ON ENDOMORPHISM ALGEBRAS OF STRING ALMOST GENTLE ALGEBRAS

YU-ZHE LIU AND PANYUE ZHOU*

ABSTRACT. For any arbitrary string almost gentle algebra, we consider specific subsets of its quiver's arrow set, denoted by \mathcal{R} . For each such \mathcal{R} , we introduce the finitely generated module $M_{\mathcal{R}}$ and define its associated \mathcal{R} -endomorphism algebra $A_{\mathcal{R}}$. In this paper, we show that the representation type of a string gentle algebra A, the representation type of the \mathcal{R} -endomorphism algebra $A_{\mathcal{R}}$ for some \mathcal{R} , the representation types of all \mathcal{R} -algebras, and the representation type of the Cohen-Macaulay Auslander algebra A^{CMA} of A are equivalent. The results presented here reveal a deep structural connection between different classes of algebras derived from string gentle algebras. By showing the equivalence of representation types, this work offers new insights into the nature of endomorphism algebras and Cohen-Macaulay Auslander algebras, contributing to a broader understanding of their algebraic properties and classification.

CONTENTS

1. Introduction	1
2. String algebras, SAG-algebras, and their module categories	3
2.1. String algebras and SAG-algebras	3
2.2. The module categories of string algebras	4
2.3. Cycles	6
3. The module αA	7
3.1. αA is an indecomposable module	7
3.2. Homomorphisms starting (resp. ending) with αA	7
4. \mathcal{R} -endomorphism algebras	13
4.1. \mathcal{R} -bound quiver	14
4.2. \mathcal{R} -endomorphism algebra	15
5. On representation types of SAG-algebras	17
5.1. Representation types of SAG-algebras and \mathcal{R} -endomorphism algebras	17
5.2. Representation types of SAG-algebras and CM-Auslander algebras	18
5.2.1. Perfect forbidden cycles	19
Acknowledgements	21
References	22

1. INTRODUCTION

String almost gentle algebras (abbreviated as SAG-algebras), a special class of string algebras, play an important role in representation theory and were first introduced by Green and Schroll in [GS18]. The systematic study of string algebras can

²⁰²⁰ Mathematics Subject Classification. 16G60; 05E10.

Key words and phrases. gentle algebra; endomorphism algebra; representation type; Cohen-Macaulay Auslander algebra.

^{*}Corresponding author.

be traced back to the work on finitely generated module categories over string algebras in [BR87], where Butler and Ringel provided descriptions of indecomposable modules using strings and bands on the bound quivers of string algebras. Furthermore, by applying the Brauer-Thrall theorem (see, for example, [ASS06, Chapter IV, Section IV.5]), it is understood that the representation types of string and gentle algebras are characterized by the existence of bands.

In [Pla19], Plamondon shows that all (support) τ -tilting finite gentle algebras are representation-finite, which partially answers the Brauer-Thrall Problem within the context of τ -tilting theory—specifically, whether a τ -tilting finite algebra is necessarily representation-finite. Building on these results, Mousavand investigated the relationship between representation types and τ -tilting finiteness in biserial algebras in [Mou23], providing examples of finite-dimensional algebras where the representation type and τ -tilting finiteness do not coincide. Furthermore, in [LZH22], the authors offer an alternative description of gentle algebras using Gorenstein projective support τ -tilting modules (abbreviated as GPS τ -tilting modules), based on the work of [Kal15]. The concept of $GPS\tau$ -tilting modules, introduced by Xie and Zhang in [XZ21], refers to modules that are both Gorenstein projective and support τ -tilting. The authors demonstrate that a gentle algebra, denoted as Λ , is representation-finite if and only if, for any $\text{GPS}\tau$ -tilting module M, the endomorphism algebra $\operatorname{End}_A(M)$ is also representation-finite. This result establishes a significant connection between Gorenstein projective modules, τ -tilting modules, and the representation types of gentle algebras. The proof hinges on the fact that any Gorenstein projective module over a gentle algebra Λ is isomorphic to $\alpha \Lambda$, where α is an arrow satisfying certain special conditions. Consequently, the Cohen-Macaulay Auslander algebra (abbreviated as CM Auslander algebra) $\Lambda^{\check{C}MA}$ of Λ takes the form $\operatorname{End}_{\Lambda}(\Lambda \oplus \bigoplus_{\alpha} \alpha \Lambda)$. It is worth noting that, in [CL17, CL19], Chen and Lu revealed that the representation types of (skew-)gentle algebras and their CM Auslander algebras coincide. However, for an algebra $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$, not all αA are Gorenstein projective. Thus, this naturally raises the following question.

Question 1.1. Is there a subset \mathcal{R} of the arrow set of \mathcal{Q} such that the representation types of A and $\operatorname{End}_A(A \oplus \bigoplus_{\alpha \in \mathcal{R}} \alpha A)$ coincide?

We will address the above questions in the case where A is an SAG-algebra. Throughout this paper, we assume that \Bbbk is an algebraically closed field, and we define a quiver as a quadruple $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$, where \mathcal{Q}_0 is the set of vertices, \mathcal{Q}_1 is the set of arrows, and \mathfrak{s} and \mathfrak{t} are functions $\mathcal{Q}_1 \to \mathcal{Q}_0$ that assign to each arrow $a \in \mathcal{Q}_1$ its source and target, respectively. Furthermore, we denote by \mathcal{Q}_ℓ the set of all paths of length ℓ (hence, \mathcal{Q}_0 naturally corresponds to the set of all paths of length zero, and \mathcal{Q}_1 to the set of all paths of length one). If a and b are arrows such that $\mathfrak{t}(a) = \mathfrak{s}(b)$, the composition of a and b is denoted by ab. All algebras considered in this paper are finite-dimensional \Bbbk -algebras, and for any algebra A, all modules under consideration are finitely generated right A-modules.

Let A be an SAG-algebra with bound quiver $(\mathcal{Q}, \mathcal{I})$. The main results of this paper are summarized as follows.

Theorem 1.2 (Theorem 4.4). There exists at least one subset \mathcal{R} of \mathcal{Q}_1 (note that the module αA with $\alpha \in \mathcal{R}$ may not be Gorenstein projective; see Example 5.8) such that the following statements hold:

- (1) The bound quiver $(\mathcal{R}(\mathcal{Q}), \mathcal{R}(\mathcal{I}))$ of $A_{\mathcal{R}} := \operatorname{End}_A (A \oplus \bigoplus_{\alpha \in \mathcal{R}} \alpha A)$ can be described by Steps 1-6 in Subsection 4.1;
- (2) $A_{\mathcal{R}}$ is an SAG-algebra.

Indeed, the subset \mathcal{R} in the above theorem is the set of certain left forbidden arrows on $(\mathcal{Q}, \mathcal{I})$, referred to as a left forbidden arrow index; see Definition 4.1. In the case where A is a gentle algebra, \mathcal{R} can be equal to

 $\mathcal{G} = \alpha \in \mathcal{Q}_1 \mid \alpha A$ is both non-projective and Gorenstein projective

or another left forbidden arrow index. In particular, when $\mathcal{R} = \mathcal{G}$, we have $A_{\mathcal{G}} = A_{\mathcal{R}} \cong A^{\text{CMA}}$, as stated in [CL19, Theorem 3.5]. The following result extends the findings of [CL19, Theorem 3.5] to SAG-algebras.

Theorem 1.3 (Theorem 5.6). Let C_1, \ldots, C_t be perfect forbidden cycles on the bound quiver (Q, \mathcal{I}) of an SAG-algebra A. Then $A_{\mathcal{R}_p}$ is isomorphic to the CM-Auslander algebra A^{CMA} of A.

The following theorem provide some descriptions of the representation types of SAG-algebras.

Theorem 1.4. An SAG-algebra $A = \mathbb{k}Q/\mathcal{I}$ is representation-finite if and only if either of the following statements holds.

- (1) (Theorem 5.2) There exists a left forbidden arrow index \mathcal{R} such that $A_{\mathcal{R}}$ is representation-finite.
- (2) (Corollary 5.3) For all left forbidden arrow indices \mathcal{R} , the \mathcal{R} -endomorphism algebras $A_{\mathcal{R}}$ is representation-finite.
- (3) (Corollary 5.7) The CM-Auslander algebra A^{CMA} of A is representation-finite.
- 2. String Algebras, SAG-Algebras, and their module categories

2.1. String algebras and SAG-algebras. A monomial algebra is a finite dimensional k-algebra which is Morita equivalent to kQ/\mathcal{I} such that \mathcal{I} is generated by some paths of length ≥ 2 . String algebras are special monomial algebras. In this part, we recall some concepts for string algebras.

Let \mathcal{Q} be a quiver and \mathcal{I} be an ideal of $\mathbb{k}\mathcal{Q}$ such that $\mathbb{k}\mathcal{Q}/\mathcal{I}$ is a monomial algebra. We say a bound quiver $(\mathcal{Q}, \mathcal{I})$ is a *string pair* if it satisfies the following conditions.

- (S1) Any vertex of Q is the source of at most two arrows and the target of at most two arrows.
- $(S2)_{R}$ For each arrow $\alpha : x \to y$, there is at most one arrow β whose source $\mathfrak{s}(\beta)$ is y such that $\alpha \beta \notin \mathcal{I}$.
- $(S2)_L$ For each arrow $\alpha : x \to y$, there is at most one arrow γ whose target $\mathfrak{t}(\gamma)$ is x such that $\gamma \alpha \notin \mathcal{I}$.

We say that a bound quiver $(\mathcal{Q}, \mathcal{I})$ of a monomial algebra is a *almost gentle pair* if it satisfies the following conditions.

(AG1) $(S2)_R$ and $(S2)_L$ holds.

(AG2) All generators of the ideal \mathcal{I} are paths of length two.

Now we recall the definitions of string algebra, almost gentle algebra, and string almost gentle algebra.

Definition 2.1. Let A be a finite-dimensional algebra. We call that A is a:

- (1) string (resp., almost gentle) algebra, if A is Morita equivalent to $\mathbb{k}\mathcal{Q}/\mathcal{I}$ such that $(\mathcal{Q},\mathcal{I})$ is a string (resp., almost gentle) pair;
- (2) string almost gentle algebra (=SAG-algebra), if A is both string and almost gentle.

Example 2.2. Let $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be an algebra whose bound quiver $(\mathcal{Q}, \mathcal{I})$ is shown in FIGURE 2.1, where

> $\mathcal{I} = \langle ab, bc, ca, dd', ee', ff', a'b', b'c', c'a',$ $e'f, e'c', f'd, f'a', d'e, d'b', a'eb, b'fc, c'da\rangle$.

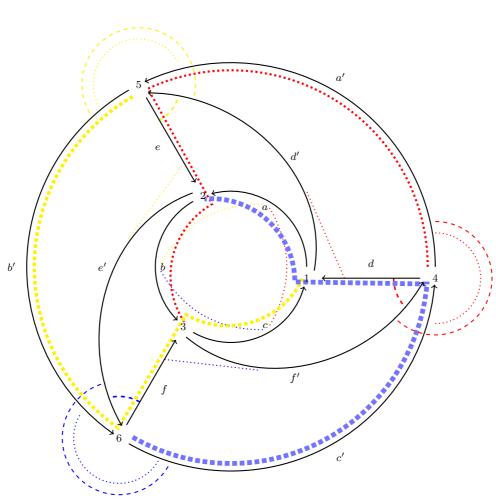


FIGURE 2.1. The bound quiver of the string algebra given in Example 2.2(The dashed lines represent the relations in \mathcal{I})

Then A is a string algebra. In this case, A is not a SAG-algebra because the lengths of relations a'eb, b'fc, and c'da are 3.

2.2. The module categories of string algebras. In [BR87], Butler and Ringel have described all indecomposable modules over string algebra. In this subsection we recall strings, bands, string modules, and band modules.

For any arrow $a \in \mathcal{Q}_1$, we denote by a^{-1} the *formal inverse* of a. Then $\mathfrak{s}(a^{-1}) = \mathfrak{t}(a)$ and $\mathfrak{t}(a^{-1}) = \mathfrak{s}(a)$. Define $\mathcal{Q}_1^{-1} := \{a^{-1} \mid a \in \mathcal{Q}_1\}$ be the set of all formal inverses of arrows. Then any path $p = a_1 a_2 \cdots a_\ell$ on a bound quiver $(\mathcal{Q}, \mathcal{I})$ naturally provides a formal inverse path $p^{-1} = a_\ell^{-1} a_{\ell-1}^{-1} \cdots a_1^{-1}$ of p. In particular, for any path ε_v of length zero corresponding to $v \in \mathcal{Q}_0$, we define $\varepsilon_v^{-1} = \varepsilon_v$.

Definition 2.3. A *string* on a bound quiver $(\mathcal{Q}, \mathcal{I})$ is a sequence $s = (\wp_1, \wp_2, \dots, \wp_n)$, where $\wp_i = a_{i,1} \cdots a_{i,l_i}$, $1 \le i \le n$, and $a_{i,j} \in \mathcal{Q}_1 \cup \mathcal{Q}_1^{-1}$, $1 \le j \le l_i$, such that:

- (Str1) for any $1 \leq i \leq n$, φ_i or φ_i^{-1} is a path on $(\mathcal{Q}, \mathcal{I})$; (Str2) if φ_i is a path, then φ_{i+1} is a formal inverse path, and $a_{i,l_i} \neq a_{i+1,1}^{-1}$;

- (Str3) if \wp_i is a formal inverse path, then \wp_{i+1} is a path, and $a_{i,l_i}^{-1} \neq a_{i+1,1}$;
- (Str4) $\mathfrak{t}(\wp_i) = \mathfrak{s}(\wp_{i+1})$ holds for all $1 \leq i \leq n-1$, which are called *turning* points.
- A band $b = (\wp_1, \wp_2, \dots, \wp_n)$ is a string such that:
 - (Band1) $\mathfrak{t}(\wp_n) = \mathfrak{s}(\wp_1)$, and if \wp_n and \wp_1 are paths then $\wp_n \wp_1 \notin \mathcal{I}$, if \wp_n and \wp_1 are formal inverse paths then $(\wp_n \wp_1)^{-1} \notin \mathcal{I}$;
 - (Band2) b is not a non-trivial power of some string, i.e., there is no string s such that $b = s^m$ for some $m \ge 2$.
- A vertex v on a string s is called a *source* if one of the following condition holds:
 - (1) v is a turning point $\mathfrak{t}(\wp_i) = \mathfrak{s}(\wp_{i+1})$ such that \wp_i is a formal inverse path and \wp_{i+1} is a path;
 - (2) \wp_1 is a path, and $v = \mathfrak{s}(s) = \mathfrak{s}(\wp_1)$;
 - (3) \wp_n is a formal inverse path, and $v = \mathfrak{t}(s) = \mathfrak{t}(\wp_n)$.

We can define *sink* by dual way.

Remark 2.4. We can define the *substring* by removing interconnected arrows on both sides of string.

Definition 2.5.

- (1) s is called a *trivial string* if it is an empty;
- (2) two strings s and s' are called *equivalent* if s' = s or $s' = s^{-1}$;
- (3) two bands $b = \alpha_1 \cdots \alpha_n$ and $b' = \alpha'_1 \cdots \alpha'_t$ are called *equivalent* if b[t] = b' or $b[t]^{-1} = b'$, where $b[t] = \alpha_{1+t} \cdots \alpha_n \alpha_1 \cdots \alpha_{1+t-1}$.

We denote by Str(A) (resp., Ban(A)) the set of all equivalent classes of strings (resp., bands) on the bound quiver of A, respectively.

The following result is first shown by Butler and Ringel.

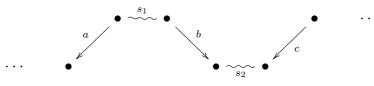
Theorem 2.6 (Butler-Ringel [BR87, Section 3]). All indecomposable objects in category mod(A) of a string algebra A can be described by the following bijection

$$\mathbb{M}: \mathrm{Str}(A) \cup (\mathrm{Ban}(A) \times \mathscr{J}) \to \mathsf{ind}(\mathsf{mod}(A)),$$

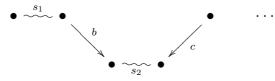
where ind(mod(A)) is the set of all isoclasses of indecomposable A-modules and \mathcal{J} is the set of all Jordan block with non-zero eigenvalue.

Notice that we can define strings and bands on any monomial pair $(\mathcal{Q}, \mathcal{I})$ and each indecomposable module corresponded by string and band is called a *string module* and *band module*, respectively. However, the set ind(modA), where $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$, of all isoclasses of indecomposable A-modules

A string s can be written as



up to equivalence by using arrows $a \in Q_1$ on bound quiver (Q, \mathcal{I}) . In this case, the substring s_1 (resp., s_2) is said to be the *factor substring* (resp., *image substring*) of s (respect to the pair (a, b) (resp., (b, c))). In particular, if a does not exist, that is, s is of the form

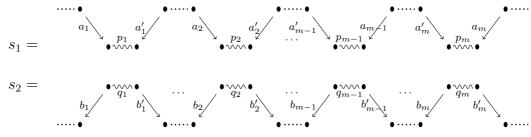


then s_1 is said to be the factor substring of s respect to the pair (0, b). We can define factor substring of s respect to the pair (a, 0), image substring of s respect to the pair (b, 0), and image substring of s respect to the pair (0, c) by similar way.

Factor and image substrings can be used to describe the homomorphisms between two string modules as the following result, see for example [Kra91, Theorem in page 191] and [Lak16, Chapter 2, Section 2, 2.4.2].

Theorem 2.7.

- (1) For two string modules corresponded by strings s_1 and s_2 , $\operatorname{Hom}_A(\mathbb{M}(s_2), \mathbb{M}(s_1)) \neq 0$ if and only if there is a factor substring q of s_2 and an image substring p of s_1 such that q and p coincide.
- (2) Furthermore, take two string s_1 and s_2 as following:



where all p_r are image substrings respect to (a_r, a'_r) of s_1 , and all q_r are factor substrings respect to (b_r, b'_r) of s_2 $(1 \le r \le m)$. If

- $-p_1 = q_1, p_2 = q_2, \ldots, p_m = q_m,$
- and for other image substring p of s_1 which does not is a substring of any p_r , there is no factor substring q of s_2 such that p = q,

then

 $\dim_{\mathbb{K}} \operatorname{Hom}_{A}(\mathbb{M}(s_{2}), \mathbb{M}(s_{1})) = m.$

(All pairs (p_r, q_r) describe the basis of Hom_A($\mathbb{M}(s_2), \mathbb{M}(s_1)$) as k-linear space.)

2.3. Cycles. A path $p = a_1 \cdots a_n$ on a quiver Q is said to be *forbidden* if $a_i a_{i+1} \in \mathcal{I}$ holds for all $1 \leq i \leq n-1$. The arrows a_1, \ldots, a_{n-1} are called *left forbidden arrows* and the arrows a_2, \ldots, a_n are called *right forbidden arrows*.

Next, we recall the definition of forbidden cycle.

Definition 2.8. Let $\overline{\mathcal{Q}}$ be the underlying graph¹ of \mathcal{Q} . A cycle \mathscr{C} (of length n) on n vertices $v_1, \ldots, v_n \in \mathcal{Q}_0$ is a sequence of n edges $\overline{c}_1, \ldots, \overline{c}_n$ of $\overline{\mathcal{Q}}$ such that the vertices of \mathscr{C} can be arranged in a cyclic sequence in such a way that two vertices v_i and v_{i+1} are adjacent connected by the arrow c_i if they are consecutive in the sequence, and are nonadjacent otherwise (the indices i are taken modulo n if necessary). An oriented cycle is a cycle $\mathscr{C} = c_1 \cdots c_n$ with $\mathfrak{t}(c_i) = \mathfrak{s}(c_{i+1})$ ($1 \leq i < n$) such that $\mathfrak{t}(\mathscr{C}) = \mathfrak{t}(c_n) = \mathfrak{s}(c_1) = \mathfrak{s}(\mathscr{C})$ holds. Furthermore, \mathscr{C} is called a forbidden cycle if there are relations $r_0, r_1, \cdots, r_{d-1}$ of \mathcal{I} such that $c_1c_2, \ldots, c_{n-1}c_n, c_nc_1 \in \mathscr{C}$. A cycle without relation is a cycle \mathscr{C} such that all paths on \mathscr{C} are not in \mathcal{I} .

Remark 2.9.

- (1) Forbidden paths are introduced by Avella-Alaminos and Geiss in [AAG08] which are used to describe AG-invariants of gentle algebras. The terminology "forbidden cycle" and "forbidden arrow" come from forbidden path.
- (2) Each cycle without relation provide a band.

¹Recall that the *underlying graph* $\overline{\mathcal{Q}}$ of \mathcal{Q} is obtained from \mathcal{Q} by forgetting the orientation of the arrows. Each $\overline{\alpha}$, the arrow α forgetting orientation, is called an *edges* of $\overline{\mathcal{Q}}$.

3. The module αA

In this section, we consider the A-module αA , where α is an arrow on the string pair $(\mathcal{Q}, \mathcal{I})$.

3.1. αA is an indecomposable module. We introduce the module αA and show that it is an indecomposable module in this part.

Lemma 3.1. For any arrow $\alpha \in Q_1$ on a string pair (Q, I), we have:

- (1) $\alpha A \leq_{\oplus} \operatorname{rad}(e_{\mathfrak{s}(\alpha)}A)$, where $e_{\mathfrak{s}(\alpha)}$ is the idempotent corresponded by $\mathfrak{s}(\alpha)$, and
- (2) αA is an indecomposable module.

Proof. First of all, we show that there exists an injection

$$\sigma: \alpha A = \sum_{\substack{\wp \in \mathcal{Q}_{\geq 0} \\ \mathfrak{t}(\alpha) = \mathfrak{s}(\wp)}} \mathbb{k} \alpha \wp \xrightarrow{\subseteq} e_{\mathfrak{s}(\alpha)} A = \sum_{\substack{\tilde{\wp} \in \mathcal{Q}_{\geq 0} \\ \mathfrak{s}(\tilde{\wp}) = \mathfrak{s}(\alpha)}} \mathbb{k} \tilde{\wp}.$$

Any path $\alpha \wp$ in αA is a path with source $\mathfrak{s}(\alpha)$. By the definition of string pair, $\mathfrak{s}(\alpha)$ is a source of at most two arrows, and then we obtain two cases as follows.

(1) There are two arrows a_1 and a'_1 such that $\mathfrak{s}(a_1) = \mathfrak{s}(a'_1) = \mathfrak{s}(\alpha)$ (α equals to either a_1 or a'_1). In this case, $e_{\mathfrak{s}(\alpha)}A$ is the indecomposable module corresponding to some string which is of the form

$$\bullet \stackrel{a'_m}{\longleftarrow} \bullet \cdots \bullet \stackrel{a'_2}{\longleftarrow} \bullet \stackrel{a'_1}{\longleftarrow} \bullet \stackrel{a_1}{\longrightarrow} \bullet \stackrel{a_2}{\longrightarrow} \bullet \cdots \bullet \stackrel{a_n}{\longrightarrow} \bullet$$

and satisfies the following conditions:

- $-\mathfrak{t}(a'_m)$ is a sink point of \mathcal{Q} , or there is an integer $1 \leq i \leq m$ such that $a'_i a'_{i+1} \cdots a'_m a'_{m+1} \in \mathcal{I}$ holds for any arrow a'_{m+1} with source $\mathfrak{s}(a'_{m+1}) = \mathfrak{t}(a'_m)$;
- $-\mathfrak{t}(a_n)$ is a sink point of \mathcal{Q} , or there is an integer $1 \leq j \leq n$ such that $a_j a_{j+1} \cdots a_n a_{n+1} \in \mathcal{I}$ holds for any arrow a_{n+1} with source $\mathfrak{s}(a_{n+1}) = \mathfrak{t}(a_n)$.

Without loss of generality, assume that $\alpha = a_1$, then αA is the module corresponding to the string

$$\bullet \xrightarrow{a_2} \bullet \cdots \bullet \xrightarrow{a_n} \bullet$$

which is a direct summand of $\operatorname{rad}(e_{\mathfrak{s}(\alpha)}A)$.

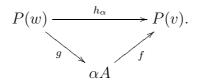
(2) The arrow α , written as a, is a unique arrow with source $\mathfrak{s}(\alpha)$. In this case, $e_{\mathfrak{s}(\alpha)}A$ is the indecomposable module corresponding to some string which is of the form in this case

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \cdots \bullet \xrightarrow{a_n} \bullet.$$

Thus, $\alpha A \leq_{\oplus} \operatorname{rad}(e_{\mathfrak{s}(\alpha)}A)$ can be given by the string $a_2 \cdots a_n$ corresponding to αA . This case can be seen as the case (1) with m = 0.

By the above two cases, it is easy to see that αA is an indecomposable module. \Box

3.2. Homomorphisms starting (resp. ending) with αA . A module is said to be an (*indecomposable*) arrowed module if it isomorphic to αA for some $\alpha \in Q_1$. Let $\operatorname{arr}(A)$ be the set of all arrowed modules. The following lemma shows that any homomorphism h_{α} induced by α between two indecomposable projective modules $P(\mathfrak{s}(\alpha))$ and $P(\mathfrak{t}(\alpha))$ is a morphism crossing αA . **Lemma 3.2.** For arbitrary arrow $\alpha \in Q_1$ with source $\mathfrak{s}(\alpha) = v$ and target $\mathfrak{t}(\alpha) = w$, the morphism $h_{\alpha} : P(w) \to P(v)$ induced by α has a decomposition



Proof. For arbitrary $a \in A$, the homomorphism h_{α} induced by $\alpha \in Q_1$ sends any $\varepsilon_w a \in \varepsilon_w A$ to $\alpha \cdot \varepsilon_w a = \varepsilon_v \cdot \alpha a \in \varepsilon_v A$. It follows a decomposition $\varepsilon_w A \xrightarrow{g} \alpha A \xrightarrow{f} \varepsilon_v A$ of h_{α} satisfying $\varepsilon_w a \xrightarrow{g} \alpha a \xrightarrow{f} \varepsilon_v \cdot \alpha a$ as required.

Lemma 3.3. Keep the notations from Lemma 3.2. The homomorphisms f and g can not be decomposed through any indecomposable projective module $P(u) \ (\not\cong P(w))$.

Proof. Assume that the string corresponding to P(w) is

$$\mathbb{M}^{-1}(P(w)) = w'_{m_0} \stackrel{a'_{m_0-1}}{\leftarrow} \cdots \stackrel{a'_2}{\leftarrow} w'_2 \stackrel{a'_1}{\leftarrow} w_1 \stackrel{a_1}{\longrightarrow} w_2 \stackrel{a_2}{\longrightarrow} \cdots \longrightarrow w_{m-1} \stackrel{a_{m-1}}{\longrightarrow} w_m.$$

Here, $w_1 = w$. Then the string corresponding to P(v) is of the form

$$\mathbb{M}^{-1}(P(v)) = v_n \stackrel{b_{n-1}}{\longleftarrow} \cdots \stackrel{b_2}{\longleftarrow} v_2 \stackrel{b_1}{\longleftarrow} v_1 \stackrel{\alpha}{\longrightarrow} w_1 \longrightarrow \cdots$$

By the definition of string algebra, we have $\alpha a'_1 \in \mathcal{I}$, see Figure 3.1.

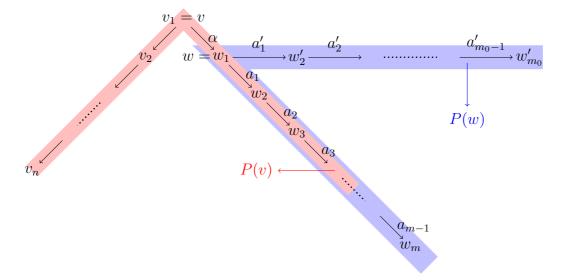
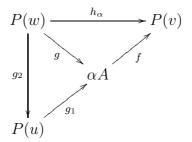


FIGURE 3.1. The strings respectively corresponding to P(v) and P(w)

Next, we show that g is a homomorphism does not through any indecomposable projective module $P \ (\not\cong P(w))$. To do this, we assume that g has a decomposition $g = g_1g_2$ such that the following diagram



commutes. Then there is a path \wp on $(\mathcal{Q}, \mathcal{I})$ whose source and sink respectively are u and w, such that $g_2 : P(w) = \varepsilon_w A \to P(u) = \varepsilon_u A$, written as h_{\wp} , sends each $\varepsilon_w a \in \varepsilon_w A$ to $\wp \cdot \varepsilon_w a = \varepsilon_u(\wp a) \in \varepsilon_u A$, i.e.,

$$g_2(\varepsilon_w a) = \wp a \tag{3.1}$$

Notice that we have the following two facts.

(1) The module αA is a string module satisfying

$$\mathbb{M}^{-1}(\alpha A) = w_1 \xrightarrow{a_1} w_2 \cdots \xrightarrow{a_{\tilde{m}-1}} w_{\tilde{m}} \ (1 \leqslant \tilde{m} \leqslant m),$$

and the factor substring of $\mathbb{M}^{-1}(\alpha A)$ respect to $(0, a_1)$ is ε_w , see the mark (I) in FIGURE 3.2;

(2) The string corresponded by P(u) is of the form

$$\mathbb{M}^{-1}(P(u)) = \cdots \longleftarrow u \xrightarrow{\wp} w \xrightarrow{a_1'} w_2' \longrightarrow \cdots \xrightarrow{a_{\tilde{m}_0}'} w_{\tilde{m}_0}'$$

where $0 \leq \tilde{m}_0 \leq m_0$, and $\wp = c_1 \cdots c_l \ (c_1, \ldots, c_l \in Q_1)$ is a path such that, by the definition of string algebra and $\alpha a_1 \notin \mathcal{I}$, we have

$$c_l a_1 \in \mathcal{I} \tag{3.2}$$

see the mark (II) in FIGURE 3.2.

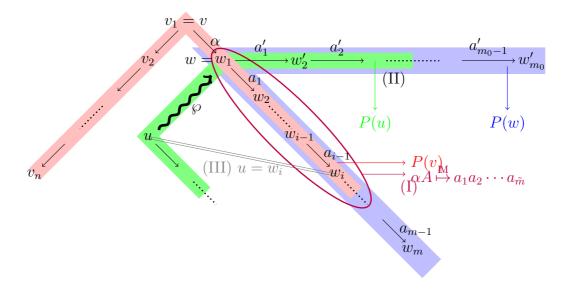


FIGURE 3.2. If g can be decomposed through $P(u) \equiv P(w)$

Now we consider the canonical decomposition of $g_1 : P(u) \to \alpha A$, it is easy to see that the image $\text{Im}(g_1)$ is a string module whose string is a factor substring of $\mathbb{M}^{-1}(P(u))$ which is of the form

$$\cdots \longleftarrow u \xrightarrow{\wp} w \xrightarrow{a_1'} w_2' \longrightarrow \cdots \xrightarrow{a_{\tilde{m}_1}'} w_{\tilde{m}_1}'$$

where $0 \leq \tilde{m}_1 \leq \tilde{m}_0 \ (\leq m_0)$, see FIGURE 3.2. We obtain two cases as follows.

- (A) The vertex u is not a vertex on the string $\mathbb{M}^{-1}(\alpha A)$;
- (B) The vertex u is a vertex on the string $\mathbb{M}^{-1}(\alpha A)$.

In case (A), we obviously have $g_1 = 0$ by Theorem 2.7 (1). This is a contradiction.

Now we show that (B) admits a contradiction and end my proof. In this case, we obtain $u = w_i$ for some $1 \le i \le \tilde{m}$. It follows that a_{i-1} is an arrow ending with u, see the mark (III) in FIGURE 3.2. One can check that $\mathbb{M}^{-1}(P(u))$ has a factor substring coincides with an image substring of $\mathbb{M}^{-1}(\alpha A)$, it describe the homomorphism

$$g_1: P(u) = \varepsilon_u A \to \alpha A, \, \varepsilon_u a \mapsto \alpha a_1 \cdots a_{i-1} \varepsilon_{w_i} a \, (\forall a \in A)$$

which is non-zero. However, g_2 sends each element $\varepsilon_w a \ (\forall a \in A)$ in P(w) to the element $\wp a$ in P(v), see (3.1). Thus,

$$g(\varepsilon_w a) = g_1 g_2(\varepsilon_w a) = g_1(\wp a) = \alpha a_1 \cdots a_{i-1} \wp a$$

It follows that

$$fg(a_1 \cdots a_{i-1}) = f(\alpha a_1 \cdots a_{i-1} \wp a_1 \cdots a_{i-1})$$
$$= \alpha a_1 \cdots a_{i-1} \wp a_1 \cdots a_{i-1} = 0 \text{ (by (3.2))}$$

By using $h_{\alpha} = fg$, we have

$$fg(a_1 \cdots a_{i-1}) = h_\alpha(a_1 \cdots a_{i-1})$$
$$= \alpha a_1 \cdots a_{i-1} \neq 0.$$

We obtain a contradiction.

We can show that f can not be decomposed through any indecomposable projective module by similar way.

Lemma 3.4. Keep the notations from Lemma 3.2. For any $\beta \in Q_1$, the homomorphisms f and g can not be decomposed through $\beta A \ (\not\cong \beta A)$.

The proof of the above proposition is similar to that of Lemma 3.4. In the proof of Lemma 3.4, we prove that g can not be decomposed through any indecomposable projective module $P(u) \ (\not\cong P(w))$. Now, we show that f can not be decomposed through any $\beta A \ (\not\cong \alpha A)$.

Proof. Similar to the proof of Lemma 3.3, assume

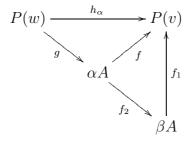
$$\mathbb{M}^{-1}(P(w)) = w'_{m_0} \stackrel{a'_{m_0-1}}{\leftarrow} \cdots \stackrel{a'_2}{\leftarrow} w'_2 \stackrel{a'_1}{\leftarrow} w_1 \stackrel{a_1}{\longrightarrow} w_2 \stackrel{a_2}{\longrightarrow} \cdots \longrightarrow w_{m-1} \stackrel{a_{m-1}}{\longrightarrow} w_m$$

and

$$\mathbb{M}^{-1}(P(v)) = v_n \stackrel{b_{n-1}}{\longleftrightarrow} \cdots \stackrel{b_2}{\longleftrightarrow} v_2 \stackrel{a_1}{\longleftrightarrow} v_1 \stackrel{a_2}{\longrightarrow} w_1 \stackrel{a_1}{\longrightarrow} \cdots \stackrel{a_{m'-1}}{\longrightarrow} w_{m'}$$

 $(1 \leq m' \leq m)$ which are shown in FIGURE 3.1.

Next, we show that f is a homomorphism does not through any βA ($\forall \beta \in Q_1$, $\beta \neq \alpha$). To do this, we assume that f has a decomposition $f = f_1 f_2$ such that the following diagram



commutes. Notice that the string $\mathbb{M}^{-1}(\beta A)$ is of the form

$$u_1 \xrightarrow{c_1} u_2 \xrightarrow{c_2} \cdots \xrightarrow{c_{l-1}} u_l$$

then any image substring of it is of the form $c_{l'}c_{l'+1}\cdots c_{l-1}$ $(1 \leq l' \leq l-1, \mathfrak{t}(\beta) = \mathfrak{s}(c_{l'})$, in the case of l' = l-1 we take image substring is ε_{u_l} , and, by Theorem 2.7 (1) and $f_2 \neq 0$, there is a factor substring of $\mathbb{M}^{-1}(\alpha A) = a_1 a_2 \cdots a_{\tilde{m}}$, written as $a_1 a_2 \cdots a_j$ $(0 \leq j \leq \tilde{m})$, coincides with some image substring of $\mathbb{M}^{-1}(\beta A)$. It follows that

 $c_{l'} = a_1, c_{l'+1} = a_2, \dots, c_l = a_j$ hold for some $0 \leq j \leq \tilde{m}$, where j = l - l' + 1(see mark "Case (A)" in FIGURE 3.3).

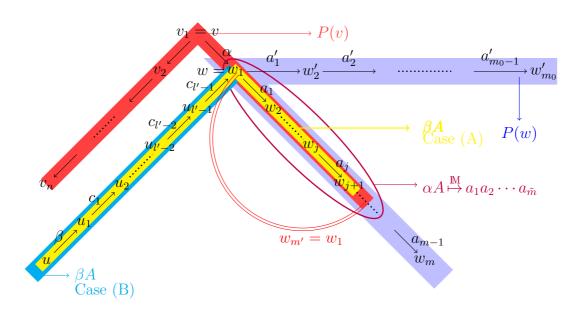


FIGURE 3.3. If f can be decomposed through $\beta A \ (\not\cong \alpha A)$

We obtain two cases as follows.

- (A) l' < l-1, that is, the length of $c_{l'}c_{l'+1}\cdots c_{l-1}$ is great than or equal to 1, see mark "Case (A)" in FIGURE 3.3.
- (B) l' = l 1, that is, $\mathbb{M}(c_{l'}c_{l'+1}\cdots c_{l-1}) = \mathbb{M}(\varepsilon_{u_l})$ is isomorphic to the simple module $S(u_l)$, see mark "Case (B)" in FIGURE 3.3.

In Case (A), $\alpha a_1 \notin \mathcal{I}$ admits $c_{l'-1}a_1 \in \mathcal{I}$ by the definition of string algebra, i.e., $c_{l'-1}c_{l'} \in \mathcal{I}$, this is a contradiction since $c_1c_2\cdots c_{l-1}$ is a string.

In Case (B), we have $\text{Im}(f_2) \cong S(w_1)$ is a simple module which must be a submodule of P(v) since $f_1 : \beta A \to P(v)$ is non-zero. It follows that $w_{m'} = w_1$ by using Theorem 2.7 (1), then, by using the definition of string algebra, we have

(B.1) $a_{m'-1}$ coincides with the arrow α ;

(B.2) or $a_{m'-1}$ coincides with the arrow $c_{l'-1}$ (= c_l).

On the other hand, by $f_2 \neq 0$, there is a factor substring of $\mathbb{M}^{-1}(\beta A) = c_1 \cdots c_{l'-1}$, say

$$c_1 \cdots c_{l''} \ (0 \leq l'' \leq l' - 1),$$

coincides with some image substring

$$a_{m''}\cdots a_{m'-1} \ (0\leqslant m''\leqslant m'-1)$$

of $M^{-1}(P(v))$, i.e.,

$$c_1 \cdots c_{l''} = a_{m''} \cdots a_{m'-1} \ (l' = m' - m''). \tag{3.3}$$

In (B.1): We get f_2 is of the following form

$$f_2: \alpha A \to \beta A, \ \alpha a \mapsto \beta a_{m''} \cdots a_{m'-2} \cdot \alpha a,$$
 (3.4)

by using (3.3), and get f_1 is of the following form

 $f_1: \beta A \to \alpha A, \ \beta a \mapsto \alpha a_1 \cdots a_{m''-1} a.$ (3.5)

Consider the path a_1 as an element in A, we have

$$h_{\alpha}(a_{1}) = f_{1}(f_{2}(g(a_{1}))) = f_{1}(f_{2}(\alpha a_{1}))$$

$$\stackrel{(3.4)}{=} f_{1}(\beta a_{m''} \cdots a_{m'-2} \alpha a_{1})$$

$$\stackrel{(3.5)}{=} \alpha a_{1} \cdots a_{m''-1} a_{m''} \cdots a_{m'-2} \alpha a_{1}$$

$$= \alpha a_{1} \cdots a_{m'-2} \alpha a_{1}$$

$$\neq \alpha a_{1} = h_{\alpha}(a_{1}),$$

a contradiction.

In (B.2): $a_{m'-1} = c_{l'-1}$ admits $a_{m'-2}c_{l'-1} \in \mathcal{I}$ by using the definition of string algebra. Then $a_{m'-2}c_{l'-1} = a_{m'-2}a_{m'-1}$, as an element in A, is zero. It contradicts with $a_1 \cdots a_{m'-2}a_{m'-1}$ is a string.

The contradictions given by Cases (A) and (B) show that this proposition holds. $\hfill \Box$

Lemma 3.5. Let $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be a string algebra and α be an arrow on \mathcal{Q} . Any homomorphism $h_{\alpha} : P(v) \to P(w)$ induced by the arrow $\alpha : w \to v$ can not be decomposed through arbitrary $\beta A \ (\ncong \alpha A)$.

Proof. It is well-known that each base h_p element of $\operatorname{Hom}_A(P(v), P(w))$ is described by the path p from w to v on $(\mathcal{Q}, \mathcal{I})$, that is, $h_p : e_v a \mapsto p \cdot e_v a = pa$. If h_p can be decomposed through βA , then, by Theorem 2.7, one can check that β is an arrow such that:

- (1) $\mathfrak{t}(\beta) = \mathfrak{s}(\mathbb{M}^{-1}(P(v))),$
- (2) p is a factor substring of $M^{-1}(P(w))$,
- (3) β is an arrow on p.

If $p = \alpha$ is an arrow, then β must be coincided with α , it contradicts with $\alpha A \ncong \beta A$ as required.

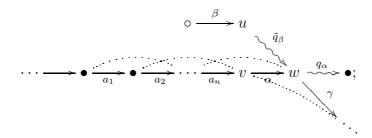
Lemma 3.6. Let $A = \mathbb{k}Q/\mathcal{I}$ be an SAG-algebra, α be a left forbidden arrow, and β be an arbitrary arrow. If $\operatorname{Hom}_A(\alpha A, \beta A) \neq 0$, then it can be decomposed through some indecomposable projective module.

Proof. Assume f is a non-zero homomorphism which can be decomposed through γA , here, $\gamma A \not\cong \alpha A$ and $\gamma A \not\cong \beta A$. Assume α is an arrow on the path

$$p = \cdots \longrightarrow \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \cdots \xrightarrow{a_n} v \xrightarrow{\alpha} w \xrightarrow{q_\alpha} \bullet$$

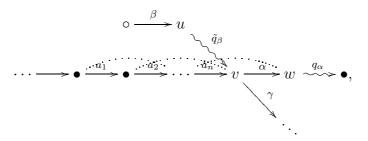
such that $a_1a_2, \ldots a_{n-1}a_n, a_n\alpha \in \mathcal{I}$ and $\mathbb{M}(q_\alpha) \cong \alpha A$. By Theorem 2.7 and $f \neq 0$, we obtain that q_α has a factor substring which coincides with an image substring of $q_\beta := \mathbb{M}^{-1}(\beta A)$ (then $\mathfrak{t}(\beta) = \mathfrak{s}(q_\beta)$ holds). Thus, $\mathfrak{t}(q_\beta)$ is a vertex on αq_α , and the positional relationship of q_α and q_β is one of the following forms:

Case 1.



where q_{β} is of the form $\tilde{q}_{\beta}q'_{\alpha}$, and q'_{α} is a factor image of q_{α} .

Case 1.



where q_{β} is of the form $\tilde{q}_{\beta}\alpha q'_{\alpha}$, and q'_{α} is a factor image of q_{α} .

In Case 1, we assume

$$q_{\alpha} = w \xrightarrow{q_{\alpha,1}} \bullet \xrightarrow{q_{\alpha,2}} \cdots \xrightarrow{q_{\alpha,l}} \bullet ; \text{ and } \tilde{q}_{\beta} = u \xrightarrow{q_{\beta,1}} \circ \xrightarrow{q_{\beta,2}} \cdots \xrightarrow{q_{\beta,\ell}} w ;$$

then $q_{\beta,\ell}q_{\alpha,1} \in \mathcal{I}$ since A is a string algebra. We know that q_{α} has a factor substring, say r, coinciding with an image substring of q_{β} , there are two subcases as follows.

Subcase 2.1. l = 0. Then, $\tilde{q}_{\beta} = q_{\beta}$, and $r = q'_{\alpha} = q_{\alpha} = \varepsilon_w$ is both a factor substring of q_{α} and an image substring of q_{β} respect to $q_{\beta,\ell}$. It follows that $\mathfrak{t}(q_{\beta}) = w$, and $\operatorname{Im}(f) \cong S(w)$ is simple. On the other hand, A is an SAGalgebra, then all relations in \mathcal{I} are paths of length two, and so, $\beta q_{\beta} \gamma =$ 0 admits that $q_{\beta,\ell} \gamma \in \mathcal{I}$. Thus, S(w) is a direct summand of the socle of $P(S(w)) = P(\mathfrak{s}(q_{\beta,1})) = P(u)$. That is, for any $a \in A$, we obtain a decomposition

$$\alpha A \xrightarrow[f_1]{f_1} P(u) \xrightarrow[f_2]{f_2} \beta A$$

of f such that $f(\alpha a) = \beta q_{\beta,1} \cdots q_{\beta,\ell-1} \cdot f_1(\alpha a)$, where f_1 sends αa to an element in P(u) which is of the form $\varepsilon_u a'$, and, for any $x \in A$, f_2 sends each $\varepsilon_u x$ to $\beta q_{\beta,1} \cdots q_{\beta,l-1} \cdot \varepsilon_u x$.

Subcase 2.2. $l \ge 1$. We can show that f can be decomposed by P(u) by the method similar to the proof of Subcase 2.1.

In Case 2, we assume

$$q_{\alpha} = w \xrightarrow{q_{\alpha,1}} \bullet \xrightarrow{q_{\alpha,2}} \cdots \xrightarrow{q_{\alpha,l}} \bullet ; \text{ and } \tilde{q}_{\beta} = u \xrightarrow{q_{\beta,1}} \circ \xrightarrow{q_{\beta,2}} \cdots \xrightarrow{q_{\beta,\ell}} v ;$$

then $q_{\beta,\ell} \alpha \notin \mathcal{I}$, which admits $q_{\beta,\ell} \gamma \in \mathcal{I}$ by using A to be a string algebra. Assume $q'_{\alpha} = q_{\alpha,1} \cdots q_{\alpha,l'}, 1 \leq l \leq l'$, then, for any $a \in A$, the homomorphism f described by q'_{α} sends each element αa to $\beta \tilde{q}_{\beta} \alpha a \ (\in \beta A)$. Notice that

$$\tilde{q}_{\beta}\alpha a = \varepsilon_{\mathfrak{s}(\tilde{q}_{\beta-1})}\tilde{q}_{\beta}\alpha a = \varepsilon_u \tilde{q}_{\beta}\alpha a \in \varepsilon_u A = P(u),$$

now one can check that f can be decomposed through P(u).

4. \mathcal{R} -endomorphism algebras

For a string algebra A, let \mathcal{R} be a subset of \mathcal{Q}_1 whose all elements are left forbidden arrows in this section.

Definition 4.1. An *R*-summed module is the direct sum of A and all arrowed modules $\alpha A \ (\alpha \in \mathcal{R})$, that is,

$$M_{\mathcal{R}} := A \oplus \bigoplus_{\alpha \in \mathcal{R}} \alpha A,$$

and its endomorphism algebra

$$A_{\mathcal{R}} := \operatorname{End}_A(M_{\mathcal{R}})$$

is called an \mathcal{R} -endomorphism algebra. The set \mathcal{R} is called an *left forbidden arrow* index (of $(\mathcal{Q}, \mathcal{I})$). In particular, there are two remarks as follows.

- (1) In the case of $\mathcal{R} = \emptyset$, the \mathcal{R} -summed module is 0.
- (2) Every arrowed module αA is an \mathcal{R} -summed module with $\mathcal{R} = \{\alpha\}$.

In this section, we provide a method to compute the bound quiver of $A_{\mathcal{R}}$ in the case of A to be an SAG-algebra.

4.1. *R*-bound quiver. Let $(\mathcal{Q}, \mathcal{I})$ be a bound quiver of a string algebra. We define its *R*-bound quiver is $(\mathcal{R}(\mathcal{Q}), \mathcal{R}(\mathcal{I}))$, where $\mathcal{R}(\mathcal{Q}) = (\mathcal{R}(\mathcal{Q})_0, \mathcal{R}(\mathcal{Q})_1, \mathcal{R}(\mathfrak{s}), \mathcal{R}(\mathfrak{t}))$ is given by the following Steps 1–4, and $\mathcal{R}(\mathcal{I})$ is given by the following Steps 5–6.

Step 1 $\mathcal{R}(\mathcal{Q})_0 := \mathcal{Q}_0 \cup \mathcal{Q}_0^{\times}$, where \mathcal{Q}_0^{\times} is a finite set such that the bijection

$$\mathfrak{v}: \{\alpha A \mid \alpha \in \mathcal{R}\} \to \mathcal{Q}_0^{\times}$$

exist.

Step 2 $\mathcal{R}(\mathcal{Q})_1 := (\mathcal{Q}_1 \setminus \mathcal{R}) \cup \mathcal{R}^{\times}$, where $\mathcal{R}^{\times} := \{ \alpha_L, \alpha_R \mid \alpha \in \mathcal{R} \}$.

- Step 3 $\mathcal{R}(\mathfrak{s}) : \mathcal{R}(\mathcal{Q})_1 \to \mathcal{R}(\mathcal{Q})_0$ sends any arrow $a \in \mathcal{Q}_1 \setminus \mathcal{R}$ to its source $\mathfrak{s}(a)$, sends any arrow $\alpha_{\mathrm{L}} \in \mathcal{R}^{\times}$ to the source $\mathfrak{s}(\alpha)$ of α , and sends any arrow $\alpha_{\mathrm{R}} \in \mathcal{R}^{\times}$ to the vertex $\mathfrak{v}(\alpha A)$.
- Step $4 \mathcal{R}(\mathfrak{t}) : \mathcal{R}(\mathcal{Q})_1 \to \mathcal{R}(\mathcal{Q})_0$ sends any arrow $a \in \mathcal{Q}_1 \setminus \mathcal{R}$ to its sink $\mathfrak{t}(a)$, sends any arrow $\alpha_{\mathrm{L}} \in \mathcal{R}^{\times}$ to the vertex $\mathfrak{v}(\alpha A)$, and sends any arrow $\alpha_{\mathrm{R}} \in \mathcal{R}^{\times}$ to the sink $\mathfrak{t}(\alpha)$ of α .

Step 5 For any arrow $a \in \mathcal{Q}_1$, define

$$a^{\times} = \begin{cases} a, & \text{if } a \in \mathcal{Q}_1 \backslash \mathcal{R}; \\ a_{\mathrm{L}} a_{\mathrm{R}}, & \text{if } a \in \mathcal{R}, \end{cases}$$
$$a_{\mathrm{L}}^{\times} = \begin{cases} a, & \text{if } a \in \mathcal{Q}_1 \backslash \mathcal{R}; \\ a_{\mathrm{L}}, & \text{if } a \in \mathcal{R}, \end{cases}$$
and
$$a_{\mathrm{R}}^{\times} = \begin{cases} a, & \text{if } a \in \mathcal{Q}_1 \backslash \mathcal{R}; \\ a_{\mathrm{R}}, & \text{if } a \in \mathcal{R}. \end{cases}$$

For any path $p = a_1 a_2 \cdots a_n$ on $(\mathcal{Q}, \mathcal{I})$, we define

$$p^{\times} = (a_1)_{\mathbf{R}}^{\times} a_2^{\times} \cdots a_{n-1}^{\times} (a_n)_{\mathbf{L}}^{\times}.$$

Step 6 $\mathcal{R}(\mathcal{I}) := \langle p^{\times} | p \in \mathcal{I} \rangle$ which is naturally induced by \mathcal{I} and Step 5.

Remark 4.2.

- The finite-dimensional algebra given by $(\mathcal{R}(\mathcal{Q}), \mathcal{R}(\mathcal{I}))$ is written as $\mathcal{R}(A)$. If A is a string algebra (resp., an SAG-algebra), then so is $\mathcal{R}(A)$.
- It is clear that $\alpha_{\rm L}\alpha_{\rm R} \notin \mathcal{I}(A)$.
- If $\mathcal{R} = \emptyset$, then it is trivial that $\mathcal{R}(A)$ and A coincide.

Example 4.3. Let $A = \mathbb{k}Q/\mathcal{I}$ be the string algebra given by Example 2.2. Take $\mathcal{R} = \{a, d, a'\}$, then the \mathcal{R} -bound quiver $(\mathcal{R}(Q), \mathcal{R}(\mathcal{I}))$ is shown in FIGURE 4.1, where

$$\mathcal{R}(\mathcal{Q})_0 = \{1, 2, 3, 4, 5, 6, aA, dA, a'A\}$$

= $\mathcal{Q}_0 \cup \{v_a = \mathfrak{v}(aA), v_d = \mathfrak{v}(dA), v_{a'} = \mathfrak{v}(a'A)\};$
$$\mathcal{R}(\mathcal{Q})_1 = (\mathcal{Q}_1 \setminus \mathcal{R}) \cup \{a_L, d_L, a'_L, a_R, d_R, a'_R\};$$

and, since

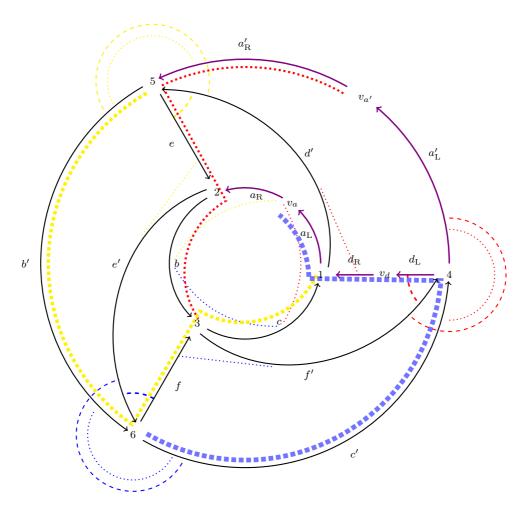


FIGURE 4.1. The bound quiver $(\mathcal{R}(\mathcal{Q}), \mathcal{R}(\mathcal{I}))$ of $\mathcal{R}(A)$

 $\mathcal{I} = \langle ab, bc, ca, dd', ee', ff', a'b', b'c', c'a', e'f, e'c', f'd, f'a', d'e, d'b', a'eb, b'fc, c'da \rangle,$

we have

$$\mathcal{R}(\mathcal{I}) = \langle a_{\mathrm{R}}b, bc, ca_{\mathrm{L}}, d_{\mathrm{R}}d', ee', ff', a'_{\mathrm{R}}b', b'c', c'a'_{\mathrm{L}}, e'f, e'c', f'd_{\mathrm{L}}, f'a'_{\mathrm{L}}, d'e, d'b', a'_{\mathrm{R}}eb, b'fc, c'd_{\mathrm{L}}d_{\mathrm{R}}a \rangle,$$

see the dashed lines in FIGURE 4.1.

4.2. \mathcal{R} -endomorphism algebra. The following result provide a method to compute the \mathcal{R} -endomorphism algebra $A_{\mathcal{R}}$ of an \mathcal{R} -summed module $M_{\mathcal{R}}$ over an SAG-algebra A.

Theorem 4.4. Let A be an SAG-algebra whose bound quiver is $(\mathcal{Q}, \mathcal{I})$ and \mathcal{R} be an arbitrary left forbidden arrow index of \mathcal{Q}_1 . Then $A_{\mathcal{R}} \cong \mathcal{R}(A)$ (= $\mathbb{k}\mathcal{R}(\mathcal{Q})/\mathcal{R}(\mathcal{I})$) is an SAG-algebra.

Proof. Assume $A_{\mathcal{R}} = \mathbb{k}\mathcal{Q}_{\mathcal{R}}/\mathcal{I}_{\mathcal{R}}$, where $\mathcal{Q}_{\mathcal{R}} = ((\mathcal{Q}_{\mathcal{R}})_0, (\mathcal{Q}_{\mathcal{R}})_1, \mathfrak{s}_{\mathcal{R}}, \mathfrak{t}_{\mathcal{R}})$. Let \mathcal{X} be the full subcategory of $\mathsf{mod}(A_{\mathcal{R}})$ generated by \mathfrak{X} ; $\mathrm{Irr}_{\mathcal{X}}(X_1, X_2)$ be the set of all irreducible homomorphisms in \mathcal{X} from X_1 to X_2 (X_1 and X_2 are indecomposable A-modules); $\mathfrak{B}(X_1, X_2)$ be a basis of $\mathrm{Irr}_{\mathcal{X}}(X_1, X_2)$ as a k-linear space; and $\mathrm{Irr}(\mathcal{X}) :=$

 $\bigcup_{X_1, X_2 \in \mathsf{ind}(\mathcal{X})} \mathfrak{B}(X_1, X_2).$

We only prove $\mathcal{R}(\mathcal{Q}) = \mathcal{Q}_{\mathcal{R}}$. Indeed, we will provide a one-to-one correspondence between $\mathcal{R}(\mathcal{Q})$ and $\operatorname{Irr}(\mathcal{X})$ in this proof, and it admits a one-to-one correspondence between the generators of $\mathcal{I}_{\mathcal{R}}$ and the generators of $\mathcal{R}(\mathcal{I})$.

First of all, by the definition of $A_{\mathcal{R}} = \operatorname{End}_A(A \oplus \bigoplus_{\alpha \in \mathcal{R}} \alpha A)$, we have a one-to-one correspondence

$$(\mathcal{Q}_{\mathcal{R}})_0 \stackrel{\bullet}{\to} \mathfrak{X} := \{ \varepsilon_i A \mid i \in \mathcal{Q}_0 \} \cup \{ \alpha A \mid \alpha \in \mathcal{R} \} \stackrel{\bullet}{\to} \mathcal{Q}_0 \cup \mathcal{Q}_0^{\times} = \mathcal{R}(\mathcal{Q})_0$$
(4.1)

from $(\mathcal{Q}_{\mathcal{R}})_0$ to $\mathcal{R}(\mathcal{Q})_0$, where \blacklozenge is obtained by $\{ \mathrm{id}_X \in \mathcal{R}(A) \mid X \in \mathfrak{X} \}$ is a complete set of primitive orthogonal idempotents of $\mathcal{R}(A)$, and \clubsuit is obtained by Step 1.

Second, we have a one-to-one correspondence

$$(\mathcal{Q}_{\mathcal{R}})_1 \xrightarrow{\mathbf{o}} \operatorname{Irr}(\mathcal{X}).$$

By Lemma 3.2, if $\alpha \in \mathcal{R}$, then $h_{\alpha} : P(\mathfrak{t}(\alpha)) \to P(\mathfrak{s}(\alpha))$ has a decomposition $h_{\alpha} = fg$ through αA . By Lemma 3.3, $g : P(\mathfrak{t}(\alpha)) \to \alpha A$ can not be decomposed through any indecomposable projective module which does not isomorphic to $P(\mathfrak{t}(\alpha))$, and $f : \alpha A \to P(\mathfrak{s}(\alpha))$ can not be decomposed through any indecomposable projective module which does not isomorphic to $P(\mathfrak{s}(\alpha))$. By Lemma 3.4, g and f can not be decomposed through any $\beta A \ (\not\cong \alpha A, \beta \in \mathcal{Q}_1)$. That is, f and g can be seen as two base vectors of $\operatorname{Irr}_{\mathcal{X}}(\alpha A, P(\mathfrak{s}(\alpha)))$ and $\operatorname{Irr}_{\mathcal{X}}(P(\mathfrak{t}(\alpha), \alpha A))$, and then, they are corresponded by two arrows $\mathfrak{v}(\alpha A) \to \mathfrak{t}(\alpha)$ and $\mathfrak{s}(\alpha) \to \mathfrak{v}(\alpha A)$ in $\mathcal{R}(\mathcal{Q})_1$ under the correspondence \heartsuit , respectively.

On the other hand, if $\alpha \in \mathcal{Q}_1 \setminus \mathcal{R}$, then, by Lemma 3.5, we obtain that h_α is irreducible in \mathcal{X} , and by Lemma 3.6, for arbitrary two left forbidden arrows in \mathcal{R} , each non-zero homomorphism in $\operatorname{Hom}_{\mathcal{X}}(\alpha A, \beta A)$ is not irreducible. Therefore, we have

$$\mathcal{R}(\mathcal{Q})_{1} \stackrel{\diamondsuit}{\underset{1-1}{\longrightarrow}} \operatorname{Irr}(\mathcal{X})$$

$$= \bigcup_{X_{1}, X_{2} \in \operatorname{ind}(\mathcal{X})} \mathfrak{B}(X_{1}, X_{2})$$

$$= \bigcup_{v \in \mathcal{Q}_{0}, \alpha \in \mathcal{R}} \mathfrak{B}(P(v), \alpha A) \bigcup \bigcup_{v \in \mathcal{Q}_{0}, \alpha \in \mathcal{R}} \mathfrak{B}(\alpha A, P(v))$$

$$\bigcup \bigcup_{\alpha \in \mathcal{Q}_{1} \setminus \mathcal{R}} \mathfrak{B}(P(\mathfrak{t}(\alpha)), P(\mathfrak{s}(\alpha))),$$

where $\mathfrak{B}(P(v), \alpha A)$, $\mathfrak{B}(\alpha A, P(v))$, and $\mathfrak{B}(P(\mathfrak{t}(\alpha)), P(\mathfrak{s}(\alpha)))$ are described by Lemmas 3.3, 3.4, and 3.5. Therefore, we obtain a one-to-one correspondence

$$(\mathcal{Q}_{\mathcal{R}})_1 \stackrel{1-1}{\to} (\mathcal{R}(\mathcal{Q}))_1 \tag{4.2}$$

by \heartsuit and \diamondsuit . Moreover, it is easy to see that four correspondences \blacklozenge , \clubsuit , \heartsuit , and \diamondsuit show that the following two diagrams

commute. Thus, $Q_{\mathcal{R}} = \mathcal{R}(Q)$.

Finally, $\mathcal{R}(\mathcal{Q})$ to be an SAG-algebra is shown in by Remark 4.2.

Notice that if A is a string algebra, then for some left forbidden arrow index \mathcal{R} , it may be holds that $A_{\mathcal{R}} \cong \mathcal{R}(A)$. For example, the string algebra A given by Example 2.2, and \mathcal{R} the left forbidden arrow index given by Example 4.3, one can check that $A_{\mathcal{R}} \cong \mathcal{R}(A)$ in this instance.

5. ON REPRESENTATION TYPES OF SAG-ALGEBRAS

Recall that a finite-dimensional Algebra A is said to be *representation-finite* (resp. *representation-infinite*) if the set ind(mod(A)) of all isoclasses of indecomposable A-modules is a finite (resp. infinite) set.

5.1. Representation types of SAG-algebras and \mathcal{R} -endomorphism algebras. Theorem 2.6 admits that the following lemma.

Lemma 5.1. A string algebra is representation-infinite if and only if its bound quiver has at least one band.

It can be shown by Brauer-Thrall Theorem, see for example, [ASS06, Chapter IV, Section IV.5].

Theorem 5.2. Let $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be an SAG-algebra. Then A is representation-finite if and only if, for all left forbidden arrow indices \mathcal{R} , the \mathcal{R} -endomorphism algebra $A_{\mathcal{R}}$ is representation-finite.

Proof. If, for arbitrary left forbidden arrow index \mathcal{R} , $A_{\mathcal{R}}$ is always representation-finite, then A is representation-finite which can be proved by the trivial case $\mathcal{R} = \emptyset$.

Next, assume that A is representation-finite. If there is a forbidden left arrow index \mathcal{R} such that $A_{\mathcal{R}}$ is representation-infinity, then, by Lemma 5.1, the bound quiver $(\mathcal{Q}_{\mathcal{R}}, \mathcal{I}_{\mathcal{R}})$ of $A_{\mathcal{R}}$ contains a band b. By $(\mathcal{Q}_{\mathcal{R}})_1 = (\mathcal{Q}_1 \setminus \mathcal{R}) \cup \mathcal{R}^{\times}$, all arrows on b can be divided to three classes:

- (1) the arrows lying in $(\mathcal{Q}_1 \setminus \mathcal{R}) \cup \mathcal{R}^{\times}$;
- (2) the arrows lying in \mathcal{R}^{\times} which are of the form $\alpha_{\rm L}$;
- (3) the arrows lying in \mathcal{R}^{\times} which are of the form $\alpha_{\rm R}$.

If there is an arrow on b which is of the form $\alpha_{\rm R} : \mathfrak{v}(\alpha A) \to \mathfrak{t}(\alpha)$, then $\alpha_{\rm L} : \mathfrak{s}(\alpha) \to \mathfrak{v}(\alpha A)$ is also an arrow on b. Otherwise, since b can be seen as a cycle without relation on $(\mathcal{Q}_{\mathcal{R}}, \mathcal{I}_{\mathcal{R}})$, we have two cases as following:

- (A) there exists an arrow β on b such that $\mathfrak{t}(\beta) = \mathfrak{v}(\alpha A)$, cf. FIGURE 5.1 (I);
- (B) there exists an arrow β on b such that $\mathfrak{s}(\beta) = \mathfrak{v}(\alpha A)$, cf. FIGURE 5.1 (II).

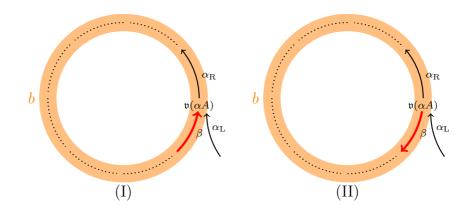


FIGURE 5.1.

In the case (A), we have $\beta \alpha_{\rm R} \in \mathcal{I}$ by using the definition of SAG-algebra, it contradicts with *b* to be a band. In the case (B), we have $\alpha_{\rm L}\alpha_{\rm R} \in \mathcal{I}$. However, by Remark 4.2 and Theorem 4.4, it contradicts with $\alpha_{\rm L}\alpha_{\rm R} \notin \mathcal{I}_R = \mathcal{R}(\mathcal{I})$.

Corollary 5.3. Let $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be an SAG-algebra such that, for some left forbidden arrow index $\mathcal{R} \subseteq \mathcal{Q}_1$, the \mathcal{R} -endomorphism algebra $A_{\mathcal{R}}$ is representation-finite. Then A is representation-finite.

Proof. Assume than $A_{\mathcal{R}}$ is representation-finite. We show that A is representation-finite. If A is representation-infinite, then by Lemma 5.1, the bound quiver $(\mathcal{Q}, \mathcal{I})$ of A contains a band $b = b_1 b_2 \cdots b_n$, here, $b_i \in \mathcal{Q}_1 \cup \mathcal{Q}_1^{-1}$, $\mathfrak{s}(b_i) = v_i$ $(1 \le i \le n)$, $\mathfrak{t}(b_n) = \mathfrak{s}(b_1)$. By using Theorem 4.4, $(\mathcal{Q}_{\mathcal{R}}, \mathcal{I}_{\mathcal{R}})$ contains a band which is of the form

$$b^{\times} = b_1^{\times} b_2^{\times} \cdots b_n^{\times}$$

where

$$b_i^{\times} = \begin{cases} b_i^{\times}, & \text{if } b_i \in \mathcal{Q}_1; \\ (b_i)_{\mathrm{R}}^{-1} (b_i)_{\mathrm{L}}^{-1}, & \text{if } b_i \in \mathcal{Q}_1^{-1} \text{ and } b_i \in \mathcal{R}; \\ b_i^{-1}, & \text{if } b_i \in \mathcal{Q}_1^{-1} \text{ and } b_i \notin \mathcal{R}. \end{cases}$$

It contradicts with Lemma 5.1 as required.

Corollary 5.4. Let $A = \mathbb{k}Q/\mathcal{I}$ be an SAG-algebra. Then the following statements are equivalent:

- (1) A is representation-finite;
- (2) there is a left forbidden arrow index \mathcal{R} such that $A_{\mathcal{R}}$ is representation-finite;
- (3) for arbitrary left forbidden arrow index \mathcal{R} such that $A_{\mathcal{R}}$ is representationfinite.

Proof. The statements (1) and (3) are equivalent by using Theorem 5.2. Moreover, we have that (2) admits (1) by using Corollary 5.3, and it is trivial that (3) admits (1). Then this corollary holds. \Box

5.2. Representation types of SAG-algebras and CM-Auslander algebras. Recall that a Gorenstein-projective (say G-projective for short) A-module G is a module with complete projective resolution, that is, there is an exact sequence

 $\cdots \xrightarrow{p_{-2}} P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} \cdots$

such that

- it is $\operatorname{Hom}_A(-, A)$ -exact;
- $G \cong \operatorname{Ker}(p_1) = \operatorname{Im}(p_0)$ holds.

We denote $G\operatorname{-proj}(A)$ the full subcategory of $\operatorname{mod}(A)$ generated by all G-projective modules over A, and denote $\operatorname{ind}(G\operatorname{-proj}(A))$ the set of all indecomposable G-projective modules over A (up to isomorphism).

In [Kal15, CSZ18, etc], Kalck and Chen-Shen-Zhou respectively provide the descriptions of G-projective modules over gentle algebra and monomial algebra. Then we obtain that SAG-algebras are CM-finite, that is, the number of isoclasses of indecomposable G-projective modules is finite. Thus, we can compute the Cohen-Macaulay Auslander algebra, defined as

$$A^{\mathrm{CMA}} := \mathrm{End}_A \left(\bigoplus_{G \in \mathsf{ind}(\mathsf{G-proj}(A))} G \right),$$

of A by using results in [Kal15, CSZ18], see for example, [CL17, CL19, LZ24, etc].

5.2.1. Perfect forbidden cycles. A forbidden cycle $\mathscr{C} = c_1 \cdots c_l$ ($\mathfrak{s}(c_i) = i, i = 1, 2, \ldots, l$) on string pair $(\mathcal{Q}, \mathcal{I})$ is said to be *perfect* if it satisfies the following two conditions.

- For any arrow α ending with some vertex t on \mathscr{C} , we have $\alpha c_t \notin \mathcal{I}$;
- For any arrow β starting with some vertex t on \mathscr{C} , we have $c_{t-1}\beta \notin \mathcal{I}$.

Perfect forbidden cycles can be used to describe all non-projective indecomposable Gorenstein-projective modules over SAG-algebra. In particular, the set $\mathcal{R}_{\rm p}$ of all arrows on all perfect forbidden cycles is a left forbidden arrow index, and call it a *perfect index*. The term "perfect" originates from "perfect path" and "perfect pair" which is first introduced by Chen–Shen–Zhou in [CSZ18]. The following result is a direct corollary of [CSZ18, Proposition 5.1]

Corollary 5.5 ([CSZ18, Proposition 5.1]). An arrowed module αA over an SAGalgebra $A = \mathbb{k}Q/\mathcal{I}$ is a non-projective indecomposable G-projective module if and only if $\alpha \in \mathcal{R}_p$.

Proof. Recall that a *perfect pair* on a perfect forbidden cycle $\mathscr{C} = c_0 c_1 \cdots c_{n-1}$ of length n is defined as a sequence which is of the following form

$$\wp[t] = (c_{\overline{1+t}}, c_{\overline{2+t}}, \dots, c_{\overline{n-1+t}}, c_{\overline{1+t}}),$$

where, for any $m \in \mathbb{N}$, \overline{m} defined as m modulo n, see [CSZ18, Definition 3.3] or cf. [LZ24, Definition 3.1]. Then, by using the definition of SAG-algebra and [CSZ18, Proposition 5.1], we obtain this corollary.

Theorem 5.6. Let $A = \mathbb{k}\mathcal{Q}/\mathcal{I}$ be an SAG-algebra, $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be perfect forbidden cycles on $(\mathcal{Q}, \mathcal{I})$. Then $A_{\mathcal{R}_p}$ is isomorphic to the CM-Auslander algebra of A.

Proof. By Corollary 5.5, an indecomposable module is a non-projective indecomposable G-projective module if and only if it is isomorphic to αA with $\alpha \in \mathcal{R}_p$. Thus, we obtain

$$A_{\mathcal{R}_{p}} = \operatorname{End}_{A}\left(A \oplus \bigoplus_{\alpha \in \mathcal{R}_{p}} \alpha A\right) \cong \operatorname{End}_{A}\left(A \oplus \bigoplus_{\substack{G \in \operatorname{ind}(G\operatorname{-proj}(A))\\G \text{ is non-projective}}} G\right)$$
$$\cong \operatorname{End}_{A}\left(\bigoplus_{G \in \operatorname{ind}(G\operatorname{-proj}(A))} G\right) = A^{\operatorname{CMA}}.$$

Furthermore, we have the following result.

Corollary 5.7. An SAG-algebra is representation-finite if and only if so is its CM-Auslander algebra.

Proof. Let A be an SAG-algebra. Notice that \mathcal{R}_{p} is a left forbidden arrow index, then, by Corollary 5.4 (1) and (2), we have the representation types of A and $A_{\mathcal{R}_{p}}$ coincide. By Theorem 5.6, we have the representation types of A and A^{CMA} coincide.

Example 5.8. Let $A = \mathbb{k}\mathcal{T}/\mathcal{J}$ is given by the bound quiver $(\mathcal{T}, \mathcal{J})$, where \mathcal{T} is the quiver of the string algebra given in Example 2.2 and

$$\begin{aligned} \mathcal{J} &= \langle ab, bc, ca, dd', ee', ff', a'b', b'c', c'a', \\ e'f, e'c', f'd, f'a', d'e, d'b', a'e, b'f, c'd \rangle. \end{aligned}$$

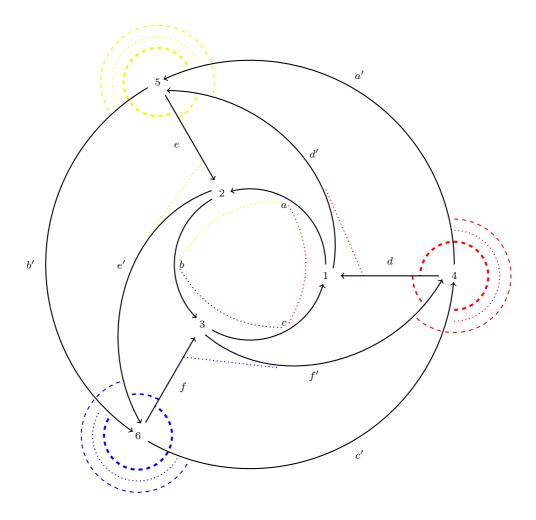


FIGURE 5.2. The bound quiver of the SAG-algebra given in Example 5.2 (The dashed lines represent the relations in \mathcal{I})

see FIGURE 5.2. Then A is an SAG-algebra, and aA, bA, and cA are both non-projective and G-projective since abc is a perfect forbidden cycle.

(1) Notice that a'b'c' is not a perfect forbidden cycle, then a'A, b'A, and c'A are not G-projective by using Corollary 5.5 (or [CSZ18, Proposition 5.1]). It follows that αA may be not G-projective.

(2) Now we provide an instance for Theorem 5.6 and Corollary 5.7. Take $\mathcal{R} = \{a, b, c\} = \mathcal{R}_{p}$, then the bound quiver of $A_{\mathcal{R}}$ is shown in FIGURE 5.3 which is isomorphic to the CM-Auslander algebra $A^{\text{CMA}} = \mathbb{k}\mathcal{T}^{\text{CMA}}/\mathcal{J}^{\text{CMA}}$ of A, where

$$\mathcal{J}^{\text{CMA}} = \langle a_{\text{R}}b_{\text{L}}, b_{\text{R}}c_{\text{L}}, c_{\text{R}}a_{\text{L}}, dd', ee', ff', a'b', b'c', c'a', \\ e'f, e'c', f'd, f'a', d'e, d'b', a'e, b'f, c'd \rangle.$$

Moreover, since the bound quiver $(\mathcal{T}, \mathcal{J})$ has a band

$$B = a'd'^{-1}ae^{-1}b'e'^{-1}bf^{-1}c'f'^{-1}cd^{-1},$$

we obtain that A is representation-infinite by using Lemma 5.1. Notice that B corresponds to the band

$$B^{\times} = a'd'^{-1}a_{\rm L}a_{\rm R}e^{-1}b'e'^{-1}b_{\rm L}b_{\rm R}f^{-1}c'f'^{-1}c_{\rm L}c_{\rm R}d^{-1}$$

on the bound quiver $(\mathcal{T}^{\text{CMA}}, \mathcal{J}^{\text{CMA}})$, then A^{CMA} is also a representation-infinite SGA-algebra.

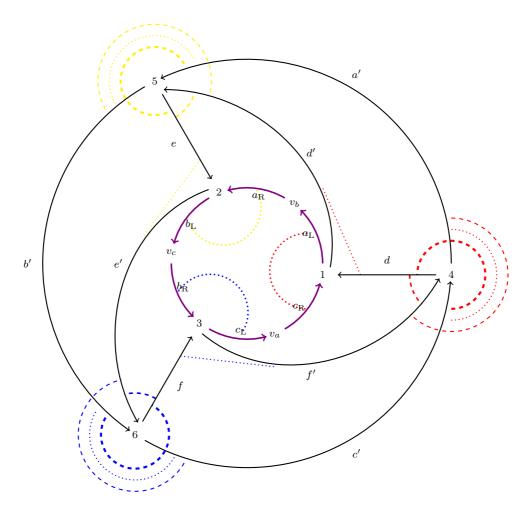


FIGURE 5.3. The bound quiver of the CM-Auslander algebra A^{CMA} , where A is the SAG-algebra in Example 5.8 (The dashed lines represent the relations in \mathcal{I}^{CMA})

Acknowledgements

- ▷ Yu-Zhe Liu is supported by the National Natural Science Foundation of China (Grant No. 12401042), Guizhou Provincial Basic Research Program (Natural Science) (Grant No. ZK[2024]YiBan066) and Scientific Research Foundation of Guizhou University (Grant Nos. [2022]53, [2022]65, [2023]16).
- ▷ Panyue Zhou is supported by the National Natural Science Foundation of China (Grant No. 12371034) and the Hunan Provincial Natural Science Foundation of China (Grant No. 2023JJ30008).

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of Interests The authors declare that they have no conflicts of interest to this work.

References

- [AAG08] Diana Avella-Alaminos and Christof Geiss. Combinatorial derived invariants for gentle algebras. J. Pure Appl. Algebra, 212(1):228–243, 2008. doi:10.1016/j.jpaa.2007.05.014.
- [ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. Elements of the Representation Theory of Associative Algebras, Volume 1 Techniques of Representation Theory. Cambridge University Press, The Edinburgh Building, Cambridge, UK, 2006.
- [BR87] Michael C.R. Butler and Claus Michael Ringel. Auslander-Reiten sequences with few middle terms and applications to string algebras. Commun. Algebra, 15(1-2):145–179, 1987. doi:10.1080/00927878708823416.
- [CL17] Xinhong Chen and Ming Lu. Cohen-Macaulay Auslander algebras of skewed-gentle algebras. Commun. Algebra, 45(2):849–865, 2017. doi:10.1080/00927872.2016.1175601 (preprint in 2015, arXiv:1502.03948).
- [CL19] Xinhong Chen and Ming Lu. Cohen-Macaulay Auslander algebras of gentle algebras. Commun. Algebra, 47(9):3597–3613, 2019. doi:10.1080/00927872.2019.1570225.
- [CSZ18] Xiao-Wu Chen, Dawei Shen, and Guodong Zhou. The Gorenstein-projective modules over a monomial algebra. P. Roy. Soc. Edinb. A, 148A(1):1115–1134, 2018. doi:10.1017/S0308210518000185.
- [GS18] Edward L. Green and Sibylle Schroll. Almost gentle algebras and their trivial extensions. P. Edinburgh. Math. Soc., 62(2):489–504, 2018. doi:10.1017/S001309151800055X.
- [Kal15] Martin Kalck. Singularity categories of gentle algebras. B. Lond. Math. Soc., 47(1):65–74, 2015. doi:10.1112/blms/bdu093.
- [Kra91] Henning Krause. Maps between tree and band modules. J. Algebra, 137(1):186–194, 1991. doi:10.1016/0021-8693(91)90088-P.
- [Lak16] Rosanna Laking. String algebras in representation theory. UK, The University of Manchester, 2016.
- [LZ24] Yu-Zhe Liu and Chao Zhang. The Cohen-Macaulay Auslander algebras of string algebras. Applied Categorical Structures, 32:no.17, 2024. doi:10.1007/s10485-024-09779-8.
- [LZH22] Yu-Zhe Liu, Yafeng Zhang, and Zhaoyong Huang. Gorenstein projective support τ -tilting modules over gentle algebras. In preparation, 2022.
- [Mou23] Kaveh Mousavand. τ-tilting finiteness of biserial algebras. Algebr. Represent. Theory, 26(1):2485–2522, 2023. https://doi.org/10.1007/s10468-022-10170-1doi:10.1007/s10468-022-10170-1.
- [Pla19] Pierre-Guy Plamondon. τ-tilting finite gentle algebras are representation-finite. Pacific Journal of Mathematics, 302(2):709–716, 2019. doi:10.2140/pjm.2019.302.709.
- [XZ21] Zongzhen Xie and Xiaojin Zhang. A bijection theorem for gorenstein projective\tautilting modules. arXiv:2109.01248, 2021.

School of Mathematics and Statistics, Guizhou University, 550025 Guiyang, Guizhou, P. R. China

Email address: liuyz@gzu.edu.cn / yzliu3@163.com

School of Mathematics and Statistics, Changsha University of Science and Technology, 410114 Changsha, Hunan, P. R. China

Email address: panyuezhou@163.com