcal{M}_d$.

In Section 3, we discuss the notion of opers with apparent singularities and how the transformation rules of the associated residue parameters define an affine bundle modeled over T^*X. The m-fold symmetric product $\mathcal{M}_{\mathrm{op},\lambda}^m$ of this affine bundle parametrises opers with m simple apparent singularities up to a choice of quadratic differential vanishing at these singularities. This follows from the characterisation of opers in terms of the positions and residue parameters of apparent singularities.

Section 4 is for the construction of SoV_λ and the proof of Poisson property. We show how the projectivisation of the data (E, L, ϕ) defines an oper with apparent singularities. We recall the construction of Baker-Akhiezer (BA) divisors from triples (E, L, ϕ) from [7] to show that the induced apparent singularities and residue parameters are indeed analogues of BA divisors. The map SoV_λ which encode these data is hence a generalisation of SoV. The proof of Poisson property of SoV_λ is analogous to that of SoV.

In Section 5, we discuss the relation between wobbly bundles, i.e. stable bundles admitting nonzero nilpotent Higgs fields, and the loci in $\mathcal{M}_{\mathrm{op},\lambda}^{3g-3}$ that needs to be removed to have a regular moduli space of opers. We then discuss a Lagrangian subspace of $\mathcal{M}_H$ that is mapped via SoV to a Lagrangian in $(T^*X)^{[3g-3]}$ defined by fixing a divisor on X, and the analogue of this in $\mathcal{M}_\lambda$. These subspaces to some extent resemble the Lagrangian leaves in $\mathcal{M}_H$ and $\mathcal{M}_\lambda$ associated to $\mathbb{C}^*$-fixed point $(E, \phi) \in \mathcal{M}_H^{\mathbb{C}^*}$ for E unstable [24].

2 Rank-2 λ-connections and sub-line bundles

In this section, we discuss how $SL_2(\mathbb{C})$ λ-connections (E, ∇_λ) and the associated moduli spaces can be understood more explicitly if we consider them together with subbundles L of degree d of E. In particular, we construct a moduli space $\mathcal{M}_\lambda^d$ that captures important aspects of the triples (E, L, ∇_λ). To this end, we start by reviewing an explicit description of the moduli space $\mathcal{N}_d$ of pairs (E, L) and its cotangent space $T^*\mathcal{N}_d$, which is the analogue of $\mathcal{M}_\lambda^d$ for triples (E, L, ϕ).

2.1 Moduli space of pairs (E, L) and its cotangent spaces

We denote by $\mathscr{M}_d$ the moduli space of nowhere-vanishing morphisms $L \hookrightarrow E$ where L is a line bundle of degree d on X and E a rank-2 bundle of trivial determinant. Alternatively, a point in $\mathscr{M}_d$ is an equivalence class of extensions of the form

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \longrightarrow 0. \tag{2.1}$$

The forgetful map that picks out L makes $\mathscr{M}_d$ a vector bundle over the Picard component Pic^d parametrizing line bundles of degree d on X, with the fiber over $L \in \mathrm{Pic}^d$ the space $\mathrm{Ext}(L^{-1}, L) \simeq H^1(X, L^2)$ of extension classes of L^{-1} by L. We will abuse the notation by denoting by $(L \hookrightarrow E)$ for both the point in $H^1(X, L^2)$ and the point in $\mathscr{M}_d$ defined by an extension of the form (2.1). Later when we equip coordinates $\boldsymbol{x}$ for $H^1(X, L^2)$, we will also denote these points by $\boldsymbol{x}$.

The projectivisation of $\mathscr{M}_d$ is the moduli space $\mathscr{N}_d$ of pairs (E, L) where L is a subbundle of E. It is a projective fiber bundle over Pic^d with fiber $\mathbb{P}H^1(L^2)$ over L. The diagram

$$\begin{array}{ccc} \mathscr{M}_d & \xrightarrow{\ \mathrm{pr}\ } & \mathscr{N}_d \\ & {\scriptstyle I}\searrow \quad \swarrow{\scriptstyle i} & \\ & \mathscr{N} & \end{array} \tag{2.2}$$

is commutative, where pr is the projectivisation map that sends $(L \hookrightarrow E)$ to the class $[L \hookrightarrow E] \equiv (E, L)$, and I and i are forgetful maps that pick out only isomorphism classes of E.

Coordinates on $\mathscr{M}_d$ Given $(L \hookrightarrow E) \in \mathscr{M}_d$, one can introduce local coordinates on a neighborhood of this point by fixing some reference divisor on X and describe extension classes of line bundles explicitly. The general idea goes back to Tyurin [27], while the specific introduction of these coordinates used in this paper was discussed also in [21]. We refer to our paper [7] for a more complete discussion.

First, recall that there is an embedding of X in $\mathbb{P}H^1(L^2)$ via the linear system associated to KL^2. We then choose a divisor $\mathbf{p} = \sum_{r=1}^{N} p_r$ on X, where $N = h^1(L^2)$, such that spanning the embedding of p_r gives us $\mathbb{P}H^1(L^2)$. A generic divisor on X would satisfy this condition. Next, choose a divisor $\check{\mathbf{q}} = \sum_{i=1}^{g-d} \check{q}_i$ such that the divisor $\mathbf{q} = \sum_{i=1}^{g} q_i$ satisfying $L \simeq \mathscr{O}_X(\mathbf{q} - \check{\mathbf{q}})$ is not exceptional, and that $\mathbf{q} + \mathbf{p} + \check{\mathbf{q}}$ has no point with multiplicity > 1. A divisor $\mathbf{p} + \check{\mathbf{q}}$ satisfying these conditions will be referred to as a reference divisor for $(L \hookrightarrow E) \in \mathscr{M}_d$.

Let us now also choose local coordinates $\underline{z}_i$, $\check{\underline{z}}_j$ and w_r on neighborhoods of q_i, $\check{q}_j$ and p_r respectively. Let $z_i = \underline{z}_i(q_i)$ and assume that $\check{\underline{z}}_i(\check{q}_i) = 0 = w_r(p_r)$. Let $X_{\mathbf{q}}$ be the complement of $\mathrm{supp}(\mathbf{q} + \mathbf{p} + \check{\mathbf{q}})$. Then the extension class $(L \hookrightarrow E) \in \mathscr{M}_d$ can be represented by transition functions

$$\begin{pmatrix} \underline{z}_i - z_i & 0 \\ 0 & \left(\underline{z}_i - z_i\right)^{-1} \end{pmatrix}, \qquad \begin{pmatrix} \check{\underline{z}}_j^{-1} & 0 \\ 0 & \check{\underline{z}}_j \end{pmatrix}, \qquad \begin{pmatrix} 1 & x_r w_r^{-1} \\ 0 & 1 \end{pmatrix}, \tag{2.3}$$

of E when transiting from $X_{\mathbf{q}}$ to neighborhoods of points in $\mathbf{q} + \mathbf{p} + \check{\mathbf{q}}$. Here x_r are complex numbers and hence are coordinates on $H^1(L^2)$; we are able to represent $(L \hookrightarrow E)$ by such transition functions thanks to our assumption on $\mathbf{p}$. Thanks to our assumption on $\mathbf{q}$ and $\check{\mathbf{q}}$, varying $\mathbf{q}$ then defines a neighborhood of L in Pic^d. In conclusion, upon choosing local coordinates and reference divisor on X, we have local coordinates $\boldsymbol{z} = (z_1(q_1), \ldots, z_g(q_g))$ on Pic^d and $\boldsymbol{x} = (x_1, \ldots, x_N)$ on the fibers of $\mathscr{M}_d$ over Pic^d.

In fact, there is a more natural set of local coordinates on Pic^d defined via the Abel map. Let $(\omega_i)_{i=1}^g$ be a basis of holomorphic differentials w.r.t. a canonical basis of cycles on X. Then

$$\boldsymbol{z} \mapsto A(\boldsymbol{z}) = \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_g), \qquad \lambda_i(\mathbf{q}) := \sum_{j=1}^{g} \int_{x_0}^{q_j} \omega_i \tag{2.4}$$

defines a change of local coordinates on Pic^d. The differential $\omega_i(z_j(q_j))$ of this change of coordinates is invertible as $\mathbf{q}$ is not exceptional by our assumption.

Higgs fields in terms of meromorphic differentials The reference divisor $\mathbf{p} + \check{\mathbf{q}}$ allows one to represent Higgs fields in terms of Abelian differentials. Suppose in local frames adapted to the embedding $\boldsymbol{x} \equiv (L \hookrightarrow E)$ over $X_\mathbf{q}$ a Higgs field ϕ on E takes the form

$$\phi \mid_{X_\mathbf{q}} = \begin{pmatrix} \phi_0 & \phi_- \\ \phi_+ & \phi_+ \end{pmatrix}. \tag{2.5}$$

For a divisor D on X, let Ω_D be the space of meromorphic differentials with divisor bounded below by $-D$. The components ϕ_0 and $\phi_\pm$ are Abelian differentials on X with singular properties depending on the coordinates $\boldsymbol{x} = (x_1, \ldots, x_N)$ of $(L \hookrightarrow E) \in \mathscr{M}_d$.

PROPOSITION 2.1. *[7] With the setup as above, ϕ_0, $\phi_\pm$ satisfy*

- *(i) ϕ_+ is an element of $\Omega_{-2\mathbf{q}+2\check{\mathbf{q}}} \simeq H^0(X, KL^{-2})$;*
- *(ii) ϕ_0 is an element of $\Omega_\mathbf{p}$ with $\operatorname{Res}_{p_r} \phi_0 = -x_r \phi_+(p_r)$ at each p_r;*
- *(iii) $\phi_- = (-\det(\phi) - \phi_0^2)/\phi_+$ is an element of $\Omega_{2\mathbf{p}+2\mathbf{q}-\mathbf{r}}$, with the singular parts at each p_r fully determined in terms of $\boldsymbol{x}$, ϕ_0 and ϕ_+.*

The Abelian differential ϕ_+ can be identified with the composition

$$c_{L \hookrightarrow E}(\phi) \equiv c_\boldsymbol{x}(\phi) : L \hookrightarrow E \xrightarrow{\phi} E \otimes K \to L^{-1} \tag{2.6}$$

and defines an element of $H^0(X, KL^{-2})$, which by Serre duality is dual to the fiber $H^1(X, L^2)$. When the scaling of the embedding $L \hookrightarrow E$ is not important, we will simply write $c(E, L, \phi)$. The following condition follows immediately from the singular properties of ϕ_0.

COROLLARY 2.2. *[7] Given a fixed extension class $(L \hookrightarrow E) \equiv \boldsymbol{x} \in H^1(X, L^2)$ and a Higgs field ϕ on E, $c_\boldsymbol{x}(\phi)$ is contained in the hyperplane $\ker(\boldsymbol{x}) \subset H^0(X, KL^{-2})$, namely*

$$\langle c_\boldsymbol{x}(\phi), \boldsymbol{x} \rangle = 0.$$

The space of all elements $c_\boldsymbol{x}(\phi)$ is the image of the map to $H^0(X, KL^{-2})$ in the long exact sequence

$$0 \longrightarrow H^0(X, E^*LK) \longrightarrow H^0(X, \mathrm{End}_0(E) \otimes K) \xrightarrow{c_x} H^0(X, KL^{-2}) \longrightarrow H^1(X, E^*LK) \longrightarrow \ldots \tag{2.7}$$

which is induced by the embedding of the bundle E^*LK of L-invariant Higgs fields on E to the bundle of trace-free Higgs fields $\mathrm{End}_0(E) \otimes K$.

PROPOSITION 2.3. *If E is stable and there exists a unique embedding $L \hookrightarrow E$ up to scaling, then $\mathrm{im}(c_\boldsymbol{x}) = \ker(\boldsymbol{x})$. In particular, if L is a subbundle of maximal degree in E then $\mathrm{im}(c_\boldsymbol{x}) = \ker(\boldsymbol{x})$.*

Proof. The proof follows from Riemann-Roch computation and exactness of the sequence (2.7). Indeed, the dimension of $\operatorname{im}(c_{\boldsymbol{x}})$ is $h^0(X, \operatorname{End}_0(E) \otimes K) - h^0(X, E^* L K_X)$, which for E stable has codimension $h^0(X, L^{-1}E)$. The second statement follows since a maximal subbundle has a unique embedding up to scaling [18]. ☐

In the range $1 \le -2d = -2\deg(L) \le 2g-1$, for a generic L and extension $\boldsymbol{x} \in H^1(X, L^2)$, we have $h^0(X, L^{-1}E) = 1$. Proposition 2.3 says that the generic situation in this range allows us to pick any element in the hyperplane $\ker(\boldsymbol{x})$ and find a corresponding lower-left component ϕ_+ of some Higgs fields on E.

Symplectic structures on $T^*\mathcal{M}_d$ and $T^*\mathcal{N}_d$ The cotangent bundle $T^*\mathcal{M}_d$ carries a canonical symplectic structure. Let $\check{\boldsymbol{z}}^H = (\check{z}_1^H, \ldots, \check{z}_g^H)$, $\boldsymbol{\kappa}^H = (\kappa_1^H, \ldots, \kappa_g^H)$ and $\boldsymbol{k}^H = (k_1^H, \ldots, k_N^H)$ be the conjugate coordinates [1] to $\boldsymbol{z}$, $\boldsymbol{\lambda}$ and $\boldsymbol{x}$ respectively, i.e. the canonical symplectic form on $T^*\mathcal{M}_d$ takes the local form

$$\tilde{\omega} = \sum_{i=1}^{g} dz_i \wedge d\check{z}_i^H + \sum_{r=1}^{N} dx_r \wedge dk_r^H = \sum_{i=1}^{g} d\lambda_i \wedge d\kappa_i^H + \sum_{r=1}^{N} dx_r \wedge dk_r^H. \tag{2.8}$$

One key step to the main results in [7] is the relation between Darboux coordinates on $T^*\mathcal{M}_d$ and Abelian differentials representing Higgs fields, which we now recall.

PROPOSITION 2.4. *[7] Suppose a Higgs field ϕ on E takes the local form $\begin{pmatrix} \phi_0 & \phi_- \\ \phi_+ & -\phi_0 \end{pmatrix}$ on $X_{\mathbf{q}}$ as in* (2.5). *Then*

$$\begin{aligned} \check{z}_i^H(\phi) &= -2\phi_0(q_i), \quad i = 1, \ldots, g, \\ k_r^H(\phi) &= \phi_+(p_r), \quad r = 1, \ldots, N, \end{aligned} \qquad \kappa_i^H(\phi) = -2\sum_{j=1}^{g} (dA^{-1})_{ij}\phi_0(q_j).$$

where the evaluations of Abelian differentials are w.r.t. chosen local coordinates.

There is a canonical $\mathbb{C}^*$-action on $\mathcal{M}_d$ which acts along the fibers over Pic^d and scales the extension classes. The induced action on $T^*\mathcal{M}_d$ is defined by pull-back. In terms of Darboux coordinates,

$$\epsilon.(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{k}^H, \boldsymbol{\kappa}^H) = (\epsilon\boldsymbol{x}, \boldsymbol{\lambda}, \epsilon^{-1}\boldsymbol{k}^H, \boldsymbol{\kappa}^H), \qquad \epsilon \in \mathbb{C}. \tag{2.9}$$

The corresponding moment map $H : T^*\mathcal{M}_d \to \mathbb{C}$ is nothing but the Serre duality pairing of $(L \hookrightarrow E) \in H^1(X, L^2)$ with $c_{\boldsymbol{x}}(\phi) \in H^0(X, KL^{-2})$. In terms of Darboux coordinates,

$$H\big((\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{k}^H, \boldsymbol{\kappa}^H)\big) = \boldsymbol{x}.\boldsymbol{k}^H = \sum_{r=1}^{N} x_r k_r^H. \tag{2.10}$$

By Corollary 2.2, the pull-back of a Higgs field $\phi \in T_E^*\mathcal{N}$ to $T^*_{(L\hookrightarrow E)}\mathcal{M}_d$ is contained in the level set $H^{-1}(0)$. These pull-backs are $\mathbb{C}^*$-equivariant and commute with the pull-backs to $T^*_{[L\hookrightarrow E]}\mathcal{N}_d$. In fact, an open dense subset of $T^*\mathcal{N}_d$ is symplectomorphic to the quotient of an open dense subset of $H^{-1}(0)$ by the $\mathbb{C}^*$-action, i.e. a symplectic reduction of an open dense subset of $T^*\mathcal{M}_d$ (cf. the Appendix B in [7]).

[1]The upper script "H" is meant to suggest that these Darboux coordinates on $T^*\mathcal{M}_d$ can be regarded as coordinates of (pull-backs of) Higgs bundles. We will use notations without this upper script for Darboux coordinates in the λ-connections setting (cf. subsection 2.3).

Fibers of $\mathcal{N}_d \dashrightarrow \mathcal{N}$ The forgetful map that picks out only E defines rational maps from $\mathcal{M}_d$ and $\mathcal{N}_d$ to the moduli space $\mathcal{N}$ of rank-2 stable bundles with trivial determinant. Denote by i the map $\mathcal{N}_d \dashrightarrow \mathcal{N}$. For $1 \le -2d \le g-1$, a dense subspace of the moduli space $\mathcal{N}_d$ is defined by pairs (E,L) with L a subbundle of maximal degree of E. The bundle E in this case is stable and lies in the stratum

$$S_d := \left\{ E \in \mathcal{N} \mid \max_{L \subset E} \deg(L) \ge d \right\}$$

of the Segre stratification on $\mathcal{N}$ [18]. In particular, $\mathcal{N} = S_d$ for $-2d = g-1$. The following results are well-known.

Proposition 2.5. *1. For $-2d = g-1$, there is an open dense subset of $\mathcal{N} = S_d$ elements of which admit finitely many maximal subbundles [18]. In particular, very-stable bundles, which are those that admit no nonzero nilpotent Higgs fields and define an open dense subset of $\mathcal{N}$, admit exactly 2^g maximal subbundles [14].*

2. For $1 \le -2d \le g-2$, there is an open dense subset of S_d elements of which admit exactly 1 maximal subbundle [18].

It follows from Proposition 2.5 that for $1 \le -2d \le g-1$, there is a dense subset $\mathcal{N}_d^{\text{im}} \subset \mathcal{N}_d$ consisting of pairs (E,L) where E is stable and the restriction of the forgetful map $i : \mathcal{N}_d \dashrightarrow \mathcal{N}$ to a neighborhood of (E,L) is an immersion (and an isomorphism if $-2d = g-1$). We denote by $\mathcal{N}^{\text{im}}$ the image of $\mathcal{N}_d^{\text{im}}$.

Proposition 2.6. *For $1 \le -2d \le g-1$, the map i is not an immersion at $[\boldsymbol{x}] \equiv (E,L)$ if and only if the line bundle $K_X L^2$ defines an exceptional divisor on X, i.e. $h^0(X, K_X L^2)$ is larger than its expected value.*

Proof. It follows from Proposition 2.4 that the kernel of $i^*_{(E,L)} : T^*_E \mathcal{N} \to T^*_{(E,L)} \mathcal{N}_d$ is isomorphic to the space $H^0(X, K_X L^2)$ of nilpotent Higgs fields on E with kernel L. But the expected value of $h^0(X, K_X L^2)$ is $g-1+2d$, which is equal to $\dim(\mathcal{N}) - \dim(\mathcal{N}_d)$. □

It follows from Proposition 2.6 that, for $-2d = g-1$, an open dense subset of $T^*\mathcal{N}_d$ is the moduli space of triples (E, L, ϕ) with E stable (modulo the loci of unstable bundles and the loci $(E,L) \in \mathcal{N}_d$ where E has nilpotent Higgs fields with kernel L). For $1 \le -2d < g-1$, Propositions 2.4 and 2.6 essentially say that, away from the loci $(E,L) \in \mathcal{N}_d$ where E has more nilpotent Higgs fields with kernel L than expected, $T^*\mathcal{N}_d$ is the moduli space of the "lower-triangular part" of the Higgs fields in local frames adapted to L.

2.2 λ-connections in terms of Abelian differentials

An $SL_2(\mathbb{C})$ λ-connection is a pair (E, ∇_λ) where E is a holomorphic bundle of trivial determinant and ∇_λ is a holomorphic map $\nabla_\lambda : E \to E \otimes K$ such that

- ∇_λ induces the trivial connection on $\mathcal{O}_X$;
- ∇_λ satisfies a twisted Leibniz rule $\nabla_\lambda(fs) = \lambda\, df \otimes s + f \nabla(s)$ for all local holomorphic functions f and holomorphic sections s of E.

Note that the case $\lambda = 0$ reduces to that of Higgs bundles. Note also that given a fixed holomorphic bundle E, the space of λ-connections on E is an affine space modeled over the space $H^0(X, \text{End}_0(E) \otimes K)$ of $SL_2(\mathbb{C})$-Higgs fields.

The following results concerning λ-connections can be regarded as the generalisation of results for Higgs bundles that we have reviewed to the case $\lambda \neq 0$. Given $(L \hookrightarrow E) \equiv \boldsymbol{x} \in H^1(X, L^2)$ and a λ-connection (E, ∇_λ), the composition

$$c_{\boldsymbol{x}}(\nabla_\lambda) : L \hookrightarrow E \xrightarrow{\nabla_\lambda} E \otimes K \longrightarrow L^{-1}K \tag{2.11}$$

is $\mathcal{O}_X$-linear and defines a section of KL^{-2}. It can be regarded as the generalisation of $c_{\boldsymbol{x}}(\phi)$ defined in (2.6) for the λ-connection setting. When the scaling of the embedding $L \hookrightarrow E$ is not important, we will also write $c(E, L, \nabla_\lambda)$. The generalisation of Proposition 2.1 is as follows. Suppose ∇_λ takes the form

$$\nabla_\lambda |_{X_{\mathbf{q}}} = \lambda \partial + \begin{pmatrix} \omega_0 & \omega_- \\ \omega_+ & -\omega_0 \end{pmatrix} \tag{2.12}$$

in some frames adapted to L over $X_{\mathbf{q}}$. Then ω_0 and $\omega_\pm$ are Abelian differentials that are holomorphic on $X_{\mathbf{q}}$ and satisfy the following conditions.

PROPOSITION 2.7. *With the above setup, ω_0, ω_- and ω_+ are Abelian differentials satisfying the following properties:*

- $\omega_+ \in \Omega_{-2\mathbf{q}+2\check{\mathbf{q}}}$ *can be identified with* $c_{\boldsymbol{x}}(\nabla_\lambda)$.
- $\omega_0 \in \Omega_{\mathbf{q}+\check{\mathbf{q}}+\mathbf{p}}$ *with residues*

$$\operatorname*{Res}_y \omega_0 = \begin{cases} -\lambda & \text{for } y \in \check{\mathbf{q}}, \\ \lambda & \text{for } y \in \mathbf{q}, \\ -x_r \omega_+(p_r) & \text{for } y \in \mathbf{p}. \end{cases} \tag{2.13}$$

- $\omega_- \in \Omega_{2\mathbf{p}+2\mathbf{q}-2\check{\mathbf{q}}}$ *with the singular part at each p_r fully determined by ω_0, ω_+ and λ.*

Proof. Using the transition functions (2.3), one obtains the local form $\lambda\partial + A$ of ∇_λ in neighborhoods of points in $\mathbf{p} + \mathbf{q} + \check{\mathbf{q}}$ where A is the local 1-form

$$\begin{pmatrix} \omega_0 + \lambda \underline{\check{z}}_i^{-1} & \underline{\check{z}}_i^{-2}\omega_- \\ \underline{\check{z}}_i^{2}\omega_+ & -\omega_0 - \lambda\underline{\check{z}}_i^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega_0 - \lambda(\underline{z}_j - z_j)^{-1} & (\underline{z}_j - z_j)^2\omega_- \\ (\underline{z}_j - z_j)^{-2}\omega_+ & -\omega_0 + \lambda(\underline{z}_j - z_j)^{-1} \end{pmatrix} \tag{2.14a}$$

around $\check{q}_i$ and w_r respectively, and

$$\begin{pmatrix} \omega_0 - x_r\omega_+ w_r^{-1} & \omega_- - 2x_r\omega_0 w_r^{-1} - x_r^2\omega_+ w_r^{-2} + \lambda x_r w_r^{-2} \\ \omega_+ & -\omega_0 + x_r\omega_+ w_r^{-1} \end{pmatrix} \tag{2.14b}$$

around p_r. The proof now follows from regularity of A at $\mathbf{p} + \mathbf{q} + \check{\mathbf{q}}$. □

The vanishing of sum of residues of ω_0 immediately implies the following.

COROLLARY 2.8. *The Serre duality pairing of $c_{\boldsymbol{x}}(\nabla_\lambda)$ and $\boldsymbol{x} \equiv (L \hookrightarrow E)$ is*

$$\langle c_{\boldsymbol{x}}(\nabla_\lambda), \boldsymbol{x} \rangle = \lambda d. \tag{2.15}$$

The pairing (2.15) is the generalisation of the vanishing pairing in Corollary 2.2 to $\lambda \neq 0$. Note that if E is strictly semi-stable with destablising subbundle L then the pairing (2.15) also vanishes. The following Proposition, which is the affine analogue of Proposition 2.3, follows immediately.

PROPOSITION 2.9. *If E is stable and L has a unique embedding into E up to scaling, then the projection from the space of λ-connections on E to the affine hyperplane in $H^0(X, K_X L^{-2})$ defined by (2.15) is surjective. In other words, in such a situation, given any element $c \in H^0(X, KL^{-2})$ satisfying $\langle c, \boldsymbol{x} \rangle = \lambda d$, then there exists a λ-connection (E, ∇_λ) such that $c = c_{\boldsymbol{x}}(\nabla_\lambda)$.*

2.3 Symplectic affine bundles on $\mathcal{N}_d$ and $\mathcal{M}_d$

Affine bundle on $\mathcal{M}_d$ We now define an affine bundle $\widetilde{\mathcal{M}_\lambda^d}$ on $\mathcal{M}_d$ modeled over $T^*\mathcal{M}_d$. It will turn out that this affine bundle $\widetilde{\mathcal{M}_\lambda^d}$ is an analogue of the Hodge moduli space $\mathcal{M}_{Hod}$, with the moment map H defined in (2.10) the analogue of the twistor parameter λ.

The idea of construction is rather straightforward. Recall the moment map $H : T^*\mathcal{M}_d \to \mathbb{C}$, $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{\kappa}) \mapsto \boldsymbol{x}.\boldsymbol{k}$, of the $\mathbb{C}^*$-action. For a fixed $\lambda \in \mathbb{C}^*$, consider the affine bundle $\mathcal{M}_\lambda^s \to \mathcal{N}$ which is the moduli space of irreducible λ-connections with stable underlying bundles. It follows from Corollary 2.8 that the pull-back along $I : \mathcal{M}_d \dashrightarrow \mathcal{N}$ of $\mathcal{M}_\lambda^s$ defines an affine bundle on $\mathcal{M}_d$ with fibers identified with $H^{-1}(\lambda d) \cap T^*_{(L\hookrightarrow E)}\mathcal{M}_d$. By varying λ, we can define a bundle with fibers identified with the complement of $H^{-1}(0)$ in $T^*_{(L\hookrightarrow E)}\mathcal{M}_d$. The extension of the associated affine isomorphisms to $H^{-1}(0)$ then defines an affine bundle modeled over $T^*\mathcal{M}_d$.

In the following, we are going to explain this construction by first introducing the generalisation of the Darboux coordinates $\boldsymbol{k}^H$, $\check{\boldsymbol{z}}^H$ and $\boldsymbol{\kappa}^H$ on $T^*\mathcal{M}_d$ to the $\lambda \neq 0$ setting (cf. Proposition 2.4). Given $(L \hookrightarrow E) \in \mathcal{M}_d$ and an irreducible λ-connection ∇_λ on E, the data consisting of a reference divisor $\boldsymbol{r} = \mathbf{p} + \check{\mathbf{q}}$ and local coordinates around $\boldsymbol{r} + \mathbf{q}$ let us define the vectors $\boldsymbol{k} = (k_1, \dots, k_N) \in \mathbb{C}^N$ and $\check{\boldsymbol{z}} = (\check{z}_i, \dots, \check{z}_g) \in \mathbb{C}^g$ where

$$k_r := \omega_+(p_r), \qquad \check{z}_i := -2\left(\omega_0 - \frac{\lambda}{\underline{z}_i - z_i}\right)(q_i). \tag{2.16a}$$

These quantities are well-defined due to regularity of the 1-form (2.14). We also define $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_g) \in \mathbb{C}^g$ where

$$\kappa_j := \sum_{i=1}^{g} (dA^{-1})_{ji} \check{z}_i, \qquad i = 1, \dots, g; \tag{2.16b}$$

here dA^{-1} is the differential of A^{-1} evaluated at $\boldsymbol{\lambda} = A(\boldsymbol{z})$ (cf. (2.4)). By Corollary 2.8, these vectors define a map

$$I_{\lambda,\boldsymbol{r}} : \mathcal{M}_\lambda \mid_E \longrightarrow H^{-1}(\lambda d) \cap T^*_{(L\hookrightarrow E)}\mathcal{M}_d$$

where $\mathcal{M}_\lambda \mid_E$ is the space of irreducible λ-connections on E and $H^{-1}(\lambda d)$ is the (λd)-level set of the moment map on $T^*\mathcal{M}_d$ (cf. (2.10)). The image of $I_{\lambda,\boldsymbol{r}}$ is an affine space modeled over the image of the pull-back

$$I_E^* : T^*\mathcal{N}_d \longrightarrow H^{-1}(0) \cap T^*_{(L\hookrightarrow E)}\mathcal{M}_d.$$

It follows from Proposition 2.6 that $I_{\lambda,\boldsymbol{r}}$ is surjective, i.e. $E \in \mathcal{N}^{\text{im}}$, if and only if $h^0(X, KL^2)$ is equal to the expected value $g - 1 + 2d$.

Suppose now $E \in \mathcal{N}^{\text{im}}$ and so $I_{\lambda,\boldsymbol{r}}$ is surjective. Consider a change of reference divisors $\boldsymbol{r} \mapsto \boldsymbol{r}' = \mathbf{p}' + \check{\mathbf{q}}'$. In terms of $\boldsymbol{r}'$, let $(L' \hookrightarrow E')$ be the representative of the same point in $\mathcal{M}_d$ defined by $(L \hookrightarrow E)$. This means E' is defined by some transition functions of the form (2.3) with $\boldsymbol{r} \to \boldsymbol{r}'$ and $\mathbf{q} \to \mathbf{q}' = q_1' + \cdots + q_g'$, where $L' = \mathcal{O}_X(\mathbf{q}' - \check{\mathbf{q}}') \simeq \mathcal{O}_X(\mathbf{q} - \check{\mathbf{q}})$ is a subbundle of E'. Let ∇_λ' be the λ-connection on E' corresponding to ∇_λ via an isomorphism $E' \simeq E$. By considering a local form of ∇_λ' in local frames adapted to L' over $X \setminus \operatorname{supp}(\mathbf{p}' + \mathbf{q}' + \check{\mathbf{q}}')$ as in (2.12), one can similarly define a map $I_{\lambda,\boldsymbol{r}'} : \mathcal{M}_\lambda \mid_{E'} \longrightarrow H^{-1}(\lambda d)$ that associates to ∇_λ' a set of vectors $\boldsymbol{k}'$ and $\check{\boldsymbol{z}}'$. For a fixed $\lambda \in \mathbb{C}^*$, there is a unique affine automorphism of $H^{-1}(\lambda d)$ that relate the two sets of tuples $(\boldsymbol{k}, \check{\boldsymbol{z}})$ and $(\boldsymbol{k}', \check{\boldsymbol{z}}')$ for all irreducible λ-connections on E. To see this, let us choose an irreducible λ-connection ∇_{ref} on E, and let $(\boldsymbol{k}_{\text{ref}}, \check{\boldsymbol{z}}_{\text{ref}})$ and $(\boldsymbol{k}'_{\text{ref}}, \check{\boldsymbol{z}}'_{\text{ref}})$ be its associated vectors w.r.t. the two reference divisors. By Proposition 2.4, the pull-back to $T^*\mathcal{M}_d$ of the Higgs field $\nabla_\lambda - \nabla_{\text{ref}}$ has respective fiber coordinates $(\boldsymbol{k}-\boldsymbol{k}_{\text{ref}}, \check{\boldsymbol{z}}-\check{\boldsymbol{z}}_{\text{ref}})$ and $(\boldsymbol{k}'-\boldsymbol{k}'_{\text{ref}}, \check{\boldsymbol{z}}'-\check{\boldsymbol{z}}'_{\text{ref}})$ which are contained in $H^{-1}(0)$. One then obtains a unique affine isomorphism $I_{\lambda,\boldsymbol{r},\boldsymbol{r}'}$ associated to the transition function of $T^*\mathcal{M}_d$ and maps $(\boldsymbol{k}_{\text{ref}}, \check{\boldsymbol{z}}_{\text{ref}}) \longmapsto (\boldsymbol{k}'_{\text{ref}}, \check{\boldsymbol{z}}'_{\text{ref}})$.

$$\begin{array}{ccccc} \mathcal{M}_\lambda\,|_E & \xrightarrow{I_{\lambda,r}} & H^{-1}(\lambda d)\cap T^*_{(L\hookrightarrow E)}\mathcal{M}_d & \hookrightarrow & T^*_{(L\hookrightarrow E)}\mathcal{M}_d \\ \downarrow\sim & & \downarrow I_{\lambda,r,r'} & & \downarrow = \\ \mathcal{M}_\lambda\,|_{E'} & \xrightarrow{I_{\lambda,r'}} & H^{-1}(\lambda d)\cap T^*_{(L'\hookrightarrow E')}\mathcal{M}_d & \hookrightarrow & T^*_{(L'\hookrightarrow E')}\mathcal{M}_d \end{array} \quad . \tag{2.17}$$

Clearly a different choice of the reference connection ∇_{ref} induces the same affine isomorphism.

Consider now the Hodge moduli space $\mathcal{M}_{Hod} \to \mathbb{C}$ of irreducible $SL_2(\mathbb{C})$ λ-connections on X. The fiber $\mathcal{M}_0$ over 0 can be identified with the Hitchin moduli space $\mathcal{M}_H$, and over a fixed $\lambda \in \mathbb{C}^*$ the moduli space $\mathcal{M}_\lambda$ of irreducible λ-connections (E, ∇_λ). Varying $\lambda \in \mathbb{C}^*$ and requiring stability of E defines an open dense subset $\widetilde{\mathcal{M}^s_\lambda} \subset \mathcal{M}_{Hod}\,|_{\mathbb{C}^*}$ which is a vector bundle

$$\begin{array}{c} \widetilde{\mathcal{M}^s_\lambda} \\ \downarrow \\ \mathbb{C}^* \times \mathcal{N} \end{array} \tag{2.18}$$

whose fiber over $\mathbb{C}^* \times \{E\}$ is the space $\widetilde{\mathcal{M}^s_\lambda}\,|_E$ of irreducible λ-connections on E modulo isomorphisms for all $\lambda \in \mathbb{C}^*$. For each $(L \hookrightarrow E) \in \mathcal{M}_d$ with E stable, upon a choice of reference divisor $\boldsymbol{r}$ and local coordinates, the vectors defined in (2.16) define a map

$$I_{\boldsymbol{r}} : \widetilde{\mathcal{M}^s_\lambda}\,|_E \longrightarrow T^*_{(L\hookrightarrow E)}\mathcal{M}_d \setminus H^{-1}(0).$$

A map $I_{\boldsymbol{r}'}$ defined by a different reference divisor $\boldsymbol{r}'$ commutes with $I_{\boldsymbol{r}}$ via the collection of affine automorphisms $I_{\lambda,\boldsymbol{r},\boldsymbol{r}'}$ on the level set $H^{-1}(\lambda d)$ as defined in (2.17). These collection of affine automorphisms extend to an affine automorphism $I_{\boldsymbol{r},\boldsymbol{r}'}$ of $T^*_{(L\hookrightarrow E)}\mathcal{M}_d$ that restricts to the vector space isomorphism defined by transition function of $T^*\mathcal{M}_d$ on $H^{-1}(0)$. Let $\{\tilde{U}_\alpha\}_{\alpha\in\mathcal{I}}$ be an atlas on $\mathcal{M}_d$ such that

- each $\tilde{U}_\alpha$ can be equipped coordinates using some reference divisor $\boldsymbol{r}_\alpha$;
- the images of overlaps $U_\alpha \cap U_\beta$ along I are contained in $\mathcal{N}^{\text{im}}$.

We then can use the affine automorphisms $I_{\boldsymbol{r}_\alpha,\boldsymbol{r}_\beta}$ of $T^*\mathcal{M}_d$ to define an affine bundle over $\mathcal{M}_d$ modeled over $T^*\mathcal{M}_d$. We denote this affine bundle by $\widetilde{\mathcal{M}^d_\lambda}$. It follows from (2.17) that the restriction of $\widetilde{\mathcal{M}^d_\lambda}$ to the complement of $H^{-1}(0)$ is simply the pull-back of $\widetilde{\mathcal{M}^s_\lambda}$ to $\mathcal{M}_d$. Hence $\widetilde{\mathcal{M}^d_\lambda}$ plays the role of an analogue of the Hodge moduli space $\mathcal{M}_{Hod}$, with the moment map H encoding the twistor coordinate λ.

Symplectic structure It follows from the local identification of $\widetilde{\mathcal{M}^d_\lambda}$ with $T^*\mathcal{M}_d$ that one can define local symplectic form by pulling-back the canonical symplectic form on $T^*\mathcal{M}_d$. A priori this local symplectic form on $\widetilde{\mathcal{M}^d_\lambda}$ is not unique, i.e. it depends on the local identifications. Using Proposition 2.4 and (2.16), we can write this canonical symplectic form as

$$\sum_{i=1}^{g} dz_i \wedge d\check{z}_i + \sum_{r=1}^{N} dx_r \wedge dk_r = \sum_{i=1}^{g} d\lambda_i \wedge d\kappa_i + \sum_{r=1}^{N} dx_r \wedge dk_r \tag{2.19}$$

in a local neighborhood of $(L \hookrightarrow E)$ with $E \in \mathcal{N}^{\mathrm{im}}$. It is easy to see that

$$\sum_{r=1}^{N} dx_r \wedge dk_r = \sum_{r=1}^{N} dx'_r \wedge dk'_r, \qquad \sum_{i=1}^{g} dz_i \wedge d\check{z}_i = \sum_{i=1}^{g} dz'_i \wedge d\check{z}'_i \tag{2.20}$$

upon a change of reference divisor $\boldsymbol{r} \to \boldsymbol{r}'$. Hence we have a global symplectic form $\widetilde{\omega}_\lambda$ on $\widetilde{\mathcal{M}_\lambda^d}$ with local Darboux coordinates $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{k}, \check{\boldsymbol{z}})$ or alternatively $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{k}, \boldsymbol{\kappa})$ defined through (2.3) and (2.16) by choosing reference divisors.

We note that for Riemann surfaces with punctures, Pinchbeck in [21] considered a similar symplectic structure and showed that it coincides with the pull-back of the canonical symplectic structure on the de Rham moduli space of holomorphic connections. We expect by adapting the strategy in [21], a similar result in the current setting of compact Riemann surfaces can be proved.

Symplectic affine bundle on $\mathcal{N}_d$ The restriction of $\widetilde{\mathcal{M}_\lambda^s}$ to $\{\lambda\} \times \mathcal{N}$ is the moduli space $\mathcal{M}_\lambda^s$ of irreducible λ-connections with stable underlying bundles. We denote by $\mathcal{M}_\lambda^d$ the pull-back of $\mathcal{M}_\lambda^s$ along $\mathcal{N}_d \dashrightarrow \mathcal{N}$. Clearly $\mathcal{M}_\lambda^d$ is an affine bundle modeled over $T^*\mathcal{N}_d$, and one can regard this as an analogue of how $\widetilde{\mathcal{M}_\lambda^s}$ is an affine bundle modeled over $T^*\mathcal{N}$.

Recall that an open dense subset of $T^*\mathcal{N}_d$ is a symplectic reduction of an open dense subset of $T^*\mathcal{M}_d$ by $\mathbb{C}^*$-action. Here, we also have an open dense subset of $\mathcal{M}_\lambda^d$ as a symplectic reduction of an open dense subset of $\widetilde{\mathcal{M}_\lambda^d}$. More precisely, the $\mathbb{C}^*$-quotient of the level set $H^{-1}(\lambda d)$ is isomorphic to an open dense subset in $\mathcal{M}_\lambda^d$ as complex manifolds, such that the pull-back of the local symplectic structure defined by local identification with $T^*\mathcal{N}_d$ to $H^{-1}(\lambda d)$ coincides with the restriction of $\widetilde{\omega}_\lambda$ to $H^{-1}(\lambda d)$.

Maps from $\mathcal{M}_\lambda^s$ Denote by $\mathcal{M}_\lambda^s\,|_E$ the fiber of $\mathcal{M}_\lambda^s$ over $E \in \mathcal{N}$: in other words, this is the space of isomorphism classes of irreducible λ-connections on E. It is an affine space modeled over $H^0(X, \mathrm{End}_0(X) \otimes K_X) \simeq \mathbb{C}^{3g-3}$. It follows from the construction of $\widetilde{\mathcal{M}_\lambda^d}$ that, given $\boldsymbol{x} \equiv (L \hookrightarrow E) \in \mathcal{M}_d$, the diagram (2.17) defines a map

$$I_\lambda\,|_{\boldsymbol{x}} \colon \mathcal{M}_\lambda^s\,|_E \to \widetilde{\mathcal{M}_\lambda^d}\,|_{\boldsymbol{x}}$$

which upon choosing a reference divisor $\boldsymbol{r}$ coincides with $I_{\lambda,\boldsymbol{r}}$. As we vary $\boldsymbol{x}$ along its $\mathbb{C}^*$-orbit, the collections of such maps are equivariant w.r.t. the $\mathbb{C}^*$-action on $\widetilde{\mathcal{M}_\lambda^d}$ and hence descends to a map

$$i_\lambda\,|_{[\boldsymbol{x}]} \colon \mathcal{M}_\lambda^s\,|_E \to \mathcal{M}_\lambda^d\,|_{[\boldsymbol{x}]}$$

where $[\boldsymbol{x}] = (L, E) \in \mathcal{N}_d$ is the class of $\boldsymbol{x}$ upon modulo scaling. These two maps are affine analogues of the point-wise pull-backs

$$I^*\,|_{\boldsymbol{x}} \colon T_E^*\mathcal{N} \longrightarrow T_{\boldsymbol{x}}^*\mathcal{M}_d, \qquad i^*\,|_{[\boldsymbol{x}]} \colon T_E^*\mathcal{N} \longrightarrow T_{[\boldsymbol{x}]}^*\mathcal{N}_d.$$

PROPOSITION 2.10. *Let d be in the range $0 < -2d \leq g-1$. Let ∇_λ and ∇'_λ be two λ-connections on E, and $\boldsymbol{x} \equiv (L \hookrightarrow E) \in \mathcal{M}_\lambda^d$. Then*

$$I_\lambda(\nabla_\lambda) = I_\lambda(\nabla'_\lambda), \qquad i_\lambda(\nabla_\lambda) = I_\lambda(\nabla'_\lambda)$$

if and only if $\nabla_\lambda - \nabla'_\lambda$ is a nilpotent Higgs fields with kernel L. In particular, $i_\lambda\,|_{[\boldsymbol{x}]}$ is not surjective if and only if i is not an immersion at $[\boldsymbol{x}]$, which occurs if and only if $K_x L^2$ defines an exceptional divisor on X.

Proof. By construction, we have $I_\lambda(\nabla_\lambda)-I_\lambda(\nabla'_\lambda)=I^*\mid_{\boldsymbol{x}}(\nabla_\lambda-\nabla'_\lambda)$. The statements now follows from Proposition 2.6. □

It follows that for $-2d=g-1$, the maps $i_\lambda\mid_{[\boldsymbol{x}]}$ are isomorphisms on the open dense subset $\mathcal{N}_d^{\mathrm{im}}\subset\mathcal{N}_d$. Consequently, in this case, similarly to how the restriction of $T^*\mathcal{N}_d$ to $\mathcal{N}_d^{\mathrm{im}}$ is the moduli space of triples (E,L,ϕ) for $(E,L)\in\mathcal{N}_d^{\mathrm{im}}$, we have the restriction of $\mathcal{M}_\lambda^d$ to $\mathcal{N}_d^{\mathrm{im}}$ is the moduli space of triples (E,L,∇_λ) for $(E,L)\in\mathcal{N}_d^{\mathrm{im}}$.

2.4 Components of λ-connections revisited

We have seen from (2.17) and (2.19) how the evaluations of Abelian differentials $\left(\begin{smallmatrix}\omega_0 & \omega_-\\ \omega_+ & -\omega_0\end{smallmatrix}\right)$ in (2.12) relate to Darboux coordinates of the symplectic form $\widetilde{\omega}_\lambda$ on $\widetilde{\mathcal{M}_\lambda^d}$. To prepare for the proof of the main theorem in Section 4.3, in the following we express ω_0 in terms of the Darboux coordinates.

Given a pair of distinct points $p_\pm$ on X, recall that the Abelian differential of the third kind $\omega_{p_+-p_-}$ has simple poles at $p_\pm$ with respective residues ±1 and is holomorphic elsewhere. It is the unique meromorphic differential with this property with vanishing A-cycles. Let us now similarly define a unique Abelian differential of the third kind $\omega^{\mathbf{q}}_{p_+-p_-}$ by imposing a different normalisation condition. Namely, we require that $\omega^{\mathbf{q}}_{p_+-p_-}$ has simple poles at $p_\pm$ with respective residues ±1, is holomorphic elsewhere, and in addition its 0-th Laurent coefficient w.r.t. the local coordinate z_i around each $q_i<\mathbf{q}$ is 0. In other words, our normalisation condition is that

$$\omega^{\mathbf{q}}_{p_+-p_-}(q_i)=0 \text{ for } q_i\notin\{p_\pm\} \tag{2.21a}$$

and

$$\omega^{\mathbf{q}}_{p_+-p_-}=\left(\frac{\pm1}{z_i}+\mathcal{O}(z_i)\right)dz_i \text{ for } q_i\in\{p_\pm\}. \tag{2.21b}$$

Note that while condition (2.21a) is coordinate-independent, the vanishing of the 0-th Laurent coefficient in (2.21b) is not. Nevertheless, one can find an explicit relation between the two sets of normalised Abelian differentials. Let $\omega^0_{p_+-p_-}(q_i)$ be the 0-th Laurent coefficient w.r.t. z_i of $\omega_{p_+-p_-}(q_i)$, namely

$$\omega^0_{p_+-p_-}(q_i)=\begin{cases}\omega_{p_+-p_-}(z_i(q_i)) & \text{for } q_i\notin\{p_\pm\}\\ \left(\omega_{p_+-p_-}\mp\frac{dz_i}{z_i}\right)(z_i(q_i)) & \text{for } q_i\in\{p_\pm\}.\end{cases}$$

Then it is straightforward to check that

$$\omega^{\mathbf{q}}_{p_+-p_-}=\omega_{p_+-p_-}-\sum_{i,n=1}^{g}\left(dA_{\boldsymbol{\lambda}}^{-1}\right)_{ni}\omega^0_{p_+-p_-}(q_i)\omega_n. \tag{2.22}$$

Knowing the residues of ω_0 and using the definition of $\check{z}_i$ and κ_i from (2.16), we can write

$$\omega_0=-\sum_{r=2}^{N}x_rk_r\omega^{\mathbf{q}}_{p_r-p_1}+\lambda\sum_{i=1}^{g}\omega^{\mathbf{q}}_{q_i-p_1}-\lambda\sum_{i=1}^{g-d}\omega^{\mathbf{q}}_{\check{q}_i-p_1}-\frac{1}{2}\sum_{i=1}^{g}\kappa_i\omega_i. \tag{2.23}$$

This formula gives a concrete way to express ω_0 in terms of the coordinates on $\widetilde{\mathcal{M}_\lambda^d}$.

We can also express ω_+ in terms of its zeroes and poles. Suppose $\boldsymbol{u} = u_1 + \cdots + u_m$ where $m = \deg(KL^{-2}) = 2g - 2 - 2d$ is its zero divisor. It follows from Proposition 2.7 that

$$\omega_+(x) = u_0 \frac{\prod_{i=1}^{g} E(x, q_i(\boldsymbol{u}))^2 \prod_{k=1}^{N+g-1} E(x, u_k)}{\prod_{j=1}^{g-d} E(x, \check{q}_k)^2} (\sigma(x))^2. \tag{2.24}$$

Here $u_0 \in \mathbb{C}^*$ is a scaling factor, and $E(p,q)$ is the prime form on $\tilde{X} \times \tilde{X}$, where $\tilde{X}$ is a fundamental domain of X obtained by cutting along a basis of canonical cycles. The definition of $\sigma(x)$, which is a multi-valued $(g/2)$-differential, in terms of the Theta function and prime forms can be found in Appendix B of [7].

3 Opers with apparent singularities and their moduli

3.1 Basic definition and properties

A branched projective structure subordinate to the Riemann surface X is a collection $\{(U_\alpha, w_\alpha\}$ where $\{U_\alpha\}$ is a covering of X and $\{w_\alpha\}$ a collection of local holomorphic maps from X to $\mathbb{P}^1$ whose values are related by Möbius transform. The ramification points of w_α are called apparent singularities. Two branched projective structures are equivalent if their union is also a branched projective structures. The analytic continuation of any local map w_α is also called a developing map, which induces a monodromy representation $\pi_1 \to PSL_2(\mathbb{C})$. It is clear that the equivalence class of a branched projective structure is determined by such a developing map.

An equivalent description of branched projective structures is that of a $PSL_2(\mathbb{C})$-oper with apparent singularities, which we from now on will call *oper* for short. An oper is a collection $\mathscr{D}$ of compatible local Schrödinger differential operators $\mathscr{D} = \{(U_\alpha, D_\alpha)\}$ where

$$D_\alpha = \lambda \partial_{z_\alpha}^2 + q_\alpha(z_\alpha), \qquad \lambda \in \mathbb{C}^*.$$

Here, "compatible" means that solutions to D_α are sections of a line bundle N of degree $1-g$ (such as $K_X^{-1/2}$), namely if f_α is a solution to D_α then $N_{\beta\alpha} f_\alpha$ is a solution to D_β [16, 17]. The two equivalent definitions of branched projective structures are related by taking the ratio $f_{1,\alpha}/f_{2,\alpha}$ of two linearly independent solutions to D_α to be w_α and by noting that

$$q_\alpha(z_\alpha) = \frac{\lambda^2}{2} \{w_\alpha(z_\alpha), z_\alpha\} = \frac{\lambda^2}{2} \left\{ \frac{f_{1,\alpha}(z_\alpha)}{f_{2,\alpha}(z_\alpha)}, z_\alpha \right\} \tag{3.1}$$

where $\{g(z), z\} := \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$ is the Schwarzian derivative of a function $g(z)$. It follows from the transformation rule of a Schwarzian derivative upon a change of coordinates $z_\alpha \to z_\beta(z_\alpha)$ that the potentials of the local Schrödinger equations transform as

$$q_\beta(z_\beta)(z_\beta')^2 = q_\alpha(z_\alpha) - \frac{\lambda^2}{2} \{z_\beta, z_\alpha\}. \tag{3.2}$$

We see that these potentials transform almost as quadratic differentials plus a correction term scaled with λ^2. Note that this correction term vanishes if $z_\beta(z_\alpha)$ is a Möbius transform. Consequently, w.r.t. a coordinate atlas where the coordinate changes are all Möbius transform, the potentials q_α indeed define a quadratic differential (Example 3.1 below gives examples of such atlases).

One can show by direct computation from (3.1) that the singularities of $q_\alpha(z_\alpha)$, i.e. the ramification points of $w_\alpha(z_\alpha)$, have particular Laurent tail expansion, which ensure that even though solutions to D_α are singular and might have non-trivial monodromy around these singularities, their ratios are holomorphic there. Consequently, the developing monodromy in $PSL_2(\mathbb{C})$ do not "see" these singularities, hence the name "apparent singularities". The order of an apparent singularity is the ramification order of w_α. In this paper, we will restrict ourselves to opers with *simple* apparent singularities, namely they are of order 1. It follows from (3.1) and the Taylor expansion $w_\alpha(z_\alpha) = \sum_{k\geq 1} w_k z_\alpha^k$ at a simple apparent singularity that the Laurent expansion of $q_\alpha(z_\alpha)$ takes the form

$$\frac{1}{\lambda^2} q_\alpha(z_\alpha) = -\frac{3}{4z_\alpha^2} + \frac{\nu_\alpha}{z_\alpha} + q_{\alpha,0} + \mathcal{O}(z_\alpha), \qquad \nu_\alpha^2 + q_{\alpha,0} = 0. \tag{3.3}$$

We call ν_α the *residue parameter* of the apparent singularity $z_\alpha = 0$ of the oper $\mathcal{D}$ w.r.t. local coordinate z_α.

EXAMPLE 3.1. The space of opers without apparent singularities on X is an affine space modeled over $H^0(X, K_X^2) \simeq \mathbb{C}^{3g-3}$. Let $\{U_\alpha, z_\alpha\}$ be a collection of coordinates charts induced by the universal covering of X, i.e. the upper-half plane, via the uniformisation theorem. In these coordinates, the "uniformising" oper takes the local form $D_\alpha = \lambda \partial_{z_\alpha}^2$. If a quadratic differential has local expression of the form $q_\alpha(z_\alpha) dz_\alpha^2$ then the collection of differential operators $\lambda \partial_{z_\alpha}^2 + q_\alpha(z_\alpha)$ define a meromorphic oper without apparent singularities. Note that the ratios w_α of linearly independent solutions of any oper without apparent singularity define coordinate atlas $\{(U_\alpha, w_\alpha)\}$ with coordinates changes being Möbius transforms. In such an atlas, an oper with simple apparent singularity is equivalent to a meromorphic quadratic differential with poles of order 2 satisfying (3.3).

3.2 Moduli of opers

Transformation rules of residue parameters Since the Schwarzian derivative of local coordinates w.r.t. each other is regular, it follows from (3.2) that upon a change of coordinates $z_\beta \to z_\alpha(z_\beta)$, the transform $\nu_\beta \to \nu_\alpha$ of the residue parameters associated to a simple apparent singularity u_n is the same as the transform of residue parameters of a quadratic differential with a double pole at u_n. By a direct computation one can show that

$$\nu_\alpha = \frac{\nu_\beta}{z_\alpha'(u)} + \frac{3}{4} \frac{z_\alpha''(u)}{(z_\alpha'(u))^2} \tag{3.4}$$

where we have evaluated the derivatives of $z_\alpha(z_\beta)$ at $z_\beta(u)$. (The invariance of the leading Laurent series coefficient $-3/4$ is on the other hand characteristic of quadratic differentials with double poles.) It will turn out later to be more convenient to use the λ-scaled residue parameter

$$\nu_{\lambda,n} := \lambda \nu_n. \tag{3.5}$$

In terms of these parameters, the transformation rule is that of fiber coordinates of T^*X plus a correction term linear in λ, namely

$$\nu_{\lambda,\alpha} = \frac{\nu_{\lambda,\beta}}{z_\alpha'(u)} + \frac{3\lambda}{4} \frac{z_\alpha''(u)}{(z_\alpha'(u))^2}. \tag{3.6}$$

Opers in terms of apparent singularities and residue parameters On the other hand, one can characterise opers with $3g-3$ apparent singularities $\mathbf{u}=\sum_{n=1}^{3g-3} u_n$ in terms of the positions and residue parameters of these singularities, provided that these there is no quadratic differential vanishing at $\mathbf{u}$. The idea in the proof of the following proposition, which uses "building blocks" of quadratic differentials to control the residue parameters, was first used by Iwasaki in [16].

PROPOSITION 3.2. *[5, 16] Let $\sum_{n=1}^{3g-3} u_n$ be a reduced divisor on X such that there is no quadratic differential vanishing at $\mathbf{u}$. Then there exists coordinate neighborhoods (U_n, z_n) of each u_n, $n \in \{1, ..., 3g-3\}$, and an injective map of sets*

$$\begin{aligned}\left(U_1 \times ... \times U_{3g-3}\right) \times (\mathbb{C})^{3g-3} &\longrightarrow \{\textit{opers with } 3g-3 \textit{ simple apparent singularities}\}/\sim \\ (\vec{u}', \vec{v}) &\longmapsto \mathscr{D}(\vec{u}', \vec{v})\end{aligned}$$

such that the oper $\mathscr{D}(\vec{u}', \vec{v})$ associated to $(\vec{u}', \vec{v}) = (u'_1, ..., u'_{3g-3}, v_1, ..., v_{3g-3})$ has a simple apparent singularity at each u'_n and associated residue parameters v_n w.r.t. the local coordinates z_n.

Proof. The idea is that due to our hypothesis on $\sum_{n=1}^{3g-3} u_n$, we can find local neighborhoods U_n of u_n such that points in $U_1 \times ... \times U_{3g-3}$ also define divisors $u'_1 + \cdots + u'_{3g-3}$ at which no quadratic differential vanish. Choose an oper without apparent singularity and equip U_n with local coordinate z_n induced by this oper (cf. Example 3.1). We now describe $\mathscr{D}(\vec{u}', \vec{v})$ by a quadratic differentials with double poles $u'_n \in U_n$ and Laurent expansion in z_n satisfying (3.3). To this end, we can use quadratic differentials with Laurent tails

$$q^{(2)}_{u'_n} = \frac{1}{(z_n - z_n(u'_n))^2} + \mathscr{O}(z_n - z_n(u'_n)), \qquad q^{(1)}_{u'_n} = \frac{1}{z_n - z_n(u'_n)} + \mathscr{O}(z_n - z_n(u'_n))$$

around u'_n and vanish at u'_k for $k \neq n$. These quadratic differentials are actually unique. The quadratic differential

$$-\frac{3\lambda^2}{4} \sum_{r=1}^{3g-3} q^{(2)}_{u'_n} + \sum_{r=1}^{3g-3} v_r q^{(1)}_{u'_n} + q^{(0)}_{(\vec{u}', \vec{v})}$$

where $q^{(0)}_{(\vec{u}', \vec{v})}$ is the unique holomorphic quadratic differential that solves the nondegenerate linear system

$$q^{(0)}_{(\vec{u}', \vec{v})}(z_n)\,|_{z_n(u'_n)} + v_n^2 = 0, \qquad n = 1, \ldots, 3g-3, \tag{3.7}$$

then defines $\mathscr{D}_{(\vec{u}', \vec{v})}$. The injectivity of the assignment $(\vec{u}', \vec{v}) \mapsto \mathscr{D}_{(\vec{u}', \vec{v})}$ is obvious. □

Q-generic and Q-special divisors The hypothesis on $\mathbf{u}$ in Proposition 3.2 is a case of the notion of Q-generic divisors [5, 6]. We say that an effective divisor $\mathbf{u} = \sum_{n=1}^{m} u_n$ on X is Q-generic if the space

$$Q_{\mathbf{u}} = \{q \in H^0(X, K_X^2) \mid \mathbf{u} < \mathrm{div}(q)\} \cup \{0\}$$

of quadratic differentials vanishing at $\mathbf{u}$, with multiplicity counted, is of expected, i.e. minimal, dimension

$$\max\{0, 3g-3-\deg(\mathbf{u})\}.$$

We say that $\mathbf{u}$ is Q-special otherwise. For $\deg(\mathbf{u}) \geq 3g-3$, this means $\mathbf{u}$ is Q-special if and only if there is a quadratic differential vanishing at $\mathbf{u}$. Note that the space of solutions to (3.7), if they exist, is an affine space modeled over $Q_{\mathbf{u}}$.

It follows from the constructive proof in Propostion 3.2 that whenever $\mathbf{u}$ is reduced and $\dim(Q_{\mathbf{u}}) > 0$, there exist families of opers with simple apparent singularities at $\mathbf{u}$ and the same residue parameters. Such families are affine spaces modeled over $Q_{\mathbf{u}}$. For example, such are generic cases for $\deg(\mathbf{u}) < 3g-3$. The following proposition follows by combinning this observation with the proof of Proposition 3.2.

PROPOSITION 3.3. *[5, 16] For $m \le 3g-3$, let $\sum_{n=1}^{m} u_n$ be a reduced Q-generic divisor on X. Then there exists local coordinates z_n around u_n and a non-canonical injective map of sets*

$$\begin{aligned}(\mathbb{C})^{3g-3} \times Q_{\mathbf{u}} &\longrightarrow \{\textit{opers with } 3g-3 \textit{ simple apparent singularities at } \mathbf{u}\}/\sim\\ (\vec{\nu}, \Delta q) &\longmapsto \mathscr{D}(\vec{\nu}, \Delta q)\end{aligned}$$

such that the oper $\mathscr{D}(\vec{\nu}, \Delta q)$ has associated residue parameters ν_n w.r.t. the local coordinates z_n.

REMARK 3.4.
1. Proposition 3.2 does not hold for a Q-special divisor $\boldsymbol{u}' = \sum_{n=1}^{3g-3} u_n'$: in fact, for most vectors $\vec{\nu} \in \mathbb{C}^{3g-3}$, there is no oper with apparent singularities at $\boldsymbol{u}'$ with residue parameters defined by $\vec{\nu}$ since the system (3.7) has no solution. On the other hand, there exist families of opers with apparent singularities at $\boldsymbol{u}'$ with the same residue parameters. These families are affine spaces modeled over the vector space $Q_{\boldsymbol{u}'}$.
2. It can be checked using (3.4) that upon a Möbius transform of local coordinates, the residue parameters can be put to 0. This means that fixing an oper without apparent singularity in the proof of Proposition 3.2 does not yet fix the choice of the oper $\mathscr{D}(\vec{u'}, \vec{0})$ for each fixed $\vec{u'}$. Rather, $\mathscr{D}(\vec{u'}, \vec{0})$ depends on the choices of the coordinates z_n themselves.
3. In contrast with the Riemann-Hilbert correspondence between isomorphism classes of flat connections on vector bundles and conjugacy classes of monodromy representations in e.g. $SL_2(\mathbb{C})$, in general there exist non-equivalent opers with apparent singularities with the same monodromy representation in $PSL_2(\mathbb{C})$.

Moduli of opers with apparent singularities Proposition 3.2 and the transformation rule (3.6) motivate us to consider the affine bundle $\mathscr{M}'_{\mathrm{op}} \to X$ modeled over T^*X with transition function defined as in (3.6). Let

$$\mathscr{M}^m_{\mathrm{op}} := \left(\mathscr{M}'_{\mathrm{op}}\right)^m / S_m. \tag{3.8}$$

One can regard $\mathscr{M}^m_{\mathrm{op}}$ as an affine analogue of the symmetric product $(T^*X)^{[m]}$, which is recovered at the limit $\lambda \to 0$.

For $m = 3g-3$, it follows from Proposition 3.2 that, away from the diagonals

$$\{[(u_1, \nu_1), \dots, (u_{3g-3}, \nu_{3g-3})] \mid u_n = u_k \text{ for some } n \neq k\} \subset \mathscr{M}^m_{\mathrm{op},\lambda} \tag{3.9}$$

and the loci defined by Q-special divisors

$$\{[(u_1, \nu_1), \dots, (u_{3g-3}, \nu_{3g-3})] \mid \sum_{n=1}^{3g-3} u_n \text{ is } Q\text{-special}\} \subset \mathscr{M}^m_{\mathrm{op},\lambda}, \tag{3.10}$$

we have $\mathscr{M}^m_{\mathrm{op},\lambda}$ as the moduli space of opers with $3g-3$ simple apparent singularities which do not form Q-special divisors. The locus defined in (3.10) signifies the presence of wobbly bundles, which are stable bundles that admit nonzero nilpotent Higgs fields; we refer to section 5.1 for a more elaborate discussion. For $2g-2 < m < 3g-3$, it follows from Proposition 3.3 that away from the loci (3.9) and (3.10), up to a choice of a quadratic differential, $\mathscr{M}^m_{\mathrm{op},\lambda}$ is the moduli space of opers with m simple apparent singularities which do not form Q-special divisors.

Poisson structure There exists a natural closed 2-form on $\mathscr{M}^m_{\mathrm{op},\lambda}$ defined away from the diagonals which locally take the form

$$\sum_{n=1}^{m} dz_n \wedge d\nu_n \tag{3.11}$$

in the coordinates defining $\mathscr{M}^m_{\mathrm{op},\lambda}$. It is easy to see from (3.4) that the local 2-form (3.11) is invariant upon a change of coordinates and hence extends to a global closed 2-form on $\mathscr{M}^m_{\mathrm{op},\lambda}$ away from the diagonals (this observation was first made by Iwasaki [16]).

Note that in the non-compact Riemann surface setting, Iwasaki [17] showed that (3.11) defines a symplectic form which coincides with the pull-back of the canonical symplectic form in the character variety

$$\mathrm{Hom}(\pi_1, PSL_2(\mathbb{C}))/\sim$$

along the monodromy map. We expect an analogous result holds in the compact Riemann surface setting.

4 Separation of Variables as a Poisson map

4.1 From triples (E, L, ∇_λ) to opers

Two methods of inducing opers Given data $(L \hookrightarrow F, \nabla_\lambda)$, we discuss in the following two methods to induce an oper with apparent singularities. The opers defined by these two methods are equivalent.

The first way is geometric: by projectivising the flat bundle F^{∇_λ} defined by (F, ∇_λ), one gets a flat $PSL_2(\mathbb{C})$-bundle $\mathbb{P}(F^{\nabla_\lambda})$ with $\mathbb{P}^1$-fibers. projectivising further the sub-line bundle of F^{∇_λ} induced by L yields a section of $\mathbb{P}(F^{\nabla_\lambda})$. This section provides local maps to $\mathbb{P}^1$ whose values are related by constant Möbius transform.

The second method is analytic and gives explicit expression of the induced opers in terms of local Schrödinger equations w.r.t. local coordinates. Suppose in local frames adapted to L and in local coordinate z we have

$$\nabla_\lambda(z) = \lambda\partial_z + \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}. \tag{4.1}$$

Consider the local function $g_\lambda(z) = a(z) - \frac{\lambda}{2}\frac{c'(z)}{c(z)}$ and the local differential operator $\lambda^2\partial_z^2 + q(z)$ where

$$\begin{aligned} q(z) &= b(z)c(z) + (g_\lambda(z))^2 + \lambda g'_\lambda(z) \\ &= b(z)c(z) + \left(a(z) - \frac{\lambda}{2}\frac{\partial_z c(z)}{c(z)}\right)^2 + \lambda\,\partial_z\left(a(z) - \frac{\lambda}{2}\frac{\partial_z c(z)}{c(z)}\right). \end{aligned} \tag{4.2}$$

It is straightforward to check that $q(z)$ is invariant upon a change of local frames adapted to L w.r.t. the local 1-forms $a(z)$, $b(z)$ and $c(z)$ are defined. One can also check that, upon a change of coordinates and transformations of 1-forms

$$a(z_\alpha) \to a(z_\alpha(z_\beta))z'_\alpha(z_\beta), \qquad b(z_\alpha) \to b(z_\alpha(z_\beta))z'_\alpha(z_\beta), \qquad c(z_\alpha) \to c(z_\alpha(z_\beta))z'_\alpha(z_\beta),$$

we have $q(z)$ transform as in (3.2). We conclude that the collection of local differential operators $\lambda^2\partial_z^2 + q(z)$ defines an oper on X. It is straightforward to check that the apparent singularities of this oper are defined by the zeroes of $c(L \hookrightarrow E, \nabla_\lambda)$ with mulitiplicities counted.

PROPOSITION 4.1. *Consider an irreducible λ-holomorphic connection (E,∇_λ) together with a subbundle $L\hookrightarrow E$. Then the oper $\mathcal{D}=\{\lambda^2\partial_z^2+q(z)\}$ with $q(z)$ defined in (4.2) is equivalent to the oper defined by projectivising the data $(L\hookrightarrow E,\nabla_\lambda)$. In particular, for $\lambda=1$, the $PSL_2(\mathbb{C})$-monodromy representation of $\mathcal{D}$ is the projection of the $SL_2(\mathbb{C})$-monodromy representation of (E,∇_λ) up to conjugation.*

Proof. It suffices to prove equivalence in a neighborhood $U\subset X$ that contains no apparent singularity. Let w be the ratio of local solutions to $\lambda^2\partial_z^2+q(z)$. Refining U if necessary, we can use w as a local coordinate on U w.r.t. which the oper can be represented by the differential operator ∂_w^2. On the other hand, upon choosing a square-root $\sqrt{c(w)}$, observe that the effect of the holomorphic gauge transformation $G_\lambda=\left(\begin{smallmatrix} c(w)^{-1/2} & 0\\ 0 & c(w)^{1/2}\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & g_\lambda(w)\\ 0 & 1\end{smallmatrix}\right)$ is that

$$\nabla_\lambda(w)=\partial_w+\begin{pmatrix} a(w) & b(w)\\ c(w) & -a(w)\end{pmatrix}\longmapsto \partial_w+\begin{pmatrix} 0 & -q(w)\\ 1 & 0\end{pmatrix} \tag{4.3}$$

where $q(w)$ is defined by the same formula as in (4.2). This implies that $q(w)=0$. Then $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-1\\w\end{smallmatrix}\right)$ are local parallel sections. In the local frame defined by these two sections, a generator of L^{∇_λ} takes the form

$$\begin{pmatrix} 0 & -1\\ 1 & w\end{pmatrix}^{-1} G^{-1}\begin{pmatrix}1\\0\end{pmatrix}=c(w)^{1/2}\begin{pmatrix}-w\\1\end{pmatrix}.$$

The local function defined by $\mathbb{P}(L^{\nabla_\lambda})$ is hence $-w$. □

REMARK 4.2.
1. It is well-known that gauge transformations of the form G_λ put holomorphic connections to the canonical form (4.3) (In the $g=0$ setting this defines a map preserving the canonical symplectic structures related to the Schlesinger and Garnier systems [10]). In fact it is this gauge transformation that motivates our definition of induced opers in (4.2). Note that while we might need to pick a system of branch cuts to define the gauge transformation around the zeroes of $c(E,L,\nabla_\lambda)$, our definition of oper in (4.2), does not depend on such choices.
2. Twisting the data $(L\hookrightarrow E,\nabla_\lambda)$ by a square-root of $\mathcal{O}_X$ induce opers up to equivalence.

Induced residue parameters At a simple zero u_n of $c(E,L,\nabla_\lambda)$ the potential $q(z)$ has the Laurent expansion

$$\frac{-3\lambda^2}{4}\frac{1}{(z-z(u_n))^2}+\lambda\left[a(u_n)-\lambda\frac{c''(u_n)}{4c'(u_n)}\right]\frac{1}{z-z(u_n)}-\left[a(u_n)-\lambda\frac{c''(u_n)}{4c'(u_n)}\right]^2+\mathcal{O}(z-z(u_n)). \tag{4.4}$$

In the conventions of (3.3) and (3.5), we have

$$\nu_{\lambda,n}:=\lambda\nu_n=a(u_n)-\lambda\frac{c''(u_n)}{4c'(u_n)}. \tag{4.5}$$

Note that a scaling of the embedding $L\hookrightarrow E$ does not change the residue parameter. It follows from Proposition 3.2 that from the data of a generic point $\boldsymbol{x}=(L\hookrightarrow E)\in\mathcal{M}_d$ together with a λ-connection ∇_λ on E one can induce a unique oper which is invariant along the $\mathbb{C}^*$-orbit.

Inverse construction The construction of opers from triples (E, L, ∇_λ) can also be inverted.

PROPOSITION 4.3. *For an even positive integer m, let $\mathscr{D} = \{D_\alpha = \lambda^2 \partial_{z_\alpha}^2 + q_\alpha(z_\alpha)\}$ be an oper with m simple apparent singularities. Then there exists a triple (E, L, ∇_λ) that induces an oper equivalent to $\mathscr{D}$ by the operation described above. Furthermore, for $\lambda = 1$, if (E, ∇_λ) is irreducible then such a triple is unique up to tensoring with a square-root of $\mathscr{O}_X$.*

Proof. (Sketch) See Proposition 5.6 in [5] for a detailed constructive proof for the case $\lambda = 1$. The case for $\lambda \in \mathbb{C}^*$ can be generalised in a straight-forward manner. The idea is to apply Hecke transformation of bundles to a split bundle at the apparent singularities of $\mathscr{D}$. The choices of the flags at which we do Hecke transformation can be chosen such that the residue parameters and the components $a(z)$, $c(z)$ of ∇_λ satisfy (4.5). This defines the bundle E and the embedding of L into E. The component $b(z)$ of ∇_λ can be defined by inverting (4.2). The uniqueness statement follows from the Riemann-Hilbert correspondence for $G = SL_2(\mathbb{C})$ and the fact that the monodromy representation of $\mathscr{D}$ has 2^{2g} lifts to $SL_2(\mathbb{C})$. □

We note that the triple (E, L, ∇_λ) constructed from $\mathscr{D}$ is such that ∇_λ is not L-invariant, but a priori is not necessarily irreducible. However, for a fixed reduced divisor $\mathbf{u} = \sum_{n=1}^m u_n$ with $m \leq 3g - 3$, then a generic [2] choice of opers with apparent singularities at $\mathbf{u}$ will be induced by (E, L, ∇_λ) with irreducible (E, ∇_λ).

4.2 Construction of the Separation of Variables maps

We now briefly review construction of Baker-Akhiezer divisors from triples (E, L, ϕ) and the map SoV for Higgs bundles from [7]. The purpose is to show that the apparent singularities and residue parameters of opers constructed from triples (E, L, ∇_λ) are analogues of Baker-Akhiezer divisors. We then construct SoV_λ, the analogue of SoV for holomorphic connections.

Baker-Akhiezer divisors of triples (E, L, ϕ) Let (E, ϕ) be a rank-2 $SL_2(\mathbb{C})$-Higgs bundle and suppose the quadratic differential $q = \det(\phi)$ has only simple zeroes. The spectral curve $S \xrightarrow{\pi} X$ is a subset of T^*X defined by taking the square-roots of q, i.e. solving for the eigenvalues of ϕ. There is the involution $\sigma : S \to S$ that acts on the fibers of π exchanging the two square-roots. By solving for the eigenvectors of ϕ, one can define a line bundle $\mathscr{L}_{(E,\phi)}$ on S called the eigen-line bundle. One can recover the Higgs bundle by taking the direct image of $\mathscr{L}_{(E,\phi)} \otimes \pi^*(K)$ along π. Hence up to isomorphisms, Higgs bundles (E, ϕ) with $\det(\phi) = q$ are in 1-1 correspondence with line bundles on S (of appropriate degrees) – this is known as Hitchin's spectral correspondence for smooth spectral curves [15].

Given a line bundle L with an injection $L \to E$ (which possibly has zeroes), we can consider on S the composition

$$\pi^*(L) \longrightarrow \pi^*(E) \longrightarrow \mathscr{L}_{(E,\phi)}^{-1} \pi^*(\det(E)) \tag{4.6}$$

where the second map is the quotient of the embedding $\mathscr{L}_{(E,\phi)} \hookrightarrow \pi^*(E)$. The zero divisor of this composition consists of the pull-back of the zero divisor of $L \to E$ and points where $\pi^*(L)$ coincide with $\mathscr{L}_{(E,\phi)}$ as subbundles of $\pi^*(E)$.

[2] Recall Proposition 3.3: opers with apparent singularities at $\mathbf{u}$ can be classified by their residue parameters and holomorphic quadratic differentials vanishing at $\mathbf{u}$. Genericity then can be defined w.r.t. this classification.

DEFINITION 4.4. Let (E,ϕ) be a Higgs bundle on X with non-degenerate spectral curve $S \xrightarrow{\pi} X$ and L a line bundle with an injection $L \to E$. The *Baker-Akhiezer (BA) divisor* associated to the data $(L \to E, \phi)$ is the involution of the zero divisor of the composition $\pi^*(L) \to \pi^*(E) \to \mathscr{L}^{-1}_{(E,\phi)}\pi^*(\det(E))$.

BA divisors can be characterised concretely if L is a subbundle of E, namely the injection $L \to E$ is nowhere vanishing. Recall the section $c(E,L,\phi)$ of the line bundle $KL^{-2}\det(E)$ from (2.6), and suppose ϕ takes the form $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & -a_\alpha \end{smallmatrix}\right)$ in local frames adapted to L. The preimages along $S \xrightarrow{\pi} X$ of a zero u_n of $c_\alpha \equiv c(E,L,\phi)$ consist of two points (counted with multiplicity), labeled by the eigenvalues $\pm a_\alpha(u_n)$ of ϕ at u_n. Then the BA divisor of (E,L,ϕ) is $\tilde{\mathbf{u}} = \sum_n \tilde{u}_n$ where $\tilde{u}_n$ is the point labeled by

$$v_n = a_\alpha(u_n). \tag{4.7}$$

We see that the apparent singularities and the projection to X of BA divisors are respectively the zeroes of $c(E,L,\nabla_\lambda)$ and $c(E,L,\phi)$. The residue parameter ν_n in (3.5) can be regarded as a deformation of v_n by a term linear in λ.

We refer to [5, 7] for the proof of the following proposition.

PROPOSITION 4.5. *Let $\tilde{\mathbf{u}}$ be the BA divisor of $(L \to E, \phi)$ on a nondegenerate spectral curve $S \xrightarrow{\pi} X$, and $\mathbf{u} = \pi(\tilde{\mathbf{u}})$. Then:*

1. *$\tilde{\mathbf{u}}$ has a summand of the form $\pi^*(\mathbf{u}')$ if and only if $L \to E$ vanishes at $\mathbf{u}'$, counted with multiplicity. In particular, $\tilde{\mathbf{u}}$ contains no summand equal to the pull-back of a divisor on X if and only if L is a subbundle of E.*
2. *The projection $\mathbf{u}$ satisfies $\mathscr{O}_X(\mathbf{u}) \simeq KL^{-2}\det(E)$. If L in particular is a subbundle of E, then $\mathbf{u}$ coincides with the zero divisor of $c_i(\phi)$.*
3. *The eigen-line bundle $\mathscr{L}_{(E,\phi)}$ of (E,ϕ) is isomorphic to $\pi^*\left(LK^{-1}\right)\otimes \mathscr{O}_S(\tilde{\mathbf{u}})$. In other words, (E,ϕ) is isomorphic to the direct image of $\pi^*(L)\otimes\mathscr{O}_S(\tilde{\mathbf{u}})$.*

Let us define an isomorphism class $[L \to E, \phi]$ of the input data of BA divisors by saying that two representative data are isomorphic if there are isomorphisms of the underlying bundles and line bundles that commute with the injections and Higgs fields [3]. Clearly BA divisors defined by isomorphic data coincide. The following theorem summarizes the invertible properties of the construction of BA divisors. We refer to the discussion following the proof of theorem 8.1 in Hitchin's original work [15] for an abstract proof and [5, 7] for a more explicit proof in terms of Abelian differentials.

THEOREM 4.6. *Let q be a quadratic differential whose zeroes are all simple and $S \xrightarrow{\pi} X$ its corresponding spectral curve. Then the construction of BA divisors and remembering the line bundles define a bijection*

$$\left\{[L \to E,\phi] \mid \det(\phi) = q\ \right\} \longleftrightarrow \left\{([L],\tilde{\mathbf{u}}) \;\middle|\; \begin{array}{l} \tilde{\mathbf{u}} \text{ effective on } S, \\ KL^{-2}\det(E) \simeq \mathscr{O}_X(\pi(\tilde{\mathbf{u}})) \end{array}\right\}.$$

In particular, in the inverse direction, the injection

$$L \longrightarrow E \simeq L \otimes \pi_*(\mathscr{O}_S(\tilde{\mathbf{u}})), \tag{4.8}$$

is obtained by taking the direct image of the canonical section of $\mathscr{O}_S(\tilde{\mathbf{u}})$.

[3] Since scalings are isomorphisms of line bundles, scaling the injections from line bundles to rank-2 bundles will define the same isomorphism class $[L \to E, \phi]$.

The map SoV For $0 < -2d \le g-1$, consider a cotangent vector on $\mathcal{N}_d$ at (E,L) which is the pull-back of some Higgs field ϕ on E via $i^* : T^*_E\mathcal{N} \to T^*_{(E,L)}\mathcal{N}_d$ where the spectral curve S_ϕ associated to (E,ϕ) is smooth. We then can assign a BA divisor on S_ϕ which defines a configuration of m points in T^*X. If $i^*(\phi) = i^*(\phi')$ for another Higgs field ϕ' on E, then $\phi - \phi'$ is a nilpotent Higgs field. The BA divisor on $S_{\phi'}$ then defines the same configuration of points in T^*X. Restricting to the locus of $T^*\mathcal{N}_d$ where the zeroes of the corresponding $c(E,L,\phi)$ are all simple, we can define a rational map

$$\mathrm{SoV} : T^*\mathcal{N}_d \dashrightarrow (T^*X)^{[m]}, \qquad m = 2g-2-2d.$$

The following is the main result in [7].

THEOREM 4.7. *(Theorem 1.1)* SoV *is a dominant Poisson map w.r.t. canonical symplectic structures whose generic fibers are* $2^{2g} : 1$.

On one hand, the map SoV is the high genus generalisation of the Separation of Variables technique applied to the classical Gaudin model, which is a variant of the Hitchin moduli space for $g = 0$ [25, 26]. On the other hand, it can be regarded as the classical limit of Drinfeld's approach to the geometric Langlands correspondence adapted to $G = SL_2(\mathbb{C})$ setting [7, 9, 12].

The map SoV_λ We have seen that from the data (E,L,∇_λ) where $c = c(E,L,\nabla_\lambda)$ has simple zeroes, one can induce an oper with simple apparent singularities at div(c) and residue parameters (4.5). By the same argument in defining SoV (cf. Proposition 2.10), for $0 < -2d \le g-1$ and fixed $\lambda \in \mathbb{C}^*$, we can define a rational map

$$\mathrm{SoV}_\lambda : \mathcal{M}^d_\lambda \dashrightarrow \mathcal{M}^m_{\mathrm{op},\lambda}, \qquad m = 2g-2-2d.$$

In the next subsection, we will show that SoV_λ is a Poisson map w.r.t. the Poisson structures we have defined on $\mathcal{M}^d_\lambda$ and $\mathcal{M}^m_{\mathrm{op},\lambda}$. To this end, it is more convenient to consider its lift to $H^{-1}(\lambda d)$, namely

$$\begin{array}{ccc} \widetilde{\mathcal{M}^d_\lambda} \supset H^{-1}(\lambda d) & \overset{\widetilde{\mathrm{SoV}_\lambda}}{\dashrightarrow} & \mathcal{M}^m_{\mathrm{op},\lambda} \\ \downarrow & \overset{\mathrm{SoV}_\lambda}{\nearrow} & \\ \mathcal{M}^d_\lambda & & \end{array} . \tag{4.9}$$

4.3 Proof of Poisson property

Let us state again the main theorem of this paper.

THEOREM 4.8. *(Theorem 1.2) For* $0 < -2d \le g-1$*, the map*

$$\mathrm{SoV}_\lambda : \mathcal{M}^d_\lambda \dashrightarrow \mathcal{M}^m_{\mathrm{op},\lambda}, \qquad m = 2g-2-2d, \tag{4.10}$$

is a dominant Poisson map.

The proof of Theorem 4.8 closely mirrors that of Theorem 1.1 in [7], except that there are additional λ-dependent terms in several formulas. In the following we shall outline these formulas to keep track of these terms and to show that in the key step (Lemma 4.9), these terms do not pose any significant challenges.

The idea of the proof is to write the map SoV_λ rather explicitly, upon choosing a reference divisor $\boldsymbol{r}$, in terms of the Darboux coordinates $(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{k},\boldsymbol{\kappa})$ of $\mathscr{M}^d_\lambda$. To this end, we again express the data (E,L,∇_λ) in terms of the restriction of ∇_λ to the open dense set $X_{\mathbf{q}}\subset X$,

$$\nabla_\lambda|_{X_{\mathbf{q}}}=\lambda\partial+\begin{pmatrix}\omega_0 & \omega_-\\ \omega_+ & -\omega_0\end{pmatrix},$$

(cf. (2.12) and Proposition 2.7). Recall from section 2.4 that the Abelian differential ω_0 can be expressed as

$$\omega_0=-\sum_{r=2}^{N}x_rk_r\omega^{\mathbf{q}}_{p_r-p_1}+\lambda\sum_{i=1}^{g}\omega^{\mathbf{q}}_{q_i-p_1}-\lambda\sum_{i=1}^{g-d}\omega^{\mathbf{q}}_{\check{q}_i-p_1}-\frac{1}{2}\sum_{i=1}^{g}\kappa_i\omega_i \tag{4.11}$$

where $\omega^{\mathbf{q}}_{x-y}$ is an Abelian differential of the third kind whose 0-th Laurent coefficient at q_i vanishes (cf. (2.21)). The scaled residue parameter $\nu_{\lambda,n}$ associated to the apparent singularity u_n is defined by

$$\nu_{\lambda,n}=\omega_0(u_n)-\lambda\frac{\omega_+''(u_n)}{4\omega_+'(u_n)},\qquad n=1,\dots,m, \tag{4.12}$$

where have taken derivative and evaluations of ω_+ using some chosen local coordinates around u_n. For this, we can write $\omega_0(u_n)$ even more explicitly by plugging the chain rule

$$\frac{\partial q_j(\mathbf{u})}{\partial u_n}=\sum_{i=1}^{g}\frac{\partial q_j}{\partial\lambda_i}\bigg|_{\boldsymbol{\lambda}(\mathbf{u})}\frac{\partial\lambda_i}{\partial u_n}\bigg|_{u_n}=-\frac{1}{2}\sum_{i=1}^{g}\left(dA^{-1}|_{\boldsymbol{\lambda}}\right)_{ij}\omega_i(u_n), \tag{4.13}$$

in equation (2.22) that relates $\omega^{\mathbf{q}}_{x-y}$ with the canonical Abelian differentials of the third kind ω_{x-y} whose A-cycles vanish. We then have

$$\begin{aligned}\omega_0(u_n)=&-\sum_{r=2}^{N}k_rx_r\left(\omega_{p_r-p_1}(u_n)+2\sum_{j=1}^{g}\omega_{p_r-p_1}(q_j)\frac{\partial q_j}{\partial u_n}\right)\\ &+\lambda\sum_{i=1}^{g}\omega^{\mathbf{q}}_{q_i-p_1}-\lambda\sum_{i=1}^{g-d}\omega^{\mathbf{q}}_{\check{q}_i-p_1}-\frac{1}{2}\sum_{i=1}^{g}\kappa_i\omega_i.\end{aligned} \tag{4.14}$$

Recall from (2.24) that the Abelian differential ω_+ can be expressed in terms of the prime forms on a fundamental domain of X from its zero divisor $\boldsymbol{u}=\sum_{n=1}^{m}u_n$ and pole structure (cf. Proposition 2.7). Its evaluation at p_r hence is

$$k_r=u_0\frac{\prod_{i=1}^{g}E(p_r,q_i(\boldsymbol{u}))^2\prod_{k=1}^{N+g-1}E(p_r,u_k)}{\prod_{j=1}^{g-d}E(p_r,\check{q}_k)^2}(\sigma(p_r))^2 \tag{4.15}$$

The key step in the proof is the following lemma.

LEMMA 4.9. *Let F be a complex-valued function on an open set in $\widetilde{\mathscr{M}^d_\lambda}$ equipped with local Darboux coordinates $(\boldsymbol{\lambda},\boldsymbol{x},\boldsymbol{\kappa},\boldsymbol{k})$ satisfying $\{k_r,F\}=0$ and $\{\lambda_\ell,F\}=0$. Then*

$$\begin{aligned}\frac{\partial}{\partial u_n}F(\boldsymbol{\lambda}(\mathbf{u}),\boldsymbol{k}(\mathbf{u}))&=-\left\{\nu_{\lambda,n},F\right\},\quad n=1,\dots,m;\\ u_0\frac{\partial}{\partial u_0}F(\boldsymbol{\lambda}(\mathbf{u}),\boldsymbol{k}(\mathbf{u}))&=\left\{\nu_{\lambda,0},F\right\},\quad \nu_{\lambda,0}:=\lambda d.\end{aligned} \tag{4.16}$$

Proof. On one hand, it follows from (4.15) that

$$\frac{1}{k_r}\frac{\partial k_r}{\partial u_n} = \frac{\partial \log E(p_r,u_n)}{\partial u_n} + 2\sum_{i=1}^{g}\frac{\partial \log E(p_r,q_i)}{\partial q_i}\frac{\partial q_i(u_n)}{\partial u_n}.$$

We then can write

$$\begin{aligned}\frac{\partial F}{\partial u_n} &= -\sum_{r=1}^{N}\frac{\partial k_r}{\partial u_n}\frac{\partial F}{\partial k_r} - \sum_{i=1}^{g}\frac{\partial \lambda_i}{\partial u_n}\frac{\partial F}{\partial \lambda_i} = -\sum_{r=1}^{N}\frac{\partial k_r}{\partial u_n}\frac{\partial F}{\partial k_r} + \frac{1}{2}\sum_{i=1}^{g}\omega_i(u_n)\frac{\partial F}{\partial \lambda_i} \\ &= \sum_{r=1}^{N}\left(\frac{\partial \log E(p_r,u_n)}{\partial u_n} + 2\sum_{i=1}^{g}\frac{\partial \log E(p_r,q_i)}{\partial q_i}\frac{\partial q_i}{\partial u_n}\right)k_r\frac{\partial F}{\partial k_r} - \frac{1}{2}\sum_{i=1}^{g}\omega_i(u_n)\frac{\partial F}{\partial \lambda_i}. \end{aligned} \tag{4.17}$$

Note that this means the LHS of the first relation in (4.16) is not λ-dependent.

On the other hand, in computing $\{\nu_{\lambda,n},F\}$ we can make use of the relations $\sum_{r=1}^{N} x_r k_r = \lambda d$ and $\omega_{p_+-p_-}(x) = d_x \log E(p_+,x) - d_x \log E(p_-,x)$ to rewrite the first term in (4.14) as

$$-\sum_{r=2}^{N} k_r x_r \omega_{p_r-p_1}(u_n) = -\sum_{r=1}^{N} k_r x_r d_x \log E(p_r,x)\,|_{x=u_n} + (\lambda d)\, d_x \log E(p_1,x)\,|_{x=u_n}\,.$$

Note also that when computing $\{\nu_{\lambda,n},F\}$ using (4.12) and (4.14), the only contribution comes from the Poisson brackets

$$\frac{\partial F}{\partial k_r} = \{\, x_r\,,\, F\,\}, \qquad -\frac{\partial F}{\partial \lambda_i} = \{\, \kappa_i\,,\, F\,\}.$$

All λ-dependent terms in $\nu_{\lambda,n}$ hence do not contribute to $\{\nu_{\lambda,n},F\}$. The first relation in (4.16) now follows from direct comparison. For the second relation, note that $\frac{\partial k_r}{\partial u_0} = \frac{k_r}{u_0}$, and hence

$$\{\lambda d, F\} = \left\{\sum_{r=1}^{N} x_r k_r, F\right\} = \sum_{r=1}^{N} k_r \frac{\partial F}{\partial k_r} = \sum_{r=1}^{N} u_0 \frac{\partial k_r}{\partial u_0}\frac{\partial F}{\partial k_r} = u_0 \frac{\partial F}{\partial u_0}.$$

□

Proof. (Proof of the main theorem) The relations

$$\{u_n, \nu_{\lambda,m}\} = \delta_{nm}, \qquad \{u_0, \nu_{\lambda,0}\} = u_0$$

follow from Lemma 4.9. The relations $\{u_n,u_m\} = 0$ follow from observation that $u_n = u_n(\boldsymbol{\lambda},\boldsymbol{k})$ while $\lambda_1,\dots,\lambda_g,k_1,\dots,k_N$ are Poisson commuting. The relations $\{\nu_{\lambda,n},\nu_{\lambda,m}\} = 0$ follow from an argument using grading on the algebra of polynomial functions in variables k_r and κ_i (cf. [7]). This shows that $\widetilde{\mathrm{SoV}}_\lambda$ and consequently SoV_λ are Poisson maps. That these maps are dominant follows from Proposition 4.3. □

5 Discussion

5.1 Relation to wobbly bundles

In constructing an analytic space that parametrises opers with $n+3g-3$ apparent singularities on Riemann surfaces with n punctures, Iwasaki identified and removed the singular loci. This locus is defined by effective divisors of degree $n+3g-3$ such that there exist quadratic differentials vanishing there and having poles at the specified punctures (cf. equation (6.3) in [16]). In the compact Riemann surfaces setting, the analogous locus is formed by

$$\mathscr{M}_{\mathrm{op}}^{3g-3}(Q) := \{[(u_1,v_1),\dots,(u_{3g-3},v_{3g-3})] \mid \sum_{n=1}^{3g-3} u_n \text{ is } Q\text{-special}\} \subset \mathscr{M}_{\mathrm{op}}^{3g-3}. \tag{5.1}$$

This is a regular locus in the affine bundle $\mathscr{M}^m_{\mathrm{op},\lambda}$ defined explicitly via (3.4) but needs to be removed if one wants to have a regular space that parametrises opers with $3g-3$ apparent singularities. A quick argument to convince oneself is to observe that the nonhomogeneous linear system (3.7) becomes degenerate at this locus: not all vectors $\vec{v}$ give opers with corresponding residue parameters, and in case there is an oper with such residue parameters it is not unique.

The locus $\mathscr{M}_{\mathrm{op}}^{3g-3}(Q)$ defined in (5.1) is related to one component of the so-called wobbly divisor on $\mathscr{N}$. A bundle E is called very stable if it does not admit nonzero nilpotent Higgs fields, and wobbly if it is stable but not very stable [8]. Laumon showed that very stable bundles are stable, and it was conjectured by Drinfeld that the wobbly locus $\mathscr{W}$ is a divisor on the moduli space of stable G-bundles for general Lie group G. Pal-Pauly [20] proved this conjecture for $G=SL_2(\mathbb{C})$, namely for the case of stable rank-2 bundles having fixed determinant Λ which can be assumed to be of degree 0 or 1. They furthermore showed that

$$\mathscr{W} = \begin{cases} \mathscr{W}_{\deg(\Lambda)} \cup \mathscr{W}_{\deg(\Lambda)+2} \cup \cdots \cup \mathscr{W}_g & \text{for } g \equiv \deg(\Lambda) \bmod 2, \\ \mathscr{W}_{\deg(\Lambda)} \cup \mathscr{W}_{\deg(\Lambda)+2} \cup \cdots \cup \mathscr{W}_{g-1} & \text{for } g \equiv \deg(\Lambda)-1 \bmod 2, \end{cases} \tag{5.2}$$

where $\mathscr{W}_k$ is the closure in $\mathscr{N}_\Lambda$ of the locus of bundles admitting some subbundle L such that $\deg(KL^2\Lambda^{-1})=k$. All $\mathscr{W}_k$ are irreducible divisors except for $\mathscr{W}_0$ which is an union of 2^{2g} irreducible divisors. The following result from [6] gives a criterion for producing wobbly bundles by taking direct images of distinguished line bundles from a spectral curve.

THEOREM 5.1. *[6] Consider a smooth spectral curve $S \xrightarrow{\pi} X$ defined by a quadratic differential on X, and $\widetilde{\mathbf{u}}$ an effective divisor on X such that $\mathbf{u}=\pi(\widetilde{\mathbf{u}})$ is Q-special and $2g-2<\deg(\mathbf{u})\leq 4g-4$. Then the rank-2 bundle $\pi_*(\mathscr{O}_S(\widetilde{\mathbf{u}}))$ on X is wobbly, provided that it is stable (which is the generic case). In this case, upon a twist by a line bundle to adjust the determinant, $\pi_*(\mathscr{O}_S(\widetilde{\mathbf{u}}))$ is contained in $\mathscr{W}_{4g-4-\deg(\mathbf{u})}$.*

For $S \subset \mathscr{M}^m_{\mathrm{op},\lambda}$ denote by $i\circ \mathrm{SoV}_\lambda{}^{-1}(S)$ the locus in $\mathscr{N}$ defined by projecting $\mathrm{SoV}_\lambda{}^{-1}(S)$ to $\mathscr{N}_d$ and then to $\mathscr{N}$. We have the following.

PROPOSITION 5.2.

1. *If (E,L,∇_λ) produces an oper with $3g-3$ simple apparent singularities that are zeroes of a quadratic differential, then E is wobbly with L the kernel of some nilpotent Higgs fields on E.*
2. *The closure in $\mathscr{N}$ of the locus $i\circ \mathrm{SoV}_\lambda{}^{-1}(\mathscr{M}_{\mathrm{op}}^{3g-3}(Q))$ is the wobbly component $\mathscr{W}_{g-1}$.*

5.2 Families of λ-connections

In this section, we will attempt to have a rather preliminary discussion on the space of Higgs bundles $(E,\phi)\in\mathscr{M}_H$ (λ-connections $(E,\nabla_\lambda)\in\mathscr{M}_\lambda$) that is the preimage along the map SoV (respectively, SoV_λ) of the subset of $(T^*X)^{[3g-3]}$ (respectively, $\mathscr{M}_{\mathrm{op}}^{3g-3}$) defined by fixing a divisor $\mathbf{u}$ on X. We will sketch a relation between these two spaces and draw comparison to the Lagrangian leaves defined by the $\mathbb{C}^*$ on the Hodge moduli space $\mathscr{M}_{Hod}$ of λ-connections.

5.2.1 Lagrangian leaves and the conformal limit

Lagrangian upward flows in Hitchin moduli space Recall from [15] that the Hitchin moduli space $\mathscr{M}_H$ admits a $\mathbb{C}^*$-action that scales the Higgs fields. A Higgs bundle $\mathscr{E} = (E_0, \phi_0) \in \mathscr{M}_H^{\mathbb{C}^*}$ has an upward flow $W^0_{\mathscr{E}}$ consisting of Higgs bundles (E, ϕ) with $\lim_{\epsilon \to 0}[(E, \epsilon\phi)] = [\mathscr{E}]$. For example, for $\mathscr{E} = (E_0, 0)$ with E_0 stable, we have $W^0_{\mathscr{E}}$ consist of Higgs fields on E_0, and for $\mathscr{E} = (E_0, \phi_0)$ with E_0 destabilised by a subbundle L_0, we have $W^0_{\mathscr{E}}$ consist of Higgs bundles (E, ϕ) where E is also destabilised by L_0 and the zero divisor of $c(E, L_0, \phi)$ coincides with that of $c(E_0, L_0, \phi_0)$.

On the other hand, the $\mathbb{C}^*$-action acts linearly on $T_{\mathscr{E}}\mathscr{M}_H$. Consider the weight decomposition $T_{\mathscr{E}}\mathscr{M}_H = \oplus_{k \in \mathbb{Z}}(T_{\mathscr{E}}\mathscr{M}_H)_k$ where $(T_{\mathscr{E}}\mathscr{M}_H)_k$ is the weight space in which $\epsilon \in \mathbb{C}^*$ acts as ϵ^k, and let $V_{\mathscr{E}} := \oplus_{k>0}(T_{\mathscr{E}}\mathscr{M}_H)_k$ be the positive weight subspace.

PROPOSITION 5.3. *[1, ?] If $\mathscr{E}$ is stable then $W^0_{\mathscr{E}} \simeq V_{\mathscr{E}} \simeq \mathbb{C}^{3g-3}$ as varieties with $\mathbb{C}^*$-action. Furthermore, $W^0_{\mathscr{E}}$ is a Lagrangian in $\mathscr{M}_H$.*

Note in particular that the $\mathbb{C}^*$-fixed point $\mathscr{E}$ is naturally identified with the origin in $V_{\mathscr{E}} \simeq \mathbb{C}^{3g-3}$.

Lagrangians in moduli space $\mathscr{M}_\lambda$ of λ-connections Simpson studied the $\mathbb{C}^*$-action on the Hodge moduli space $\mathscr{M}_{Hod}$ of λ-connections with varying $\lambda \in \mathbb{C}$ and similarly define the upward flows $W_{\mathscr{E}}$ to $\mathbb{C}^*$-fixed points $\mathscr{E} = (E_0, \phi_0)$ in $\mathscr{M}_H \equiv \mathscr{M}_{\lambda=0}$. For E_0 stable and $\phi_0 = 0$, $W_{\mathscr{E}}$ consists of all irreducible λ-connections on E_0, and for E_0 destabilised by subbundle L, $W_{\mathscr{E}}$ consists of (E, ∇_λ) where E is also destabilised by L and the zero divisors of $c(L, E, \nabla_\lambda)$ and $c(L, E_0, \phi_0)$ coincide. The restriction of $W_{\mathscr{E}}$ to $\mathscr{M}_H \equiv \mathscr{M}_{\lambda=0}$ defines the upward flows $W^0_{\mathscr{E}}$; we similarly denote by $W^\lambda_{\mathscr{E}}$ the restriction of $W_{\mathscr{E}}$ to the moduli space $\mathscr{M}_\lambda$ of λ-connections with fixed $\lambda \in \mathbb{C}^*$. For $\lambda \neq 0$, Simpson conjectured that the leaves $W^\lambda_{\mathscr{E}}$ are closed [4] in $\mathscr{M}_\lambda \equiv \mathscr{M}_{dR}$.

Conformal limit and biholomorphism between Lagrangian leaves Let us recall from [3, 22] that the nonabelian Hodge correspondence is a diffeomorphism between the Hitchin and de Rham moduli spaces

$$\mathrm{NAH} : \mathscr{M}_H \equiv \mathscr{M}_0 \longrightarrow \mathscr{M}_{dR} \equiv \mathscr{M}_1$$

obtained by solving Hitchin's self-duality equations which define stable Higgs bundles. It is known that for a $\mathbb{C}^*$-fixed stable Higgs bundle $\mathscr{E}$,

$$\mathrm{NAH}(\mathscr{E}) \in W^1_{\mathscr{E}}.$$

In [2], Collier-Wentworth showed that there is a biholomorphic embedding

$$p_{Hod} : V_{\mathscr{E}} \times \mathbb{C} \xrightarrow{\sim} W_{\mathscr{E}}. \tag{5.3}$$

The restrictions p^0_{Hod} and p^λ_{Hod} of p_{Hod} to $V_{\mathscr{E}} \times \{0\}$ and $V_{\mathscr{E}} \times \{\lambda\}$ for some $\lambda \in \mathbb{C}^*$

$$\begin{array}{ccccc} & & V_{\mathscr{E}} & & \\ & \swarrow^{p^0_{Hod}} & & \searrow^{p^\lambda_{Hod}} & \\ W^0_{\mathscr{E}} & & & & W^\lambda_{\mathscr{E}} \end{array} \tag{5.4}$$

are biholomorphisms satisfying

[4] The construction of Lagrangian leaves hold for all ranks. Simpson's conjecture on the closed-ness of $W^\lambda_{\mathscr{E}}$ for $\lambda \neq 0$ was proved for the rank-2 cases in [4].

- p^0_{Hod} is the isomorphism $V_{\mathscr{E}} \simeq W_{\mathscr{E}}$ in Proposition 5.3;
- $p^\lambda_{Hod} \circ (p^0_{Hod})^{-1}(\mathscr{E})$ corresponds to NAH($\mathscr{E}$) scaled by λ.

We then see that the map

$$\mathrm{CL}_\lambda := p^\lambda_{Hod} \circ (p^0_{Hod})^{-1} : W^0_{\mathscr{E}} \longrightarrow W^\lambda_{\mathscr{E}}$$

is a biholomorphism that can be constructed in two steps:

1. Use (the scaling of) NAH($\mathscr{E}$) as a “reference connection”, i.e. a counter-part of the “origin” $\mathscr{E}$;
2. Use $V_{\mathscr{E}}$ as a deformation space on both $\mathscr{M}_H$ and $\mathscr{M}_\lambda$ to cover $W^0_{\mathscr{E}}$ and $W^\lambda_{\mathscr{E}}$ respectively.

We have used the notation CL_λ as this map is related to the $\hbar$-conformal limit defined by Gaiotto [13, 11], which is the limit of a family of flat connections, i.e. elements of $\mathscr{M}_{dR} \equiv \mathscr{M}_1$, induced by the nonabelian Hodge correspondence [5] of a Higgs bundle (E,ϕ). Collier-Wentworth showed that for any Higgs bundle (E,ϕ) with stable $\mathbb{C}^*$-limit, its $\hbar$-conformal limit is equal to $\hbar^{-1}.\mathrm{CL}_\hbar((E,\phi))$. Note that in the formulation of the conformal limit, the technical step is to show existence of the limit of the families of flat connections and then compute this limit [2, 11]. On the other hand, in the work of Collier-Wentworth [2], the technical part was to show existence of a deformation space $V_{\mathscr{E}}$ that can be embedded in both $\mathscr{M}_H$ and $\mathscr{M}_{dR}$ to carry out Step 2 above.

Examples of conformal limit

EXAMPLE 5.4. Let $E_\lambda = E$ a fixed stable bundle for all $\lambda \in \mathbb{C}$, and ∇^{ref}_1 the holomorphic connection defined by the nonabelian Hodge correspondence of the Higgs bundle $(E,0)$. For each $\lambda \in \mathbb{C}$, define $\nabla^{\mathrm{ref}}_\lambda = \lambda.\nabla^{\mathrm{ref}}_1$; in particular ∇^{ref}_0 is the Higgs bundle $\mathscr{E} = (E,0)$. The deformation space is $V_{\mathscr{E}} = H^0(X, \mathrm{End}_0(E)) \simeq \mathbb{C}^{3g-3}$, and so the deformation $\delta \in V_{\mathscr{E}}$ are trace-free Higgs fields on E. The assignment

$$\begin{aligned} \mathrm{CL}_\lambda : W^0_{\mathscr{E}} &\longrightarrow W^\lambda_{\mathscr{E}} \\ (E,\delta) &\longmapsto \left(E, \nabla^{\mathrm{ref}}_\lambda + \delta\right) \end{aligned}$$

then defines a biholomorphic map between the space of Higgs fields on E and the space of irreducible λ-connections on E.

EXAMPLE 5.5. For $\lambda \in \mathbb{C}^*$, let E' be the bundle realised as the non-trivial extension of $K_X^{-1/2}$ by $K_X^{1/2}$, which is unique up to scaling, and let $E_0 = K_X^{1/2} \oplus K_X^{-1/2}$. Let $\mathscr{E} = (E_0, \phi_0)$ with $\phi_0 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and denote by $(E', \nabla^{\mathrm{ref}}_\lambda)$ the λ-scaling of the nonabelian Hodge correspondence $(E', \nabla^{\mathrm{ref}}_1)$ of $\mathscr{E}$. Note that $(E', \nabla^{\mathrm{ref}}_1)$ is often called the “uniformising oper”, which in coordinates on X induced by coordinates on its universal covering – the upper-half plane – takes the form $\nabla^{\mathrm{ref}}_1 = \partial + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ in local frames adapted to $K_X^{1/2}$. Let $V_{\mathscr{E}}$ be the space of quadratic differentials on X and, for $q \in V_{\mathscr{E}}$, abusing the notation denote by $\delta_q = \left(\begin{smallmatrix} 0 & q \\ 0 & 0 \end{smallmatrix}\right)$ both the corresponding Higgs fields on E and E'. The assignment

$$\begin{aligned} \mathrm{CL}_\lambda : W^0_{\mathscr{E}} &\longrightarrow W^\lambda_{\mathscr{E}} \\ \left(E_0, \phi_0 + \delta_q\right) &\longmapsto \left(E', \nabla^{\mathrm{ref}}_\lambda + \delta_q\right) \end{aligned}$$

[5] More precisely, the family of flat connections is induced by applying two scaling to the nonabelian Hodge correspondence: one by the $\mathbb{C}^*$-action on $\mathscr{M}_H$ and one by the so-called real twistor line in $\mathscr{M}_{Hod}$.

defines a biholomorphic map between the Hitchin section and the space of λ-opers

$$W^0_{\mathscr{E}} = \left\{\left(E_0, \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right)\right\} \longrightarrow W^\lambda_{\mathscr{E}} = \left\{\left(E', \lambda\partial + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}\right)\right\} \tag{5.5}$$

(upon choosing a square-root $K_X^{1/2}$).

5.2.2 Families of λ-connections with fixed zero divisor $c(E, L, \nabla_\lambda)$

Examples 5.5 and 5.4 demonstrate the fact that for a family $\{(E_\lambda, \nabla_\lambda)\}_{\lambda\in\mathbb{C}}$ of λ-connections, if one wants to keep fixed one of the data

- the zero divisor of $c \equiv c(L_\lambda \hookrightarrow E_\lambda, \nabla_\lambda)$,
- the isomorphism class of E_λ

for all $\lambda \in \mathbb{C}$, then the other data has a "jump" as $\lambda \to 0$ (cf. (2.15)). In particular, if the family is defined by $\mathbb{C}^*$-action, then $c \neq 0$ and $\operatorname{div}(c)$ is fixed only if E_λ are destabilised by L.

Let us observe that, however, we could define a family $\{(E_\lambda, \nabla_\lambda)\}_{\lambda\in\mathbb{C}}$ where $\operatorname{div}(c)$ is fixed and E_λ are generically stable for all $\lambda \in \mathbb{C}$ provided that we are willing to let E_λ vary. The following is a simple example.

EXAMPLE 5.6. Let $E \equiv E_0$ be a stable bundle, L a maximal subbundle of E_0, and ϕ a Higgs field on E. Denote by $\boldsymbol{x}_0 \equiv (L \hookrightarrow E) \in H^1(X, L^2)$ an extension class realising E and $c \equiv c(\boldsymbol{x}_0, \phi)$. Pick an extension class $\boldsymbol{x}_1 \in H^1(X, L^2)$ which satisfies $\langle \boldsymbol{x}_1, c\rangle = 1$ (so in particular it not contained in the hyperplane $\ker(c) \subset H^1(X, L^2)$). For $\lambda \in \mathbb{C}$, consider a family of bundles E_λ realised by extension classes $\boldsymbol{x}_\lambda$ where

$$\boldsymbol{x}_\lambda := \boldsymbol{x}_0 + \lambda \deg(L)\boldsymbol{x}_1, \qquad \lambda \in \mathbb{C}. \tag{5.6}$$

A generic element E_λ in this family is stable and admit L as a maximal subbundle [18]; in this case, E_λ is in particular not "overcounted" in $H^1(X, L^2)$, namely $h^0(X, L^{-1}E_\lambda) = 1$. Suppose that the condition $h^0(X, L^{-1}E_\lambda) = 1$ holds for all $\lambda \in \mathbb{C}$. It follows from Proposition 2.9 that we can define a family $\{(E_\lambda, \nabla_\lambda)\}_{\lambda\in\mathbb{C}}$ where $c(E_\lambda, L, \nabla_\lambda) = c$.

Note that this family is not unique as the choice of $\boldsymbol{x}_1$ is not unique – one can in fact form a basis for $H^1(X, L^2)$ consisting of such elements – and even upon fixing such $\boldsymbol{x}_1$ there exist different choices of ∇_λ whose difference is an L-invariant Higgs field – these are of dimension $h^0(X, E^*LK)$ (cf. Propositions 2.3 and 2.9)

Deformation spaces in the Hitchin moduli space Consider a reduced divisor $\mathbf{u} = \sum_{n=1}^{3g-3} u_n$ on X, i.e. all u_n have multiplicity 1. The deformation space associated to $\mathbf{u}$ is the space $\mathscr{M}_H(\mathbf{u}) \subset \mathscr{M}_H$ defined by

$$\mathscr{M}_H(\boldsymbol{u}) := \{(E, \phi) \in \mathscr{M}_H \mid \exists L \hookrightarrow E,\ \operatorname{div}(c(E, L, \phi)) = \boldsymbol{u}\}.$$

The subbundle L in the definition of $\mathscr{M}_H(\mathbf{u})$ is a square-root of $K_X \otimes \mathscr{O}_X(-\mathbf{u})$. For each $(E, \phi) \in \mathscr{M}_H(\mathbf{u})$ there exists a unique subbundle L of E such that $c(E, L, \phi)$ has zero divisor $\mathbf{u}$, and so $\mathscr{M}_H(\mathbf{u})$ has 2^{2g} connected components each corresponding to a choice of the square-root. Consider the dense subset $\mathscr{M}^s_H(\mathbf{u}) \subset \mathscr{M}_H(\mathbf{u})$ defined by smooth spectral curves. The composition of SoV with the pull-back of $(E, \phi) \in \mathscr{M}_H(\mathbf{u})$ to $T^*_{(E,L)}\mathscr{N}_d$ defines a map

$$\mathrm{SoV}(\mathbf{u}) : \mathscr{M}^s_H(\mathbf{u}) \longrightarrow \mathbb{C}_{\mathbf{u}}$$

where

$$\mathbb{C}_{\mathbf{u}} := \prod_{i=1}^{3g-3} T^*_{u_i} X \simeq \mathbb{C}^{3g-3}$$

is a Lagrangian in $(T^*X)^{[3g-3]}$. Denote by $\mathscr{M}_H^{ss}(\mathbf{u})$ the open dense subset in $\mathscr{M}_H^{s}(\mathbf{u})$ defined by Higgs bundles (E,ϕ) where E is stable.

PROPOSITION 5.7. *Let* $\mathbf{u} = \sum_{n=1}^{3g-3} u_n$ *be a Q-generic divisor. Then*

1. *The map* $\mathrm{SoV}(\mathbf{u})$ *is a* $2^{2g} : 1$ *unbranched holomorphic covering onto its image which is dense in* $\mathbb{C}_{\mathbf{u}}$.
2. $\mathscr{M}_H^{ss}(\mathbf{u})$ *is a Lagrangian in* $\mathscr{M}_H$.

Proof. 1. Observe that any point in $\mathbb{C}_{\mathbf{u}}$ defines a spectral curve, and in particular an open dense subset of $\mathbb{C}_{\mathbf{u}}$ defines smooth spectral curves. The claim now comes from the invertibility of the construction of Baker- Akhiezer divisors (cf. Theorem 4.6)) up to tensoring with square-roots of $\mathscr{O}_X$.

2. The symplectic structure of $\mathscr{M}_H$ at the loci defined by (E,ϕ) where E is stable coincides with the canonical symplectic structure of $T^*\mathscr{N}$. The claim now follows from the Theorem 1.1 and the fact that $i : \mathscr{N}_d \dashrightarrow \mathscr{N}$ is a local isomorphism away from the loci that makes $\mathbf{u}$ a Q-special divisor (cf. Proposition 2.6).

□

REMARK 5.8. 1. Given a Higgs bundle $(E,\phi) \in \mathscr{M}_H^{ss}(\mathbf{u})$, its scaling $\epsilon.(E,\phi)$ for some $\epsilon \in \mathbb{C}^*$ is also contained in $\mathscr{M}_H^{ss}(\mathbf{u})$. The closure of $\mathscr{M}_H^{ss}(\mathbf{u})$ in $\mathscr{M}_H$ therefore contains the $\mathbb{C}^*$-fixed point $(E,0)$ in the nilpotent cone.

2. It is natural to expect that the Lagrangian property of $\mathscr{M}_H^{ss}(\mathbf{u})$ extends to all of $\mathscr{M}_H(\mathbf{u})$.

Analogues of deformation spaces in the moduli space of λ-connections? For a fixed $\lambda \neq 0$, in the moduli space $\mathscr{M}_\lambda$ of λ-connections one can consider an analogous subspace

$$\mathscr{M}_\lambda(\boldsymbol{u}) := \{(E,\nabla_\lambda) \in \mathscr{M}_\lambda \mid \exists L \hookrightarrow E,\ \mathrm{div}(c(E,L,\nabla_\lambda)) = \boldsymbol{u}\},$$

which also has 2^{2g} connected components corresponding to square-roots of $K_X \otimes \mathscr{O}_X(-\mathbf{u})$. The map SoV_λ then allows us to define the analogue of $\mathrm{SoV}(\mathbf{u})$, namely

$$\mathrm{SoV}_\lambda(\mathbf{u}) : \mathscr{M}_\lambda(\mathbf{u}) \longrightarrow \mathbb{C}_{\lambda,\mathbf{u}} := \prod_{i=1}^{3g-3} \mathscr{M}'_{\mathrm{op}}\,|_{u_i}$$

where $\mathscr{M}'_{\mathrm{op}}$ is the affine bundle on X modeled over T^*X defined via (3.6). By construction, the target space $\mathbb{C}_{\lambda,\mathbf{u}}$ is an affine space modeled over the vector space $\mathbb{C}_{\mathbf{u}} \simeq \mathbb{C}^{3g-3}$ and defines a Lagrangian in $\mathscr{M}_{\mathrm{op}}^{3g-3}$.

Suppose now $\mathbf{u}$ is Q-generic. Note that by Proposition 3.2, a point in $\mathbb{C}_{\lambda,\mathbf{u}}$ corresponds uniquely to an oper with apparent singularities at $\mathbf{u}$. By Proposition 4.3, one can find a λ-connection (E,∇_λ) that induces such an oper, and if $(E,\nabla_\lambda) \in \mathscr{M}_\lambda$ (i.e. it is irreducible) then there are exactly 2^{2g} such λ-connections, which differ from each other by a twist by a square-root of $\mathscr{O}_X$. Denote by $\mathbb{C}^s_{\mathbf{u}} \subset \mathbb{C}_{\mathbf{u}}$ the image of $\mathscr{M}_H^s(\mathbf{u})$ along $\mathrm{SoV}(\mathbf{u})$ (note that $0 \notin \mathbb{C}^s_{\mathbf{u}}$). We expect that there exists a dense subspace $Y \subset \mathbb{C}_{\lambda,\mathbf{u}}$ such that

- Y is biholomorphic to $\mathbb{C}^s_{\mathbf{u}}$;
- the preimage $\mathscr{M}'_\lambda(\mathbf{u}) := (\mathrm{SoV}_\lambda(\mathbf{u}))^{-1}(Y)$ is a Lagrangian in $\mathscr{M}_\lambda$ and is biholomorphic to $\mathscr{M}^s_H(\mathbf{u})$.

If this is true, then we can regard $\mathbb{C}^s_{\mathbf{u}}$ as a deformation space which is holomorphically embedded in both $\mathscr{M}_H$ and $\mathscr{M}_\lambda$. A (non-canonical) choice of family of λ-connections as in Example 5.6 would then define an analogue of the embedding p_{Hod} defined in (5.3).

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References

[1] Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) 98, 480–497 (1973).

[2] B. Collier, R. Wentworth, *Conformal limits and the Białynicki-Birula stratification of the space of λ-connections*, Advances in Mathematics 350(3):1193-1225 (2019).

[3] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom., 28(3):361–382 (1988).

[4] P. Dimakis, S. Schulz, *On a conjecture of Simpson*, arXiv:2410.12945 (2024).

[5] Đinh D. *Stratified description of the moduli spaces of Higgs bundles and connections*, PhD thesis, *electronic publication* (2023).

[6] Đinh D., *Rank-2 wobbly bundles from special divisors on spectral curves*, arXiv:2406.04224 (2024).

[7] Đinh D., J. Teschner, *Classical limit of the geometric Langlands correspondence for $SL_2(\mathbb{C})$*, arXiv:2312.13393 (2023).

[8] R. Donagi, T. Pantev, *Geometric Langlands and non-abelian Hodge theory*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, volume 13 of Surv. Differ. Geom., p85–116. Int. Press, Somerville, MA (2009).

[9] V. Drinfeld, *Two-dimensional l-Adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL_2*, American Journal of Mathematics, vol. 105, no. 1, 85-114 (1983).

[10] B. Dubrovin, M. Mazzocco, *Canonical Structure and Symmetries of the Schlesinger Equations*, Comm. Math. Phys. 271, 289–373 (2007).

[11] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, A. Neitzke, *From the Hitchin section to opers through Nonabelian Hodge*, Journal of Differential Geometry, vol. 117, no. 2, 223 - 253 (2021).

[12] E. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, Proceedings of the International Congress of Mathematical Physics, Paris, 1994 (D. Iagolnitzer, ed.), 606–642. International Press (1995).

[13] D. Gaiotto, *Opers and TBA*, arXiv:1403.6137 [hep-th] (2014).

[14] M. Gronow, *Extension maps and the moduli spaces of rank 2 vector bundles over an algebraic curve*, PhD Thesis, Durham University (1997).

[15] N. Hitchin, *The Self-Duality Equations On A Riemann Surface*, Proc. London Math. Soc. 55, 59-126 (1987).

[16] K. Iwasaki, *Moduli and deformation for Fuchsian projective connections on a Riemann surface*, J. Fac. Sci. Univ. Tokyo, 38, 431-531 (1991).

[17] K. Iwasaki, *Fuchsian moduli on a Riemann surface–its Poisson structure and Poincaré-Lefschetz duality*, Pacific J. Math. 155(2), 319-340 (1992).

[18] H. Lange, M. S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Mathematische Annalen, vol. 266, 55–72 (1983).

[19] S. Oblezin, *Isomonodromic Deformations of* $sl(2)$ *Fuchsian Systems on the Riemann Sphere*, Moscow Mathematical Journal, volume 5, issue 2, pages 415-441 (2005).

[20] S. Pal, C. Pauly, *The wobbly divisors of the moduli space of rank-2 vector bundles*, Advances in Geometry 21 (4), 473–482 (2018).

[21] D. Pinchbeck, *Isomonodromic flows for Fuchsian connections on Riemann surfaces*, International Mathematics Research Notices, Volume 2005, Issue 40, Pages 2473–2497 (2005).

[22] C. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc., 1(4):867–918 (1988).

[23] C. Simpson, *The Hodge filtration on nonabelian cohomology*, Algebraic geometry—Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pages 217–281. Amer. Math. Soc., Providence, RI (1997).

[24] C. Simpson, *Iterated destabilizing modifications for vector bundles with connection*, Vector bundles and complex geometry, volume 522 of Contemp. Math., pages 183–206. Amer. Math. Soc., Providence, RI (2010).

[25] E. Sklyanin, *Separation of variables in the Gaudin model*, J. Soviet Math. 47, 2473–2488 (1989).

[26] J. Teschner, *Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence*, Geometry and Physics, Festschrift in honour of Nigel Hitchin, Vol. I, eds. Dancer, J.E. Andersen, and O. Garcia-Prada, pp. 347–375, Oxford University Press, arXiv:1707.07873 (2018).

[27] A. Tyurin, *Classification of vector bundles over an algebraic curve of arbitrary genus*, Amer. Math. Soc. Translat., II, Ser. 63 (1967), 245-79 (English); Izv. Ak. Nauk. SSSR, Ser. Mat. 29, 657-88 (Russian) (1965).