

LOWER BOUNDS ON THE ESSENTIAL DIMENSION OF REDUCTIVE GROUPS

DANNY OFEK

ABSTRACT. We introduce a new technique for proving lower bounds on the essential dimension of split reductive groups. As an application, we strengthen the best previously known lower bounds for various split simple algebraic groups, most notably for the exceptional group E_8 . In the case of the projective linear group PGL_n , we recover A. Merkurjev's celebrated lower bound with a simplified proof. Our technique relies on decompositions of loop torsors over valued fields due to P. Gille and A. Pianzola.

CONTENTS

1. Introduction	1
2. Notation	5
3. Preliminaries about Henselian valued fields	7
4. Preliminaries about anisotropic torsors	11
5. Loop torsors	14
6. A theorem of Gille and Pianzola	15
7. Functoriality of decompositions of loop torsors	18
8. Almost all torsors are loop torsors	21
9. Proof of Theorem 1.2: First reduction	24
10. Conclusion of the proof of Theorem 1.2	26
11. Reductive subgroups of maximal rank	29
12. Abelian subgroups arising from gradings on the character lattice	31
13. Proof of Theorem 1.5: The overall strategy	34
14. Essential dimension of PGL_{p^n} at p	35
15. Essential dimension of PGO_{2n}^+ at 2	38
16. Essential dimension of HSpin_{16} at 2	41
17. Essential dimension of E_6 at 3	45
References	48

1. INTRODUCTION

Let G be a smooth linear algebraic group over a field k_0 and $\gamma \in H^1(L, G)$ a G -torsor over a field $k_0 \subset L$. A *field of definition* for γ is a subfield $k_0 \subset F \subset L$ such that γ lies in the image of the natural map:

$$H^1(F, G) \rightarrow H^1(L, G).$$

Danny Ofek was partially supported by a graduate fellowship from the University of British Columbia.

The *essential dimension* of γ is the minimal number of parameters needed to define γ . It is given by the formula:

$$\text{ed}(\gamma) = \min \left\{ \text{trdeg}_{k_0}(F) \mid F \text{ is a field of definition of } \gamma \right\}.$$

The *essential dimension* of G is defined as the supremum $\text{ed}(G) = \sup\{\text{ed}(\gamma)\}$ taken over all torsors $\gamma \in H^1(L, G)$ and all fields $k_0 \subset L$. It should be thought of as the minimal number of algebraically independent parameters needed to define an arbitrary G -torsor. Often G -torsors correspond bijectively to a class of algebraic objects, in which case $\text{ed}(G)$ is the minimal number of independent parameters needed to define a member of that class. For example:

- The essential dimension of the symmetric group S_d is the number of independent variables required to define an arbitrary field extension of degree d . In this form, mathematicians have tried to compute it as early as the 17th century. For an overview, see [22],[10] and [9].
- The essential dimension of PGL_d is the number of independent variables required to define an arbitrary central division algebra of degree d . This quantity has been studied since generic division algebras were first defined by C. Procesi [51, Section 2].

The problem of computing $\text{ed}(G)$ for a general algebraic group G has been studied by many authors since it was first posed by J. Buhler-Z. Reichstein [9] and Reichstein [52]. For a comprehensive survey of the developments in the field, we refer the reader to [44].

Let p be a prime integer. The *essential dimension at p* of G , denoted $\text{ed}(G; p)$, measures how many parameters are required to construct an arbitrary G -torsor up to prime-to- p extensions. See Section 2 for a precise definition. The inequality $\text{ed}(G) \geq \text{ed}(G; p)$ always holds, and almost all existing techniques to prove lower bounds on $\text{ed}(G)$ apply to $\text{ed}(G; p)$ as well; See [53, Section 5]. The same is true of the new technique introduced in this paper.

1.1. Overview of previous techniques. B. Youssin-Z. Reichstein gave the first systematic way to prove lower bounds on the essential dimension of algebraic groups. We recall their main theorem, commonly referred to as “the fixed-point method” because it was proven by an analysis of fixed points on generically free G -varieties. See [27],[14] for generalizations to positive characteristic.

Theorem 1.1. [55, Theorem 7.7] *Assume G is defined over an algebraically closed field of characteristic zero k_0 and G° is semisimple. Let $A \subset G(k_0)$ be a finite abelian group and p a prime number.*

- (1) *If $C_G(A \cap G^\circ)$ is finite, then $\text{ed}(G) \geq \text{rank}(A)$.*
- (2) *If A is a p -group and $C_G(A \cap G^\circ)$ is finite, then $\text{ed}(G; p) \geq \text{rank}(A)$.*

While Theorem 1.1 applies to many groups, it usually gives bounds which are far from tight. This is partially explained by the fact that the G -torsors witnessing the lower bound can be constructed over an iterated Laurent series field $k_0((t_1)) \dots ((t_r))$ which is a relatively simple field when k_0 is algebraically closed (here $r = \text{rank}(A)$).

P. Brosnan-A. Vistoli-Reichstein [8], [6] and later A. Merkurjev-N. Karpenko [32] introduced stack-theoretic techniques to construct G -torsors of high essential dimension over function fields of Severi-Brauer varieties. Stack-theoretic techniques give much stronger lower bounds than is possible using Theorem 1.1 for some groups, like finite p -groups and algebraic tori

[32],[36]. However, they give trivial lower bounds for most semisimple groups, including all adjoint simple groups. In [43], Merkurjev overcame this limitation of the stack-theoretic methods for adjoint groups of type A_n by proving $\text{ed}(\text{PGL}_n; p) \geq \text{ed}(T; p)$ for a certain torus T . He then computed $\text{ed}(T; p)$ using [36] to obtain the lower bounds

$$(1.1) \quad \text{ed}(\text{PGL}_{p^r}; p) \geq (r - 1)p^r + 1$$

which are orders of magnitude stronger than the lower bounds previously obtained by Theorem 1.1. Merkurjev's arguments are specific to groups of type A_n because they rely on explicit computations in the Brauer group (see [13] for generalizations to SL_n/μ_d).

1.2. Main results. In this paper, we introduce a new technique to prove lower bounds on $\text{ed}(G)$, $\text{ed}(G; p)$, which applies whenever G° is split reductive. We proceed in two steps:

- (1) We first prove $\text{ed}(G) \geq \text{ed}(C_G(A))$ for any finite split diagonalizable subgroup $A \subset G$ satisfying certain conditions.
- (2) We choose A in a systematic way, so that $C_G(A)$ is an extension of a torus by a finite group. This allows us to apply the results of [37] to give a strong lower bound on $\text{ed}(C_G(A))$.

In this way we obtain new lower bounds on the essential dimension of some simple groups as well as recover (1.1), see Theorem 1.5 below and Section 14. The next theorem gives sufficient conditions for the inequality $\text{ed}(G) \geq \text{ed}(C_G(A))$ to hold. Note that we do not assume G° is split. Recall that a G -torsor $[c_\sigma] \in H^1(F, G)$ is called *anisotropic* if the twisted group cG contains no copy of \mathbb{G}_m . A finite algebraic group A over k_0 is called *split-diagonalizable*, if it is isomorphic to $\mu_{n_1} \times \cdots \times \mu_{n_r}$ for some n_1, \dots, n_r coprime to $\text{char } k_0$.

Theorem 1.2. *Let G be a smooth linear algebraic group over a field k_0 . Assume either k_0 is perfect or G° is reductive. Let $A \subset G$ be a finite split-diagonalizable subgroup.*

- (1) *Let $p \neq \text{char } k_0$ be a prime. If A is a p -group and $C_G(A)$ admits an anisotropic torsor over some p -closed field $k_0 \subset k$, then we have:*

$$\text{ed}(G; p) \geq \text{ed}(C_G(A); p).$$

- (2) *Assume $\text{char } k_0$ is good for G (see Definition 2.1). If $C_G(A)$ admits an anisotropic torsor over some field $k_0 \subset k$, then we have:*

$$\text{ed}(G) \geq \text{ed}(C_G(A)).$$

Let F be a Henselian valued field with value group of finite rank. Our proof of Theorem 1.2 relies on the decompositions of *loop torsors* over F . Loop torsors and their decompositions were introduced by P. Gille-A. Pianzola for iterated Laurent series over a characteristic zero field in the context of the classification of loop algebras [25]. We will use both [25] and the recent generalizations to valuation rings of positive characteristic obtained by Gille [24].

Remark 1.3. Theorem 1.1 is a special case of Theorem 1.2. Indeed, if $C_G(A \cap G^\circ)$ is finite, then so is $C_G(A)$. Combining [9, Lemma 4.1] and [9, Theorem 6] gives:

$$\text{ed}(C_G(A)) \geq \text{ed}(A) = \text{rank}(A).$$

Since $C_G(A)$ is finite, all $C_G(A)$ -torsors are anisotropic. Therefore Theorem 1.2 implies:

$$\text{ed}(G) \geq \text{ed}(C_G(A)) \geq \text{rank}(A).$$

Note that the above argument works under the assumption $|C_G(A)| < \infty$, which is strictly weaker than $|C_G(A \cap G^\circ)| < \infty$.

As noted in the previous remark, any $C_G(A)$ -torsor is anisotropic when $C_G(A)$ is finite. However, in general, there is no simple criterion to determine whether $C_G(A)$ admits anisotropic torsors. J. Tits classified the simple simply connected split groups that do not admit anisotropic torsors [63]. For a generalization of his work to arbitrary simple groups and for semisimple groups of certain types, see [48]. We have no examples where the inequalities of Theorem 1.2 fail, so it is natural to ask if one may improve the theorem as follows:

Question 1.4. *Is the conclusion of Theorem 1.2 true without the assumption that $C_G(A)$ admits anisotropic torsors?*

After the proof of Theorem 1.2 in Section 10, we give an example where all $C_G(A)$ -torsors are isotropic and our proof breaks down. However, we have no reason to expect a negative answer to Question 1.4.

In Section 12, we give a streamlined root-theoretic approach to choosing split-diagonalizable subgroups of split groups that satisfy the conditions of Theorem 1.2. This leads to the following new lower bounds:

Theorem 1.5. *Assume $\text{char } k_0 \neq 2, 3$.*

(1) $\text{ed}(E_8; 2) = \text{ed}(\text{HSpin}_{16}; 2) \geq 56$

(2) $\text{ed}(E_8; 3) \geq 13$

(3) $\text{ed}(E_6^{\text{ad}}; 3) \geq 6$

(4) $\text{ed}(E_7^{\text{ad}}; 2) \geq 19$

(5) *If $n = 2^r m \geq 4$ for $r \geq 1$ and m is odd, then*

$$\text{ed}(\text{PGO}_{2n}^+; 2) \geq (r-1)2^{r+1} + n.$$

(6) *If $n \geq 3$ is odd, then*

$$\text{ed}(\text{PGO}_{2n}^+; 2) \geq 3n - 4.$$

The inequalities in Theorem 1.5 improve on long-standing lower bounds. To put them into context, we include a list indicating the best lower and upper bounds for these groups that were known prior to this paper.

Previously known bounds. *Assume $\text{char } k_0 \neq 2, 3$.*

(1) $120 \geq \text{ed}(E_8; 2) \geq 9$

(2) $73 \geq \text{ed}(E_8; 3) \geq 5$

(3) $21 \geq \text{ed}(E_6^{\text{ad}}; 3) \geq 4$

(4) $57 \geq \text{ed}(E_7^{\text{ad}}; 2) \geq 8$

(5) *If $n = 2^r m$ for some $r \geq 0$ and $m \geq 3$ is odd, then*

$$(m-1)2^{2(r+1)} - n \geq \text{ed}(\text{PGO}_{2n}^+; 2) \geq 2n - 2.$$

(6) *If $n = 2^r$ for some $r \geq 2$, then*

$$n^2 - n \geq \text{ed}(\text{PGO}_{2n}^+; 2) \geq r2^r.$$

All of the lower bounds in (1)-(4) follow from the characteristic-free versions of Theorem 1.1; See [14] and [27]. For references for the upper bounds in (1)-(4), see [44, 3h]. The

upper bounds in (5) and (6) are due to M. Macdonald [39, Section 0.2]. For the lower bounds in (5),(6), see [14, Theorem 13] and [1, Lemma 2.6] respectively. We note that (5) implies

$$3n - 4 \geq \text{ed}(\text{PGO}_{2n}^+; 2)$$

for odd n ; See also [1, Corollary 3.2]. Therefore the inequality in Theorem 1.5(6) is tight.

Recall that the *reductive rank* of G is the dimension of a maximal torus in $T \subset G$. Clearly if $H \subset G$ is a subgroup, then $\text{rank}(H) \leq \text{rank}(G)$. A subgroup H is said to be of *maximal rank* if $\text{rank}(H) = \text{rank}(G)$. In Section 11, we use Theorem 1.2 to prove

$$\text{ed}(G) \geq \text{ed}(H), \quad \text{ed}(G; p) \geq \text{ed}(H; p)$$

for reductive subgroups $H \subset G$ of maximal rank under certain conditions; See Corollary 11.1. We use this corollary to establish Parts (1),(2) and (4) of Theorem 1.5. The proof relies on Borel–de Siebenthal’s theorem on reductive subgroups of maximal rank [4].

1.3. Outline of the paper. The rest of the paper is structured as follows. Sections 2-4 deal with notation and preliminaries. In Section 5 we introduce the definition of loop torsors and their decompositions. In Section 6 we adapt of a theorem of Gille-Pianzola about the uniqueness of decompositions of anisotropic loop torsors to our needs. Section 7 deals with the functoriality of decompositions of loop torsors with respect to extension of scalars. In Section 8 we give criteria to determine whether a G -torsor is a loop torsor. The proof of Theorem 1.2 is completed in Section 10. Section 12 contains a method of choosing subgroups $A \subset G$ that satisfy the conditions of Theorem 1.2 and such that $C_G(A)^\circ$ is a split torus. The last four sections contain the computations needed to prove Theorem 1.5.

1.4. Acknowledgments. I am grateful to A. Soofiani for many fruitful discussions, and to V. Chernousov for pointing out the connection between my ideas and his work with P. Gille and A. Pianzola. This connection was further explained to me by Gille and it simplified this paper greatly. Finally, I would like to thank my supervisor Z. Reichstein for many useful meetings, suggestions and advice throughout the work on this project.

2. NOTATION

Throughout this paper, G will denote a smooth linear algebraic group over a base field k_0 . All fields and rings we consider will contain k_0 . We will sometimes assume G and k_0 satisfy the following additional assumptions.

Definition 2.1. Let $R_u(G) \subset G^\circ$ denote the unipotent radical of G and set $\overline{G} = G/R_u(G)$; See [46, Definition 6.44] for the definition of $R_u(G)$. We say that the characteristic of k_0 is *good for G* if the following hold:

- (1) The group \overline{G}° is reductive.
- (2) There exists a maximal torus $T \subset \overline{G}$ split by a prime-to-char k_0 extension of k_0 and such that $|N_{\overline{G}}(T)/T|$ is prime to char k_0 (note that $|N_{\overline{G}}(T)/T|$ is finite because \overline{G}° is reductive).

Note that condition (1) is automatic if G° is reductive or k_0 is perfect [46, Proposition 19.11].

We fix a system of compatible primitive roots of unity $\zeta_n \in k_{0,\text{sep}}$ for all n not divisible by char k_0 . Let $k_0 \subset k$ be a field. We will denote the cyclotomic character of k by:

$$\theta : \text{Gal}(k) \rightarrow (\hat{\mathbb{Z}}')^*.$$

Here $\hat{\mathbb{Z}}' = \prod_{p \neq \text{char } k_0} \hat{\mathbb{Z}}_p$ is the prime-to-char k_0 part of the profinite integers $\hat{\mathbb{Z}}$. The cyclotomic character is the unique homomorphism satisfying:

$$\sigma \zeta_n = \zeta_n^{\theta(\sigma)}$$

for all n not divisible by $\text{char } k_0$ and $\sigma \in \text{Gal}(k)$. We will identify étale groups A over k with their $\text{Gal}(k)$ -group of points $A(k_{\text{sep}})$. This will be particularly convenient when working with finite split-diagonalizable subgroups $A \subset G$. Recall that a *finite split-diagonalizable* group A over k is an étale group scheme isomorphic to $\mu_{n_1} \times \cdots \times \mu_{n_r}$ for some n_1, \dots, n_r coprime to $\text{char } k_0$. The following characterization of split-diagonalizable groups will be used implicitly throughout the paper. We leave the proof as an exercise to the reader.

Fact 2.2. *Let A be an abelian étale group over k such that $|A|$ is prime to $\text{char } k$. Then A is split-diagonalizable if and only if for any $a \in A$ and $\sigma \in \text{Gal}(k)$:*

$$(2.1) \quad \sigma a = a^{\theta(\sigma)}.$$

The letter \mathcal{O} will denote a Henselian valuation ring containing k_0 . We set $F = \text{Frac}(\mathcal{O})$ and let $\nu : F^* \rightarrow \Gamma_F$ be the corresponding valuation. The residue field of ν will be denoted k . Note there exists a natural embedding $k_0 \subset k$ induced from the inclusion $k_0 \subset \mathcal{O}$. Since ν is Henselian, it admits a unique extension to any algebraic extension $F \subset L$. We will denote the extension of ν to L by ν again by abuse of notation. Unless explicitly stated we always assume:

Assumption 2.3. The value group Γ_F is a finitely generated (free) abelian group.

We will say (F, ν) is an *iterated Laurent series field*, if it is isomorphic to $k((t_1)) \cdots ((t_r))$ equipped with the usual (t_1, \dots, t_r) -adic valuation for some $r \geq 0$; See [62, Section 1.1]. Let $F_{\text{in}} \subset F_{\text{tr}}$ be the maximal inertial and tamely ramified extensions of F inside a fixed separable closure F_{sep} . The corresponding Henselian valuation rings are denoted $\mathcal{O}_{\text{in}} \subset \mathcal{O}_{\text{tr}}$. We will use the following notation for the tamely ramified part of the absolute Galois group of F :

$$\text{Gal}_{\text{tr}}(F) = \text{Gal}(F_{\text{tr}}/F).$$

We will denote the homomorphism $G(\mathcal{O}_{\text{tr}}) \rightarrow G(k_{\text{sep}})$ induced from the residue homomorphism $\mathcal{O}_{\text{tr}} \rightarrow k_{\text{sep}}$ by $g \mapsto \bar{g}$. We refer the reader to [18], [62, Appendix A] for general results on valuation theory.

Remark 2.4. All of $\mathcal{O}, \mathcal{O}_{\text{in}}, \mathcal{O}_{\text{tr}}$ are stabilized by $\text{Gal}_{\text{tr}}(F)$ because the extension of ν to F_{tr} is preserved by $\text{Gal}_{\text{tr}}(F)$. Moreover, $\text{Gal}_{\text{tr}}(F)$ stabilizes the maximal ideal of \mathcal{O}_{tr} and so it acts on its residue field k_{sep} . Therefore $\text{Gal}_{\text{tr}}(F)$ acts on $G(\mathcal{O}_{\text{in}}), G(\mathcal{O}_{\text{tr}}), G(k_{\text{sep}})$ and the induced map $G(\mathcal{O}_{\text{tr}}) \rightarrow G(k_{\text{sep}})$ is $\text{Gal}_{\text{tr}}(F)$ -equivariant.

The symbol $H^1(F, G)$ will stand for the Galois cohomology set $H^1(\text{Gal}(F), G(F_{\text{sep}}))$. We will denote cohomology classes in $H^1(F, G)$ by $[c_\sigma]$ where $c_\sigma \in Z^1(\text{Gal}(F), G(F_{\text{sep}}))$ is a $\text{Gal}(F)$ -cocycle. Let $F \subset L$ be a field extension and assume $F_{\text{sep}} \subset L_{\text{sep}}$. The restriction map of absolute Galois groups

$$\text{Gal}(L) \rightarrow \text{Gal}(F), \quad \sigma \mapsto \sigma|_{F_{\text{sep}}}$$

gives rise to inflation maps

$$H^1(F, G) \rightarrow H^1(L, G), \quad \gamma \mapsto \gamma_L.$$

Here γ_L is the cohomology class of the inflation of c_σ to L given by

$$\text{Inf}_{L/F}(c)_\sigma = c_{\sigma|_{F_{\text{sep}}}}.$$

The set of *tamely ramified torsors* will be denoted:

$$H_{\text{tr}}^1(F, G) = H^1(\text{Gal}_{\text{tr}}(F), G(F_{\text{tr}})).$$

There is always a natural inclusion $H_{\text{tr}}^1(F, G) \subset H^1(F, G)$ which is a bijection if G° is reductive and $\text{char } k_0$ is good for G in the sense of Definition 2.1 (see Proposition 8.1).

The morphism $\text{Spec } \mathcal{O}_{\text{in}} \rightarrow \text{Spec } \mathcal{O}$ is the universal pro-étale cover of $\text{Spec } \mathcal{O}$ and therefore there is a natural identification $\pi_1(\text{Spec } \mathcal{O}) = \text{Gal}(F_{\text{in}}/F)$ [45, Example 5.2.(d)]. For any cocycle $a_\sigma \in Z^1(\text{Gal}(F_{\text{in}}/F), G(\mathcal{O}_{\text{in}}))$, we will write ${}_a G$ for the twist of $G_{\mathcal{O}}$ by the G -torsor defined by a_σ over \mathcal{O} ; See [23, Section 2.2].

A variety X over k_0 is a reduced (but possibly reducible) quasi-projective scheme of finite type. Finally, we recall the definition of the essential dimension of G at a prime.

Definition 2.5. Let $\gamma \in H^1(F, G)$ be a G -torsor over a field $k_0 \subset F$ and p a prime integer. An algebraic extension $F \subset L$ is called a *prime-to- p* extension if any finite subextension $F \subset L' \subset L$ is of degree prime to p . The *essential dimension at p* of γ is the minimal number of parameters required to define γ if one ignores prime-to- p extensions:

$$\text{ed}(\gamma; p) = \min \left\{ \text{ed}(\gamma_L) \mid F \subset L \text{ is a prime-to-} p \text{ field extension} \right\}.$$

We define the *essential dimension at p* of G by $\text{ed}(G; p) = \sup\{\text{ed}(\gamma; p)\}$, the supremum taken over all G -torsors as before.

3. PRELIMINARIES ABOUT HENSELIAN VALUED FIELDS

3.1. A fundamental exact sequence. Recall that the Galois extensions $F \subset F_{\text{in}} \subset F_{\text{tr}}$ give us the exact sequence

$$(3.1) \quad 1 \rightarrow \text{Gal}_{\text{tr}}(F_{\text{in}}) \rightarrow \text{Gal}_{\text{tr}}(F) \rightarrow \text{Gal}(F_{\text{in}}/F) \rightarrow 1.$$

The groups $\text{Gal}_{\text{tr}}(F_{\text{in}})$ and $\text{Gal}(F_{\text{in}}/F)$ can be understood in terms of the value group Γ_F and the residue field k respectively. There is a natural isomorphism $\text{Gal}(F_{\text{in}}/F) \cong \text{Gal}(k)$ given by taking an automorphism $\sigma \in \text{Gal}(F_{\text{in}}/F)$ to the unique automorphism $\bar{\sigma} \in \text{Gal}(k)$ satisfying for all $x \in \mathcal{O}$:

$$(3.2) \quad \bar{\sigma}(\bar{x}) = \overline{\sigma(x)},$$

where $\bar{x} \in k$ is the residue class of x [62, Theorems A.23].

Remark 3.1. We will often identify $\text{Gal}(F_{\text{in}}/F)$ and $\text{Gal}(k)$ using this isomorphism. For example, we might act on $G(F_{\text{in}})$ using $\text{Gal}(k)$ or act on $G(k_{\text{sep}})$ using $\text{Gal}(F_{\text{in}}/F)$.

The group $\text{Gal}_{\text{tr}}(F_{\text{in}})$ is the *tame inertia group* of F . Let $\hat{\mathbb{Z}}' = \prod_{p \neq \text{char } k} \hat{\mathbb{Z}}_p$ be the prime-to-char k part of $\hat{\mathbb{Z}}$ and set $\Gamma_F^\vee := \text{Hom}(\Gamma_F, \hat{\mathbb{Z}}')$. There is a natural isomorphism (depending only on our choice of roots of unity)

$$(3.3) \quad \Phi : \text{Gal}_{\text{tr}}(F_{\text{in}}) \xrightarrow{\sim} \Gamma_F^\vee.$$

The isomorphism Φ is uniquely determined by the following equation, which holds for all n prime to $\text{char } k$, $x \in F^*$ and $\sigma \in \text{Gal}_{\text{tr}}(F_{\text{in}})$:

$$(3.4) \quad \frac{\sigma(x^{1/n})}{x^{1/n}} = \zeta_n^{\Phi(\sigma)(\nu(x))}.$$

See [62, Theorem A.24] for example. Note that the above expression does not depend on a choice of an n -th root of x because F_{in} contains all n -th roots of unity.

Remark 3.2. Since we are assuming $\Gamma_F \cong \mathbb{Z}^r$ for some r , we have $\Gamma_F^\vee \cong \text{Hom}(\mathbb{Z}^r, \hat{\mathbb{Z}}') \cong \hat{\mathbb{Z}}'^r$. In particular, any finite quotient of Γ_F^\vee is an abelian group of order prime to $\text{char } k$.

Next we introduce uniformizers. They will help us describe splittings of (3.1) in a systematic way.

Definition 3.3. A left inverse $\pi : \Gamma_F \rightarrow F^*$ to $\nu : F^* \rightarrow \Gamma_F$ will be called a *uniformizing parameter*. Since the group operation in Γ_F is written additively while F^* is written multiplicatively, it will be convenient for us to use the exponential notation π^γ in place of $\pi(\gamma)$, for any $\gamma \in \Gamma_F$.

The next proposition gives a convenient splitting of (3.1). It is originally due to J. Neukirch [47, Satz 2].

Proposition 3.4. *Let $\pi : \Gamma_{F_{\text{tr}}} \rightarrow F_{\text{tr}}^*$ be a uniformizer such that $\pi^\gamma \in F$ for all $\gamma \in \Gamma_F$.*

- (1) *The field $F_\pi := F(\pi^\gamma; \gamma \in \Gamma_{F_{\text{tr}}})$ is a complement of F_{in} in F_{tr} . That is, it satisfies $F_\pi F_{\text{in}} = F_{\text{tr}}$ and $F_\pi \cap F_{\text{in}} = F$.*
- (2) *There exists a unique section $s_\pi : \text{Gal}(F_{\text{in}}/F) \rightarrow \text{Gal}_{\text{tr}}(F)$ of the homomorphism $\text{Gal}_{\text{tr}}(F) \rightarrow \text{Gal}(F_{\text{in}}/F)$ in (3.1) whose image fixes F_π .*
- (3) *Let $\Phi : \text{Gal}_{\text{tr}}(F_{\text{in}}) \rightarrow \Gamma_F^\vee$ be the isomorphism (3.3) and denote by*

$$\bar{s}_\pi : \text{Gal}(k) \rightarrow \text{Gal}_{\text{tr}}(F)$$

the composition of s_π with the isomorphism $\text{Gal}(k) \xrightarrow{\sim} \text{Gal}(F_{\text{in}}/F)$. There exists an isomorphism:

$$\Psi_\pi : \text{Gal}(k) \times \Gamma_F^\vee \xrightarrow{\sim} \text{Gal}_{\text{tr}}(F), \quad (\sigma, f) \mapsto \bar{s}_\pi(\sigma)\Phi^{-1}(f)$$

Here the $\text{Gal}(k)$ -action on Γ_F^\vee is given by pointwise multiplication with the cyclotomic character $\theta : \text{Gal}(k) \rightarrow (\hat{\mathbb{Z}}')^$.*

- (4) *For any $(\sigma, f) \in \text{Gal}(k) \times \Gamma_F^\vee$, $u \in F_{\text{in}}$, m coprime to $\text{char}(k)$ and $\gamma \in \Gamma_F$, we have:*

$$(3.5) \quad \Psi_\pi(\sigma, f)(u\pi^{\gamma/m}) = \sigma(u)\zeta_m^{\theta(\sigma)f(\gamma)}\pi^{\gamma/m}.$$

Here we let $\text{Gal}(k)$ act on F_{in} as in Remark 3.1.

Proof. Let $\gamma_1, \dots, \gamma_r$ be a \mathbb{Z} -basis for Γ_F . The field $F_n = F(\pi_1^{\gamma_1/n}, \dots, \pi_r^{\gamma_r/n})$ is totally tamely ramified with value group $\frac{1}{n}\Gamma_F$ for all n prime to $\text{char } k$. Since $\Gamma_{F_{\text{tr}}} = \bigcup_{n \text{ prime to } p} \frac{1}{n}\Gamma_F$ [62, Theorem A.24], we have:

$$F_\pi = \bigcup_{n \text{ prime to } p} F_n.$$

Therefore residue field of F_π is k and $\Gamma_{F_\pi} = \Gamma_{F_{\text{tr}}}$. This implies Parts (1) and (2) by [47, Satz 2] and the discussion following it; See also the proof of [18, Theorem 22.1.1]. Part (3)

follows from the fact that s_π splits the exact sequence (3.1); See [18, Corollary 22.1.2]. To prove Part (4), we compute using Part (3):

$$\begin{aligned}\Psi_\pi(\sigma, f)(u\pi^{\gamma/m}) &= \bar{s}_\pi(\sigma)\Phi^{-1}(u\pi^{\gamma/m}) \\ &= \bar{s}_\pi(\sigma)(u\zeta_m^{f(\gamma)}\pi^{\gamma/m}) \\ &= \sigma(u\zeta_m^{f(\gamma)})\bar{s}_\pi(\sigma)(\pi^{\gamma/m}) \\ &= \sigma(u)\zeta_m^{\theta(\sigma)f(\gamma)}\pi^{\gamma/m}.\end{aligned}$$

Here the last three equalities follow from (3.4), (3.2) and the fact that $\bar{s}_\pi(\sigma)$ fixes F_π . \square

Remark 3.5. From now on, whenever we use a uniformizer π of F_{tr} , we will always assume $\pi^\gamma \in F$ for all $\gamma \in \Gamma_F$. For any $\pi_1, \dots, \pi_r \in F^*$ such that $\nu(\pi_1), \dots, \nu(\pi_r)$ is a basis for Γ_F there exists a uniformizer π for F such that for all $1 \leq i \leq r$:

$$\pi^{\nu(\pi_i)} = \pi_i.$$

Moreover, π may be extended to a uniformizer of F_{tr} . To see this pick a compatible system of roots $\pi^{1/n}$ for all n prime to $q = \text{char } k_0$ as in [54, Lemma 3.2]. By [18, Section 16.2], $\Gamma_{F_{\text{tr}}} = \mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} \Gamma_F$, where:

$$\mathbb{Z}_{(q)} = \left\{ \frac{m}{n} \in \mathbb{Q} \mid n \text{ prime to } q \right\}.$$

Therefore $\nu(\pi_1), \dots, \nu(\pi_r) \in \Gamma_{F_{\text{tr}}}$ is a $\mathbb{Z}_{(q)}$ basis and the homomorphism

$$\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} \Gamma_F \rightarrow F_{\text{tr}}, \quad \sum_{i=1, \dots, r} \frac{m_i}{n_i} \nu(\pi_i) \mapsto \prod_{i=1, \dots, r} \pi_i^{m_i/n_i},$$

is a uniformizer for F_{tr} extending π .

3.2. Extensions of Henselian fields. Let L/F be an extension of Henselian fields and denote the corresponding extension of residue fields by l/k . Pick separable closures such that $F_{\text{sep}} \subset L_{\text{sep}}$. Then we have $F_{\text{tr}} \subset L_{\text{tr}}$ and $F_{\text{in}} \subset L_{\text{in}}$. Therefore there are well-defined restriction maps:

$$\text{Gal}_{\text{tr}}(L) \rightarrow \text{Gal}_{\text{tr}}(F), \quad \text{Gal}(L_{\text{in}}/L) \rightarrow \text{Gal}(F_{\text{in}}/F), \quad \text{Gal}_{\text{tr}}(L_{\text{in}}) \rightarrow \text{Gal}_{\text{tr}}(F_{\text{in}}).$$

There is also a restriction map $\Gamma_L^\vee \rightarrow \Gamma_F^\vee, f \mapsto f|_F$, given by restricting a homomorphism $f \in \Gamma_L^\vee$ to $\Gamma_F \subset \Gamma_L$. Next we state a functoriality lemma for the decomposition of $\text{Gal}_{\text{tr}}(F)$.

Lemma 3.6. *Let $F \subset L$ be an extension of Henselian valued fields and denote by ν the valuation on L . Denote the residue field of F, L by k and l respectively. Let π, τ be uniformizers for F_{tr} and L_{tr} respectively. Denote the induced section $\bar{s}_\tau : \text{Gal}(l) \rightarrow \text{Gal}_{\text{tr}}(L)$ and isomorphisms as in Proposition 3.4 by*

$$\Psi_\pi^F : \text{Gal}(k) \times \Gamma_F^\vee \rightarrow \text{Gal}_{\text{tr}}(F), \quad \Psi_\tau^L : \text{Gal}(l) \times \Gamma_L^\vee \rightarrow \text{Gal}_{\text{tr}}(L).$$

For any $\gamma \in \Gamma_{F_{\text{tr}}}$ set $u^\gamma = \pi^\gamma \tau^{-\gamma}$. For any $\sigma \in \text{Gal}(l)$, there exists a unique homomorphism $\chi_\sigma \in \Gamma_F^\vee$ satisfying the equation:

$$(3.6) \quad \frac{\bar{s}_\tau(\sigma)(u^{\gamma/n})}{u^{\gamma/n}} = \zeta_n^{\chi_\sigma(\gamma)}$$

for all $\gamma \in \Gamma_F$ and n prime to $\text{char } k$. We have for all $\sigma \in \text{Gal}(l), f \in \Gamma_L^\vee$:

$$\Psi_\tau^L(\sigma, f)|_{F_{\text{tr}}} = \Psi_\pi^F(\sigma|_{k_{\text{sep}}}, f|_F + \theta(\sigma)^{-1}\chi_\sigma).$$

Proof. For any $\gamma \in \Gamma_F$ and n prime to $\text{char } k$, $u^{\gamma/n}$ is an n -th root of $u^\gamma \in L_{\text{in}}$. Since u^γ is a unit and the residue field of L_{in} is separably closed, Hensel's lemma implies $u^{\gamma/n} \in L_{\text{in}}$. In particular, we have:

$$\bar{s}_\tau(\sigma)(u^{\gamma/n}) = \sigma(u^{\gamma/n}).$$

Therefore for any $v \in F_{\text{in}}, \gamma \in \Gamma_F$ applying (3.5) twice gives:

$$\begin{aligned} \Psi_\tau^L(\sigma, f)(v\pi^{\gamma/n}) &= \Psi_\tau^L(\sigma, f)(vu^{\gamma/n}\tau^{\gamma/n}) \\ &\stackrel{(3.5)}{=} \sigma(v)\sigma(u^{\gamma/n})\zeta_n^{\theta(\sigma)f(\gamma)}\tau^{\gamma/n} \\ &= \sigma(v)\frac{\sigma(u^{\gamma/n})}{u^{\gamma/n}}\zeta_n^{\theta(\sigma)f(\gamma)}u^{\gamma/n}\tau^{\gamma/n} \\ &= \sigma(v)\zeta_n^{\chi_\sigma(\gamma)}\zeta_n^{\theta(\sigma)f(\gamma)}\pi^{\gamma/n} \\ &= \sigma(v)\zeta_n^{\theta(\sigma)f(\gamma)+\chi_\sigma(\gamma)}\pi^{\gamma/n} \\ &\stackrel{(3.5)}{=} \Psi_\pi^F(\sigma_{k_{\text{sep}}}, f|_F + \theta(\sigma)^{-1}\chi_\sigma)(v\pi^{\gamma/n}) \end{aligned}$$

The claim follows because elements of the form $v\pi^{\gamma/n}$ generate F_{tr} over F by Proposition 3.4(1). \square

Next, we recall A. Ostrowski's foundational theorem in valuation theory. It will be used often and sometimes implicitly; See [18, Theorem 17.2.1] for a modern proof.

Theorem 3.7. [49] *Let $F \subset L$ be a finite extension of Henselian valued fields with value groups $\Gamma_F \subset \Gamma_L$ and residue fields $k \subset l$. There exists an integer δ such that:*

$$[L : F] = \delta[\Gamma_L : \Gamma_F][l : k].$$

If $\text{char } k > 0$, then δ is a power of $\text{char } k$ and $\delta = 1$ otherwise. In particular, both $[\Gamma_L : \Gamma_F]$ and $[l : k]$ divide $[L : F]$.

We finish this section with a technical lemma that will be used to avoid working with infinite algebraic extensions.

Lemma 3.8. *Let (F, ν) be an ascending union of a countable chain of valued field $F_1 \subset F_2 \subset \dots$ and let L/F be a tamely ramified Galois extension. For all large enough i there exists a tamely ramified Galois extension L_i/F_i such that L is a compositum $L = L_iF$ and the restriction map $\text{Gal}(L/F) \rightarrow \text{Gal}(L_i/F_i)$ is an isomorphism.*

Proof. Let $x \in L$ be an element such that $L = F(x)$ and let $P(t) \in F[t]$ be the minimal polynomial of x . For large enough i , $P(t) \in F_i[t]$ and so $P(t)$ is also the minimal polynomial of x over F_i . Set $L_i := F_i(x)$ and note

$$[L_i : F_i] = \deg P = [L : F].$$

By enlarging i further we can make $P(t)$ factor fully over L_i so that L_i/F_i is Galois. Let l_i/k_i denote the residue field extension of L_i/F_i . A similar argument shows that l_i/k_i is Galois of degree $[l : k]$ for large enough i . One also easily checks that for large enough i we have $[\Gamma_{L_i} : \Gamma_{F_i}] = [\Gamma_L : \Gamma_F]$. Combining all of this together, we find that for large i :

$$[L_i : F_i] = [L : F] = [\Gamma_L : \Gamma_F][l : k] = [\Gamma_{L_i} : \Gamma_{F_i}][l_i : k_i].$$

Therefore L_i/F_i is defectless and tamely ramified. The restriction map $\text{Gal}(L/F) \rightarrow \text{Gal}(L_i/F_i)$ is an isomorphism because it is injective and

$$|\text{Gal}(L/F)| = [L : F] = [L_i : F_i] = |\text{Gal}(L_i/F_i)|.$$

Indeed, any two different elements $\sigma, \tau \in \text{Gal}(L/F)$ differ on L_i for large enough i . \square

4. PRELIMINARIES ABOUT ANISOTROPIC TORSORS

In this final section of preliminaries, we record a couple of facts about versal torsors and anisotropic torsors. We first recall the definition of a versal torsor.

Definition 4.1. A versal torsor $\gamma \in H^1(l, G)$ is the generic fiber of a G -torsor for the étale topology $T \rightarrow B$ such that:

- (1) B is an irreducible smooth variety over k_0 .
- (2) For any field extension E/k_0 with $|E| = \infty$ and open dense subset $U \subset B$, there exists $u \in U(E)$ such that:

$$\eta \cong T_u := T \times_u \text{Spec } E.$$

Note that there exists a versal G -torsor [59, Section 5.3]. A G -torsor $T \rightarrow B$ is called isotropic if there exists an embedding $\mathbb{G}_m \rightarrow {}_T G$ over B , where ${}_T G$ denotes the twisted group defined by T ; See [23, Page 6]. The following technical lemma shows the notion of isotropy plays well with inductive limits of base rings.

Lemma 4.2. *Let $T \rightarrow B$ be a G -torsor over an irreducible variety B over k_0 .*

- (1) *If T is isotropic, then so is T_u for any k_0 -algebra R and $u \in B(R)$.*
- (2) *Let (Λ, \leq) be a partially ordered filtered set with minimal element $0 \in \Lambda$. Let $(R_\lambda)_{\lambda \in \Lambda}$ be an inductive system of k_0 -algebras over Λ with transition maps $\sigma_{\lambda\mu} : R_\lambda \rightarrow R_\mu$ for all $\lambda \leq \mu$. Let $u_0 : \text{Spec } R_0 \rightarrow B$ be a point and set $u_\lambda = u_0 \circ \sigma_{0\lambda}$ for any $\lambda \in \Lambda$. Assume $(R_\lambda)_{\lambda \in \Lambda}$ has an inductive limit R and let $u = u_0 \circ \iota$, where $\iota : \text{Spec } R \rightarrow \text{Spec } R_0$ is the canonical morphism. If T_u is isotropic, then T_{u_λ} is isotropic for some $\lambda \in \Lambda$.*
- (3) *Let $u \in B(L)$ be the generic point of B . If T_u is isotropic, then there exists an open $U \subset B$ such that T_U is isotropic.*
- (4) *Let $\gamma \in H^1(k, G)$ be a torsor over a field extension k/k_0 . If γ_l is isotropic for some l/k , then $\gamma_{l'}$ is isotropic for some finitely generated subextension $k \subset l' \subset l$.*

Proof. Let $G' = {}_T G$ be the twisted group over B defined by T . For any $u \in B(R)$, let $G'_u = {}_{T_u} G$ denote the group $G' \times_u \text{Spec } R$ over $\text{Spec } R$.

(1) If T is isotropic, then there exists an embedding $\mathbb{G}_{m,B} \subset G'$. This embedding specializes to an embedding $\mathbb{G}_{m,R} \subset G'_u$ and so T_u is isotropic.

(2) Since T_u is isotropic, there exists an embedding $\mathbb{G}_{m,R} \subset G'_u$. This embedding is induced from an embedding $\mathbb{G}_{m,R_\lambda} \subset G'_{u_\lambda}$ for some $\lambda \in \Lambda$ by [30, Lemma 10.62]. Therefore T_{u_λ} is isotropic.

(3) Assume without loss of generality that $B = \text{Spec } R$ for some integral domain R of finite type over k_0 . Let L be the fraction field of R . The generic point u corresponds to the inclusion $R \subset L$. Since L is the inductive limit of all localization $R[f^{-1}]$ for $f \in R \setminus \{0\}$, the result follows from Part (2).

(4) The result follows immediately from Part (2) because l is the inductive limit of all finitely generated subextensions $k \subset l' \subset l$. \square

Before the next lemma, recall that a field k is called p -closed for some prime p if it admits no proper prime-to- p extension. An algebraic extension $k^{(p)}/k$ is called a p -closure of k if $k^{(p)}$ is p -closed and any finite subextension $k \subset k' \subset k^{(p)}$ is of degree prime-to- p . Any field k admits p -closures for any $p \neq \text{char } k$. We refer the reader to [19, Proposition 101.16] for more details.

Proposition 4.3. *Let $\gamma \in H^1(k, G)$ be a versal torsor over some field k/k_0 .*

- (1) *If G admits an anisotropic torsor η over some other field $k_0 \subset l$, then γ is anisotropic.*
- (2) *Let $p \neq \text{char } k_0$ be a prime. If G admits an anisotropic torsor η over some a p -closed field $k_0 \subset l$, then $\gamma_{k^{(p)}}$ is anisotropic for any p -closure $k^{(p)}/k$.*

Proof. Assume γ is the generic fiber of a G -torsor $T \rightarrow B$ as in Definition 4.1. Let η be an arbitrary G -torsor over a field extension $k_0 \subset l$.

(1) Let $u \in B(k)$ be a generic point such that $\gamma = T_u$ is isotropic. By Lemma 4.2(3), there exists a dense open $U \subset B$ such that T_U is isotropic. We may assume $|l| = \infty$ because passing to $l((t))$ does not affect whether η is anisotropic or not; See e.g. [24, Proposition 4.8]. By versality, there exists a point $v \in U(l)$ such that $T_v = \eta$. Therefore η is isotropic by Lemma 4.2(1). This proves Part (1).

(2) Assume l is p -closed. If $\gamma_{k^{(p)}}$ is isotropic, then $\gamma_{k'}$ is isotropic for some prime-to- p extension $k \subset k' \subset k^{(p)}$ by Lemma 4.2(4). The extension $k \subset k'$ is induced from a morphism $f : V \rightarrow B$ of varieties for some irreducible variety V over k_0 with function field $k_0(V) = k'$. By Lemma 4.2(3), we may replace V with an open subset to assume $T_V = T \times_f V$ is isotropic. Since f is generically finite, there exist dense opens $U \subset B$, $W \subset V$ such that $f(W) \subset U$ and the restriction $f : W \rightarrow U$ is finite and flat [60, Tag 02NX]. Replace B by U and V by W to assume f is finite and flat. Now let $u \in B(l)$ be a point such that $T_u = \eta$ (note that l is infinite because it contains all roots of unity of order prime to p and $\text{char } k_0$). Since f is flat, finite of degree prime-to- p and l is p -closed, we can lift u to a point $v \in V(l)$ such that

$$f(v) = u.$$

See Lemma 4.4 below. Associativity of fiber products gives a canonical G -equivariant isomorphism:

$$T_u = T \times_u \text{Spec } l \cong T_V \times_v \text{Spec } l.$$

Therefore

$$\eta = T_u = T_V \times_v \text{Spec } l$$

is isotropic by Lemma 4.2(1). \square

Lemma 4.4. *Let $k_0 \subset l$ be a p -closed field for some prime $p \neq \text{char } k_0$. Let $f : V \rightarrow U$ be a finite flat map of varieties over k_0 . If the degree of f is prime-to- p , then the induced map $V(l) \rightarrow U(l)$ is surjective.*

Proof. Let $u : \text{Spec } l \rightarrow U$ be a point. The scheme $\text{Spec } l \times_U V$ is finite of degree prime to p over $\text{Spec } l$ by our assumption on f . Therefore $\text{Spec } l \times_U V = \text{Spec } R$ for some finite l -algebra R of dimension prime to p . By the structure theorem for Artin rings there exist local Artin rings R_1, \dots, R_n such that:

$$R \cong R_1 \times \dots \times R_n$$

as l -algebras. Comparing dimensions, we see that $\dim_l R_i$ is prime to p for some $1 \leq i \leq n$. Let $\mathfrak{m} \subset R_i$ be its maximal ideal and $d = [R_i/\mathfrak{m} : l]$ the residue field degree. By [20, Lemma A.1.3], we have:

$$\dim_l(R_i) = d \operatorname{len}(R_i),$$

where $\operatorname{len}(R_i)$ denotes the length of R_i as a module over itself. In particular, d is prime to p . Since l has no prime-to- p extensions, it follows that $R_i/\mathfrak{m} = l$. The composition:

$$R \xrightarrow{\sim} R_1 \times \cdots \times R_n \twoheadrightarrow R_i \rightarrow R_i/\mathfrak{m} = l,$$

gives a section of the inclusion $l \subset R$. Dualizing, we obtain a section of the left column of the following Cartesian square:

$$\begin{array}{ccc} \operatorname{Spec} R = \operatorname{Spec} l \times_U V & \xrightarrow{f^*(u)} & V \\ s \uparrow \downarrow & & \downarrow f \\ \operatorname{Spec} l & \xrightarrow{u} & U \end{array}$$

The composition $f^*(u) \circ s : \operatorname{Spec} l \rightarrow V$ is a lift of u to $V(l)$. □

The next lemma will be used in Section 9 to reduce the proof of Theorem 1.2 to the case where G° is reductive.

Lemma 4.5. *Let $f : G \rightarrow H$ be a homomorphism of smooth linear algebraic groups over k_0 . Let $k_0 \subset k$ be a field and denote the induced pushforward map by $f_* : H^1(k, G) \rightarrow H^1(k, H)$. If $\gamma \in H^1(k, G)$ is anisotropic and one of the following conditions hold, then $f_*(\gamma)$ is anisotropic.*

- (1) f is an embedding and $\dim G = \dim H$.
- (2) The base field k_0 is perfect and f is a quotient map with a unipotent kernel.

Proof. Assume $\gamma = [c_\sigma]$ for some cocycle c_σ and denote its pushforward to H by $f_*(c)$. Consider the homomorphism of twisted groups defined by f :

$${}_c f : {}_c G \rightarrow {}_{f_*(c)} H.$$

(1) If f is an embedding, then so is ${}_c f$. By assumption we have:

$$\dim({}_c G) = \dim({}_{f_*(c)} H).$$

Therefore ${}_c f$ restricts to an isomorphism ${}_c G^\circ \cong {}_{f_*(c)} H^\circ$. A split torus $T \subset {}_{f_*(c)} H$ is contained in ${}_{f_*(c)} H^\circ$ because it is connected. Since ${}_c G^\circ \cong {}_{f_*(c)} H^\circ$ is anisotropic, this implies $T = \{e\}$. Therefore $[f_*(c)] = f_*(\gamma)$ is anisotropic.

(2) Assume f is a quotient map with unipotent kernel $U \subset G$. Let $T \subset {}_{f_*(c)} H$ be a split torus and denote by S the (scheme-theoretic) preimage of T under ${}_c f$. Restricting ${}_c f$ to S gives an exact sequence [46, Theorem 5.55]:

$$(4.1) \quad 1 \rightarrow U \rightarrow S \rightarrow T \rightarrow 1.$$

Since k_0 is perfect and T is connected the exact sequence (4.1) splits by [26, Expose XVII, Theorem 5.1.1]. We obtain an embedding $T \rightarrow S \subset {}_c G$. Since ${}_c G$ is anisotropic, we conclude T must be trivial. This shows $[f_*(c)] = f_*(\gamma)$ is anisotropic. □

5. LOOP TORSORS

Loop torsors were introduced by Gille and Pianzola in the monograph [25]. They can be defined in a few equivalent ways. The useful point of view for us will be:

Definition 5.1. A torsor $\gamma \in H_{\text{tr}}^1(F, G)$ is called a *loop torsor* if it can be represented by a $\text{Gal}_{\text{tr}}(F)$ -cocycle taking values in $G(\mathcal{O}_{\text{in}})$. We denote the subset of all loop torsors by $H_{\text{loop}}^1(F, G) \subset H_{\text{tr}}^1(F, G)$.

We use loop torsors for two reasons. They can be decomposed in tandem with the decomposition of $\text{Gal}_{\text{tr}}(F)$, and they are integral, so one can apply the homomorphism $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$ to obtain $G(k_{\text{sep}})$ -valued cocycles from them. We start by describing the decomposition of loop torsors introduced in [25, Section 3.3].

Let $\Psi_{\pi}^F : \text{Gal}(k) \times \Gamma_F^{\vee} \rightarrow \text{Gal}_{\text{tr}}(F)$ be the isomorphism induced by a uniformizer π as in Proposition 3.4. Any loop cocycle $c_{\tau} \in Z^1(\text{Gal}_{\text{tr}}(F), G(\mathcal{O}_{\text{in}}))$ defines a $\text{Gal}(k)$ -cocycle $a_{\sigma} \in Z^1(\text{Gal}(k), G(\mathcal{O}_{\text{in}}))$ and a homomorphism $\varphi : \Gamma_F^{\vee} \rightarrow G(\mathcal{O}_{\text{in}})$ by the formulas:

$$a_{\sigma} = c_{\Psi_{\pi}(\sigma, 0)}, \varphi(x) = c_{\Psi_{\pi}(1, x)}.$$

Note that φ is a homomorphism because the tame inertia group of F acts fixes $G(F_{\text{in}})$. The cocycle a_{σ} is called *the arithmetic part* of c_{τ} and φ is called its *geometric part*. Clearly c_{τ} is uniquely determined by a_{σ} and φ . We will use the notation:

$$c_{\tau} = \langle a_{\sigma}, \varphi \rangle_{\pi},$$

to denote a loop cocycle c_{τ} with arithmetic part a_{σ} and geometric part φ . The corresponding loop torsor will be denoted:

$$[a_{\sigma}, \varphi]_{\pi} := [\langle a_{\sigma}, \varphi \rangle_{\pi}] \in H_{\text{loop}}^1(F, G).$$

Any torsor $\gamma \in H_{\text{loop}}^1(F, G)$ is by definition of the form $\gamma = [a_{\sigma}, \varphi]_{\pi}$ for some a_{σ}, φ , but the symbol $\langle a_{\sigma}, \varphi \rangle_{\pi}$ is not defined for an arbitrary pair a_{σ}, φ .

Definition 5.2. We will call a cocycle $a_{\sigma} \in Z^1(\text{Gal}(k), G(\mathcal{O}_{\text{in}}))$ and homomorphism $\varphi : \Gamma_F^{\vee} \rightarrow G(\mathcal{O}_{\text{in}})$ *compatible* if there exists a cocycle $c_{\tau} \in Z^1(\text{Gal}_{\text{tr}}(F), G(\mathcal{O}_{\text{in}}))$ such that $c_{\tau} = \langle a_{\sigma}, \varphi \rangle_{\pi}$.

Let $\theta : \text{Gal}(k) \rightarrow (\hat{\mathbb{Z}}')^*$ be the cyclotomic character. One can check that a cocycle a_{σ} and a homomorphism φ as above are compatible if and only if they satisfy:

$$(5.1) \quad a_{\sigma}^{\sigma} \varphi(f) a_{\sigma}^{-1} = \varphi(f)^{\theta(\sigma)},$$

for all $f \in \Gamma_F^{\vee}, \sigma \in \text{Gal}(k)$; See [25, Lemma 3.7] for a similar computation. Using this, we see that centralizers of finite abelian subgroups are a natural source for loop torsors.

Example 5.3. Let $\varphi : \Gamma_F^{\vee} \rightarrow G(\mathcal{O}_{\text{in}})$ be a continuous homomorphism onto a finite split-diagonalizable subgroup $A \subset G(\mathcal{O}_{\text{in}})$ and let $a_{\sigma} \in Z^1(\text{Gal}(k), C_G(A)(\mathcal{O}_{\text{in}}))$ a cocycle. The formula:

$$c_{\Psi_{\pi}(\sigma, f)} = a_{\sigma}^{\sigma} \varphi(f)$$

defines a loop cocycle $c_{\tau} = \langle a_{\sigma}, \varphi \rangle_{\pi}$ because

$${}^{\sigma} \varphi(f) = \varphi(f)^{\theta(\sigma)}$$

for all $\sigma \in \text{Gal}(k), f \in \Gamma_F^{\vee}$ by (2.1). That is, a_{σ} and φ are compatible in the sense of Definition 5.2.

One can often assume a loop torsor is of the form given in Example 5.3 using the next lemma.

Lemma 5.4. *Let $c_\tau = \langle a_\sigma, \varphi \rangle_\pi$ be a loop cocycle and set $A = \text{im } \varphi$. If the group*

$$\overline{A} := \{\overline{a} \in G(k_{\text{sep}}) \mid a \in A\}$$

is split-diagonalizable, then $\overline{a_\sigma} \in C_G(A)(k_{\text{sep}})$ for all $\sigma \in \text{Gal}(k)$.

Proof. Let $\theta : \text{Gal}(k) \rightarrow (\hat{\mathbb{Z}}')^*$ denote the cyclotomic character. Since c_τ is a cocycle, (5.1) gives:

$$a_\sigma^\sigma \varphi(f) a_\sigma^{-1} = \varphi(f)^{\theta(\sigma)}.$$

Reducing modulo the maximal ideal of \mathcal{O}_{in} we get:

$$\overline{a_\sigma}^\sigma \overline{\varphi(f)} \overline{a_\sigma}^{-1} = \overline{\varphi(f)}^{\theta(\sigma)}.$$

We have

$$\sigma \overline{\varphi(f)} = \overline{\varphi(f)}^{\theta(\sigma)}$$

because \overline{A} is split-diagonalizable by Fact 2.2. Therefore $\overline{a_\sigma}$ centralizes $\sigma \overline{\varphi(f)}$. Since $f \in \Gamma_F^\vee$ was arbitrary, we conclude that $\overline{a_\sigma} \in C_G(A)(k_{\text{sep}})$. \square

We finish this section by examining when do two loop cocycles $\langle a_\sigma, \varphi \rangle_\pi, \langle a_\sigma, \varphi \rangle_\pi$ give rise to the same class in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$.

Lemma 5.5. *Two loop cocycle $\langle a_\sigma, \varphi \rangle_\pi, \langle b_\sigma, \psi \rangle_\pi$ are cohomologous in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$ if and only if there exists $s \in G(\mathcal{O}_{\text{in}})$ such that:*

$$(5.2) \quad s^{-1} a_\sigma^\sigma s = b_\sigma, \quad s^{-1} \varphi s = \psi.$$

Proof. For any $s \in G(\mathcal{O}_{\text{in}}), \sigma \in \text{Gal}(k), f \in \Gamma_F^\vee, \tau := \Psi_\pi(\sigma, f)$ we have:

$$\begin{aligned} s^{-1} \langle a_\sigma, \varphi \rangle_\pi(\tau)^\tau s &= s^{-1} a_\sigma^\sigma \varphi(f)^\sigma s \\ &= s^{-1} a_\sigma^\sigma s^\sigma (s^{-1} \varphi(f) s) = \langle s^{-1} a_\sigma^\sigma s, s^{-1} \varphi s \rangle_\pi(\tau). \end{aligned}$$

This proves the claim. \square

6. A THEOREM OF GILLE AND PIANZOLA

Recall that a valued field (F, ν) is an iterated Laurent series field, if it is isomorphic to $k((t_1)) \dots ((t_r))$ equipped with the usual (t_1, \dots, t_r) -adic valuation. In this section we prove the following adaptation of a theorem of Gille and Pianzola.

Theorem 6.1. *Let (F, ν) be an iterated Laurent series field and assume G° is reductive. Two anisotropic loop torsors $[a_\sigma, \varphi]_\pi, [b_\sigma, \psi]_\pi \in H_{\text{loop}}^1(F, G)_{\text{an}}$ are cohomologous if and only if there exists $s \in G(k_{\text{sep}})$ such that:*

$$(6.1) \quad s^{-1} \overline{\varphi(f)} s = \overline{\psi(f)}, \quad s^{-1} \overline{a_\sigma}^\sigma s = \overline{b_\sigma}$$

for all $f \in \Gamma_F^\vee, \sigma \in \text{Gal}(k)$.

Since F is an iterated Laurent series field, it admits a coefficient field. That is, there exists an inclusion $k \subset \mathcal{O}$ splitting the map onto the residue field $\mathcal{O} \rightarrow k$. This inclusion extends to an embedding $k_{\text{sep}} \subset \mathcal{O}_{\text{in}}$ splitting the homomorphism $\mathcal{O}_{\text{in}} \rightarrow k_{\text{sep}}$, which in turn gives a section $G(k_{\text{sep}}) \subset G(\mathcal{O}_{\text{in}})$ of the homomorphism $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$. Gille proved Theorem 6.1 in case $a_\sigma, \varphi, b_\sigma, \psi$ all take values in $G(k_{\text{sep}}) \subset G(\mathcal{O}_{\text{in}})$ [24, Corollary 4.11] (before that Gille-Pianzola proved it under the additional assumption $\text{char } k = 0$ in [25]). Thus in order to prove Theorem 6.1 it suffices to establish the following proposition:

Proposition 6.2. *Let $G(k_{\text{sep}}) \subset G(\mathcal{O}_{\text{in}})$ denote the inclusion induced from the inclusion of a coefficient field $k_{\text{sep}} \subset \mathcal{O}_{\text{in}}$. The corresponding map of cohomology sets*

$$(6.2) \quad H_{\text{tr}}^1(F, G(k_{\text{sep}})) \rightarrow H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$$

is a bijection.

For the proof of this proposition we will need the following consequences of Hensel's lemma.

Lemma 6.3. *Let $\varphi, \psi : \Gamma_F^\vee \rightarrow G(\mathcal{O}_{\text{in}})$ be two continuous homomorphisms and let $\overline{\varphi}, \overline{\psi} : \Gamma_F^\vee \rightarrow G(k_{\text{sep}})$ be the composition of φ, ψ with the reduction homomorphism $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$, $g \mapsto \overline{g}$. If $s\overline{\varphi}s^{-1} = \overline{\psi}$ for some $s \in G(k_{\text{sep}})$, then $\tilde{s}\varphi\tilde{s}^{-1} = \psi$ for some $\tilde{s} \in G(\mathcal{O}_{\text{in}})$.*

Proof. Let $\Gamma_F^\vee \rightarrow A$ be a large enough finite quotient of Γ_F^\vee so that both φ and ψ factor through A . Let $\varphi', \psi' : A \rightarrow G_{\mathcal{O}_{\text{in}}}$ denote the induced homomorphisms and denote by $\text{Tran}_G(\varphi', \psi')$ the transporter of φ' and ψ' . This is a closed subscheme of $G_{\mathcal{O}_{\text{in}}}$ such that for any ring homomorphism $\mathcal{O}_{\text{in}} \rightarrow R$:

$$\text{Tran}_G(\varphi', \psi')(R) = \{g \in G(R) \mid g\varphi'_R g^{-1} = \psi'_R\}.$$

Note that A is a finite abelian group of order prime to $\text{char } k$ by Remark 3.2. Therefore $\text{Tran}_G(\varphi', \psi')$ is smooth because A is of multiplicative type; See [26, Expose XI, Corollaire 5.2]. By assumption $\text{Tran}_G(\varphi', \psi')(k_{\text{sep}}) \neq \emptyset$. It follows from Hensel's lemma that $\text{Tran}_G(\varphi', \psi')(\mathcal{O}_{\text{in}}) \neq \emptyset$ and the result follows. \square

Lemma 6.4. *For any smooth algebraic group C over \mathcal{O} , the reduction homomorphism $C(\mathcal{O}_{\text{in}}) \rightarrow C(k_{\text{sep}})$ induces a bijection on Galois cohomology sets:*

$$H^1(\text{Gal}(k), C(\mathcal{O}_{\text{in}})) \rightarrow H^1(\text{Gal}(k), C(k_{\text{sep}})).$$

Proof. Any C -torsor for the étale topology over $\text{Spec } \mathcal{O}$ is split by the universal cover $\text{Spec } \mathcal{O}_{\text{in}} \rightarrow \text{Spec } \mathcal{O}$ because $\text{Spec } \mathcal{O}$ is local. The fundamental group of $\text{Spec } \mathcal{O}$ is $\text{Gal}(k)$ by [45, Section 5, Example 5.2.d]. Therefore [23, Section 2.2] gives isomorphisms

$$H^1(\text{Gal}(k), C(\mathcal{O}_{\text{in}})) \xrightarrow{\sim} H_{\text{étale}}^1(\text{Spec } \mathcal{O}, C), \quad H^1(\text{Gal}(k), C(k_{\text{sep}})) \xrightarrow{\sim} H_{\text{étale}}^1(\text{Spec } k, C).$$

Now the lemma follows from [26, Expose XXIV, Proposition 8.1], which states that the bottom row in the following commutative square is a bijection:

$$\begin{array}{ccc} H^1(\text{Gal}(k), C(\mathcal{O}_{\text{in}})) & \longrightarrow & H^1(\text{Gal}(k), C(k_{\text{sep}})) \\ \downarrow & & \downarrow \\ H_{\text{étale}}^1(\text{Spec } \mathcal{O}, C) & \longrightarrow & H_{\text{étale}}^1(\text{Spec } k, C) \end{array}$$

\square

Next we prove that any loop torsor is represented by a cocycle taking values in $G(k_{\text{sep}})$. Note that $\text{Gal}_{\text{tr}}(F)$ acts on the coefficient field $k_{\text{sep}} \subset \mathcal{O}_{\text{in}}$ and on $G(k_{\text{sep}}) \subset G(\mathcal{O}_{\text{in}})$ because $k \subset F$ is fixed by $\text{Gal}_{\text{tr}}(F)$ and k_{sep}/k is Galois. Let $[a_\sigma, \varphi]_\pi \in H_{\text{loop}}^1(F, G)$ be a loop torsor. Denote the images of a_σ under the reduction homomorphism $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$ by \bar{a}_σ and define $\bar{\varphi} : \Gamma_F^\vee \rightarrow G(k_{\text{sep}})$ by the formula

$$\bar{\varphi}(f) = \overline{\varphi(f)}$$

for all $f \in \Gamma_F^\vee$. By Remark 2.4, the map :

$$\langle \bar{a}_\sigma, \bar{\varphi} \rangle_\pi : \text{Gal}(k) \rtimes \Gamma_F^\vee \rightarrow G(\mathcal{O}_{\text{in}}), (\sigma, f) \mapsto \bar{a}_\sigma {}^\sigma \bar{\varphi}(f)$$

is a $\text{Gal}_{\text{tr}}(F)$ -cocycle in $G(\mathcal{O}_{\text{in}})$. Therefore $\bar{a}_\sigma, \bar{\varphi}$ are compatible in the sense of Definition 5.2 and the loop torsor $[\bar{a}_\sigma, \bar{\varphi}]_\pi \in H_{\text{loop}}^1(F, G)$ is defined.

Lemma 6.5. *Let $G(k_{\text{sep}}) \subset G(\mathcal{O}_{\text{in}})$ denote the inclusion induced from the inclusion of a coefficient field $k_{\text{sep}} \subset \mathcal{O}_{\text{in}}$. Any loop cocycle $\langle a_\sigma, \varphi \rangle_\pi$ is cohomologous in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$ to $\langle \bar{a}_\sigma, \bar{\varphi} \rangle_\pi$. In particular, we have:*

$$[a_\sigma, \varphi]_\pi = [\bar{a}_\sigma, \bar{\varphi}]_\pi.$$

Proof. Let $\langle a_\sigma, \varphi \rangle_\pi$ be a loop cocycle. We need to prove $\langle a_\sigma, \varphi \rangle_\pi$ and $\langle \bar{a}_\sigma, \bar{\varphi} \rangle_\pi$ are cohomologous in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$. By Lemma 5.5, it suffices to find $s \in G(\mathcal{O}_{\text{in}})$ such that:

$$(6.3) \quad s^{-1} a_\sigma {}^\sigma s = \bar{a}_\sigma, \quad s^{-1} \varphi(f) s = \bar{\varphi}(f)$$

for all $\sigma \in \text{Gal}(k), f \in \Gamma_F^\vee$. We start by tackling the special case where $\varphi(f) \in G(k_{\text{sep}})$ for all f and so $\bar{\varphi} = \varphi$. Define for all $\sigma \in \text{Gal}(k)$:

$$c_\sigma = \bar{a}_\sigma a_\sigma^{-1}.$$

It is simple to check that c_σ is a cocycle in ${}_a G(\mathcal{O}_{\text{in}})$ (here ${}_a G$ is the twisted group defined by a , see Section 2). We check c_σ centralizes $\text{im } \varphi$. Since both a_σ, φ and \bar{a}_σ, φ are compatible (5.1) gives for any $f \in \Gamma_F^\vee$:

$$a_{\sigma^{-1}} {}^{\sigma^{-1}} \varphi(f) a_{\sigma^{-1}} = \varphi(f)^{\theta(\sigma^{-1})} = \overline{a_{\sigma^{-1}}} {}^{\sigma^{-1}} \varphi(f) \overline{a_{\sigma^{-1}}}^{-1}.$$

Therefore $\overline{a_{\sigma^{-1}}}^{-1} a_{\sigma^{-1}}$ centralizes ${}^{\sigma^{-1}} \varphi(f)$. Applying $\sigma(\cdot)$ to both sides and using the cocycle identity we get:

$$C_{{}_a G}(\varphi(f))(\mathcal{O}_{\text{in}}) \ni \sigma(\overline{a_{\sigma^{-1}}}^{-1} a_{\sigma^{-1}}) = \sigma \overline{a_{\sigma^{-1}}}^{-1} {}^\sigma a_{\sigma^{-1}} = \bar{a}_\sigma a_\sigma^{-1} = c_\sigma.$$

Since $f \in \Gamma_F^\vee$ was arbitrary this shows $c_\sigma \in C_{{}_a G}(\text{im } \varphi)$. The class $[c_\sigma] \in H^1(\text{Gal}(k), C_{{}_a G}(\varphi)(\mathcal{O}_{\text{in}}))$ clearly goes to zero under the map

$$H^1(\text{Gal}(k), C_{{}_a G}(\varphi)(\mathcal{O}_{\text{in}})) \rightarrow H^1(\text{Gal}(k), C_{{}_a G}(\varphi)(k_{\text{sep}})).$$

This implies $[c_\sigma]$ is split in $H^1(\text{Gal}(k), C_{{}_a G}(\varphi)(\mathcal{O}_{\text{in}}))$ by Lemma 6.4. Therefore there exists $s \in C_{{}_a G}(\varphi)(\mathcal{O}_{\text{in}})$ such that for all $\sigma \in \text{Gal}(k)$:

$$\bar{a}_\sigma a_\sigma^{-1} = c_\sigma = s^{-1} a_\sigma {}^\sigma s a_\sigma^{-1}.$$

Multiply by a_σ on the right to obtain

$$s^{-1} a_\sigma {}^\sigma s = \bar{a}_\sigma.$$

Since s centralizes $\varphi = \bar{\varphi}$, we conclude that it satisfies (6.3).

We now tackle the case where $\varphi \neq \bar{\varphi}$. By Lemma 6.3, φ and $\bar{\varphi}$ are conjugate by an element $s \in G(\mathcal{O}_{\text{in}})$. That is, for any $f \in \Gamma_F^\vee$ we have:

$$s^{-1}\varphi(f)s = \bar{\varphi}(f).$$

Applying the reduction map $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$, $g \mapsto \bar{g}$ to the above equation we get:

$$\bar{s}^{-1}\bar{\varphi}(f)\bar{s} = \bar{\varphi}(f).$$

Therefore the element $t = s\bar{s}^{-1} \in G(\mathcal{O}_{\text{in}})$ satisfies $\bar{t} = 1$ and

$$(6.4) \quad t^{-1}\varphi(f)t = \bar{s}s^{-1}(f)s\bar{s}^{-1} = \bar{s}\bar{\varphi}(f)\bar{s}^{-1} = \bar{\varphi}(f).$$

By Lemma 5.5, (6.4) implies $\langle a_\sigma, \varphi \rangle_\pi$ and $\langle t^{-1}a_\sigma^\sigma t, \bar{\varphi} \rangle_\pi$ represent the same class in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$. By the first case we tackled and because $\bar{t} = 1$, we conclude:

$$[\langle t^{-1}a_\sigma^\sigma t, \bar{\varphi} \rangle_\pi] = [\langle \bar{t}^{-1}\bar{a}_\sigma^\sigma \bar{t}, \bar{\varphi} \rangle_\pi] = [\langle \bar{a}_\sigma, \bar{\varphi} \rangle_\pi] \text{ in } H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}})).$$

Therefore $\langle \bar{a}_\sigma, \bar{\varphi} \rangle_\pi$ and $\langle a_\sigma, \varphi \rangle_\pi$ represent the same class in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$. This finishes the proof. \square

We now prove Proposition 6.2, which completes the proof of Theorem 6.1.

Proof. The map (6.2) is injective because it has a left inverse induced by the reduction homomorphism $G(\mathcal{O}_{\text{in}}) \rightarrow G(k_{\text{sep}})$, $g \mapsto \bar{g}$. To prove surjectivity, one needs to show any loop cocycle is cohomologous in $H_{\text{tr}}^1(F, G(\mathcal{O}_{\text{in}}))$ to a cocycle taking values in $G(k_{\text{sep}})$. This is immediate from Lemma 6.5. \square

7. FUNCTORIALITY OF DECOMPOSITIONS OF LOOP TORSORS

Let L/F be an extension of Henselian valued fields with uniformizers π and τ . It is clear from the definition that γ_L is a loop torsor for any $\gamma \in H_{\text{loop}}^1(F, G)$. Therefore there is an induced map $H_{\text{loop}}^1(F, G) \rightarrow H_{\text{loop}}^1(L, G)$. Our goal in this section is to understand the functoriality of the decompositions $\gamma = [a_\sigma, \varphi]_\pi$. That is, we wish to write $\gamma_L = [b_\sigma, \psi]_\tau$ for some compatible b_σ, ψ explicitly described in terms of a_σ and φ . The following lemma shows we have to account for the choice of uniformizers π, τ .

Lemma 7.1. *For any $\sigma \in \text{Gal}(l)$, let $\chi_\sigma \in \Gamma_F^\vee$ be as in (3.6) (note that χ_σ depends on π and τ). The following holds for any compatible pair a_σ, φ .*

$$([a_\sigma, \varphi]_\pi)_L = [\varphi(\chi_\sigma) \text{Inf}_{l/k}(a)_\sigma, \varphi|_L]_\tau.$$

Here $\text{Inf}_{l/k}(a)$ is the inflation of a_σ to l and $\varphi|_L$ is the composition of φ with the restriction map $\Gamma_L^\vee \rightarrow \Gamma_F^\vee$.

Proof. Let $\Psi_\pi^F : \text{Gal}(k) \times \Gamma_F^\vee \rightarrow \text{Gal}_{\text{tr}}(F)$, $\Psi_\tau^L : \text{Gal}(l) \times \Gamma_L^\vee \rightarrow \text{Gal}_{\text{tr}}(L)$ be the isomorphisms of Proposition 3.4. Lemma 3.6 gives for any $\sigma \in \text{Gal}(l)$, $f \in \Gamma_L^\vee$:

$$\begin{aligned} \langle a_\sigma, \varphi \rangle_\pi(\Psi_\tau^L(\sigma, f)|_{F_{\text{tr}}}) &= \langle a_\sigma, \varphi \rangle_\pi(\Psi_\pi^F(\sigma|_{k_{\text{sep}}}, f|_F + \theta(\sigma)^{-1}\chi_\sigma)) \\ &= a_{\sigma|_{k_{\text{sep}}}}^\sigma(\varphi(\chi_\sigma)^{\theta(\sigma)^{-1}}\varphi(f|_F)) \\ &= a_{\sigma|_{k_{\text{sep}}}}^\sigma\varphi(\chi_\sigma)^{\theta(\sigma)^{-1}}\sigma\varphi(f|_F) \\ &\stackrel{(5.1)}{=} \varphi(\chi_\sigma)a_{\sigma|_{k_{\text{sep}}}}^\sigma\varphi(f|_F) \\ &= \langle \varphi(\chi_\sigma) \text{Inf}_{l/k}(a)_\sigma, \varphi|_L \rangle_\tau(\Psi_\tau^L(\sigma, f)). \end{aligned}$$

The result follows. \square

Corollary 7.2. *If $\Gamma_F = \Gamma_L$, then the following holds for any compatible pair a_σ, φ :*

$$([a_\sigma, \varphi]_\pi)_L = [\text{Inf}_{l/k}(a)_\sigma, \varphi]_\pi.$$

Note that π is a uniformizer for L_{tr} because $\Gamma_{F_{\text{tr}}} = \Gamma_{L_{\text{tr}}}$.

Proof. The result is immediate from Lemma 7.1 once we note that $\chi_\sigma = 1$ for all $\sigma \in \text{Gal}(l)$ by (3.6). \square

To improve Lemma 7.1 we need to understand the cocycle $\varphi(\chi_\sigma) \in Z^1(\text{Gal}(l), A)$ for arbitrary uniformizers τ of L .

Lemma 7.3. *Let $\varphi : \Gamma_F^\vee \rightarrow A$ be a continuous homomorphism onto $A = \mu_{n_1} \times \cdots \times \mu_{n_r}$ for some n_1, \dots, n_r coprime to $\text{char } k$.*

(1) *There are elements $\gamma_1, \dots, \gamma_r \in \Gamma_F$ such that for all $f \in \Gamma_F^\vee$:*

$$(7.1) \quad \varphi(f) = (\zeta_{n_1}^{f(\gamma_1)}, \dots, \zeta_{n_r}^{f(\gamma_r)}).$$

(2) *Let $u^\gamma = \pi^\gamma \tau^{-\gamma}$ for $\gamma \in \Gamma_{F_{\text{tr}}}$ as in Lemma 3.6. The class $[\varphi(\chi_\sigma)] \in H^1(l, A)$ represents $(\overline{u^{\gamma_1}}, \dots, \overline{u^{\gamma_r}})$ under the Kummer isomorphism $H^1(l, A) \cong l^*/l^{*n_1} \times \cdots \times l^*/l^{*n_r}$.*

Proof. One proves Part (1) by choosing a basis for Γ_F . The details are left to the reader. We prove Part (2). Recall that by the definition of χ_σ in (3.6), we have:

$$\zeta_{n_i}^{\chi_\sigma(\gamma_i)} = \frac{\overline{s}_\tau(\sigma)(u^{\gamma_i/n_i})}{u^{\gamma_i/n_i}},$$

for any $1 \leq i \leq r$. Note that $\nu(u^{\gamma_i/n_i}) = 0$, so the residue class $\overline{u^{\gamma_i/n_i}} \in l_{\text{sep}}$ is defined. Since \overline{s}_τ is a section of $\text{Gal}_{\text{tr}}(L) \rightarrow \text{Gal}(l)$, applying (3.2) gives:

$$(7.2) \quad \zeta_{n_i}^{\chi_\sigma(\gamma_i)} = \overline{\zeta_{n_i}^{\chi_\sigma(\gamma_i)}} = \frac{\sigma(\overline{u^{\gamma_i/n_i}})}{u^{\gamma_i/n_i}}.$$

Since $\overline{u^{\gamma_i/n_i}}$ is an n_i -th root of $\overline{u^{\gamma_i}}$, the right hand side in (7.2) is a cocycle representing $\overline{u^{\gamma_i}}$ under the classical kummer isomorphism $H^1(l, \mu_{n_i}) \cong l^*/l^{*n_i}$. Now Part (2) follows from Part (1) which states:

$$\varphi(\chi_\sigma) = (\zeta_{n_1}^{\chi_\sigma(\gamma_1)}, \dots, \zeta_{n_r}^{\chi_\sigma(\gamma_r)}).$$

\square

Corollary 7.4. *Let π be a uniformizer for F_{tr} and $\varphi : \Gamma_F^\vee \rightarrow G(\mathcal{O}_{\text{in}})$ a continuous homomorphism onto a split-diagonalizable subgroup $A \subset G(\mathcal{O}_{\text{in}})$. If $[L : F]$ and $|A|$ are coprime, then there exists a uniformizer $\tau : \Gamma_{L_{\text{tr}}} \rightarrow L_{\text{tr}}^*$ such that*

$$([a_\sigma, \varphi]_\pi)_L = [\text{Inf}_{l/k}(a)_\sigma, \varphi|_L]_\tau$$

for any $\text{Gal}(k)$ -cocycle $a_\sigma \in G(\mathcal{O}_{\text{in}})$ centralizing A .

Proof. There exists a basis e_1, \dots, e_r of Γ_L and integers d_1, \dots, d_r such that $d_1 e_1, \dots, d_r e_r$ is a basis for Γ_F and $\prod_i d_i = [\Gamma_L : \Gamma_F]$ [34, Theorem III.7.8]. By Ostrowski Theorem, $[\Gamma_L : \Gamma_F]$ is prime to $|A|$. Choose integers m_1, \dots, m_r such that:

$$(7.3) \quad m_i d_i \equiv 1 \pmod{|A|}.$$

Let $x_i \in L^*$ be such that $\nu(x_i) = e_i$. Set $\pi_i = \pi^{d_i e_i}$ and define

$$\tau_i := x_i(\pi_i x_i^{-d_i})^{m_i}.$$

We have for any $1 \leq i \leq r$:

$$\nu(\tau_i) = \nu(x_i) + m_i(\nu(\pi_i) - d_i \nu(x_i)) = e_i + m_i(d_i e_i - d_i e_i) = e_i.$$

Therefore there exists a uniformizer τ for L_{tr} such that $\tau^{e_i} = \tau_i$ for all i by Remark 3.5. For this uniformizer we have:

$$\pi^{d_i e_i} \tau^{-d_i e_i} = \pi_i \tau_i^{-d_i} = \pi_i x_i^{-d_i} (\pi_i x_i^{-d_i})^{-d_i m_i} = (\pi_i x_i^{-d_i})^{1-d_i m_i}.$$

Therefore $\pi^{d_i e_i} \tau^{-d_i e_i}$ is an $|A|$ -th power for all i by (7.3). This gives $[\chi_\sigma] = 1$ in $H^1(k, A)$ by Lemma 7.3(2). Let $a \in A$ be such that for all $\sigma \in \text{Gal}(k)$:

$$\chi_\sigma = a^{-1} \sigma a.$$

We use Lemma 7.1 and the fact that a_σ centralizes A to get:

$$([a_\sigma, \varphi]_\pi)_L = [a^{-1} \sigma a \text{Inf}_{l/k}(a)_\sigma, \varphi|_L]_\tau = [a^{-1} \text{Inf}_{l/k}(a)_\sigma^\sigma a, \varphi|_L]_\tau.$$

We finish the proof by noting that Lemma 5.5 gives:

$$[a^{-1} \text{Inf}_{l/k}(a)_\sigma^\sigma a, \varphi|_L]_\tau = [\text{Inf}_{l/k}(a)_\sigma, a \varphi|_L a^{-1}]_\tau = [\text{Inf}_{l/k}(a)_\sigma, \varphi|_L]_\tau.$$

□

Let F be an iterated Laurent series field and $\varphi : \Gamma_F^\vee \rightarrow G(k_{\text{sep}})$ be a continuous homomorphism onto a split-diagonalizable p -group $A \subset G$. Let $a_\sigma \in Z^1(\text{Gal}(k), C_G(A)(k_{\text{sep}}))$ be a cocycle and set

$$\gamma = [a_\sigma, \varphi]_\pi$$

as in Example 5.3. If G° is reductive, [24, Proposition 4.8] implies γ is anisotropic if and only if $[a_\sigma] \in H^1(k, C_G(A))$ is anisotropic. We will use Corollary 7.4 to upgrade this to a criterion deciding whether γ is anisotropic over a p -closure of F .

Lemma 7.5. *Assume G° is reductive. If k is p -closed, and $[a_\sigma] \in H^1(k, C_G(A))$ is anisotropic, then for any prime-to- p extension $F \subset L$:*

- (1) $\text{im } \varphi|_L = A$
- (2) γ_L is anisotropic

Proof. The residue field of L is a prime-to- p extension of k by Ostrowski's theorem and so it must be k because k is p -closed. Since A is a p -group, Corollary 7.4 implies there exists uniformizer $\tau : \Gamma_{L_{\text{tr}}} \rightarrow L_{\text{tr}}^*$ such that:

$$([a_\sigma, \varphi]_\pi)_L = [a_\sigma, \varphi|_L]_\tau.$$

The inclusion $\Gamma_F \rightarrow \Gamma_L$ induces an exact sequence:

$$\Gamma_L^\vee \rightarrow \Gamma_F^\vee \rightarrow \text{Ext}_{\mathbb{Z}}^1(\Gamma_L/\Gamma_F, \hat{\mathbb{Z}}').$$

Since $\text{Ext}_{\mathbb{Z}}^1(\Gamma_L/\Gamma_F, \hat{\mathbb{Z}}')$ is killed by $[\Gamma_L : \Gamma_F]$, so is the cokernel of $\Gamma_L^\vee \rightarrow \Gamma_F^\vee$. In particular, this cokernel is of order prime to p because $[\Gamma_L : \Gamma_F]$ divides $[L : F]$ by Ostrowski's Theorem. Since $\text{im } \varphi = A$ is a p -group, it follows that

$$\varphi|_L : \Gamma_L^\vee \rightarrow A$$

is a surjection (recall that $\varphi|_L$ is the composition of φ with the restriction map $\Gamma_L^\vee \rightarrow \Gamma_F^\vee$). The torsor $[a_\sigma] \in H^1(k, C_G(A))$ is anisotropic because $\gamma = [a_\sigma, \varphi]_\pi$ is anisotropic by [24, Proposition 4.8]. Applying [24, Proposition 4.8] once more, we get that $\gamma_L = [a_\sigma, \varphi|_L]_\tau$ is anisotropic because $C_G(\varphi|_L) = C_G(A)$. \square

Corollary 7.6. *Let $A \subset G$ be a finite split-diagonalizable p -subgroup for some prime $p \neq \text{char } k_0$. Assume G° is reductive and $C_G(A)$ admits an anisotropic torsor over some p -closed field $k_0 \subset k$. There exists a valued field $k_0 \subset F$ with residue field k such that G admits anisotropic torsors over any p -closure $F^{(p)}$ of F .*

Proof. Set $r = \text{rank}(A)$. Let $F = k((t_1)) \dots ((t_r))$ be an iterated Laurent series field equipped with its (t_1, \dots, t_r) -adic valuation and let π be a uniformizer for F_{tr} . Since $\Gamma_F = \mathbb{Z}^r$, we have

$$\Gamma_F^\vee = \text{Hom}(\Gamma_F, \hat{\mathbb{Z}}^r) = \hat{\mathbb{Z}}^r.$$

Therefore there exists a surjection $\varphi : \Gamma_F^\vee \rightarrow A$. If $[a_\sigma] \in H^1(k, C_G(A))$ is anisotropic, then

$$\gamma = [a_\sigma, \varphi]_\pi \in H_{\text{loop}}^1(F, G)$$

is anisotropic over any prime-to- p extension $F \subset F'$ by Lemma 7.5. Therefore $\gamma_{F^{(p)}}$ is anisotropic for any p -closure $F \subset F^{(p)}$ by Lemma 4.2. \square

8. ALMOST ALL TORSORS ARE LOOP TORSORS

In the proof of Theorem 1.2, we will need to show certain G -torsors are in fact loop torsors. To achieve this, in this section we collect propositions that show all torsors under reductive groups are loop torsors "up to wild ramification". In particular, when G° is reductive and the characteristic of k_0 is good for G , all G -torsors are loop torsors.

Proposition 8.1. *Assume G° is reductive. If the characteristic of k_0 is good for G (see Definition 2.1), then*

$$H_{\text{loop}}^1(F, G) = H_{\text{tr}}^1(F, G) = H^1(F, G).$$

Proof. A torsor induced from a finite subgroup $S \subset G$ of order prime to $\text{char } k_0$ is a loop torsor; See [25, Lemma 3.12]. If the characteristic of k_0 is good for G , then all G -torsors are induced from some such subgroup $S \subset G$ defined over k_0 [38, Corollary 18]. The result follows. \square

The next lemma is the only occasion on which we relax Assumption 2.3. The proof is inspired by [25, 6.2].

Lemma 8.2. *Let (F, ν) be a Henselian valued field with residue field k and value group Γ_F not necessarily finitely generated. Assume $F_{\text{tr}} = F_{\text{sep}} = F_{\text{alg}}$. If G° is a torus, then*

$$H_{\text{loop}}^1(F, G) = H_{\text{tr}}^1(F, G) = H^1(F, G).$$

Proof. Set $T = G^\circ$ and $W = G/T$. We have an exact sequence:

$$1 \rightarrow T \rightarrow G \rightarrow W \rightarrow 1.$$

Since G is smooth, W is an étale group over k_0 [26, Exposé VI, Proposition 5.5.1]. This implies that $W_{k_0, \text{sep}}$ is a constant finite group and in particular

$$W(k_0, \text{sep}) = W(\mathcal{O}_{\text{in}}) = W(F_{\text{sep}}).$$

Therefore $T(\mathcal{O}_{\text{in}}) \subset G(F_{\text{sep}})$ is a *normal* subgroup. Consider the induced exact sequence:

$$(8.1) \quad 1 \rightarrow T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}) \rightarrow N(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}) \rightarrow W(k_{0,\text{sep}}) \rightarrow 1.$$

Let $\gamma \in H^1(F, N)$ and denote by $\gamma' \in H^1(F, N(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}))$ the image of γ under the natural map:

$$H^1(F, N) \rightarrow H^1(F, N(F_{\text{sep}})/T(\mathcal{O}_{\text{in}})).$$

The torsor γ lies in the image of $H^1(F, N(\mathcal{O}_{\text{in}})) \rightarrow H^1(F, N)$ if and only if γ' lies in the image of the map

$$H^1(F, N(\mathcal{O}_{\text{in}})/T(\mathcal{O}_{\text{in}})) \rightarrow H^1(F, N(F_{\text{sep}})/T(\mathcal{O}_{\text{in}})).$$

See for example [48, Lemma 3.1].

The exact sequence (8.1) splits. Indeed, the subgroup

$$N(\mathcal{O}_{\text{in}})/T(\mathcal{O}_{\text{in}}) \subset N(F_{\text{sep}})/T(\mathcal{O}_{\text{in}})$$

intersects $T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}})$ trivially and it surjects onto $W(k_{0,\text{sep}})$ because $N(k_{0,\text{sep}}) \subset N(\mathcal{O}_{\text{in}})$. Therefore we may rewrite (8.1) as:

$$(8.2) \quad 1 \rightarrow T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}) \rightarrow T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}) \rtimes N(\mathcal{O}_{\text{in}})/T(\mathcal{O}_{\text{in}}) \rightarrow W(k_{0,\text{sep}}) \rightarrow 1.$$

Since (8.2) is split, to show that γ is induced from $H^1(F, N(\mathcal{O}_{\text{in}})/T(\mathcal{O}_{\text{in}}))$ it suffices to prove that $T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}})$ is a uniquely divisible abelian group by Lemma 8.3 below. Let $X_*(T)$ be the $\text{Gal}(F)$ -module of cocharacters of T . We have isomorphisms of $\text{Gal}(F)$ -modules:

$$T(F_{\text{sep}}) \cong X_*(T) \otimes_{\mathbb{Z}} F_{\text{sep}}^*, \quad T(\mathcal{O}_{\text{in}}) \cong X_*(T) \otimes_{\mathbb{Z}} \mathcal{O}_{\text{in}}^*.$$

Therefore to show

$$T(F_{\text{sep}})/T(\mathcal{O}_{\text{in}}) \cong X_*(T) \otimes F_{\text{sep}}^*/\mathcal{O}_{\text{in}}^*$$

is uniquely divisible it suffices to show $F_{\text{sep}}^*/\mathcal{O}_{\text{in}}^*$ is uniquely divisible. The group $F_{\text{sep}}^*/\mathcal{O}_{\text{in}}^*$ is divisible because $F_{\text{sep}}^* = F_{\text{alg}}^*$ is divisible. To show $F_{\text{sep}}^*/\mathcal{O}_{\text{in}}^*$ is torsion-free, let $\alpha \in F_{\text{sep}}^*$ be such that $\alpha^n \in \mathcal{O}_{\text{in}}^*$. Our goal is to prove $\alpha \in \mathcal{O}_{\text{in}}^*$. Let $n = q^r m$ for some m prime to $q = \text{char } k$ and integer r . The polynomial $x^m - \alpha^n$ is separable and splits fully over the residue field of k_{sep} of F_{in} . By Hensel's lemma, it follows that all roots of $x^m - \alpha^n$ lie in $\mathcal{O}_{\text{in}}^*$. In particular, $\alpha^{q^r} \in \mathcal{O}_{\text{in}}^*$. Since F is perfect, so is F_{in} . Therefore the unique q^r -th root of α^{q^r} must already lie in F_{in} . That is, $\alpha \in F_{\text{in}}$. Since $\nu(\alpha) = 0$ we conclude that $\alpha \in \mathcal{O}_{\text{in}}^*$. This shows $F_{\text{sep}}^*/\mathcal{O}_{\text{in}}^*$ is torsion-free and finishes the proof. \square

Lemma 8.3. *Let A, B be $\text{Gal}(F)$ groups for some field F such that B acts on A compatibly with the $\text{Gal}(F)$ -action. If A is a uniquely divisible abelian group, then the natural map:*

$$(8.3) \quad H^1(\text{Gal}(F), B) \rightarrow H^1(\text{Gal}(F), A \rtimes B)$$

is a bijection.

Proof. The map (8.3) is injective because it is split by the map

$$(8.4) \quad H^1(\text{Gal}(F), A \rtimes B) \rightarrow H^1(\text{Gal}(F), B)$$

induced by the projection $A \rtimes B \rightarrow B$. Moreover, (8.3) is surjective because its left inverse (8.4) is injective. Indeed, the fibers of (8.4) over $[b_\sigma] \in H^1(\text{Gal}(F), B)$ is in bijective correspondence with $H^1(\text{Gal}(F), {}_b A)$ by [58, Page 52, Corollary 2] and $H^1(\text{Gal}(F), {}_b A)$ is a singleton because A is uniquely divisible [28, Corollary 4.2.7]. This shows (8.3) is a bijection. \square

We will need the following definition from valuation theory.

Definition 8.4. Let $q = \text{char } F$ be non-zero. An extension of Henselian valued fields $F \subset L$ with value groups $\Gamma_F \subset \Gamma_L$ and residue fields $k \subset l$ is called *purely wild* if the following conditions hold:

- (1) The group Γ_L/Γ_F is a q -group.
- (2) The field extension l/k is algebraic and purely inseparable.

We will use the following proposition in the proof of part (1) of Theorem 1.2 to avoid assuming $\text{char } k$ is good for G .

Proposition 8.5. *Assume G° is reductive. For any G -torsor $\gamma \in H^1(F, G)$ there exists a purely wild extension $F \subset L$ of degree a power of $\text{char } k = q$ such that γ_L is a loop torsor.*

Proof. Let $F \subset L$ to be a maximal algebraic purely wild extension of F ; See [33, Section 4]. The field L satisfies

$$F_{\text{tr}} \cdot L = F_{\text{alg}}, \quad F_{\text{tr}} \cap L = F, \quad L_{\text{tr}} = L_{\text{sep}} = L_{\text{alg}}.$$

See [33, Theorem 4.3] and [33, Lemma 6.3] respectively. Moreover, L/F is a pro- q extension. Indeed, for any finite subextension $F \subset L' \subset L$ with residue field extension l'/k , both $[\Gamma_{L'} : \Gamma_F]$ and $[l' : k]$ are powers of q because L'/F is purely wildly ramified [33, Section 4]. Therefore $[L' : F]$ is a power of q by Ostrowski's theorem.

Let $T \subset G$ be a maximal torus defined over k_0 and $N = N_G(T)$ its normalizer. Note that N is smooth because T of multiplicative type and G is smooth [26, Expose XI, Corollaire 2.4]. By Lemma 8.2, we have

$$H_{\text{loop}}^1(L, N) = H^1(L, N).$$

Since $H^1(L, N) \rightarrow H^1(L, G)$ is surjective [29, Corollary 5.3], this gives

$$H_{\text{loop}}^1(L, G) = H^1(L, G).$$

Write L as a union $L = \bigcup L_i$ of a chain $L_1 \subset L_2 \subset \dots$ of a chain of finite purely wild extensions of $F \subset L_i$ of degree a power of q . We need to show γ_{L_i} is a loop torsor for some i . Since γ_L is a loop torsor, it is represented by a cocycle:

$$c_\sigma : \text{Gal}_{\text{tr}}(L) \rightarrow G(\mathcal{O}_{L, \text{in}}).$$

Since $c_{(\cdot)}$ is continuous it factors through $\text{Gal}(E/L)$ for some finite Galois extension E/L . By Lemma 3.8, for all large enough i , E is a compositum $E = E_i L$ for some tamely ramified Galois extension E_i/L_i such that the restriction map

$$\text{res}_i : \text{Gal}(E/L) \rightarrow \text{Gal}(E_i/L_i)$$

is an isomorphism. We have $\mathcal{O}_{L, \text{in}} = \bigcup_i \mathcal{O}_{L_i, \text{in}}$, which implies $G(\mathcal{O}_{L, \text{in}}) = \bigcup_i G(\mathcal{O}_{L_i, \text{in}})$ by [30, Lemma 10.62]. Choose an integer i large enough such that:

$$\forall \sigma \in \text{Gal}(E/L), \quad c_\sigma \in G(\mathcal{O}_{L_i, \text{in}}).$$

Therefore c_σ is the inflation of the cocycle $c_\sigma^{(i)} : \text{Gal}(E_i/L_i) \rightarrow G(\mathcal{O}_{L_i, \text{in}})$ given by:

$$c_\sigma^{(i)} = c_{\text{res}_i^{-1}(\sigma)}.$$

Set $\eta = [c_\sigma^{(i)}] \in H_{\text{loop}}^1(L_i, G)$. Since c_σ is the inflation of $c_\sigma^{(i)}$ to L , we have

$$\gamma_L = \eta_L.$$

Since $H^1(L, G) = \operatorname{colim} H^1(L_i, G)$ by [40, Theorem 2.1], this implies

$$\gamma_{L_j} = \eta_{L_j}$$

is a loop torsor for large enough $i \leq j$. \square

We have seen that in the absence of wild ramification all torsors are loop torsors. This might lead one to guess that all tamely ramified torsors are loop torsors. We finish with an example showing this is not the case.

Example 8.6. Let k be a field of characteristic 2 equipped with a surjective homomorphism

$$\varepsilon : \operatorname{Gal}(k) \rightarrow \operatorname{GL}_1(\mathbb{Z}) = \{\pm 1\}.$$

The homomorphism ε defines a 1-dimensional torus $T_\varepsilon := {}_\varepsilon\mathbb{G}_m$. The $\operatorname{Gal}(k)$ -action on an element $\alpha \in T_\varepsilon(k_{\text{sep}}) = k_{\text{sep}}^*$ is given by:

$$\sigma\alpha = \alpha^{\varepsilon(\sigma)}.$$

Let $F = k((t))$ and consider the decomposition $\Psi_t : \operatorname{Gal}(k) \times \hat{\mathbb{Z}}' \rightarrow \operatorname{Gal}_{\text{tr}}(F)$ of Proposition 3.4. We claim the following $\operatorname{Gal}_{\text{tr}}(F)$ -cocycle represents a tamely ramified T_ε -torsor which is not a loop torsor:

$$c_{\Psi_t(\sigma, n)} = \begin{cases} t & \text{if } \varepsilon(\sigma) = -1 \\ 1 & \text{if } \varepsilon(\sigma) = 1 \end{cases}$$

To see this, we assume that c_τ is cohomologous to a cocycle taking values in $T_\varepsilon(\mathcal{O}_{\text{in}})$ and reach a contradiction. Let $s \in T_\varepsilon(F_{\text{tr}}) = F_{\text{tr}}^*$ be such that for all $\tau \in \operatorname{Gal}_{\text{tr}}(F)$:

$$(8.5) \quad s^{-1}c_\tau s \in T_\varepsilon(\mathcal{O}_{\text{in}}) = \mathcal{O}_{\text{in}}^*.$$

We choose $\sigma \in \operatorname{Gal}(k)$ such that $\varepsilon(\sigma) = -1$ and compute:

$$s^{-1}c_{\Psi_t(\sigma, 0)}^{\Psi_t(\sigma, 0)} s = ts^{-1}\sigma s = ts^{-1+\varepsilon(\sigma)} = ts^{-2}.$$

Let ν be the Henselian valuation on F_{tr} extending the t -adic valuation on F . Equation (8.5) gives $\nu(ts^{-2}) = 0$ and therefore:

$$2\nu(s) = \nu(t).$$

This contradicts the fact that F_{tr}/F is a tamely ramified extension and so $\Gamma_{F_{\text{tr}}}/\Gamma_F$ contains no 2-torsion by definition [62, Appendix A.1].

9. PROOF OF THEOREM 1.2: FIRST REDUCTION

In this section we reduce Theorem 1.2 to the case where G° is reductive. We will assume k_0 is perfect throughout the section, because this is the only case in which G° being reductive is an extra assumption. The following lemma is false over imperfect base fields; See [61].

Lemma 9.1. *Let $U \subset G$ be a normal smooth and connected unipotent subgroup. For any $p \neq \operatorname{char} k_0$, we have:*

$$\operatorname{ed}(G) = \operatorname{ed}(G/U), \quad \operatorname{ed}(G; p) = \operatorname{ed}(G/U; p).$$

Proof. This follows from the fact that the induced map

$$H^1(k, G) \rightarrow H^1(k, G/U)$$

is a bijection for all fields $k_0 \subset k$ [57, Lemma 1.13]. \square

Let $A \subset G$ be a finite split-diagonalizable subgroup. Denote the unipotent radical of G by $R_u(G)$ and set $\overline{G} := G/R_u(G)$. Note that $R_u(G)$ is normal in G because it is a characteristic subgroup of G° . Let $\pi : G \rightarrow \overline{G}$ be the natural quotient map. By [3, Proposition 9.6] and the remark following it, the restriction of π to $C_G(A)$ gives a quotient map:

$$\pi|_{C_G(A)} : C_G(A) \rightarrow C$$

onto a smooth subgroup $C \subset C_{\overline{G}}(\pi(A))$ with

$$(9.1) \quad \dim(C) = \dim(C_{\overline{G}}(\pi(A))).$$

Let $R_u(G)^A = R_u(G) \cap C_G(A)$ be the subgroup fixed by the conjugation action of A on $R_u(G)$. There is a short exact sequence:

$$1 \rightarrow R_u(G)^A \rightarrow C_G(A) \xrightarrow{\pi} C \rightarrow 1.$$

The unipotent group $R_u(G)^A$ is smooth and connected by [46, Theorem 13.7] and [3, Proposition 9.4] respectively. Therefore Lemma 9.1 gives:

$$(9.2) \quad \text{ed}(C_G(A)) = \text{ed}(C), \quad \text{ed}(C_G(A); p) = \text{ed}(C; p).$$

Since $\dim(C) = \dim(C_{\overline{G}}(\pi(A)))$, [44, Proposition 3.15] and (9.2) give:

$$(9.3) \quad \text{ed}(C_{\overline{G}}(\pi(A))) \geq \text{ed}(C) = \text{ed}(C_G(A))$$

Similarly, we obtain for any prime $p \neq \text{char } k_0$:

$$(9.4) \quad \text{ed}(C_{\overline{G}}(\pi(A)); p) \geq \text{ed}(C_G(A); p)$$

To finish the reduction to the reductive case we will need the following lemma.

Lemma 9.2. *If $C_G(A)$ admits anisotropic torsors over a field $k_0 \subset k$, then so does $C_{\overline{G}}(\pi(A))$.*

Proof. Assume $\gamma \in H^1(k, C_G(A))$ is anisotropic. The push-forward $\pi_*(\gamma) \in H^1(k, C)$ is anisotropic by Lemma 4.5(2). The push-forward of $\pi_*(\gamma)$ to $H^1(k, C_{\overline{G}}(\pi(A)))$ is anisotropic by Lemma 4.5(1) because $\dim(C) = \dim(C_{\overline{G}}(\pi(A)))$. \square

Proposition 9.3. *To establish Theorem 1.2, it suffices to prove it under the additional assumption that G° is reductive.*

Proof. We assume that Theorem 1.2 holds for $G/R_u(G) = \overline{G}$ and show that it holds for G (note that $\overline{G}^\circ = G^\circ/R_u(G)$ is reductive because k_0 is perfect [46, Proposition 19.11]).

Let $A \subset C_G(A)$ be a finite split-diagonalizable subgroup such that $C_G(A)$ admits anisotropic torsors over a field $k_0 \subset k$. Then $C_{\overline{G}}(\pi(A))$ admits anisotropic torsors over k as well by Lemma 9.2. We prove the two parts of Theorem 1.2 separately.

(1) Let $p \neq \text{char } k_0$ be a prime. If k is p -closed, and A is a p -group, then Theorem 1.2(1) gives:

$$\text{ed}(\overline{G}; p) \geq \text{ed}(C_{\overline{G}}(\pi(A)); p).$$

Combining this with Lemma 9.1 and (9.4), we get:

$$\text{ed}(G; p) = \text{ed}(\overline{G}; p) \geq \text{ed}(C_{\overline{G}}(\pi(A)); p) \geq \text{ed}(C_G(A); p).$$

Therefore the conclusion of Theorem 1.2(1) holds for G .

(2) If the characteristic of k_0 is good for G , then it is good for \overline{G} as well by Definition 2.1. Therefore Theorem 1.2(2) gives:

$$\text{ed}(\overline{G}) \geq \text{ed}(C_{\overline{G}}(\pi(A))).$$

By Lemma 9.1 and (9.3), we have:

$$\mathrm{ed}(G) = \mathrm{ed}(\overline{G}) \geq \mathrm{ed}(C_{\overline{G}}(\pi(A))) \geq \mathrm{ed}(C_G(A)).$$

This shows Theorem 1.2(2) holds for G . □

10. CONCLUSION OF THE PROOF OF THEOREM 1.2

In this section, F will be an iterated Laurent series field $k((t_1)) \dots ((t_r))$ over k for some field $k_0 \subset k$. The valuation ν on F is the (t_1, \dots, t_r) -adic valuation. We can and shall assume that G° is reductive by Proposition 9.3. We start the proof of Theorem 1.2 with an elementary lemma.

Lemma 10.1. *Let $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in G(k_{\mathrm{sep}})^d$ be d -tuples of elements. Assume that x_1, \dots, x_d generate a finite abelian group $A \subset G(k_{\mathrm{sep}})$ such that $|A|$ is prime to $\mathrm{char} k$. Let G act on G^d by conjugation. If x, y are conjugate in $G(l_{\mathrm{sep}})^d$ for some field extension $k \subset l$, then x, y are conjugate in $G(k_{\mathrm{sep}})^d$.*

Proof. Let $X \subset G^d$ denote the G -orbit of x under the conjugation action on G^d . By assumption $y \in X(l_{\mathrm{sep}})$. Since $y \in G^d(k_{\mathrm{sep}})$ and $X \subset G^d$ is locally closed, we get

$$y \in X(l_{\mathrm{sep}}) \cap G^d(k_{\mathrm{sep}}) = X(k_{\mathrm{sep}}).$$

The stabilizer $\mathrm{stab}_G(x)$ is $C_G(A)$ and so the orbit map induces an isomorphism $X \cong G/C_G(A)$ [46, Corollary 7.13] over k_{sep} . The homomorphism

$$G(k_{\mathrm{sep}}) \rightarrow G/C_G(A)(k_{\mathrm{sep}}), \quad g \mapsto gxg^{-1}$$

is surjective because $C_G(A)$ is smooth [46, Theorem 13.9]. Therefore $y \in X(k_{\mathrm{sep}})$ implies $y = gxg^{-1}$ for some $g \in G(k_{\mathrm{sep}})$. □

The following descent lemma is the main ingredient of the proof of Theorem 1.2.

Lemma 10.2. *Let $\varphi : \Gamma_F^\vee \rightarrow A$ be a continuous surjection onto a split-diagonalizable subgroup $A \subset G$ and $[a_\sigma] \in H_{\mathrm{an}}^1(k, C_G(A)(k_{\mathrm{sep}}))$ an anisotropic torsor. Let $F_1 \subset F$ be a Henselian subfield with $\Gamma_{F_1} = \Gamma_F$ and choose a uniformizer π for $F_{1, \mathrm{tr}}$. If $[a_\sigma, \varphi]_\pi \in H^1(F, G)$ descends to a loop torsor $\gamma_1 \in H_{\mathrm{loop}}^1(F_1, G)$, then $[a_\sigma]$ descends to the residue field of F_1 .*

Proof. Let k_1 denote the residue field of F_1 . By assumption γ_1 is a loop torsor and so we can write $\gamma_1 = [a'_\sigma, \varphi']_\pi$ for some $\varphi' : \Gamma_{F_1}^\vee \rightarrow G(\mathcal{O}_{F_1, \mathrm{in}})$, $a'_\sigma \in G(\mathcal{O}_{F_1, \mathrm{in}})$. Since $\Gamma_{F_1} = \Gamma_F$, we have by Corollary 7.2:

$$[\mathrm{Inf}_{k/k_1}(a'_\sigma)_\sigma, \varphi']_\pi = (\gamma_1)_F = \gamma = [a_\sigma, \varphi]_\pi.$$

Since $[a_\sigma]$ is anisotropic, so is γ by [24, Proposition 4]. Therefore Theorem 6.1 implies there exists $s \in G(k_{\mathrm{sep}})$ such that:

$$(10.1) \quad s^{-1} a_\sigma s = \overline{\mathrm{Inf}_{k/k_1}(a')_\sigma}, \quad s^{-1} \varphi(x) s = \overline{\varphi'(x)}.$$

Let $\mathcal{O}_{F_1, \mathrm{in}}$ denote the inertial closure of the valuation ring of F_1 . By Lemma 10.1, there exists $s_1 \in G(k_{1, \mathrm{sep}})$ such that

$$s_1^{-1} \varphi(x) s_1 = \overline{\varphi'(x)}$$

Lift s_1 to an element $h \in G(\mathcal{O}_{F_1, \text{in}})$ using Hensel's lemma to get

$$\bar{h}^{-1} \varphi(x) \bar{h} = \overline{\varphi'(x)}.$$

We replace $\langle a'_\sigma, \varphi' \rangle_\pi$ by the cohomologous cocycle $\langle h^{-1} a'_\sigma h, h^{-1} \varphi' h \rangle_\pi$ using Lemma 5.5 to assume

$$\overline{\varphi'(x)} = \varphi(x)$$

without loss of generality. Then (10.1) shows $s \in C_G(A)(k_{\text{sep}})$ because $A = \text{im } \varphi$. Therefore

$$(10.2) \quad [\text{Inf}_{k/k_1} \bar{a}'_\sigma] = [a_\sigma] \in H^1(k, C_G(g)).$$

Moreover, $\bar{a}'_\sigma \in C_G(A)(k_{0, \text{sep}})$ for all $\sigma \in \text{Gal}(k_1)$ by Lemma 5.4. Therefore (10.2) proves $[a_\sigma] \in H^1(k, C_G(A))$ descends to k_1 as we wanted to show. \square

Without the assumption $\Gamma_{F_1} = \Gamma_F$ in Lemma 10.2, we cannot say with certainty that $[a_\sigma]$ descends to the residue field of F_1 . However, we can still get a lower bound on the transcendence degree of F_1 in terms of the essential dimension of $[a_\sigma]$.

Corollary 10.3. *Let $\varphi : \Gamma_F^\vee \rightarrow A$ be a continuous surjection onto a split-diagonalizable subgroup $A \subset G$ and $\eta = [a_\sigma] \in H_{\text{an}}^1(k, C_G(A)(k_{\text{sep}}))$ be an anisotropic torsor. If $[a_\sigma, \varphi]_\pi \in H^1(F, G)$ descends to a loop torsor $\gamma_1 \in H_{\text{loop}}^1(F_1, G)$ for some Henselian subfield $F_1 \subset F$ (with respect to $\nu_{|F_1}$), then $\text{trdeg}_{k_0}(F_1) \geq \text{ed}(\eta)$.*

Proof. Set $F_2 = F_1(\pi^\gamma; \gamma \in \Gamma_F)$ and let k_2 denote the residue field of F_2 . Since $\Gamma_{F_2} = \Gamma_F$, Lemma 10.2 implies η descends to k_2 and so $\text{trdeg}_{k_0}(k_2) \geq \text{ed}(\eta)$. Therefore:

$$\text{trdeg}_{k_0}(F_1) \geq \text{trdeg}_{k_0}(F_2) - r \geq \text{trdeg}_{k_0}(k_2) \geq \text{ed}(\eta).$$

Here the first inequality follows because F_2 is generated by r elements over F_1 . The second inequality from the fact that k_2 is the residue field of a valuation of rank r on F_2 ; see [64, Chapter VI, Theorem 3, Corollary 1]. \square

We proceed to prove Theorem 1.2. Assume $C_G(A)$ admits an anisotropic torsor $[a_\sigma] = \eta \in H_{\text{an}}^1(k, C_G(g))$ over k . We may assume that η is versal by Proposition 4.3 and so:

$$(10.3) \quad \text{ed}(\eta) = \text{ed}(C_G(g)) \quad \text{and} \quad \text{ed}(\eta; p) = \text{ed}(C_G(g); p).$$

Set $r = \text{rank}(A)$, let $F = k((t_1)) \dots ((t_r))$ be a field of iterated Laurent series equipped with the (t_1, \dots, t_r) -adic valuation. Since $\Gamma_F = \mathbb{Z}^r$, we have $\Gamma_F^\vee = \text{Hom}(\Gamma_F, \hat{\mathbb{Z}}') = \hat{\mathbb{Z}}'^r$. Choose a surjection $\varphi : \Gamma_F^\vee \rightarrow A$ and set

$$\gamma = [a_\sigma, \varphi]_\pi \in H_{\text{loop}}^1(F, G).$$

Note that $[a_\sigma, \varphi]_\pi$ is well-defined by Example 5.3 and anisotropic by [24, Proposition 4]. We prove the two parts of Theorem 1.2 separately. Starting from Part (2) because it is simpler.

Proof of Theorem 1.2(2). Assume γ descends to a field $k \subset F_1 \subset F$ with:

$$\text{trdeg}_{k_0}(F_1) = \text{ed}(\gamma).$$

We may replace F_1 by its Henselization inside of F to assume F_1 is Henselian; See [62, Appendix A.3]. Under the characteristic assumptions in Part (2), γ descends to a loop torsor over F_1 by Lemma 8.1. Therefore Corollary 10.3 and (10.3) give

$$(10.4) \quad \text{ed}(\gamma) = \text{trdeg}_{k_0}(F_1) \geq \text{ed}(\eta) = \text{ed}(C_G(g)).$$

This finishes the proof because $\text{ed}(G) \geq \text{ed}(\gamma)$ by definition. \square

Proof of Theorem 1.2(1). Assume A is a p -group and k is p -closed. Let L/F be a prime-to- p extension and $k_0 \subset L_1 \subset L$ a field such that γ_L descends to $\gamma_1 \in H^1(L_1, G)$ and:

$$(10.5) \quad \text{trdeg}_{k_0}(L_1) = \text{ed}(\gamma; p).$$

We replace L_1 by its Henselization inside of L to assume it is Henselian. By Proposition 8.5, there exists a prime-to- p extension L'_1/L_1 such that $(\gamma_1)_{L'_1}$ is a loop torsor. Note that L'_1 is contained in a prime-to- p extension $L \subset L'$ [41, Lemma 6.1]. The residue field of L' is k because k is p -closed by Ostrowski's theorem. By Corollary 7.4, there exists a uniformizer τ for L'_{tr} such that:

$$\gamma_{L'} = [a_\sigma, \varphi|_{L'}]_\tau$$

Note that $\gamma_{L'}$ is anisotropic and $\text{im } \varphi|_{L'} = A$ by Lemma 7.5. The field L' is isomorphic (as a valued field) to an iterated Laurent series field because it is a finite extension of F ; See the first paragraph of [27, Page 199]. Since $\gamma_{L'}$ descends to a loop torsor over L'_1 , Corollary 10.3 implies

$$\text{trdeg}_{k_0}(L'_1) \geq \text{ed}(\eta) \geq \text{ed}(\eta; p).$$

Since $L_1 \subset L'_1$ is an algebraic extension we get from (10.5):

$$\text{ed}(\gamma; p) = \text{trdeg}_{k_0}(L_1) = \text{trdeg}_{k_0}(L'_1) \geq \text{ed}(\eta; p) = \text{ed}(C_G(A); p)$$

This finishes the proof because $\text{ed}(G; p) \geq \text{ed}(\gamma; p)$ by definition. \square

We end this section with an example showing why the assumption that $C_G(A)$ admits anisotropic torsors is required for our approach to work.

Example 10.4. Let $G = \text{PGL}(V)$ be the projective linear group associated to the vector space $V = \mathbb{C}^4 \otimes \mathbb{C}^2$. Let $A \subset G$ be the subgroup generated by the equivalence class of the matrix

$$a = \text{id}_{\mathbb{C}^4} \otimes d$$

where $d \in \text{GL}_2(\mathbb{C})$ is the diagonal matrix:

$$d = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}.$$

We will prove that for any loop torsor $[a_\sigma] \in H^1(k, C_G(A))$ and any surjection $\varphi : \Gamma_F^\vee \rightarrow A$, (10.4) fails for the loop torsor $\gamma = [a_\sigma, \varphi]_\pi$. That is, we will prove:

$$\text{ed}(\gamma) \not\leq \text{ed}(C_G(A)).$$

The failure of (10.4) for all $C_G(A)$ -torsors is explained by the fact that, as we shall see, $C_G(A)$ does not admit anisotropic torsors.

A choice of a basis $b_1, b_2 \in \mathbb{C}^2$ allows us to view matrices $g \in \text{GL}(V)$ as block matrices:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

where $g_{ij} \in \text{M}_4(\mathbb{C})$ are four by four matrices. The action of g on $u \otimes b_j$ is given by

$$gu \otimes b_j = g_{1j}u \otimes b_1 + g_{2j}u \otimes b_2.$$

The corresponding block decomposition of $a = \text{id}_{\mathbb{C}^4} \otimes d$ is

$$a = \begin{pmatrix} I_4 & 0 \\ 0 & \zeta_3 I_4 \end{pmatrix}.$$

Here $I_4 \in M_4(\mathbb{C})$ is the identity matrix. Using this, one see computes the centralizer of $A = \langle a \rangle$:

$$C_G(A) = \left\{ \left[\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \mid g_{11}, g_{22} \in \mathrm{GL}_4(\mathbb{C}) \right] \right\}.$$

We identify the right hand side with $\mathrm{GL}_4 \times \mathrm{GL}_4 / \lambda(\mathbb{G}_m)$, where $\lambda : \mathbb{G}_m \rightarrow \mathrm{GL}_4 \times \mathrm{GL}_4$ is the embedding $\lambda(s) = (sI_4, sI_4)$. Consider the diagonal embedding:

$$\Delta : \mathrm{PGL}_4 \rightarrow C_G(A), \quad [g] \mapsto [g \otimes I_2] = \left[\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \right].$$

Let $\mathbb{C} \subset F$ be a field. By [11, Theorem A.1], the quotient map $\pi : C_G(A) \rightarrow \mathrm{PGL}_4 \times \mathrm{PGL}_4$ induces an embedding

$$\pi_* : H^1(F, C_G(A)) \hookrightarrow H^1(F, \mathrm{PGL}_4)^{\times 2}$$

onto the diagonal subset consisting of all pairs (x, x) with $x \in H^1(F, \mathrm{PGL}_4)$. This subset is precisely the image of $\pi_* \circ \Delta_*$, where $\Delta_* : H^1(F, \mathrm{PGL}_4) \rightarrow H^1(F, C_G(A))$ is the map induced by Δ . We conclude that Δ_* must be an isomorphism. In particular, we get:

$$\mathrm{ed}(C_G(A)) = \mathrm{ed}(\mathrm{PGL}_4) = 5$$

by [42]. Let $\gamma \in H^1(F, G)$ be induced from a $C_G(A)$ -torsor. Since Δ_* is an isomorphism, γ is represented by a cocycle of the form $\Delta(c_\sigma)$ for some cocycle $c_\sigma \in \mathrm{PGL}_4$. Recall that PGL_n -torsors classify central simple algebras of degree n up to isomorphism; See for example [28, Chapter 2]. If the cocycle c_σ corresponds to some central simple algebra B of degree 4 over F , then the cocycle $\Delta(c_\sigma)$ corresponds to the matrix algebra $M_2(B) = B \otimes M_2(\mathbb{C})$ over B because $\Delta(c_\sigma) = c_\sigma \otimes I_2$. Both the algebra $M_2(B)$ and γ descend to a field $F_0 \subset F$ with $\mathrm{trdeg}_{\mathbb{C}}(F_0) \leq 4$ by [35, Corollary 1.4]. This gives the upper bound

$$\mathrm{ed}(\gamma) \leq 4.$$

In particular, for any loop torsor $\gamma = [a_\sigma, \varphi]_\pi$ with $\mathrm{im} \varphi = A$ over any Henselian field $\mathbb{C} \subset F$ the essential dimension of γ is strictly smaller than $\mathrm{ed}(C_G(A))$:

$$\mathrm{ed}(\gamma) \leq 4 < 5 = \mathrm{ed}(C_G(A)).$$

This shows the assumption that γ is anisotropic is necessary for our proof of Theorem 1.2 to work. Note that in this example any $C_G(A)$ -torsor $\eta = [a_\sigma]$ is isotropic because $Z(C_G(A)) \cong \mathbb{G}_m$ embeds into the twisted group ${}_{a_\sigma}C_G(A)$.

11. REDUCTIVE SUBGROUPS OF MAXIMAL RANK

Some of the inequalities in Theorem 1.5 follow from the next useful corollary of Theorem 1.2 whose proof relies on Borel–de Siebenthal theory.

Corollary 11.1. *Let G be a group over a field k_0 with G° reductive and assume $\mathrm{char} k_0 \neq 2, 3$. Let $H \subset G$ be a (connected) reductive subgroup of maximal rank. Assume that $Z(H)$ is finite and split-diagonalizable.*

- (1) *Let $p \neq \mathrm{char} k_0$ be a prime. If $Z(H)$ is a p -group and H admits an anisotropic torsor over some p -closed field $k_0 \subset k$, then we have:*

$$\mathrm{ed}(G; p) \geq \mathrm{ed}(H; p).$$

(2) Assume $\text{char } k_0$ is good for G (see Definition 2.1). If H admits an anisotropic torsor over some field $k_0 \subset k$, then we have:

$$\text{ed}(G) \geq \text{ed}(H).$$

Proof. By Borel-de Siebenthal's theorem, we have

$$C_G(Z(H))^\circ = C_{G^\circ}(Z(H))^\circ = H.$$

See [4] for the characteristic zero case and [50] for the general case. Therefore

$$\dim(C_G(Z(H))) = \dim H.$$

By [7, Lemma 2.2], we get:

$$\text{ed}(C_G(Z(H))) \geq \text{ed}(H), \quad \text{ed}(C_G(Z(H)); p) \geq \text{ed}(H; p).$$

Now the corollary follows from Theorem 1.2. □

Corollary 11.1 allows us to prove the following parts of Theorem 1.5.

Proposition 11.2. *Assume $\text{char } k_0 \neq 2, 3$.*

- (1) $\text{ed}(E_8; 2) = \text{ed}(\text{HSpin}_{16}; 2)$
- (2) $\text{ed}(E_8; 3) \geq 13$
- (3) $\text{ed}(E_7^{ad}; 2) \geq 19$

Proof. (1) There exists an embedding $\text{HSpin}_{16} \subset E_8$ [21, Example 4.3] and an anisotropic HSpin_{16} -torsor over a 2-closed field $k_0 \subset k$ by Lemma 16.2. Therefore by Corollary 11.1

$$\text{ed}(E_8; 2) \geq \text{ed}(\text{HSpin}_{16}; 2).$$

The inequality

$$\text{ed}(E_8; 2) \leq \text{ed}(\text{HSpin}_{16}; 2)$$

follows from the fact that any E_8 -torsor admits reduction of structure to HSpin_{16} after an odd degree extension; See the first paragraph of the proof of [21, Theorem 16.2].

(2) By [21, Example 4.4], SL_9/μ_3 embeds into E_8 and we have $\text{ed}(\text{SL}_9/\mu_3; 3) = 13$ by results of S. Baek-Merkurjev [2] and Chernosouv-Merkurjev [13, Theorem 1.1]. Therefore Corollary 11.1 will give:

$$\text{ed}(E_8; 3) \geq \text{ed}(\text{SL}_9/\mu_3; 3) = 13,$$

once we show SL_9/μ_3 admits anisotropic torsors over a 3-closed field. This is simple to see using the theory of central simple algebras. The reader is referred to [28] for the relevant definitions. There exists a central division algebra D over a field $k_0 \subset k$ of period 3 and index 9 [48, Lemma 4.8]. One can take k to be 3-closed because the index and period of D are unaffected by passing to prime-to-3 extensions of k [56, Theorem 3.15]. The algebra D corresponds to an anisotropic PGL_9 -torsor $\gamma \in H^1(k, \text{PGL}_9)$ by [48, Lemma 5.4]. Let $\pi_* : H^1(k, \text{SL}_9/\mu_3) \rightarrow H^1(k, \text{PGL}_9)$ be the pushforward map. There exists $\eta \in H^1(k, \text{SL}_9/\mu_3)$ such that $\pi_*(\eta) = \gamma$ by [48, Lemma 5.3] and η is anisotropic by [48, Corollary 3.4].

(3) There exists an embedding $\text{SL}_8/\mu_4 \subset E_8$; See [15, Corollary 4.2] or [17, Table 4]. We have $\text{ed}(\text{SL}_8/\mu_4; 2) = 19$ by [13, Corollary 1.2]. To deduce the inequality

$$\text{ed}(E_7^{ad}; 2) \geq \text{ed}(\text{SL}_8/\mu_4; 2) = 19$$

from Corollary 10.3, one needs to prove the existence of an anisotropic SL_8/μ_4 -torsor over some 2-closed field. As in Part (2), this is done by first constructing a central division algebra

D of index 8 and period 4 over some 2-closed field $k_0 \subset k$, and then lifting the anisotropic PGL_8 -torsor corresponding to D to an anisotropic SL_8/μ_4 -torsor using [48, Lemma 5.3]. \square

Remark 11.3. In [48], a torsor $\gamma \in H^1(k, G)$ is defined to be isotropic if there exists a proper parabolic subgroup $P \subset G$ such that γ lies in the image of the induced map $H^1(k, P) \rightarrow H^1(k, G)$. In the proof of Proposition 11.2 above, we used the fact that this alternative definition of isotropy is equivalent to our notion of isotropy when G is a quasi-split semisimple group. To prove this fact, assume that G is quasi-split and semisimple. Then γ is isotropic in the sense of [48] if and only if ${}_\gamma G$ admits proper parabolic subgroups [48, Lemma 2.2]. Since $Z({}_\gamma G)$ is finite, ${}_\gamma G$ admits proper parabolic subgroups if and only if it contains non-trivial split tori [5, Corollary 4.17]. Therefore γ is isotropic in the sense of [48] if and only if it is isotropic according to our definition.

12. ABELIAN SUBGROUPS ARISING FROM GRADINGS ON THE CHARACTER LATTICE

In this section we assume G° is a split reductive group. Let $T \subset G$ be a split maximal torus defined over k_0 . We will give a systematic root-theoretic approach to choosing split-diagonalizable p -subgroups $A \subset T$ such that:

- (1) The connected centralizer $C_G(A)^\circ$ is T .
- (2) The group A satisfies the conditions of Theorem 1.2(1).

These conditions allow us to bound $\mathrm{ed}(C_G(A); p)$ from below using [37] and to upgrade our lower bound on $\mathrm{ed}(C_G(A); p)$ to a lower bound on $\mathrm{ed}(G; p)$ using Theorem 1.2. We will obtain a combinatorial formula for lower bounds on $\mathrm{ed}(G; p)$ in terms of gradings on the character lattice $X(T)$.

Definition 12.1. Fix a prime $p \neq \mathrm{char} k_0$ and an abstract abelian p -group \mathcal{V} . A \mathcal{V} -grading on $X(T)$ is a surjective homomorphism

$$\varepsilon : X(T) \rightarrow \mathcal{V}.$$

Let $X(\mathcal{V}) = \mathrm{Hom}(\mathcal{V}, k_{0, \mathrm{sep}}^*)$ be the group of characters of \mathcal{V} . The group $X(\mathcal{V})$ is split-diagonalizable of order $|\mathcal{V}|$. Any \mathcal{V} -grading induces an embedding $\varepsilon^* : X(\mathcal{V}) \rightarrow T(k_{0, \mathrm{sep}})$ by the anti-equivalence between finitely generated abelian groups and quasi-tori; See e.g. [5, Corollary 8.3]. We will denote the image of this embedding by

$$(12.1) \quad A_\varepsilon := \varepsilon^*(X(\mathcal{V})).$$

The following lemma gives us a way to verify the condition $C_G(A_\varepsilon)^\circ = T$ holds.

Lemma 12.2. *Let $\Phi \subset X(T)$ be the root system associated to $T \subset G$. We have $C_G(A_\varepsilon)^\circ = T$ if and only if*

$$(12.2) \quad \forall \alpha \in \Phi, \quad \varepsilon(\alpha) \neq 0.$$

Proof. The inclusion $T \subset C_G(A_\varepsilon)^\circ$ holds because $A_\varepsilon \subset T$. Therefore the equality $C_G(A_\varepsilon)^\circ = T$ is equivalent to:

$$(12.3) \quad \dim T = \dim C_G(A_\varepsilon).$$

To check this equality of dimensions we can base change to $k_{0, \mathrm{alg}}$ to assume that k_0 is algebraically closed. Moreover, we may assume G is connected by replacing G with G°

because this does not affect (12.3). The proof is a variation of [31, Theorem 2.2]. We include the details for completeness. Let $H \subset G$ be the subgroup generated by $N_G(T) \cap C_G(A_\varepsilon)$ and all root subgroup U_α with $\varepsilon(\alpha) = 0$. It will suffice to prove

$$H = C_G(A_\varepsilon).$$

To prove the inclusion $H \subset C_G(A_\varepsilon)$ it suffices to prove $U_\alpha \subset C_G(A_\varepsilon)$ for any $\alpha \in \Phi$ such that

$$\varepsilon(\alpha) = 0.$$

The group A_ε commutes with U_α if and only if

$$\alpha(\varepsilon^*(\mathcal{V})) = 1.$$

Now note that $\alpha \circ \varepsilon^*$ is dual to the homomorphism $\varepsilon \circ \alpha^* : X(\mathbb{G}_m) \rightarrow \mathcal{V}$. Therefore:

$$(12.4) \quad \alpha(\varepsilon^*(A_\varepsilon)) = 1 \iff \varepsilon(\alpha) = 0 \iff U_\alpha \subset C_G(A_\varepsilon).$$

To prove $H \supset C_G(A_\varepsilon)$, choose representatives $n_w \in N_{G^\circ}(T)$ for Weyl group elements $w \in W = N_G(T)/T$. Choose an ordering of the roots Φ and let $U_+, U_- \subset G^\circ$ be the corresponding unipotent groups generated by the positive and negative root groups respectively. An element $x \in C_G(A_\varepsilon)(k_0)$ has a unique Bruhat decomposition:

$$x = un_w tv,$$

where $v \in U_+, t \in T, u \in U_+ \cap n_w U_- n_w^{-1}$. The uniqueness of the decomposition forces $u, n_w, v \in C_G(A_\varepsilon)$ because A_ε normalizes $U, N_G(T), U_-$. Therefore we can assume $x = u$ or $x = v$. The two cases are similar, we will focus on the case $x = u \in U_+$. Up to an ordering $\alpha_1, \dots, \alpha_d$ of the positive roots, u can be written uniquely as a product:

$$u = p_{\alpha_1}(t_1) \dots p_{\alpha_d}(t_d),$$

where p_α is a parameterization of U_α and $t_i \in k_0$. For any $v \in \mathcal{V}, u \in C_G(A_\varepsilon)$ implies:

$$p_{\alpha_1}(t_1) \dots p_{\alpha_d}(t_d) = \varepsilon^*(a) u \varepsilon^*(a)^{-1} = p_{\alpha_1}(\alpha_1(\varepsilon^*(a))t_1) \dots p_{\alpha_d}(\alpha_d(\varepsilon^*(a))t_d).$$

By uniqueness of the decomposition, we get for all $1 \leq i \leq d$:

$$\alpha_i(\varepsilon^*(a))t_i = t_i.$$

If $t_i \neq 0$, then $\alpha_i(\varepsilon^*(A_\varepsilon)) = 1$ and (12.4) implies $\varepsilon(\alpha) = 0$. Therefore $p_{\alpha_i}(t_i) \in H$ for all $1 \leq i \leq d$ and $u \in H$, as we wanted to show. \square

Let ε be a \mathcal{V} -grading such that $\varepsilon(\alpha) \neq 0$ for any $\alpha \in \Phi$. We denote the component group by

$$W(\varepsilon) := C_G(A_\varepsilon)/C_G(A_\varepsilon)^\circ.$$

Note that $W(\varepsilon)$ acts naturally on $C_G(A_\varepsilon)^\circ$ by conjugation. We have

$$C_G(A_\varepsilon)^\circ = T$$

by Lemma 12.2, so $C_G(A_\varepsilon)$ acts on T and on the root lattice $X(T)$. Therefore we can consider the natural inclusion into the Weyl group:

$$(12.5) \quad W(\varepsilon) \subset W = N_G(T)/T.$$

Denote the natural action of W on $\chi \in X(T)$ by $w \cdot \chi$. The subgroup $W(\varepsilon) \subset W$ can be explicitly described as the subgroup of W preserving ε :

$$(12.6) \quad W(\varepsilon) = \{w \in W \mid \varepsilon(w \cdot \chi) = \varepsilon(\chi), \text{ for any } \chi \in X(T)\}.$$

Indeed, we have $C_G(A_\varepsilon) \subset N_G(T)$ and therefore by definition:

$$C_G(A_\varepsilon) = \{n \in N_G(T) \mid \text{Ad}(n) \circ \varepsilon^* = \varepsilon^*\}.$$

Here $\text{Ad}(n) \in \text{Aut}(N_G(T))$ denotes conjugation by n . Since the functor $A \mapsto X(A)$ is an anti-equivalence between the category finitely generated abelian groups and the category of split-diagonalizable groups [5, Corollary 8.3], $\text{inn}(n) \circ \varepsilon^* = \varepsilon^*$ if and only if $nT \in W$ preserves ε .

To give a lower bound on the essential dimension of $C_G(A_\varepsilon)$, it is useful to introduce the following definition:

Definition 12.3. [39] Let S be a finite group acting on a finitely generated abelian group \mathcal{U} . A subset $\Gamma \subset \mathcal{U}$ is called *p-generating* if the subgroup generated by Γ is of finite and prime to p index in \mathcal{U} . Choose a Sylow p -subgroup $S_p \subset S$. The *symmetric p-rank* of the S -action on \mathcal{U} is the following integer:

$$\text{Rank}(S, \mathcal{U}; p) = \min\{|\Gamma| \mid \Gamma \subset \mathcal{U} \text{ is } S_p\text{-invariant and } p\text{-generating}\}.$$

The notion of p -symmetric rank is related to essential dimension at p by work of M. Macdonald-R. Löttscher-A. Meyer-Reichstein.

Proposition 12.4. [39, Theorem 1.10] *Let N be a smooth linear algebraic group over k_0 such that $N^\circ = T$ is a split torus. We have:*

$$\text{ed}(N; p) \geq \text{Rank}(N/T, X(T); p) - \dim T.$$

Here the N/T action on $X(T)$ is induced from the conjugation action of N/T on T .

Note that while N is assumed to be the normalizer of a split maximal torus in a simple algebraic group in [39, Theorem 1.10], this assumption is not used in the proof of the lower bound $\text{ed}(N; p) \geq \text{Rank}(N/T, X(T); p)$. Setting $C_G(A_\varepsilon) = N$ in Proposition 12.4, we obtain:

Corollary 12.5. *If ε satisfies (12.2), then*

$$\text{ed}(C_G(A_\varepsilon); p) \geq \text{Rank}(W(\varepsilon), X(T); p) - \dim T.$$

Next we formulate a condition on ε that ensures the hypotheses of Theorem 1.2(1) hold for A_ε and G .

Lemma 12.6. *Assume ε satisfies (12.2) and choose a Sylow p -subgroup $W(\varepsilon)_p \subset W(\varepsilon)$. If we have*

$$(12.7) \quad X(T)^{W(\varepsilon)_p} = \{0\},$$

then $C_G(A_\varepsilon)$ admits anisotropic torsors over some p -closed field $k_0 \subset k$.

Proof. Let $\pi : C_G(A_\varepsilon) \rightarrow W(\varepsilon) = C_G(A_\varepsilon)/C_G(A_\varepsilon)^\circ$ be the natural projection. There exists a finite p -group $P \subset C_G(A_\varepsilon)$ such that $\pi(P) = W(\varepsilon)_p$ by [37, Lemma 5.3]. By assumption we have:

$$(12.8) \quad X(T)^P = \{0\}.$$

Replace k_0 by its algebraic closure $k_{0, \text{alg}}$ to assume P is a constant group without loss of generality. There exists a p -closed field k containing k_0 and a surjection $\phi : \text{Gal}(k) \rightarrow P$. For example, if $k_1 = k_{0, \text{alg}}(t)$ and $k_1 \subset k$ is a p -closure of k_1 , then $\text{Gal}(k)$ is a Sylow p -subgroup of $\text{Gal}(k_1)$ [19, Proposition 101.16]. Therefore $\text{Gal}(k)$ is a free p -group of infinite

rank because its cohomological dimension is one; See [58, Page 80] and [58, Page 30]. In particular, there exists a surjection $\phi : \text{Gal}(k) \rightarrow P$. This surjection defines a $C_G(A_\varepsilon)$ -torsor $[\phi] \in H^1(k, C_G(A_\varepsilon))$ because P is constant. The connected component of the twisted group ${}_\phi C_G(A_\varepsilon)$ is the torus ${}_\phi T$ whose character $\text{Gal}(k)$ -module is $X(T)$ equipped with the $\text{Gal}(k)$ -action given by:

$$\sigma \cdot \chi = \phi(\sigma)(\chi).$$

To show $[\phi] \in H^1(k, C_G(A_\varepsilon))$ is anisotropic we assume there exists an embedding $f : \mathbb{G}_m \rightarrow {}_\phi T$ and reach a contradiction. The embedding f corresponds by duality to a P -invariant surjection:

$$f^* : X(T) \rightarrow X(\mathbb{G}_m).$$

See [5, Corollary 8.3]. Let $\chi \in X(T)$ be such that $f^*(\chi) \neq 0$. By the P -invariance of $f^* :$

$$f^*\left(\sum_{p \in P} p \cdot \chi\right) = |P|f^*(\chi) \neq 0.$$

In particular, $\sum_{p \in P} p \cdot \chi \in X(T)^P$ is non-zero, contradicting (12.8). \square

Lemma 12.6 allows us to apply Corollary 12.5 and Theorem 1.2 to get a lower bound on $\text{ed}(G; p)$.

Proposition 12.7. *Let G, T and $\varepsilon : X(T) \rightarrow \mathcal{V}$ be as above. If ε satisfies (12.2) and (12.7), then*

$$(12.9) \quad \text{ed}(G; p) \geq \text{Rank}(W(\varepsilon), X(T); p) - \dim T.$$

Proof. Lemma 12.6 and Theorem 1.2(1) imply $\text{ed}(G; p) \geq \text{ed}(C_G(A_\varepsilon); p)$. Therefore (12.9) follows from Corollary 12.5. \square

13. PROOF OF THEOREM 1.5: THE OVERALL STRATEGY

13.1. Setup for the next sections. In the next four sections we prove all parts of Theorem 1.5 except for Parts (2) and (4) which are included in Proposition 11.2. We also reprove Merkurjev's lower bound (1.1). In each section we focus on the essential dimension of a group G at a prime p . We choose a grading ε on the character lattice $X(T)$ of a maximal split torus $T \subset G$. We check that ε satisfies (12.2) and $W(\varepsilon)$ satisfies (12.7). Applying Proposition 12.7 then gives:

$$\text{ed}(G; p) \geq \text{Rank}(W(\varepsilon), X(T); p) - \dim T.$$

Finally, we prove a lower bound on $\text{Rank}(W(\varepsilon), X(T); p)$ using Lemma 13.1. This last step involves proving lower bounds on the size of $W(\varepsilon)$ -orbits. We will use the following notation:

- For any set I and ring R , R^I is the free R -module with standard basis $\{e_i | i \in I\}$. The coordinates of an element $x \in R^I$ in the standard basis are denoted $(x_i)_{i \in I}$.
- The group of permutations of a set X is denoted $\text{Sym}(X)$. If a finite group S acts on X , we denote the S -orbit of $x \in X$ by Sx and its stabilizer by $\text{Stab}_S(x)$.
- If G is an adjoint simple group of type Δ , we identify the character lattice $X(T)$ with the root lattice $Q(\Delta)$. We use the description of the roots $\Phi \subset Q(\Delta)$ and the Weyl group W action given in [16].
- We always assume $\text{char } k_0 \neq p$.

We also note that for any subgroup $S \subset W(\varepsilon)$ we have:

$$(13.1) \quad \text{Rank}(W(\varepsilon), X(T); p) \geq \text{Rank}(S, X(T); p).$$

To see this, choose Sylow p -subgroup $S_p, W(\varepsilon)_p$ of S and $W(\varepsilon)$ such that $S_p \subset W(\varepsilon)_p$. Then (13.1) follows from the fact that p -generating $W(\varepsilon)$ -invariant subset $\Gamma \subset X(T)$ is S -invariant.

13.2. A lower bound on the symmetric rank. Let p be a prime and S a finite p -group. Assume S acts on a finitely generated abelian group \mathcal{U} . The next elementary lemma gives a useful lower bound on $\text{Rank}(S, \mathcal{U}; p)$.

Lemma 13.1. *Let $\varepsilon : \mathcal{U} \rightarrow \mathbb{F}_p^d$ be an S -invariant surjective homomorphism and assume we are given a direct sum decomposition $L_1 \oplus L_2 = \mathbb{F}_p^d$. Let $\varepsilon_i : \mathcal{U} \rightarrow L_i$ be the composition of ε with the projection $\pi_i : \mathbb{F}_p^d \rightarrow L_i$ for $i = 1, 2$. If $|S\mathcal{U}| \geq C_i$ whenever $\varepsilon_i(u) \neq 0$, then:*

$$\text{Rank}(S, \mathcal{U}; p) \geq C_1 \dim_{\mathbb{F}_p} L_1 + C_2 \dim_{\mathbb{F}_p} L_2.$$

Proof. Denote $d_1 = \dim_{\mathbb{F}_p} L_1$ and $d_2 = \dim_{\mathbb{F}_p} L_2$. Let $\Gamma \subset \mathcal{U}$ be an S -invariant, p -generating subset such that

$$|\Gamma| = \text{Rank}(S, \mathcal{U}; p).$$

The image $\varepsilon(\Gamma)$ is p -generating in \mathbb{F}_p^d because ε is surjective. Because \mathbb{F}_p^d contains no proper subgroups of index prime to p , $\varepsilon(\Gamma)$ has to generate \mathbb{F}_p^d . Therefore Γ contains elements $\gamma_1, \dots, \gamma_d$ such that $\varepsilon(\gamma_1), \dots, \varepsilon(\gamma_d)$ is a basis for \mathbb{F}_p^d . We denote $b_i = \varepsilon(\gamma_i)$ for all $1 \leq i \leq d$.

An elementary linear algebra argument shows that, up to relabeling, we can assume that the projections $\pi_1(b_1), \dots, \pi_1(b_{d_1})$ to L_1 are a basis for L_1 , and the projections $\pi_2(b_{d_1+1}), \dots, \pi_2(b_{d_2})$ are a basis for L_2 . We have for all $1 \leq i \leq d_1$:

$$\varepsilon_1(\gamma_i) = \pi_1 \varepsilon(\gamma_i) = \pi_1(b_i) \neq 0.$$

By our assumptions, this implies $|S\gamma_i| \geq C_1$. Similarly, we have

$$|S\gamma_i| \geq C_2 \text{ for all } d_1 < i \leq d_2.$$

Since ε is S -invariant, the orbits $S\gamma_1, \dots, S\gamma_d$ are disjoint. We conclude:

$$\text{Rank}(S, \mathcal{U}; p) = |\Gamma| \geq |S\gamma_1| + \dots + |S\gamma_d| \geq C_1 d_1 + C_2 d_2.$$

□

14. ESSENTIAL DIMENSION OF PGL_{p^n} AT p

Our first application of Proposition 12.7 is to give a new proof of Merkurjev's lower bound on $\text{ed}(\text{PGL}_{p^n}; p)$ (1.1). By Proposition 12.7, it suffices to prove:

Proposition 14.1. *Let $p \neq \text{char } k_0$ be a prime and $n \geq 1$. Let $T \subset \text{PGL}_{p^n}$ be a maximal split torus with corresponding Weyl group W . There exists an \mathbb{F}_p^n -grading $\varepsilon : X(T) \rightarrow \mathbb{F}_p^n$ satisfying (12.2) and (12.7) such that*

$$(14.1) \quad \text{Rank}(W(\varepsilon), X(T); p) \geq np^n.$$

Definition of ε . We start by introducing the notation needed to define ε . We set $\mathcal{V} = \mathbb{F}_p^n$ and identify the character lattice $X(T)$ of the diagonal torus $T \subset \mathrm{PGL}_{p^n}$ with the following sublattice of $\mathbb{Z}^{\mathcal{V}}$ as in [16, Section 6]:

$$\mathbb{Z}_0^{\mathcal{V}} = \{x \in \mathbb{Z}^{\mathcal{V}} \mid \sum_{v \in \mathcal{V}} x_v = 0\}$$

Define $\varepsilon : \mathbb{Z}_0^{\mathcal{V}} \rightarrow \mathcal{V}$ to be the surjective homomorphism given by:

$$\varepsilon(x) = \sum_{v \in \mathcal{V}} x_v v.$$

Verification of Condition (12.2). The roots of $\Phi \subset \mathbb{Z}_0^{\mathcal{V}}$ are the vectors $e_u - e_v \in \mathbb{Z}_0^{\mathcal{V}}$ with $u \neq v$. For any such root we have:

$$\varepsilon(e_u - e_v) = u - v \neq 0.$$

Therefore ε satisfies (12.2)

Description of the Weyl group. The Weyl group W is the symmetric group $\mathrm{Sym}(\mathcal{V})$ acting on $\mathbb{Z}_0^{\mathcal{V}}$ by permuting the standard basis vectors. For any $u \in \mathcal{V}$, let $\lambda_u \in \mathrm{Sym}(\mathcal{V})$ be the translation by u . The subgroup $W(\varepsilon) \subset \mathrm{Sym}(\mathcal{V})$ of permutations preserving ε is the subgroup consisting of all such translations. Indeed, if $u \in \mathcal{V}$ and $x \in \mathbb{Z}_0^{\mathcal{V}}$ then $\lambda_u \in W(\varepsilon)$ because:

$$\varepsilon(\lambda_u(x)) = \sum_{v \in \mathcal{V}} x_v(u + v) = \sum_{v \in \mathcal{V}} x_v v + \left(\sum_{v \in \mathcal{V}} x_v\right)u = \sum_{v \in \mathcal{V}} x_v v = \varepsilon(x).$$

Here the second to last equality follows from the fact that $\sum_{v \in \mathcal{V}} x_v = 0$. Conversely, if $\sigma \in \mathrm{Sym}(\mathcal{V})$ preserves ε , then for all $v \in \mathcal{V}$:

$$v = \varepsilon(e_0 - e_v) = \varepsilon(e_{\sigma(0)} - e_{\sigma(v)}) = \sigma(0) - \sigma(v).$$

Therefore $\sigma(v) = v + \sigma(0)$. That is, $\sigma = \lambda_{\sigma(0)}$. This proves:

$$W(\varepsilon) = \{\lambda_u \mid u \in \mathcal{V}\}.$$

We will identify $W(\varepsilon)$ with \mathcal{V} using the isomorphism $\mathcal{V} \xrightarrow{\sim} W(\varepsilon), u \mapsto \lambda_u$.

Verification of Condition (12.7). Note that \mathcal{V} is its own Sylow p -subgroup. If an element $x \in \mathbb{Z}_0^{\mathcal{V}}$ is fixed by \mathcal{V} , then for any $u, v \in \mathcal{V}$ we have:

$$(14.2) \quad x_v = x_{u+v}.$$

Since $x \in \mathbb{Z}_0^{\mathcal{V}}$ this implies

$$0 = \sum_{v \in \mathcal{V}} x_v = |\mathcal{V}|x_0.$$

Therefore $x_v = x_0 = 0$ for all $v \in \mathcal{V}$ by (14.2). We conclude that $W(\varepsilon) = \mathcal{V}$ satisfies (12.7).

Proof of the lower bound on $\text{Rank}(W(\varepsilon), X(T); p)$. To finish the proof of (14.1) we will show that for any element $x \in \mathbb{Z}_0^\mathcal{V}$ with $\varepsilon(v) \neq 0$ we have

$$(14.3) \quad |\mathcal{V}x| = p^n.$$

Once we do so, Lemma 13.1 will give:

$$\text{Rank}(W(\varepsilon), X(T); p) \geq \dim_{\mathbb{F}_p} \mathcal{V} \cdot p^n = np^n.$$

By the stabilizer-orbit formula, (14.3) is equivalent to

$$\text{Stab}_{\mathcal{V}}(x) = \{0\}.$$

Therefore we need to prove that $\text{Stab}_{\mathcal{V}}(x) = \{0\}$ for all $x \in \mathbb{Z}_0^\mathcal{V}$ with $\varepsilon(x) \neq 0$. We prove a slightly more general lemma which will be used again in Section ??.

Lemma 14.2. *Let $x \in \mathbb{Z}^\mathcal{V}$. Assume either p is odd, or $\sum_{v \in \mathcal{V}} x_v$ is divisible by four. If $\text{Stab}_{\mathcal{V}}(x) \neq \{0\}$, then*

$$\sum_{v \in \mathcal{V}} x_v v = 0.$$

In particular, if $\varepsilon(x) \neq 0$ for some $x \in \mathbb{Z}_0^\mathcal{V}$, then $\text{Stab}_{\mathcal{V}}(x) = \{0\}$.

Proof. Assume that x is fixed by $0 \neq u \in \mathcal{V}$ and let $\mathcal{C} \subset \mathcal{V}$ be a set of coset representatives for the quotient $\mathcal{V}/\langle u \rangle$. Since x is fixed by u , we have for all $v \in \mathcal{V}$:

$$x_v = x_{u+v}.$$

Using this, we compute:

$$\begin{aligned} \sum_{v \in \mathcal{V}} x_v v &= \sum_{c \in \mathcal{C}} \sum_{i=0,1,\dots,p-1} x_{c+iu} (c+iu) \\ &= \sum_{c \in \mathcal{C}} x_c \sum_{i=0,1,\dots,p-1} (c+iu) \\ &= \sum_{c \in \mathcal{C}} x_c (pc + \binom{p}{2} u) \\ &= \left(\sum_{c \in \mathcal{C}} x_c \right) \binom{p}{2} u. \end{aligned}$$

Here in the last equality we used the fact that $pc = 0$ in \mathcal{V} . If p is odd, then it divides $\binom{p}{2}$ and we get

$$\sum_{v \in \mathcal{V}} x_v v = \left(\sum_{c \in \mathcal{C}} x_c \right) \binom{p}{2} u = 0.$$

Otherwise $p = 2$ and $4 \mid \sum_{v \in \mathcal{V}} x_v$. We have:

$$\sum_{v \in \mathcal{V}} x_v = \sum_{c \in \mathcal{C}} x_c + x_{c+b} = 2 \sum_{c \in \mathcal{C}} x_c.$$

Since $\sum_{v \in \mathcal{V}} x_v$ is divisible by four, this implies $\sum_{c \in \mathcal{C}} x_c$ is even. Therefore in $\mathcal{V} = \mathbb{F}_2^n$:

$$\sum_{v \in \mathcal{V}} x_v v = \left(\sum_{c \in \mathcal{C}} x_c \right) \binom{p}{2} u = 0.$$

□

15. ESSENTIAL DIMENSION OF PGO_{2n}^+ AT 2

In this section we prove Parts (5) and (6) of Theorem 1.5. By Proposition 12.7, it suffices to prove:

Proposition 15.1. *Let $n = 2^r m \geq 4$ be an integer with $r \geq 0$ and m odd. Let $T \subset \mathrm{PGO}_{2n}^+$ be a split maximal torus with corresponding Weyl group W . There exists a grading $\varepsilon : X(T) \rightarrow \mathbb{F}_2^{r+m-1}$ satisfying (12.2) and (12.7) such that*

$$(15.1) \quad \mathrm{Rank}(W(\varepsilon), X(T); 2) \geq \begin{cases} (r+m-1)2^{r+1} & \text{if } r \geq 1 \\ 4(m-1) & \text{if } r = 0. \end{cases}$$

Definition of ε . We set notation in order to describe $X(T)$ and define ε . Define $\mathcal{V} = \mathbb{F}_2^r$, $K = \mathcal{V} \times \{1, \dots, m\}$ and

$$(\mathbb{F}_2^m)_0 = \{x \in \mathbb{F}_2^m \mid \sum_{i=1, \dots, m} x_i = 0\}.$$

Since PGO_{2n}^+ is the split adjoint group of type D_n , we can choose a split maximal torus $T \subset \mathrm{PGO}_{2n}^+$ such that the character lattice $X(T) = Q(D_n)$ is identified with the following sublattice of \mathbb{Z}^K [16, Section 7.1]:

$$Q(D_n) = \{x \in \mathbb{Z}^K \mid \sum_{k \in K} x_k \text{ is even}\}.$$

Pick a basis $\{b_1, \dots, b_m\}$ for \mathbb{F}_2^m and let $\varepsilon : Q(D_n) \rightarrow \mathcal{V} \oplus (\mathbb{F}_2^m)_0$ be the grading given by

$$\varepsilon(x) = \sum_{(v,i) \in K} x_{v,i} (v + b_i) \in \mathcal{V} \oplus (\mathbb{F}_2^m)_0.$$

Note that ε is surjective because for any $i \neq j$ and $v \in \mathcal{V}$ we have:

$$\varepsilon(e_{0,i} + e_{0,j}) = b_i + b_j, \quad \varepsilon(e_{0,1} + e_{v,1}) = (v, 0).$$

Verification of Condition (12.2). The roots $\Phi \subset Q(D_n)$ are given by:

$$\Phi = \{\pm e_k \pm e_s \mid k, s \in K, k \neq s\}.$$

For any root $\alpha = \pm e_{(v,i)} \pm e_{(v',j)}$ we have:

$$\varepsilon(\alpha) = v + v' + b_i + b_j \neq 0.$$

Therefore ε satisfies (12.2).

Description of the Weyl group. The Weyl group of G is $\mathrm{Sym}(K) \times (\mathbb{F}_2^K)_0$, where $\mathrm{Sym}(K)$ is the group of permutations of K and $(\mathbb{F}_2^K)_0$ is the group:

$$(\mathbb{F}_2^K)_0 = \{\delta \in \mathbb{F}_2^K \mid \sum_k \delta_k = 0\}.$$

The action of $(\sigma, \delta) \in \mathrm{Sym}(K) \times (\mathbb{F}_2^K)_0$ on $Q(D_n)$ is the restriction of the action on \mathbb{Z}^K defined by:

$$(15.2) \quad (\sigma, \delta)e_k = (-1)^{\delta_k} e_{\sigma(k)}$$

for all $k \in K$.

Lemma 15.2. *Identify \mathcal{V} with the subgroup of $\text{Sym}(K)$ consisting of all translations $\lambda_u : K \rightarrow K$ by an element $u \in \mathcal{V}$:*

$$\lambda_u(v, i) = (u + v, i).$$

We have $W(\varepsilon) = \mathcal{V} \ltimes (\mathbb{F}_2^K)_0$.

Proof. The inclusion $(\mathbb{F}_2^K)_0 \subset W(\varepsilon)$ is easy to see. The inclusion $\mathcal{V} \subset W(\varepsilon)$ follows from the computation:

$$\begin{aligned} \varepsilon(\lambda_u(x)) &= \sum x_{v,i}(u + v + b_i) \\ &= \sum x_{v,i}(v + b_i) + \sum x_{v,i}u \\ &= \varepsilon(x) + \sum x_{v,i}u = \varepsilon(v). \end{aligned}$$

Here in the last equality we used the fact that $\sum x_{v,i}$ is even. To verify the inclusion $W(\varepsilon) \subset \mathcal{V} \ltimes (\mathbb{F}_2^K)_0$, we assume $w = (\sigma, \delta) \in W(\varepsilon)$ and show $\sigma = \lambda_u$ for some $u \in \mathcal{V}$. Since $(\mathbb{F}_2^K)_0 \subset W(\varepsilon)$, we can assume $w = \sigma$ without loss of generality. We start by denoting for any $(v, i) \in K$:

$$\sigma(v, i) = (f(v, i), g(v, i)) \in K.$$

For any $(v, i), (v', j) \in K$ we have:

$$(15.3) \quad f(v, i) + f(v', j) + b_{g(v,i)} + b_{g(v',j)} = \varepsilon(\sigma(e_{v,i} + e_{v',j})) = \varepsilon(e_{v,i} + e_{v',j}) = v + v' + b_i + b_j.$$

Assume $i \neq j$. Then if $g(v, i) \neq i$, (15.3) implies

$$g(v, i) = j.$$

Since m is odd if $i \neq j$ then $m \geq 3$. Therefore repeating the same argument with $1 \leq j' \leq m$ different from i and j gives the contradiction $g(v, i) = j'$. Therefore we see that

$$g(v, i) = i$$

for all $(v, i) \in K$. Setting $v' = 0, j = 1$ in (15.3), we get for all $(v, i) \in K$:

$$f(v, i) - v = f(0, 1).$$

Therefore $\sigma = \lambda_{f(0,1)}$ is the translation by $f(0, 1)$. This finishes the proof. \square

Verification of Condition (12.7). Let $x \in Q(D_n)^{W(\varepsilon)}$ be fixed by $W(\varepsilon)$ and let $k \in K$ be some element. Since $|K| = n \geq 3$ there exists $\delta \in (\mathbb{F}_2^K)_0$ such that $\delta_k = 1$. The equation

$$\delta x = x$$

implies $x_k = 0$ because δ flips the sign of the k -th coordinate. Since $k \in K$ was arbitrary, we conclude that $x = 0$. Therefore

$$Q(D_n)^{W(\varepsilon)} = \{0\}.$$

Since $W(\varepsilon)$ is its own Sylow 2-subgroup, (12.7) follows.

Proof of the lower bound on $\text{Rank}(W(\varepsilon), X(T); 2)$. For the proof we will need the following elementary lemma.

Lemma 15.3. *Let $Y \subset K$ be a non-empty finite subset of $K = \mathcal{V} \times \{1, \dots, m\}$. For any $u \in \mathcal{V}$ denote:*

$$u + Y = \{(u + v, i) \mid (v, i) \in Y\}.$$

There are at most $|Y|$ elements $u \in \mathcal{V}$ such that:

$$(15.4) \quad u + Y = Y.$$

Proof. Replace Y with $u + Y$ for some $u \in \mathcal{V}$ such that $(-u, i) \in Y$ to assume that $(0, i) \in Y$ without loss of generality. Then $u + Y = Y$ implies $(u, i) \in Y$ and so there are at most $|Y|$ elements satisfying (15.4). \square

Proof of (15.1). Note that $W(\varepsilon)$ is its own Sylow 2-subgroup. By Lemma 13.1, it suffices to prove that for any $x \in Q(D_n)$ with $\varepsilon(x) \neq 0$, we have

$$|W(\varepsilon)x| \geq \begin{cases} 2^{r+1} & \text{if } r \geq 1 \\ 4 & \text{if } r = 0. \end{cases}$$

Let $x \in Q(D_n)$ be such that $\varepsilon(x) \neq 0$. We define

$$Y_x = \{(v, i) \in K \mid x_{a,i} \neq 0\},$$

and split into cases.

Assume $r = 0$. We have $|Y_x| \geq 2$. Otherwise, $x = x_k e_k$ for some $k \in K, x_k \in 2\mathbb{Z}$ and so $\varepsilon(v) = 0$. Write:

$$x = x_{k_1} e_{k_1} + x_{k_2} e_{k_2} + \sum_{k \neq k_1, k_2} x_k e_k,$$

for some $x_{k_1}, x_{k_2} \neq 0$. Since $|K| = n \geq 3$, for any $(\delta_1, \delta_2) \in \mathbb{F}_2^2$ there exists $\delta \in (\mathbb{F}_2^K)_0$ such that $\delta_{k_1} = \delta_{k_2} = 1$. Therefore the orbit $W(\varepsilon)x$ contains elements of the form:

$$(-1)^{\delta_1} x_{k_1} e_{k_1} + (-1)^{\delta_2} x_{k_2} e_{k_2} + \sum_{k \neq k_1, k_2} x'_k e_k,$$

for some $x'_k \in \mathbb{Z}$. We conclude that $|W(\varepsilon)x| \geq 4$.

Assume $r \geq 1$. The stabilizer-orbit formula gives:

$$|W(\varepsilon)x| = |W(\varepsilon)| / |\text{Stab}_{W(\varepsilon)}(x)| = 2^{r+n-1} / |\text{Stab}_{W(\varepsilon)}(x)|.$$

Therefore our goal is to show:

$$(15.5) \quad |\text{Stab}_{W(\varepsilon)}(x)| \leq 2^{n-2}$$

If $w = (u, \delta) \in W(\varepsilon)$ fixes x , then:

$$u + Y_x = Y_x.$$

Therefore the projection onto the first component gives a short exact sequence:

$$0 \rightarrow \text{Stab}_{(\mathbb{F}_2^K)_0}(x) \rightarrow \text{Stab}_{W(\varepsilon)}(x) \rightarrow \text{Stab}_{\mathcal{V}}(Y_x).$$

Here $\text{Stab}_{\mathcal{V}}(Y_x)$ is the stabilizer of Y_x with respect to the action of \mathcal{V} on subsets of K by translation. Thus we get an inequality:

$$(15.6) \quad |\text{Stab}_{W(\varepsilon)}(x)| \leq |\text{Stab}_{(\mathbb{F}_2^K)_0}(x)| |\text{Stab}_{\mathcal{V}}(Y_x)|.$$

By (15.2), the stabilizer $\text{Stab}_{(\mathbb{F}_2^K)_0}(x)$ is the following subspace of $(\mathbb{F}_2^K)_0$:

$$(15.7) \quad (\mathbb{F}_2^{K \setminus Y_x})_0 := \{\delta \in (\mathbb{F}_2^K)_0 \mid \forall k \in Y_x : \delta_k = 0\}.$$

If $Y_x = K$ then $(\mathbb{F}_2^{K \setminus Y_x})_0 = \{0\}$ and (15.6) gives:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq |\text{Stab}_{\mathcal{V}}(Y_x)| = |\mathcal{V}| = 2^r.$$

One can check that $2^r \leq 2^{n-2}$ using $n \geq 3$ (recall that $n = 2^r m$). Therefore (15.5) holds if $Y_x = K$. Assume $Y_x \neq K$. We have:

$$|(\mathbb{F}_2^{K \setminus Y_x})_0| = |\mathbb{F}_2^{|K \setminus Y_x| - 1}| = 2^{n - |Y_x| - 1}.$$

Since $x \neq 0$, Y_x is non-empty. Applying Lemma 15.3 gives:

$$|\text{Stab}_{\mathcal{V}}(Y_x)| \leq |Y_x|.$$

Plugging this inequality into (15.6) we get:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq 2^{n - |Y_x| - 1} |Y_x| \leq 2^{n - |Y_x| - 1} 2^{|Y_x| - 1} = 2^{n-2}.$$

Here the last inequality follows from the inequality $n \leq 2^{n-1}$ which holds for all natural numbers n . This shows (15.5) holds and finishes the proof. \square

Remark 15.4. Assume $r \geq 1$ and let v_1, \dots, v_r be a basis for \mathcal{V} . One can check that the following is a $W(\varepsilon)$ -invariant generating set of size $(r + m - 1)2^{r+1}$:

$$\Gamma = \bigsqcup_{i=1, \dots, r} W(\varepsilon)(e_{0,1} + e_{v_i,1}) \cup \bigsqcup_{j=2, \dots, m} W(\varepsilon)(e_{0,1} + e_{v_1,j}).$$

Therefore (15.1) is in fact an equality if $r \geq 1$.

16. ESSENTIAL DIMENSION OF HSpin_{16} AT 2

In this section we prove Part (1) of Theorem 1.5. We have already proven

$$\text{ed}(E_8; 2) = \text{ed}(\text{HSpin}_{16}; 2)$$

in Proposition 11.2(1). It remains to show $\text{ed}(\text{HSpin}_{16}; 2) \geq 56$. By Proposition 12.7 it suffices to prove:

Proposition 16.1. *Let $T \subset \text{HSpin}_{16}$ be a split maximal torus with corresponding Weyl group W . There exists a grading $\varepsilon : X(T) \rightarrow \mathbb{F}_2^4$ satisfying (12.2) and (12.7) such that*

$$(16.1) \quad \text{Rank}(W(\varepsilon), X(T); 2) \geq 2^6.$$

Definition of ε . We start by introducing the notation needed to define ε . We set:

$$\mathcal{V} = \mathbb{F}_2^3 \text{ and } \nu = \frac{1}{2} \sum_{v \in \mathcal{V}} e_v \in \mathbb{Q}^{\mathcal{V}}.$$

There exists a split maximal torus $T \subset \text{HSpin}_{16}$ such that $X(T)$ is the following sublattice of $\mathbb{Q}^{\mathcal{V}}$ [39, Section 3]:

$$(16.2) \quad X(T) = Q(D_8) \cup (\nu + Q(D_8))$$

$$(16.3) \quad = \left\{ x \in \mathbb{Q}^{\mathcal{V}} \mid \begin{array}{l} \sum_{v \in \mathcal{V}} x_v \text{ is even.} \\ \text{For all } v \in \mathcal{V}, x_v \in \mathbb{Z} \text{ or} \\ \text{for all } v \in \mathcal{V}, x_v \in 1/2 + \mathbb{Z} \end{array} \right\}$$

Here, as in the previous section, $Q(D_8) \subset \mathbb{Z}^{\mathcal{V}}$ is the sublattice:

$$Q(D_8) = \{x \in \mathbb{Z}^{\mathcal{V}} \mid \sum_{v \in \mathcal{V}} x_v \text{ is even}\}.$$

Let $\varepsilon_0 : Q(D_8) \rightarrow \mathcal{V}$ be the grading from the previous section given by:

$$\varepsilon_0(y) = \sum_{v \in \mathcal{V}} y_v v.$$

Any vector $x \in X(T)$ can be written uniquely as $x = d\nu + y$, where $d \in \{0, 1\}$ and $y \in Q(D_8)$. We define $\varepsilon : X(T) \rightarrow \mathcal{V} \oplus \mathbb{F}_2$ by

$$(16.4) \quad \varepsilon(x) = \varepsilon(d\nu + y) = (\varepsilon_0(y), d) \in \mathcal{V} \oplus \mathbb{F}_2.$$

One can check that ε is a surjective homomorphism.

Verification of Condition (12.2). Any root $\alpha \in X(T)$ lies in $Q(D_8) \subset X(T)$ because it is a root of PGO_{16}^+ . We have $\varepsilon_0(\alpha) \neq 0$ because we have shown ε_0 satisfies (12.2) in the previous section. Therefore:

$$\varepsilon(\alpha) = (\varepsilon_0(\alpha), 0) \neq 0.$$

Description of the Weyl group. As in the previous section, the Weyl group of HSpin_{16} is the group

$$W = \text{Sym}(\mathcal{V}) \ltimes (\mathbb{F}_2^{\mathcal{V}})_0.$$

The action of $(\sigma, \delta) \in \text{Sym}(\mathcal{V}) \ltimes (\mathbb{F}_2^{\mathcal{V}})_0$ on $X(T)$ is the restriction of the action on $\mathbb{Q}^{\mathcal{V}}$ defined by:

$$(16.5) \quad (\sigma, \delta)e_v = (-1)^{\delta_v} e_{\sigma(v)}$$

for all $v \in \mathcal{V}$. Since $\varepsilon_0 : Q(D_8) \rightarrow \mathcal{V}$ is the grading we considered in the previous section, Lemma 15.2 implies that the subgroup of W preserving ε_0 is the 2-group:

$$W(\varepsilon_0) = \mathcal{V} \ltimes (\mathbb{F}_2^{\mathcal{V}})_0.$$

Note that $W(\varepsilon) \subset W(\varepsilon_0)$ by the definition of ε . Since ε is a homomorphism, an automorphism $w \in W(\varepsilon_0)$ is in $W(\varepsilon)$ if and only if it satisfies:

$$\varepsilon(w\nu) = \varepsilon(\nu) = (0, 1).$$

Let $w = (u, \delta) \in W(\varepsilon_0)$. We compute:

$$\varepsilon(w\nu) = \varepsilon(\nu - \sum_{\delta_v=1} e_v) = \left(\sum_{\delta_v=1} v, 1 \right).$$

Therefore $w \in W(\varepsilon)$ if and only if $\delta \in (\mathbb{F}_2^\mathcal{V})_1$, where:

$$(\mathbb{F}_2^\mathcal{V})_1 = \{\delta \in (\mathbb{F}_2^\mathcal{V})_0 \mid \sum_{\delta_v=1} v = 0\}.$$

Clearly, $(\mathbb{F}_2^\mathcal{V})_1$ is the kernel of the surjective homomorphism:

$$(\mathbb{F}_2^\mathcal{V})_0 \rightarrow \mathcal{V}, \quad \delta \mapsto \sum_{v \in \mathcal{V}} \delta_v v.$$

Therefore we have $\dim_{\mathbb{F}_2}(\mathbb{F}_2^\mathcal{V})_1 = \dim_{\mathbb{F}_2}(\mathbb{F}_2^\mathcal{V})_0 - \dim_{\mathbb{F}_2} \mathcal{V} = 7 - 3 = 4$ and:

$$|W(\varepsilon)| = |\mathcal{V} \times (\mathbb{F}_2^\mathcal{V})_1| = 2^{3+4} = 2^7.$$

Verification of Condition (12.7). The element $(1, \dots, 1) \in (\mathbb{F}_2^A)_1 \subset W(\varepsilon)$ acts on $X(T)$ by multiplication with -1 . Therefore $X(T)^{W(\varepsilon)} = \{0\}$ and (12.7) is satisfied. We note that this implies a fact we used earlier in the proof of Proposition 11.2(1).

Lemma 16.2. *There exists an anisotropic HSpin_{16} -torsor over some 2-closed field k containing k_0 .*

Proof. Let $A_\varepsilon \subset \mathrm{HSpin}_{16}$ be the finite split-diagonalizable 2-subgroup defined by ε as in (12.1). By Lemma 12.6, $C_G(A_\varepsilon)$ admits anisotropic torsors over some 2-closed field $k_0 \subset k$ because ε satisfies (12.7). Therefore the result follows from Corollary 7.6. \square

Proof of the lower bound on $\mathrm{Rank}(W(\varepsilon), X(T); 2)$.

Proof of (16.1). By Lemma 13.1, in order to prove (16.1) it suffices to prove that for any $x \in X(T)$ with $\varepsilon(x) = (v, d) \neq 0$, we have

$$|W(\varepsilon)x| \geq 2^4.$$

We handle the cases $d \neq 0$ and $v \neq 0$ separately.

Assume $d \neq 0$. Then (16.4) implies

$$x = \nu + y$$

for some $y \in Q(D_8)$. In particular, the coordinates $x_v = \frac{1}{2} + y_v$ are all half-integers and therefore non-zero. We conclude that $(\mathbb{F}_2^\mathcal{V})_1$ acts freely on x by sign changes and we have:

$$|W(\varepsilon)x| \geq |(\mathbb{F}_2^\mathcal{V})_1 x| = 2^4.$$

Assume $v \neq 0$. If $d \neq 0$, then $|W(\varepsilon)x| \geq 2^4$ by the previous case. Therefore we can assume $d = 0$, which implies $x \in Q(D_8)$ by (16.4). We set:

$$Y_x = \{v \in \mathcal{V} \mid x_v \neq 0\}.$$

The stabilizer-orbit formula gives:

$$|W(\varepsilon)x| = |W(\varepsilon)| / |\mathrm{Stab}_{W(\varepsilon)}(x)| = 2^7 / |\mathrm{Stab}_{W(\varepsilon)}(x)|.$$

Therefore to show $|W(\varepsilon)x| \geq 2^4$ it suffices to prove:

$$(16.6) \quad |\mathrm{Stab}_{W(\varepsilon)}(x)| \leq 2^3.$$

A similar argument to the proof of (15.6) shows:

$$(16.7) \quad |\mathrm{Stab}_{W(\varepsilon)}(x)| \leq |\mathrm{Stab}_{(\mathbb{F}_2^\mathcal{V})_1}(x)| |\mathrm{Stab}_{\mathcal{V}}(Y_x)|.$$

By (16.5), the stabilizer $\text{Stab}_{(\mathbb{F}_2^{\mathcal{V}})_1}(x)$ is the following subspace of $(\mathbb{F}_2^{\mathcal{V}})_1$:

$$(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 = \left\{ \delta \in (\mathbb{F}_2^{\mathcal{V}})_1 \mid \forall v \in Y_x : \delta_v = 0 \right\}.$$

The set $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1$ can also be described as the set of all relations satisfied by an even number of elements of $\mathcal{V} \setminus Y_x$ in \mathcal{V} . This gives the explicit formula:

$$(16.8) \quad |(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1| = \left| \left\{ \mathcal{C} \subset \mathcal{V} \setminus Y_x \mid \begin{array}{l} |\mathcal{C}| \text{ is even} \\ \sum_{c \in \mathcal{C}} c = 0 \end{array} \right\} \right|$$

We divide further into three cases based on $|\mathcal{V} \setminus Y_x|$. If $|\mathcal{V} \setminus Y_x| \leq 3$, then $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 = \{0\}$. Indeed, if $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 \neq \{0\}$, then (16.8) implies there exists a pair $\{u, v\} \subset \mathcal{V} \setminus Y_x$ such that

$$u + v = 0.$$

This is impossible because in characteristic two, the above equation implies $u = v$. Plugging $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 = \{0\}$ into (16.7) gives:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq |\text{Stab}_{\mathcal{V}}(Y_x)| \leq |\mathcal{V}| = 2^3.$$

Therefore (16.6) holds if $|\mathcal{V} \setminus Y_x| \leq 3$.

If $|\mathcal{V} \setminus Y_x| = 4$, then by the previous case, the only non-empty subset of $\mathcal{V} \setminus Y_x$ that might show up in the set (16.8) is $\mathcal{V} \setminus Y_x$ itself. Therefore (16.8) gives:

$$|(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1| \leq 2.$$

Together with (16.7) this implies:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq |(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1| |\text{Stab}_{\mathcal{V}}(Y_x)| \leq 2 |\text{Stab}_{\mathcal{V}}(Y_x)|.$$

Apply Lemma 15.3 to get:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq 2 |\text{Stab}_{\mathcal{V}}(Y_x)| \leq 2 |Y_x|.$$

This implies (16.6) because $|Y_x| = |\mathcal{V}| - |\mathcal{V} \setminus Y_x| = 4$.

If $|\mathcal{V} \setminus Y_x| \geq 5$, then $\mathcal{V} \setminus Y_x$ contains an \mathbb{F}_2 -basis v_1, v_2, v_3 for \mathcal{V} and two other vectors. Therefore $\mathcal{V} \setminus Y_x$ contains either 0 or $v_i + v_j$ for some $i \neq j$. Either way, the homomorphism

$$f : (\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_0 \rightarrow \mathcal{V}, \delta \mapsto \sum_{v \in \mathcal{V}} \delta_v v$$

is easily seen to be surjective in this case (here $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_0$ is defined as in (15.7)). Since $(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 = \ker f$, this gives:

$$\dim(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1 = \dim(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_0 - \dim \mathcal{V} = |\mathcal{V} \setminus Y_x| - 4.$$

In particular, we have

$$|(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1| = 2^{|\mathcal{V} \setminus Y_x| - 4}.$$

Plugging this into (16.7) and applying Lemma 15.3 we get:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq |(\mathbb{F}_2^{\mathcal{V} \setminus Y_x})_1| |\text{Stab}_{\mathcal{V}}(Y_x)| \leq 2^{|\mathcal{V} \setminus Y_x| - 4} |Y_x|.$$

Note that we can apply Lemma 15.3 because $x \neq 0$ implies $Y_x \neq \emptyset$. Using the inequality $|Y_x| \leq 2^{|Y_x|-1}$ we find:

$$|\text{Stab}_{W(\varepsilon)}(x)| \leq 2^{|\mathcal{V} \setminus Y_x| - 4} 2^{|Y_x| - 1} = 2^{|\mathcal{V}| - 5} = 2^3.$$

Therefore (16.6) holds in all cases and (16.1) follows. □

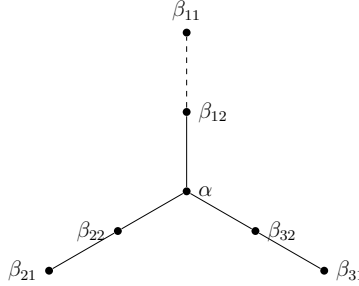
17. ESSENTIAL DIMENSION OF E_6 AT 3

In this section we prove Part (3) of Theorem 1.5. By Proposition 12.7 it suffices to prove:

Proposition 17.1. *Let $T \subset E_6^{ad}$ be a split maximal torus with corresponding Weyl group W . There exists a grading $\varepsilon : X(T) \rightarrow \mathbb{F}_3^2$ satisfying (12.2) and (12.7) such that*

$$(17.1) \quad \text{Rank}(W(\varepsilon), X(T); 3) \geq 12.$$

Definition of ε . We start by introducing the notation needed in order to define ε . We identify the character lattice $X(T)$ of a split maximal torus $T \subset E_6^{ad}$ with the root lattice $Q(E_6)$. We will use the following labeling of the extended Dynkin diagram of E_6 :



Here $-\beta_{11}$ is the highest root given by

$$-\beta_{11} = 3\alpha + \beta_{12} + 2\beta_{21} + \beta_{22} + 2\beta_{31} + \beta_{32}.$$

Since the simple roots form a basis for $Q(E_6)$, any element in $x \in Q(E_6)$ can be written uniquely as

$$(17.2) \quad x = d\alpha + \sum_{i,j} a_{ij}\beta_{ij},$$

for some $d \in \{0, 1, 2\}$ and integers $a_{ij} \in \mathbb{Z}$. We define a grading $\varepsilon : Q(E_6) \rightarrow \mathbb{F}_3^2$ by:

$$\varepsilon(x) = \varepsilon(d\alpha + \sum_{i,j} a_{ij}\beta_{ij}) = \left(\sum_{i,j} a_{ij}, d \right).$$

Note that ε is surjective.

Verification of Condition (12.2). Let $\beta \in \Phi$ be a root and express it as above:

$$\beta = d\alpha + \sum_{i,j} a_{ij}\beta_{ij},$$

for some $d \in \{0, 1, 2\}$ and integers $a_{ij} \in \mathbb{Z}$. If $d \neq 0$, then clearly

$$\varepsilon(\beta) = \left(\sum_{i,j} a_{ij}, d \right) \neq 0.$$

If $d = 0$, then β lies in the closed subsystem $A_2^{\times 3} \subset E_6$ generated by the β_{ij} 's; See [12, Section 4]. Therefore there exists $1 \leq i \leq 3$ such that:

$$\pm\beta \in \{\beta_{i1}, \beta_{i2}, \beta_{i1} + \beta_{i2}\}_{i=1,2,3}.$$

This implies $\varepsilon(\beta) \neq 0$ and so ε satisfies (12.2).

Description of the Weyl group. For any root $\beta \in \Phi \subset Q(E_6)$, let $r_\beta \in W$ be the corresponding reflection. Set $\sigma_i = r_{\beta_{i1}} r_{\beta_{i2}}$ for $1 \leq i \leq 3$ and define:

$$\sigma := \sigma_1 \sigma_2 \sigma_3.$$

By [12, Section 4], there exists an element $\tau \in W$ whose action on the set of roots Φ is determined by a $2\pi/3$ -rotation of the extended Dynkin diagram above. Explicitly, we have $\tau(\alpha) = \alpha$ and for any $1 \leq i \leq 3, 1 \leq j \leq 2$ we have:

$$(17.3) \quad \tau(\beta_{ij}) = \beta_{i+1,j},$$

where we set $\beta_{4,j} := \beta_{1,j}$.

Lemma 17.2. *The elements σ, τ generate a subgroup $\langle \sigma, \tau \rangle \subset W(\varepsilon)$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.*

Proof. Since τ fixes α and satisfies (17.3), τ commutes with σ and preserves ε . To check that σ preserves ε , one first computes:

$$\sigma_i(\alpha) = r_{\beta_{i1}} r_{\beta_{i2}}(\alpha) = r_{\beta_{i1}}(\alpha + \beta_{i2}) = \alpha + \beta_{i1} + \beta_{i2}.$$

Since $\sigma_i(\beta_{jk}) = \beta_{jk}$ for any $j \neq i$, this gives:

$$(17.4) \quad \sigma(\alpha) = \sigma_1 \sigma_2 \sigma_3(\alpha) = \alpha + \sum_{i,j} \beta_{ij}.$$

Here we are summing over all $1 \leq i \leq 3, 1 \leq j \leq 2$. Therefore in \mathbb{F}_3^2 :

$$(17.5) \quad \varepsilon(\sigma(\alpha)) = (6, 1) = (0, 1) = \varepsilon(\alpha).$$

Any vector $x \in Q(E_6)$ is of the form:

$$x = d\alpha + \sum_{i,j} a_{ij} \beta_{ij},$$

as in (17.2). By (17.5), to prove $\varepsilon(\sigma(x)) = \varepsilon(x)$ it suffices to check:

$$(17.6) \quad \varepsilon(\sigma(\sum_{i,j} a_{ij} \beta_{ij})) = \varepsilon(\sum_{i,j} a_{ij} \beta_{ij}) = (\sum_{i,j} a_{ij}, 0).$$

A computation shows for all $1 \leq i \leq 3$:

$$(17.7) \quad \sigma(\beta_{i1}) = \beta_{i2}, \quad \sigma(\beta_{i2}) = -\beta_{i1} - \beta_{i2}, \quad \sigma(-\beta_{i1} - \beta_{i2}) = \beta_{i1}.$$

Therefore we have:

$$\begin{aligned} \sigma(\sum_{i,j} a_{ij} \beta_{ij}) &= \sum_{i,j} a_{ij} \sigma(\beta_{ij}) \\ &= \sum_i -a_{i2} \beta_{i1} + (a_{i1} - a_{i2}) \beta_{i2}. \end{aligned}$$

Now (17.6) follows from the following computation in \mathbb{F}_3^2 :

$$\varepsilon(\sigma(\sum_{i,j} a_{ij}\beta_{ij})) = (\sum_i a_{i1} - 2a_{i2}, 0) = (\sum_i a_{i1} + a_{i2}, 0) = (\sum_{i,j} a_{ij}, 0)$$

□

Remark 17.3. One can show that $\langle \sigma, \tau \rangle$ is a Sylow 3-subgroup of $W(\varepsilon)$. We do not include this computation because it is not required for the proof of Theorem 1.5.

Verification of Condition (12.7). Using (17.7) one sees that for any $1 \leq i \leq 3, 1 \leq j \leq 2$:

$$\beta_{ij} + \sigma(\beta_{ij}) + \sigma^2(\beta_{ij}) = 0.$$

Since the β_{ij} 's generate a subgroup of finite index in $Q(E_6)$, it follows that:

$$(17.8) \quad \text{id}_{Q(E_6)} + \sigma + \sigma^2 = 0.$$

If $\sigma(x) = x$ for some $x \in Q(E_6)$, then (17.8) implies $x = 0$ because

$$0 = (\text{id}_{Q(E_6)} + \sigma + \sigma^2)x = 3x.$$

Therefore $Q(E_6)^{W(\varepsilon)_3} = \{0\}$ for any Sylow 3-subgroup $W(\varepsilon)_3 \subset W(\varepsilon)$ containing σ .

Proof of the lower bound on $\text{Rank}(W(\varepsilon), X(T); 3)$. We start with two lemmas that help us understand the $\langle \sigma, \tau \rangle$ -stabilizers of elements of $Q(E_6)$.

Lemma 17.4. *Let $x \in Q(E_6)$ be an element and set $\varepsilon(x) = (u, v) \in \mathbb{F}_3^2$.*

- (1) *If $\sigma(x) = x$ then $x = 0$.*
- (2) *If $\sigma(x) \in \{x, \tau(x), \tau^2(x)\}$ then $u = 0$.*
- (3) *If $\tau(x) = x$ then $u = 0$.*

Proof. We proved Part (1) during the verification of Condition (12.7) above. To prove Part (2), we can assume $\sigma(x) = \tau^k(x)$ for $k = 1$ or $k = 2$ by Part (1). Apply (17.8) to get:

$$(17.9) \quad 0 = (\text{id}_{Q(E_6)} + \sigma + \sigma^2)x = x + \tau^k(x) + \tau^{2k}(x) = x + \tau(x) + \tau^2(x).$$

Here in the last equality we used that $\tau^3 = 1$. Express x as a sum

$$x = d\alpha + \sum_{i,j} a_{ij}\beta_{ij}.$$

for some $d \in \{0, 1, 2\}$ and integers $a_{ij} \in \mathbb{Z}$. Comparing coefficients in (17.9) we see that for all $1 \leq j \leq 2$:

$$a_{1j} + a_{2j} + a_{3j} = 0.$$

Therefore:

$$(u, v) = \varepsilon(x) = (\sum_j a_{1j} + a_{2j} + a_{3j}, d) = (0, d).$$

To prove Part (3), we assume:

$$\tau(x) = x.$$

Comparing coefficients of both sides we get for all $1 \leq j \leq 2$:

$$a_{1j} = a_{2j} = a_{3j}.$$

This gives:

$$(u, v) = \varepsilon(x) = \left(\sum_j a_{1j} + a_{2j} + a_{3j}, d \right) = \left(\sum_j 3a_{1j}, d \right) = (0, d).$$

□

Lemma 17.5. *Let X be a set equipped with an $\mathcal{V} \times C$ -action for some abelian group \mathcal{V} and cyclic group $C = \langle \sigma \rangle$ of prime order p . If $\text{Stab}_{\mathcal{V}}(x) = \{0\}$ and $\sigma.x \notin \mathcal{V}x$ for some $x \in X$, then $\text{Stab}_{\mathcal{V} \times C}(x) = \{0\}$.*

Proof. Let $(v, \sigma^n) \in \mathcal{V} \times C$ be such that

$$v\sigma^n(x) = x.$$

It suffices to prove n is divisible by p because then $\sigma^n = 1$ and $v \in \text{Stab}_{\mathcal{V}}(x) = \{0\}$. If n is prime to p , then there exists m prime to p such that $mn = 1$ modulo p . By associativity of the action of $\mathcal{V} \times C$ on X :

$$\sigma(x) = \sigma^{mn}(x) = v^{-m}(x) \in \mathcal{V}x$$

This contradicts our assumption $\sigma(x) \notin \mathcal{V}x$, and so n is divisible by p . □

Corollary 17.6. *Let $x \in Q(E_6)$ be an element and denote $\varepsilon(x) = (u, v)$ for some $u, v \in \mathbb{F}_3$. We note:*

- (1) *If $u \neq 0$, then $|\langle \sigma, \tau \rangle x| = 9$.*
- (2) *If $v \neq 0$, then $|\langle \sigma, \tau \rangle x| \geq 3$.*

Proof. If $u \neq 0$, then $\tau(x) \neq x$ and $\sigma(x) \notin \{x, \tau(x), \tau^2(x)\}$ by Lemma 17.4. Applying Lemma 17.5 gives $\text{Stab}_{\langle \sigma, \tau \rangle}(x) = \{1\}$. Therefore $\langle \sigma, \tau \rangle$ acts freely on x and we have:

$$|\langle \sigma, \tau \rangle x| = 9.$$

If $v \neq 0$, then $\sigma(x) \neq x$ by Lemma 17.4(1). Therefore

$$|\langle \sigma, \tau \rangle x| \geq |\langle \sigma \rangle x| = 3.$$

□

Combining Corollary 17.6 and Lemma 13.1 gives

$$\text{Rank}(\langle \sigma, \tau \rangle, X(T); 3) \geq 9 + 3 = 12.$$

This proves (17.1) because by (13.1) we have

$$\text{Rank}(W(\varepsilon), X(T); 3) \geq \text{Rank}(\langle \sigma, \tau \rangle, X(T); 3).$$

REFERENCES

- [1] Sanghoon Baek. Essential dimension of projective orthogonal and symplectic groups of small degree. *Communications in Algebra*, 43(2):693–701, 2015.
- [2] Sanghoon Baek and Alexander S Merkurjev. Essential dimension of central simple algebras. *Acta mathematica*, 209:1–27, 2012.
- [3] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [4] Armand Borel and Jean De Siebenthal. Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Commentarii Mathematici Helvetici*, 23(1):200–221, 1949.
- [5] Armand Borel and Jacques Tits. Groupes réductifs. *Publications Mathématiques de l’IHÉS*, 27:55–151, 1965.

- [6] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension and algebraic stacks. *arXiv preprint math/0701903*, 2007.
- [7] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension, spinor groups, and quadratic forms. *Ann. of Math. (2)*, 171(1):533–544, 2010.
- [8] Patrick Brosnan, Zinovy Reichstein, and Angelo Vistoli. Essential dimension of moduli of curves and other algebraic stacks. With an appendix by N. Fakhruddin. *J. European Math. Society*, 13(4):1079–1112, 2011.
- [9] Joe Buhler and Zinovy Reichstein. On the essential dimension of a finite group. *Compositio Mathematica*, 106(2):159–179, 1997.
- [10] Joe Buhler and Zinovy Reichstein. On Tschirnhaus transformations. In *Topics in number theory (University Park, PA, 1997)*, volume 467 of *Math. Appl.*, pages 127–142. Kluwer Acad. Publ., Dordrecht, 1999.
- [11] Shane Cernele and Zinovy Reichstein. Essential dimension and error-correcting codes. *Pacific J. Math.*, 279(1-2):155–179, 2015. With an appendix by Athena Nguyen.
- [12] Vladimir Chernousov. Another proof of Totaro’s theorem on E_8 -torsors. *Canadian Mathematical Bulletin*, 49(2):196–202, 2006.
- [13] Vladimir Chernousov and Alexander Merkurjev. Essential-dimension of split simple groups of type A. *Mathematische Annalen*, 357(1):1–10, 2013.
- [14] Vladimir Chernousov and Jean-Pierre Serre. Lower bounds for essential dimensions via orthogonal representations. *Journal of Algebra*, 305(2):1055–1070, 2006.
- [15] Arjeh M Cohen and RL Griess. On finite simple subgroups of the complex Lie group of type e8. In *Proc. Symp. Pure Math*, volume 47, pages 367–405, 1987.
- [16] John Horton Conway and Neil James Alexander Sloane. *Sphere packings, lattices and groups*, volume 290. Springer Science & Business Media, 2013.
- [17] Haibao Duan and Shali Liu. The isomorphism type of the centralizer of an element in a Lie group. *Journal of algebra*, 376:25–45, 2013.
- [18] Ido Efrat. *Valuations, orderings, and Milnor K-theory*, volume 124 of *Mathematical Surveys and Monographs*. American Mathematical Soc., 2006.
- [19] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
- [20] William Fulton. *Intersection theory*, volume 2. Springer Science & Business Media, 2013.
- [21] Skip Garibaldi. E_8 , the most exceptional group. *Bulletin of the American Mathematical Society*, 53(4):643–671, 2016.
- [22] Raymond Garver. The Tschirnhaus transformation. *Annals of Mathematics*, 29(1/4):319–333, 1927.
- [23] Philippe Gille. Sur la classification des schémas en groupes semi-simples. *Panoramas et synthèses*, 47:39–110, 2016.
- [24] Philippe Gille. Loop torsors and Abhyankar’s lemma. *arXiv preprint arXiv:2406.16435*, 2024.
- [25] Philippe Gille and Arturo Pianzola. *Torsors, Reductive Group Schemes and Extended Affine Lie Algebras*. Number 1063 in *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 2013. Volume 226.
- [26] Philippe Gille and Patrick Polo, editors. *Schémas en groupes (SGA 3)*, volume 7 of *Documents Mathématiques (Paris)*. Société Mathématique de France, Paris, 2011. Séminaire de Géométrie Algébrique du Bois Marie 1962–64. A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre, Revised and annotated edition of the 1970 French original.
- [27] Philippe Gille and Zinovy Reichstein. A lower bound on the essential dimension of a connected linear group. *Commentarii Mathematici Helvetici*, 84(1):189–212, 2009.
- [28] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.
- [29] Philippe Gille, Chernousov Vladimir, and Zinovy Reichstein. Reduction of structure for torsors over semilocal rings. *Manuscripta Math.*, 126(4):465–480, 2008.

- [30] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Springer, 2010.
- [31] James E. Humphreys. *Conjugacy classes in semisimple algebraic groups*, volume 43 of *Mathematical Surveys and Monographs*. American Mathematical Soc., 1995.
- [32] Nikita A. Karpenko and Alexander S. Merkurjev. Essential dimension of finite p -groups. *Invent. Math.*, 172(3):491–508, 2008.
- [33] Franz-Viktor Kuhlmann, Matthias Pank, and Peter Roquette. Immediate and purely wild extensions of valued fields. *Manuscripta Mathematica*, 55(1):39–67, 1986.
- [34] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [35] Martin Lorenz, Zinovy Reichstein, Louis H Rowen, and David J Saltman. Fields of definition for division algebras. *Journal of the London Mathematical Society*, 68(3):651–670, 2003.
- [36] Roland Lötscher, Mark MacDonald, Aurel Meyer, and Zinovy Reichstein. Essential dimension of algebraic tori. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(677):1–13, 2013.
- [37] Roland Lötscher, Mark MacDonald, Aurel Meyer, and Zinovy Reichstein. Essential p -dimension of algebraic groups whose connected component is a torus. *Algebra Number Theory*, 7(8):1817–1840, 2013.
- [38] Giancarlo Lucchini Arteche. Groupe de Brauer non ramifié algébrique des espaces homogènes. *Transformation Groups*, 20:463–493, 2015.
- [39] Mark L. MacDonald. Essential p -dimension of the normalizer of a maximal torus. *Transform. Groups*, 16(4):1143–1171, 2011.
- [40] Benedictus Margaux. Passage to the limit in non-abelian Čech cohomology. *J. Lie Theory*, 17(3):591–596, 2007.
- [41] Alexander S. Merkurjev. Essential dimension. In *Quadratic forms—algebra, arithmetic, and geometry*, volume 493 of *Contemp. Math.*, pages 299–325. Amer. Math. Soc., Providence, RI, 2009.
- [42] Alexander S. Merkurjev. Essential p -dimension of $\mathrm{PGL}(p^2)$. *J. Amer. Math. Soc.*, 23(3):693–712, 2010.
- [43] Alexander S. Merkurjev. A lower bound on the essential dimension of simple algebras. *Algebra Number Theory*, 4(8):1055–1076, 2010.
- [44] Alexander S. Merkurjev. Essential dimension: a survey. *Transform. Groups*, 18(2):415–481, 2013.
- [45] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
- [46] James S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.
- [47] Jürgen Neukirch. Zur Verzweigungstheorie der allgemeinen Krullischen Bewertungen. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 32(3-4):207–215, 1968.
- [48] Danny Ofek. Reduction of structure to parabolic subgroups. *Documenta Mathematica*, 27:1421–1446, 2022.
- [49] Alexander Ostrowski. Untersuchungen zur arithmetischen Theorie der Körper: Die Theorie der Teilbarkeit in allgemeinen Körpern. Teil II und III. *Mathematische Zeitschrift*, 39(1):321–404, 1935.
- [50] Simon Pépin Lehalleur. Subgroups of maximal ranks of reductive groups. In *Autour des schémas en groupes, III*, volume 57 of *Panoramas et Synthèses*, pages 147–172. Société Mathématique de France, Paris, 2012.
- [51] Claudio Procesi. Non-commutative affine rings. *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8)*, 8:237–255, 1967.
- [52] Zinovy Reichstein. On the notion of essential dimension for algebraic groups. *Transformation Groups*, 5(3):265–304, 2000.
- [53] Zinovy Reichstein. Essential dimension. *Proceedings of the International Congress of Mathematicians (Vol. 2)*, 2010.
- [54] Zinovy Reichstein and Federico Scavia. The behavior of essential dimension under specialization. *Épjournal Géom. Algébrique*, 6:Art. 21, 28, 2022.
- [55] Zinovy Reichstein and Boris Youssin. Essential dimensions of algebraic groups and a resolution theorem for G -varieties. *Canad. J. Math.*, 52(5):1018–1056, 2000. With an appendix by János Kollár and Endre Szabó.

- [56] David J Saltman. *Lectures on division algebras*, volume 94 of *Regional Conference Series in Mathematics*. American Mathematical Soc., 1999.
- [57] J.-J. Sansuc. Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. *J. Reine Angew. Math.*, 327:12–80, 1981.
- [58] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.
- [59] Jean-Pierre Serre. Cohomological invariants, Witt invariants, and trace forms. In *Cohomological invariants in Galois cohomology*, volume 28 of *Univ. Lecture Ser.*, pages 1–100. Amer. Math. Soc., Providence, RI, 2003. Notes by Skip Garibaldi.
- [60] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2024.
- [61] Nguyễn Duy Tân. On the essential dimension of unipotent algebraic groups. *Journal of Pure and Applied Algebra*, 217(3):432–448, 2013.
- [62] Jean-Pierre Tignol and Adrian R Wadsworth. *Value functions on simple algebras, and associated graded rings*, volume 6. Springer, 2015.
- [63] Jacques Tits. Strongly inner anisotropic forms of simple algebraic groups. *J. Algebra*, 131(2):648–677, 1990.
- [64] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960.