

# PINCHING RIGIDITY OF MINIMAL SURFACES IN SPHERES

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ABSTRACT. In 1980, U. Simon proposed a quantization conjecture about the Gaussian curvature  $K$  of closed minimal surfaces in unit spheres: if  $K(s+1) \leq K \leq K(s)$  ( $K(s) := 2/(s(s+1))$ ,  $s \in \mathbb{N}$ ), then either  $K = K(s)$  or  $K = K(s+1)$ . Notice that the surface must be one of Calabi's standard minimal 2-spheres if the curvature is a positive constant. The cases  $s = 1$  and  $s = 2$  were proven in the 1980s by Simon and others. In this paper we give a pinching theorem of the Simon conjecture in the case  $s = 3$  and also give a new proof of the cases  $s = 1$  and  $s = 2$  by some Simons-type integral inequalities.

## 1. INTRODUCTION

In 1967, E. Calabi (cf. [3]) studied minimal immersions of  $\mathbb{S}^2$  with constant Gaussian curvature  $K$  into  $\mathbb{S}^N(1)$ . These immersions were classified up to a rigid motion with the curvature  $K$  corresponding to the following values,  $K = K(s) := \frac{2}{s(s+1)}$ ,  $s \in \mathbb{N}$  (cf. [9]). In 1980, U. Simon made the following quantization conjecture (cf. [13, 17, 25]).

**Conjecture 1** (Intrinsic version). *Let  $M$  be a closed surface minimally immersed into  $\mathbb{S}^N(1)$  such that the image is not contained in any hyperplane of  $\mathbb{R}^{N+1}$ . If  $K(s+1) \leq K \leq K(s)$  for an  $s \in \mathbb{N}$ , then either  $K = K(s+1)$  or  $K = K(s)$ , and thus the immersion is one of the Calabi's standard minimal immersions with the dimension of the ambient space  $N = 2s + 2$  or  $N = 2s$ .*

For minimal surfaces in  $\mathbb{S}^N(1)$ , the curvature  $K$  and the squared norm  $S = |h|^2$  of the second fundamental form  $h$  are related as follows:

$$2K = 2 - S.$$

It follows that, by setting

$$S(s) := \frac{2(s-1)(s+2)}{s(s+1)} = 2 - 2K(s),$$

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2010 *Mathematics Subject Classification.* 53C24, 53C42, 53C65.

*Key words and phrases.* minimal surfaces; rigidity theorem; the Simon conjecture.

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J. Q. Ge is partially supported by NSFC (No. 12171037) and the Fundamental Research Funds for the Central Universities.

F. G. Li is partially supported by NSFC (No. 12171037, 12271040) and China Postdoctoral Science Foundation (No. 2022M720261).

the Simon conjecture above can also be stated as:

**Conjecture 2** (Extrinsic version). *Let  $M$  be a closed surface minimally immersed into  $\mathbb{S}^N(1)$  such that the image is not contained in any hyperplane of  $\mathbb{R}^{N+1}$ . If  $S(s) \leq S \leq S(s+1)$  for an  $s \in \mathbb{N}$ , then either  $S = S(s)$  or  $S = S(s+1)$ , and thus the immersion is one of the Calabi's standard minimal immersions with the dimension of the ambient space  $N = 2s$  or  $N = 2s + 2$ .*

So far, the Simon conjecture has only been solved in the cases  $s = 1$  and  $s = 2$  (cf. [1, 13]). Remarkably, without the assumption of minimality of the immersion, Li and Simon gave a generalization of the Simon conjecture in the case  $s = 1$  (cf. [17]). In fact, the case  $s = 1$  has also been studied in the extrinsic version even for higher-dimensional minimal submanifolds by Simons-type integral inequalities (cf. [6, 14, 16, 18, 26], etc.). It shows the pinching rigidity and the first gap of the squared norm  $S$  of the second fundamental form, whereas the case  $s = 2$  shows the second gap as the well-known Peng-Terng-type second gap theorems for hypersurfaces in the unit sphere (cf. [5, 8, 15, 21, 31, 32], etc.). More generally, the discrete property of  $S$  of higher-dimensional minimal hypersurfaces in the unit sphere is linked to the famous Chern Conjecture (cf. [4, 27, 28, 29, 33], etc.). Please see the excellent and detailed surveys for more developments and references on this type of rigidity problems (cf. [10, 24]). To the best of our knowledge, the third gap of the Simon conjecture is far from being solved, although there are indeed many partial results for the case  $s \geq 3$  of the Simon conjecture under additional assumptions (cf. [2, 7, 12, 19, 20, 23], etc.).

In this paper, we make progress on the case  $s = 3$  of the Simon conjecture in its extrinsic version by establishing Simons-type integral inequalities. Specifically, we prove the following result:

**Theorem A.** *Let  $M$  be a closed surface minimally immersed into  $\mathbb{S}^N(1)$ .*

- (i) *If  $0 \leq S \leq \frac{4}{3}$ , then  $S = 0$  or  $S = \frac{4}{3}$ ;*
- (ii) *If  $\frac{4}{3} \leq S \leq \frac{5}{3}$ , then  $S = \frac{4}{3}$  or  $S = \frac{5}{3}$ ;*
- (iii) *If  $\frac{5}{3} \leq S \leq \frac{9}{5}$ ,  $S_{\max} = \sup_{p \in M} S(p)$  and  $S_{\min} = \inf_{p \in M} S(p)$ , then*

$$S_{\max} - S_{\min} \geq \frac{134 - 114S_{\min} + \sqrt{\mathcal{F}}}{108},$$

$$\text{where } \mathcal{F} = (134 - 114S_{\min})^2 + 864(3S_{\min} - 5)(9 - 5S_{\min}).$$

**Remark 1.** *By choosing a special frame field on  $M$ , Yang (cf. [30]) provided a proof of the cases  $s = 1, 2$  if  $M$  has a flat or nowhere flat normal bundle. We now show that the ideas there work without the assumption on the flatness of the normal bundle.*

**Remark 2.** *Okayasu (cf. [19]) proved a similar result to Theorem A by using a different method.*

By Theorem A (iii), one has

**Corollary B.** *Let  $M$  be a closed 2-dimensional Riemannian manifold. Then there exists no isometric minimal immersion  $\Phi : M \rightarrow \mathbb{S}^N$  for any  $N$  such that  $\frac{5}{3} \leq S_{\min} \leq \frac{9}{5}$  and*

$$S_{\max} < S_{\min} + \frac{134 - 114S_{\min} + \sqrt{\mathcal{F}}}{108},$$

where  $\mathcal{F} = (134 - 114S_{\min})^2 + 864(3S_{\min} - 5)(9 - 5S_{\min})$ .

**Remark 3.** *Let*

$$f(S_{\min}) = \frac{134 - 114S_{\min} + \sqrt{\mathcal{F}}}{108}, \quad \frac{5}{3} \leq S_{\min} \leq \frac{9}{5}.$$

*It should be pointed out that we cannot obtain any gaps if  $S_{\min} = \frac{5}{3}$  or  $\frac{9}{5}$ , and the gap exists when  $\frac{5}{3} < S_{\min} < \frac{9}{5}$ . Now,*

$$f'(S_{\min}) = \frac{1}{108} \left( \frac{6(3S_{\min} + 599)}{\sqrt{9S_{\min}^2 + 3594S_{\min} - 5231}} - 114 \right), \quad \frac{5}{3} \leq S_{\min} \leq \frac{9}{5}.$$

*Letting  $f'(S_{\min}) = 0$ , we obtain  $-45(S_{\min})^2 - 17970S_{\min} + 31211 = 0$ , from which we deduce that  $S_{\min} \approx 1.72935007 \in (\frac{5}{3}, \frac{9}{5})$ , at which  $f(S_{\min})$  attains its maximal value  $(f(S_{\min}))_{\max} \approx 0.00419291$ . Hence, there exists no isometric minimal immersion  $\Phi : M \rightarrow \mathbb{S}^N$  for any  $N$  such that  $1.72936 < S < 1.73355$ .*

In addition, we also obtain the following integral equations.

**Theorem C.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then we have some Simons-type identities as follows (see Theorems 5, 6 and 8):*

$$\begin{aligned} \int_M S(3S - 4) &= 2 \int_M \mathcal{B}_1 \geq 0, \\ \int_M S(3S - 4)(3S - 5) &= 2 \int_M [\mathcal{B}_2 - \frac{1}{4}S(3S - 4)^2 + \frac{1}{2}|\nabla S|^2] \geq 0, \\ \int_M S(3S - 4)(3S - 5)(5S - 9) &= 2 \int_M [\mathcal{B}_3 - \frac{1}{8}S(3S - 4)(45S^2 - 144S + 116) \\ &\quad + \frac{1}{8}(65S - 166)|\nabla S|^2 - \frac{5}{8}(\Delta S)^2], \end{aligned}$$

where  $\mathcal{B}_1 = |\nabla h|^2$ ,  $\mathcal{B}_2 = |\nabla^2 h|^2$  and  $\mathcal{B}_3 = |\nabla^3 h|^2$  are the squared lengths of the first, second and third covariant derivatives of  $h$ , respectively.

## 2. NOTATIONS AND LOCAL FORMULAS

Let  $M$  be a 2-dimensional manifold immersed in a unit sphere  $\mathbb{S}^N(1)$ . We assume the range of the indices as follows:

$$1 \leq i, j, k, \dots \leq 2; \quad 3 \leq \alpha, \beta, \gamma, \dots \leq N; \quad 1 \leq A, B, C, \dots \leq N.$$

Let  $\{e_1, \dots, e_N\}$  be a local orthonormal frame on  $T(\mathbb{S}^N(1))$  such that, when restricted to  $M$ ,  $\{e_1, e_2\}$  ( $\{e_3, \dots, e_N\}$ ) lie in the tangent bundle  $T(M)$  (normal bundle  $T^\perp(M)$ ). We take  $(\omega_A)$  and  $(\omega_{AB})$  as the metric 1-form field and connection form field associated with  $\{e_1, \dots, e_N\}$ . Let  $S_\alpha = (h_{ij}^\alpha)_{2 \times 2}$ , where  $\omega_{i\alpha} = h_{ij}^\alpha \omega_j$ . Then,  $h_{ij}^\alpha = h_{ji}^\alpha$ . In the following, we will use the Einstein summation convention. The second fundamental form of  $M$  is defined by  $h = h_{ij}^\alpha \omega_i \omega_j e_\alpha$ . The mean curvature normal vector field is defined by  $H = \frac{1}{2} h_{ii}^\alpha e_\alpha$ . If the mean curvature normal vector field of  $M$  vanishes identically, the immersion is called minimal. Now we consider minimal surfaces. Define column vectors  $a := (a^\alpha) \in \mathbb{R}^p, b := (b^\alpha) \in \mathbb{R}^p$ , where  $a^\alpha := h_{11}^\alpha = -h_{22}^\alpha, b^\alpha := h_{12}^\alpha = h_{21}^\alpha$  and  $p = N - 2$  is the codimension. We use the following notations:

$$A := (\langle S_\alpha, S_\beta \rangle) = 2aa^\top + 2bb^\top, \quad S := \text{tr}A = |h|^2, \quad \rho^\perp := \sum_{\alpha, \beta} |[S_\alpha, S_\beta]|^2.$$

The covariant derivatives  $h_{ijk}^\alpha, h_{ijkl}^\alpha, h_{ijklm}^\alpha$  and  $h_{ijklmn}^\alpha$  are defined as follows:

$$\begin{aligned} h_{ijk}^\alpha \omega_k &= dh_{ij}^\alpha + h_{mj}^\alpha \omega_{mi} + h_{im}^\alpha \omega_{mj} + h_{ij}^\beta \omega_{\beta\alpha}, \\ h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha + h_{mj}^\alpha \omega_{mi} + h_{im}^\alpha \omega_{mj} + h_{ijm}^\alpha \omega_{mk} + h_{ijk}^\beta \omega_{\beta\alpha}, \\ h_{ijklm}^\alpha \omega_m &= dh_{ijkl}^\alpha + h_{njkl}^\alpha \omega_{ni} + h_{inkl}^\alpha \omega_{nj} + h_{ijnl}^\alpha \omega_{nk} + h_{ijkn}^\alpha \omega_{nl} + h_{ijkl}^\beta \omega_{\beta\alpha}, \\ h_{ijklmn}^\alpha \omega_n &= dh_{ijklm}^\alpha + h_{pjklm}^\alpha \omega_{pi} + h_{ipklm}^\alpha \omega_{pj} + h_{ijplm}^\alpha \omega_{pk} + h_{ijkpm}^\alpha \omega_{pl} + h_{ijklp}^\alpha \omega_{pm} + h_{ijklm}^\beta \omega_{\beta\alpha}. \end{aligned}$$

For convenience, the following notations are defined by

$$a_i^\alpha := h_{11i}^\alpha, \quad a_i := (a_i^\alpha),$$

and

$$\mathcal{B}_1 := \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2, \quad \mathcal{B}_2 := \sum_{i,j,k,l,\alpha} (h_{ijkl}^\alpha)^2, \quad \mathcal{B}_3 := \sum_{i,j,k,l,m,\alpha} (h_{ijklm}^\alpha)^2.$$

The Codazzi equation and Ricci's formulas are

$$(2.1) \quad h_{ijk}^\alpha - h_{ikj}^\alpha = 0,$$

$$(2.2) \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = h_{pj}^\alpha R_{pikl} + h_{ip}^\alpha R_{pjkl} + h_{ij}^\beta R_{\beta\alpha kl},$$

$$(2.3) \quad h_{ijklm}^\alpha - h_{ijkml}^\alpha = h_{pj}^\alpha R_{pilm} + h_{ipk}^\alpha R_{pjlm} + h_{ijp}^\alpha R_{pklm} + h_{ijk}^\beta R_{\beta\alpha lm},$$

$$(2.4) \quad h_{ijklmn}^\alpha - h_{ijklnm}^\alpha = h_{pjkl}^\alpha R_{pimn} + h_{ipkl}^\alpha R_{pjmn} + h_{ijpl}^\alpha R_{pkmn}$$

$$(2.5) \quad + h_{ijkp}^\alpha R_{plmn} + h_{ijkl}^\beta R_{\beta\alpha mn}.$$

The Laplacians  $\Delta h_{ij}^\alpha$ ,  $\Delta h_{ijk}^\alpha$  and  $\Delta h_{ijkl}^\alpha$  are defined by

$$\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha, \quad \Delta h_{ijk}^\alpha = \sum_l h_{ijkl}^\alpha, \quad \Delta h_{ijkl}^\alpha = \sum_m h_{ijklmm}^\alpha.$$

From (2.1), (2.2) and (2.3), we obtain

$$(2.6) \quad \Delta h_{ij}^\alpha = h_{mmij}^\alpha + h_{pi}^\alpha R_{pmjm} + h_{mp}^\alpha R_{pijm} + h_{mi}^\delta R_{\delta\alpha jm},$$

and

$$(2.7) \quad \begin{aligned} \Delta h_{ijk}^\alpha &= (\Delta h_{ij}^\alpha)_k + 2h_{pjm}^\alpha R_{pikm} + 2h_{ipm}^\alpha R_{pjkm} + h_{ijp}^\alpha R_{pmkm} + 2h_{ijm}^\delta R_{\delta\alpha km} \\ &\quad + h_{pj}^\alpha R_{pikmm} + h_{ip}^\alpha R_{pjkm} + h_{ij}^\delta R_{\delta\alpha kmm}. \end{aligned}$$

The Riemannian curvature tensor, the normal curvature tensor and the first covariant differentials of the normal curvature tensor are given by

$$(2.8) \quad R_{ijkl} = \frac{1}{2}(2 - S)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

$$(2.9) \quad R_{\alpha\beta kl} = h_{km}^\alpha h_{ml}^\beta - h_{km}^\beta h_{ml}^\alpha,$$

$$(2.10) \quad R_{\alpha\beta 12k} = 2(b^\beta a_k^\alpha + a^\alpha h_{12k}^\beta - b^\alpha a_k^\beta - a^\beta h_{12k}^\alpha).$$

From now on, we assume that the 2-dimensional manifold  $M$  is minimally immersed in  $\mathbb{S}^N(1)$ . The Simons identity for minimal submanifolds in a unit sphere is

$$(2.11) \quad \frac{1}{2}\Delta S = \mathcal{B}_1 + 2S - |A|^2 - \rho^\perp.$$

We now establish an improved version of Proposition 2.4 of Yang (cf. [30]). The proof follows the idea there, however, without the assumption on flatness of the normal bundle of  $M$ .

**Theorem 1.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. We have*

$$(2.12) \quad \frac{1}{2}\Delta S = \mathcal{B}_1 - \frac{1}{2}S(3S - 4).$$

*Proof.* Let  $\{e_1, \dots, e_N\}$  be a local orthonormal frame field on  $M$  as before, and we have

$$(2.13) \quad \begin{aligned} (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k &= \sum_{i,j,\alpha} (\Delta h_{ij}^\alpha)^2 + h_{ijk}^\alpha h_{pi}^\alpha R_{pljl} + h_{ijk}^\alpha h_{lpk}^\alpha R_{pijl} + h_{ijk}^\alpha h_{pi}^\alpha R_{pljlk} \\ &\quad + h_{ijk}^\alpha h_{lp}^\alpha R_{pijlk} + h_{ijk}^\alpha h_{lik}^\delta R_{\delta\alpha jl} + h_{ijk}^\alpha h_{li}^\delta R_{\delta\alpha jlk}. \end{aligned}$$

By (2.6), (2.8) and (2.9), we have

$$\begin{aligned} \Delta h_{11}^\beta &= a^\beta(2 - S) + b^\delta(2a^\delta b^\beta - 2a^\beta b^\delta), \\ \Delta h_{12}^\beta &= b^\beta(2 - S) + a^\delta(2a^\beta b^\delta - 2a^\delta b^\beta). \end{aligned}$$

The first term on the right-hand side of (2.13) becomes

$$(2.14) \quad \begin{aligned} \sum_{i,j,\alpha} (\Delta h_{ij}^\alpha)^2 &= 2 \sum_{\alpha} (\Delta a^\alpha)^2 + 2 \sum_{\alpha} (\Delta b^\alpha)^2 \\ &= S(2-S)^2 + \frac{1}{2}(5S-8)(-S^2 + |A|^2 + \rho^\perp). \end{aligned}$$

By (2.8) and (2.9), the second, the third and the sixth terms are

$$(2.15) \quad \begin{aligned} &h_{ijk}^\alpha h_{pij}^\alpha R_{pljl} + h_{ijk}^\alpha h_{lpk}^\alpha R_{pijl} + h_{ijk}^\alpha h_{lik}^\delta R_{\delta\alpha jl} \\ &= (2-S)\mathcal{B}_1 + \sum_{\alpha,\delta} 4(a_1^\alpha a_2^\delta - a_2^\alpha a_1^\delta)R_{\delta\alpha 12} \\ &= (2-S)\mathcal{B}_1 + 16\langle a, a_2 \rangle \langle b, a_1 \rangle - 16\langle a, a_1 \rangle \langle b, a_2 \rangle. \end{aligned}$$

By (2.8), the fourth and the fifth terms are equal to

$$(2.16) \quad h_{ijk}^\alpha h_{pi}^\alpha R_{pljlk} + h_{ijk}^\alpha h_{lp}^\alpha R_{pijlk} = -h_{ij}^\alpha h_{ijk}^\alpha S_k = -\frac{1}{2}|\nabla S|^2.$$

By (2.10), the last term becomes

$$(2.17) \quad \begin{aligned} h_{ijk}^\alpha h_{li}^\delta R_{\delta\alpha jlk} &= 2(b^\delta a_1^\alpha - a^\delta a_2^\alpha)R_{\delta\alpha 121} + 2(a^\delta a_1^\alpha + b^\delta a_2^\alpha)R_{\delta\alpha 122} \\ &= -\frac{1}{2}S\mathcal{B}_1 + \frac{1}{4}|\nabla S|^2 + 16\langle a, a_2 \rangle \langle b, a_1 \rangle - 16\langle a, a_1 \rangle \langle b, a_2 \rangle. \end{aligned}$$

Combining (2.13), (2.14), (2.15), (2.16) and (2.17), we get

$$(2.18) \quad \begin{aligned} (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k &= \frac{1}{2}(4-3S)\mathcal{B}_1 + (2-S)^2 S + \frac{1}{2}(5S-8)(-S^2 + |A|^2 + \rho^\perp) \\ &\quad - \frac{1}{4}|\nabla S|^2 + 32\langle a, a_2 \rangle \langle b, a_1 \rangle - 32\langle a, a_1 \rangle \langle b, a_2 \rangle \\ &= \frac{1}{2}(2-S)S^2 + (2-S)(S^2 - |A|^2 - \rho^\perp) + \Delta S - \frac{3}{8}\Delta S^2 \\ &\quad + 8(\langle a, a_2 \rangle + \langle b, a_1 \rangle)^2 + 8(\langle a, a_1 \rangle - \langle b, a_2 \rangle)^2, \end{aligned}$$

where in the last equality, we used the equation (2.11) and the relation

$$S\Delta S = \frac{1}{2}\Delta S^2 - |\nabla S|^2.$$

Integration over  $M$  on both sides of (2.18) gives

$$0 = \int_M \left( \frac{1}{2}S^2(2-S) - (2-S)(-S^2 + |A|^2 + \rho^\perp) \right) + \int_M 8(\langle a, a_2 \rangle + \langle b, a_1 \rangle)^2 + 8(\langle a, a_1 \rangle - \langle b, a_2 \rangle)^2.$$

Therefore,

$$(2.19) \quad \int_M (2-S)(-S^2 + |A|^2 + \rho^\perp) \geq \int_M \frac{1}{2}(2-S)S^2.$$

On the other hand, we have the following relation by direct computation:

$$(2.20) \quad -S^2 + |A|^2 + \rho^\perp \leq \frac{1}{2}S^2,$$

where the equality holds if and only if  $a \perp b$  and  $|a| = |b|$ . The positivity of Gaussian curvature implies that  $S < 2$ . Therefore,

$$(2.21) \quad \int_M (2 - S)(-S^2 + |A|^2 + \rho^\perp) \leq \int_M \frac{1}{2}(2 - S)S^2.$$

By (2.19), (2.20) and (2.21), we get

$$(2.22) \quad -S^2 + |A|^2 + \rho^\perp = \frac{1}{2}S^2.$$

Combining (2.11) and (2.22), we complete the proof.  $\square$

**Corollary 2.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Under the foregoing assumptions and notations, we have*

$$(2.23) \quad \langle a, b \rangle = 0, \quad |a|^2 = |b|^2 = \frac{1}{4}S,$$

$$(2.24) \quad \Delta a = \frac{1}{2}a(4 - 3S), \quad \Delta b = \frac{1}{2}b(4 - 3S),$$

$$(2.25) \quad \Delta a_1 = \frac{1}{2}a_1(14 - 9S) + \frac{7}{4}(-aS_1 + bS_2),$$

$$(2.26) \quad \Delta a_2 = \frac{1}{2}a_2(14 - 9S) - \frac{7}{4}(bS_1 + aS_2),$$

$$|A|^2 = \frac{1}{2}S^2 \quad \text{and} \quad \rho^\perp = S^2.$$

*Proof.* On one hand, (2.23) is obtained by (2.18) and (2.22) in the proof of Theorem 1. On the other hand, (2.24) can be calculated directly by (2.6), (2.8), (2.9) and (2.23). Also, (2.25) and (2.26) can be calculated directly by (2.7), (2.8), (2.9), (2.10) and (2.23) in a similar way. Finally, by definition, we have

$$\begin{aligned} |A|^2 &= \sum (2a^\alpha a^\beta + 2b^\alpha b^\beta)^2 \\ &= \sum (4|a|^2|a|^2 + 4|b|^2|b|^2 + 8\langle a, b \rangle^2) \\ &= 8|a|^4 \\ &= \frac{1}{2}S^2 \end{aligned}$$

and

$$\begin{aligned} \rho^\perp &= \sum |[S_\alpha, S_\beta]|^2 = 8 \sum (a^\alpha b^\beta - a^\beta b^\alpha)^2 \\ &= 8 \sum (|a|^2|b|^2 + |a|^2|b|^2 - 2\langle a, b \rangle^2) \\ &= 16|a|^4 \\ &= S^2, \end{aligned}$$

which prove the corollary.  $\square$

**Remark 4.** *Theorem 1 can also be proved by using the method of holomorphic functions (cf. [11]). We will use this method to prove Lemma 1 and Proposition 4 in the following.*

**Lemma 1.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. We have*

$$\langle a_1, a_2 \rangle = 0 \text{ and } |a_1|^2 = |a_2|^2 = \frac{1}{8}\mathcal{B}_1.$$

*Proof.* Define

$$\phi := (|a_1|^2 - |a_2|^2 - 2i\langle a_1, a_2 \rangle)dz^6.$$

Then,  $\phi$  is a differential form of degree 6. It can be verified that  $\phi$  is independent of the choice of the vector field. We now prove that  $\phi$  is actually holomorphic by showing that it satisfies Cauchy-Riemann equations. First we have

$$\begin{aligned} & e_1(|a_1|^2 - |a_2|^2) + e_2(2\langle a_1, a_2 \rangle) \\ &= 2(\langle a_1, a_{11} \rangle - \langle a_2, a_{21} \rangle) + 2(\langle a_{12}, a_2 \rangle + \langle a_1, a_{22} \rangle) \\ &= 2(\langle a_1, \Delta a \rangle - \langle a_2, \Delta b \rangle) \\ &= 0, \end{aligned}$$

where we used (2.1) and (2.24) in the last equation. By the same process, it can be shown that  $e_2(|a_1|^2 - |a_2|^2) - e_1(2\langle a_1, a_2 \rangle) = 0$ . Therefore,  $\phi$  is holomorphic. But the holomorphic differential on a 2-dimensional sphere must be zero. The result follows.  $\square$

**Corollary 3.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Under the foregoing assumptions and notations, we have*

(i)

$$(2.27) \quad \begin{aligned} \langle a, a_1 \rangle &= \langle b, a_2 \rangle = \frac{1}{8}S_1, \\ \langle a, a_2 \rangle &= -\langle b, a_1 \rangle = \frac{1}{8}S_2, \end{aligned}$$

(ii)

$$(2.28) \quad \begin{aligned} \langle a, a_{11} \rangle &= \langle b, a_{21} \rangle = \frac{1}{8}(S_{11} - \mathcal{B}_1), \\ \langle a, a_{22} \rangle &= -\langle b, a_{12} \rangle = \frac{1}{8}(S_{22} - \mathcal{B}_1), \\ \langle a, a_{12} \rangle &= \langle b, a_{22} \rangle = \frac{1}{8}S_{12}, \\ \langle a, a_{21} \rangle &= -\langle b, a_{11} \rangle = \frac{1}{8}S_{21}, \end{aligned}$$



(iii) and

$$\begin{aligned}
(2.29) \quad & \langle a_1, a_{21} \rangle = -\langle a_2, a_{11} \rangle, \\
& \langle a_1, a_{22} \rangle = -\langle a_2, a_{12} \rangle, \\
& \langle a_1, a_{11} \rangle = \langle a_2, a_{21} \rangle = \frac{1}{16}(\mathcal{B}_1)_1, \\
& \langle a_1, a_{12} \rangle = \langle a_2, a_{22} \rangle = \frac{1}{16}(\mathcal{B}_1)_2.
\end{aligned}$$

*Proof.* First, (2.27) can be obtained by differentiating (2.23). Second, by differentiating (2.27) and using Lemma 1, we can obtain (2.28). Finally, (2.29) can be obtained by differentiating the equations in Lemma 1.  $\square$

Similar to Lemma 1, we obtain the following proposition.

**Proposition 4.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. We have*

(i)  $\langle a_{11}, a_{21} \rangle = 0$  and

$$|a_{11}|^2 = |a_{21}|^2 = \frac{1}{16}\mathcal{B}_2 - \frac{1}{32}(3S - 4)(S_{11} - S_{22}),$$

(ii)  $\langle a_{22}, a_{12} \rangle = 0$  and

$$|a_{22}|^2 = |a_{12}|^2 = \frac{1}{16}\mathcal{B}_2 + \frac{1}{32}(3S - 4)(S_{11} - S_{22}).$$

*Proof.* Define

$$\phi_1 := (|a_{11}|^2 - |a_{21}|^2 - 2i\langle a_{11}, a_{21} \rangle)dz^8.$$

Then  $\phi_1$  is a differential form of degree 8. It can be verified that  $\phi_1$  is independent of the choice of the vector field. We now prove that  $\phi_1$  is actually holomorphic by showing that it satisfies Cauchy-Riemann equations. First we have

$$\begin{aligned}
& e_1(|a_{11}|^2 - |a_{21}|^2) + e_2(2\langle a_{11}, a_{21} \rangle) \\
&= 2\langle a_{11}, a_{111} \rangle + 2\langle a_{11}, a_{212} \rangle + 2\langle a_{21}, a_{112} \rangle - 2\langle a_{21}, a_{211} \rangle \\
&= 2\langle a_{11}, \Delta a_1 \rangle - 2\langle a_{21}, \Delta a_2 \rangle \\
&= 0,
\end{aligned}$$

where we used (2.25), (2.26) and the relations

$$\begin{aligned}
a_{212}^\alpha &= a_{122}^\alpha + \frac{3}{2}a_1^\alpha S - \frac{3}{2}b^\alpha S_2 - 2a_1^\alpha, \\
-a_{112}^\alpha &= a_{222}^\alpha + \frac{3}{2}a_2^\alpha S + \frac{3}{2}a^\alpha S_2 - 2a_2^\alpha,
\end{aligned}$$

which can be obtained by using (2.2). By the same process, it can be shown that

$$e_2(|a_{11}|^2 - |a_{21}|^2) - e_1(2\langle a_{11}, a_{21} \rangle) = -2\langle a_{11}, \Delta a_2 \rangle - 2\langle a_{21}, \Delta a_1 \rangle = 0.$$

Therefore,  $\phi_1$  is holomorphic. Similarly, by defining

$$\phi_2 := (|a_{22}|^2 - |a_{12}|^2 + 2i\langle a_{22}, a_{12} \rangle) dz^8,$$

it can be shown that  $\phi_2$  is also holomorphic. But the holomorphic differential on a 2-dimensional sphere must be zero. Then we obtain  $\langle a_{11}, a_{21} \rangle = 0$ ,  $\langle a_{22}, a_{12} \rangle = 0$ ,  $|a_{11}| = |a_{21}|$  and  $|a_{22}| = |a_{12}|$ . By (2.2), we have

$$\begin{aligned} a_{12}^\alpha - a_{21}^\alpha &= h_{p1}^\alpha R_{p112} + h_{1p}^\alpha R_{p112} + h_{11}^\beta R_{\beta\alpha 12} \\ &= b^\alpha(S-2) + 2a^\beta(a^\beta b^\alpha - a^\alpha b^\beta) \\ &= \frac{1}{2}b^\alpha(3S-4). \end{aligned}$$

Then, with Theorem 1 and (2.28) we obtain

$$\begin{aligned} |a_{12}|^2 &= \langle a_{12}, a_{12} \rangle \\ &= \langle a_{21} + \frac{1}{2}b(3S-4), a_{21} + \frac{1}{2}b(3S-4) \rangle \\ &= |a_{21}|^2 + 2\langle a_{21}, \frac{1}{2}b(3S-4) \rangle + \frac{1}{4}\langle b(3S-4), b(3S-4) \rangle \\ &= |a_{21}|^2 + \frac{1}{8}(3S-4)(S_{11} - \mathcal{B}_1) + \frac{1}{16}S(3S-4)^2 \\ &= |a_{21}|^2 + \frac{1}{16}(3S-4)(S_{11} - S_{22}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{B}_2 &= \sum 4 [(h_{1111}^\alpha)^2 + (h_{1112}^\alpha)^2 + (h_{1121}^\alpha)^2 + (h_{1122}^\alpha)^2] \\ &= 8(|a_{12}|^2 + |a_{21}|^2) \\ &= 16|a_{21}|^2 + \frac{1}{2}(3S-4)(S_{11} - S_{22}) \\ &= 16|a_{12}|^2 - \frac{1}{2}(3S-4)(S_{11} - S_{22}), \end{aligned}$$

which completes the proof.  $\square$

### 3. THE SIMON CONJECTURE IN THE CASES $s = 1, 2$

#### 3.1. The first Simons-type identity.

**Theorem 5.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then*

$$\int_M S(3S-4) = 2 \int_M \mathcal{B}_1 \geq 0.$$

*In particular, if  $0 \leq S \leq \frac{4}{3}$ , then  $S = 0$  or  $S = \frac{4}{3}$ .*

*Proof.* Integrating on both sides of (2.12) and combining  $\mathcal{B}_1 \geq 0$ , we obtain

$$(3.1) \quad \frac{1}{2} \int_M S(3S - 4) = \int_M \mathcal{B}_1 \geq 0.$$

If  $0 \leq S \leq \frac{4}{3}$ , then  $S(3S - 4) \leq 0$ . Combining (3.1), we arrive at  $S(3S - 4) = 0$ . It follows that  $S = 0$  or  $S = \frac{4}{3}$ .  $\square$

### 3.2. The second Simons-type identity.

**Theorem 6.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then*

$$\int_M S(3S - 4)(3S - 5) = 2 \int_M [\mathcal{B}_2 - \frac{1}{4}S(3S - 4)^2 + \frac{1}{2}|\nabla S|^2] \geq 0.$$

In particular, if  $\frac{4}{3} \leq S \leq \frac{5}{3}$ , then  $S = \frac{4}{3}$  or  $S = \frac{5}{3}$ .

*Proof.* By (2.7), We have

$$(3.2) \quad \begin{aligned} h_{ijk}^\alpha \Delta h_{ijk}^\alpha &= (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k - \sum (\Delta h_{ij}^\alpha)^2 + 2h_{ijk}^\alpha h_{pj}^\alpha R_{pikmm} + h_{ijk}^\alpha h_{ijp}^\alpha R_{pmkm} \\ &\quad + 4h_{ijk}^\alpha h_{pjm}^\alpha R_{pikm} + 2h_{ijk}^\alpha h_{ijm}^\beta R_{\beta\alpha km} + h_{ijk}^\alpha h_{ij}^\beta R_{\beta\alpha kmm}. \end{aligned}$$

By (2.24), we get

$$(3.3) \quad \sum_\alpha (\Delta h_{ij}^\alpha)^2 = 2 \sum_\alpha [(\Delta a^\alpha)^2 + (\Delta b^\alpha)^2] = \frac{1}{4}S(3S - 4)^2.$$

By (2.8), it follows that

$$(3.4) \quad \begin{aligned} 2h_{ijk}^\alpha h_{pj}^\alpha R_{pikmm} &= -h_{ijk}^\alpha h_{pj}^\alpha (\delta_{pk}\delta_{im} - \delta_{pm}\delta_{ik})S_m \\ &= -h_{ijm}^\alpha h_{ij}^\alpha S_m \\ &= -\frac{1}{2}|\nabla S|^2, \end{aligned}$$

and

$$(3.5) \quad 4h_{ijk}^\alpha h_{pjm}^\alpha R_{pikm} + h_{ijk}^\alpha h_{ijp}^\alpha R_{pmkm} = 5(1 - \frac{S}{2})\mathcal{B}_1.$$

By (2.9), we obtain

$$(3.6) \quad 2h_{ijk}^\alpha h_{ijm}^\beta R_{\beta\alpha km} = 16(a_1^\alpha a_2^\beta - a_2^\alpha a_1^\beta)(a^\beta b^\alpha - a^\alpha b^\beta) = -\frac{1}{2}|\nabla S|^2.$$

By (2.10), we get

$$(3.7) \quad \begin{aligned} h_{ijk}^\alpha h_{ij}^\beta R_{\beta\alpha kmm} &= h_{ij1}^\alpha h_{ij}^\beta R_{\beta\alpha 122} - h_{ij2}^\alpha h_{ij}^\beta R_{\beta\alpha 121} \\ &= 4(a_1^\alpha a^\beta + a_2^\alpha b^\beta)(a_2^\beta b^\alpha - a^\beta a_1^\alpha - a_2^\alpha b^\delta + a^\alpha a_1^\beta) \\ &\quad - 4(a_2^\alpha a^\beta - a_1^\alpha b^\beta)(a_1^\beta b^\alpha + a^\beta a_2^\alpha - a_1^\alpha b^\beta - a^\alpha a_2^\beta) \\ &= -\frac{1}{2}S\mathcal{B}_1. \end{aligned}$$

Combining (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7), we obtain

$$(3.8) \quad h_{ijk}^\alpha \Delta h_{ijk}^\alpha = (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k + \frac{5}{2} \Delta S - \frac{3}{4} \Delta S^2 + \frac{1}{2} |\nabla S|^2 - \frac{1}{4} S(3S-4)(9S-14).$$

Also, with (3.3) we can give a representation of  $\mathcal{B}_2$  as

$$(3.9) \quad \begin{aligned} \mathcal{B}_2 &= 4 \sum [(a_{11}^\alpha)^2 + (a_{22}^\alpha)^2 + (a_{12}^\alpha)^2 + (a_{21}^\alpha)^2] \\ &= 2 \sum [(a_{11}^\alpha + a_{22}^\alpha)^2 + (a_{12}^\alpha - a_{21}^\alpha)^2] + 2 \sum [(a_{11}^\alpha - a_{22}^\alpha)^2 + (a_{12}^\alpha + a_{21}^\alpha)^2] \\ &= 2 \sum [(\Delta a^\alpha)^2 + (\Delta b^\alpha)^2] + 2 \sum [(a_{11}^\alpha - a_{22}^\alpha)^2 + (a_{12}^\alpha + a_{21}^\alpha)^2] \\ &= \frac{1}{4} S(3S-4)^2 + \mathcal{C}_1, \end{aligned}$$

where

$$\mathcal{C}_1 = 2 \sum [(a_{11}^\alpha - a_{22}^\alpha)^2 + (a_{12}^\alpha + a_{21}^\alpha)^2] = \mathcal{B}_2 - \frac{1}{4} S(3S-4)^2 \geq 0.$$

By (3.8) and (3.9), we conclude that

$$(3.10) \quad \begin{aligned} \frac{1}{2} \Delta \mathcal{B}_1 &= h_{ijk}^\alpha \Delta h_{ijk}^\alpha + \mathcal{B}_2 \\ &= (h_{ijk}^\alpha \Delta h_{ij}^\alpha)_k + \frac{5}{2} \Delta S - \frac{3}{4} \Delta S^2 + \frac{1}{2} |\nabla S|^2 - \frac{1}{2} S(3S-4)(3S-5) + \mathcal{C}_1. \end{aligned}$$

Integrating on both sides of (3.10), we get

$$(3.11) \quad \begin{aligned} \frac{1}{2} \int_M S(3S-4)(3S-5) &= \int_M (\mathcal{C}_1 + \frac{1}{2} |\nabla S|^2) \\ &= \int_M [\mathcal{B}_2 - \frac{1}{4} S(3S-4)^2 + \frac{1}{2} |\nabla S|^2] \\ &\geq 0. \end{aligned}$$

If  $\frac{4}{3} \leq S \leq \frac{5}{3}$ , then  $S(3S-4)(3S-5) \leq 0$ . Combining (3.11), we arrive at  $S(3S-4)(3S-5) = 0$ . It follows that  $S = \frac{4}{3}$  or  $S = \frac{5}{3}$ .  $\square$

**Remark 5.** By (2.18), (2.22), (2.23) and (3.8), we conclude that

$$(3.12) \quad \frac{1}{2} \Delta \mathcal{B}_1 = \frac{7}{2} \Delta S - \frac{9}{8} \Delta S^2 + \frac{1}{2} |\nabla S|^2 - \frac{1}{4} S(3S-4)(9S-14) + \mathcal{B}_2.$$

#### 4. THE CASE $s = 3$

We now consider the case  $s = 3$ .

##### 4.1. The lower bound of $\mathcal{B}_3$ .

**Lemma 2.** *Suppose that  $M$  is a closed surface minimally immersed in a unit sphere  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Under the foregoing assumptions and notations, we have*

$$\mathcal{B}_3 = \sum (h_{ijklm}^\alpha)^2 = \frac{1}{4} (45S^2 - 144S + 116) \mathcal{B}_1 + \frac{13}{8} (7S-8) |\nabla S|^2 + \mathcal{C}_2 + \mathcal{C}_3,$$

where  $\mathcal{C}_2 = 2 \sum [(a_{111}^\alpha - a_{122}^\alpha)^2 + (a_{211}^\alpha - a_{222}^\alpha)^2]$  and  $\mathcal{C}_3 = 2 \sum [(a_{112}^\alpha + a_{121}^\alpha)^2 + (a_{212}^\alpha + a_{221}^\alpha)^2]$ .

*Proof.* By (2.25), (2.26) and Corollary 3, we have

$$\begin{aligned} & \sum [(\Delta a_1^\alpha)^2 + (\Delta a_2^\alpha)^2] \\ &= \sum \left[ \left( \frac{1}{2} a_1^\alpha (14 - 9S) + \frac{7}{4} (-a^\alpha S_1 + b^\alpha S_2) \right)^2 + \left( \frac{1}{2} a_1^\alpha (14 - 9S) + \frac{7}{4} (-a^\alpha S_1 + b^\alpha S_2) \right)^2 \right] \\ &= \frac{1}{16} (9S - 14)^2 \mathcal{B}_1 + \frac{7}{32} (25S - 28) |\nabla S|^2. \end{aligned}$$

Then we obtain

$$\begin{aligned} (4.1) \quad & 4 \sum [(a_{111}^\alpha)^2 + (a_{122}^\alpha)^2 + (a_{211}^\alpha)^2 + (a_{222}^\alpha)^2] \\ &= 2 \sum [(a_{111}^\alpha + a_{122}^\alpha)^2 + (a_{211}^\alpha + a_{222}^\alpha)^2] + 2 \sum [(a_{111}^\alpha - a_{122}^\alpha)^2 + (a_{211}^\alpha - a_{222}^\alpha)^2] \\ &= 2 \sum [(\Delta a_1^\alpha)^2 + (\Delta a_2^\alpha)^2] + \mathcal{C}_2 \\ &= \frac{1}{8} (9S - 14)^2 \mathcal{B}_1 + \frac{7}{16} (25S - 28) |\nabla S|^2 + \mathcal{C}_2. \end{aligned}$$

By (2.3) and (2.9), we have

$$\begin{aligned} & 4 \sum [(a_{112}^\alpha)^2 + (a_{121}^\alpha)^2 + (a_{212}^\alpha)^2 + (a_{221}^\alpha)^2] \\ &= 2 \sum [(a_{112}^\alpha - a_{121}^\alpha)^2 + (a_{212}^\alpha - a_{221}^\alpha)^2] + 2 \sum [(a_{112}^\alpha + a_{121}^\alpha)^2 + (a_{212}^\alpha + a_{221}^\alpha)^2] \\ &= 2 \left[ (3h_{p11}^\alpha R_{p112} + h_{111}^\beta R_{\beta\alpha 12})^2 + (3h_{p12}^\alpha R_{p112} + h_{112}^\beta R_{\beta\alpha 12})^2 \right] + \mathcal{C}_3 \\ &= 2 \left[ (3h_{211}^\alpha R_{2112} + h_{111}^\beta R_{\beta\alpha 12})^2 + (3h_{212}^\alpha R_{2112} + h_{112}^\beta R_{\beta\alpha 12})^2 \right] + \mathcal{C}_3 \\ &= 2 \left[ (3a_2^\alpha \left(\frac{S}{2} - 1\right) + a_1^\beta R_{\beta\alpha 12})^2 + (3a_1^\alpha \left(1 - \frac{S}{2}\right) + a_2^\beta R_{\beta\alpha 12})^2 \right] + \mathcal{C}_3 \\ &= 18 \left(1 - \frac{S}{2}\right)^2 (|a_1|^2 + |a_2|^2) + 2(a_1^\beta R_{\beta\alpha 12})^2 + 2(a_2^\beta R_{\beta\alpha 12})^2 + 24a_1^\alpha a_2^\beta \left(1 - \frac{S}{2}\right) R_{\beta\alpha 12} + \mathcal{C}_3 \\ &= 18 \left(1 - \frac{S}{2}\right)^2 \frac{1}{4} \mathcal{B}_1 + \frac{1}{32} S |\nabla S|^2 + \frac{1}{32} S |\nabla S|^2 - \frac{3}{4} \left(1 - \frac{S}{2}\right) |\nabla S|^2 + \mathcal{C}_3 \\ &= \frac{9}{2} \left(1 - \frac{S}{2}\right)^2 \mathcal{B}_1 + \frac{1}{16} (7S - 12) |\nabla S|^2 + \mathcal{C}_3. \end{aligned}$$

Combining these two parts gives that

$$\begin{aligned} \mathcal{B}_3 &= \sum (h_{ijklm}^\alpha)^2 \\ &= 4 \sum [(a_{111}^\alpha)^2 + (a_{122}^\alpha)^2 + (a_{211}^\alpha)^2 + (a_{222}^\alpha)^2 + (a_{112}^\alpha)^2 + (a_{121}^\alpha)^2 + (a_{212}^\alpha)^2 + (a_{221}^\alpha)^2] \\ &= \frac{1}{8} \mathcal{B}_1 (9S - 14)^2 + \frac{7}{16} |\nabla S|^2 (25S - 28) + \frac{9}{2} \mathcal{B}_1 \left(1 - \frac{S}{2}\right)^2 + \frac{1}{16} (7S - 12) |\nabla S|^2 + \mathcal{C}_2 + \mathcal{C}_3 \\ &= \frac{1}{4} (45S^2 - 144S + 116) \mathcal{B}_1 + \frac{13}{8} (7S - 8) |\nabla S|^2 + \mathcal{C}_2 + \mathcal{C}_3, \end{aligned}$$

which completes the proof.  $\square$

## 4.2. Laplacian of $\mathcal{B}_2$ .

**Lemma 3.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then we have*

$$(4.2) \quad h_{ijkl}^\alpha h_{ijlk}^\alpha = \mathcal{B}_2 - \frac{1}{4}S(3S - 4)^2,$$

$$(4.3) \quad h_{ijkk}^\alpha h_{ijll}^\alpha = \frac{1}{4}S(3S - 4)^2.$$

*Proof.* The second identity can be obtained by (3.3) directly. For the first identity, by using (2.2) we obtain

$$\begin{aligned} h_{ijkl}^\alpha h_{ijlk}^\alpha &= h_{ijkl}^\alpha h_{ijkl}^\alpha + h_{ijkl}^\alpha (2h_{ip}^\alpha R_{pjlk} + h_{ij}^\beta R_{\beta\alpha lk}) \\ &= \mathcal{B}_2 + 2(h_{ij12}^\alpha - h_{ij21}^\alpha)h_{ip}^\alpha R_{pj21} + h_{ij}^\beta (h_{ij21}^\alpha - h_{ij12}^\alpha)R_{\beta\alpha 12} \\ &= \mathcal{B}_2 + 2(2h_{iq}^\alpha R_{qj12} + h_{ij}^\beta R_{\beta\alpha 12})h_{ip}^\alpha R_{pj21} + h_{ij}^\beta (2h_{ip}^\alpha R_{pj21} + h_{ij}^\gamma R_{\gamma\alpha 21})R_{\gamma\alpha 12} \\ &= \mathcal{B}_2 + S(1 - \frac{S}{2})(3S - 4) - \frac{1}{4}S^2(3S - 4) \\ &= \mathcal{B}_2 - \frac{1}{4}S(3S - 4)^2, \end{aligned}$$

which completes the proof.  $\square$

We now derive the Laplacian of  $\mathcal{B}_2$ .

**Theorem 7.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then*

$$\begin{aligned} \frac{1}{2}\Delta\mathcal{B}_2 &= (h_{ijkl}^\alpha \Delta h_{ijjk}^\alpha)_l - (21S^2 - 64S + 49)\mathcal{B}_1 + 7(1 - \frac{S}{2})\mathcal{B}_2 + \frac{1}{4}S(3S - 4)^2(7S - 12) \\ &\quad - \frac{7}{2}(7S - 8)|\nabla S|^2 - \langle \nabla \mathcal{B}_1, \nabla S \rangle + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2 + \mathcal{B}_3. \end{aligned}$$

*Proof.* By (2.5), (2.6) and (2.7), we get

$$\begin{aligned} \Delta h_{ijkl}^\alpha &= h_{ijklm}^\alpha \\ &= (h_{ijkml}^\alpha + h_{pj}^\alpha R_{pilm} + h_{ipk}^\alpha R_{pjlm} + h_{ijp}^\alpha R_{pklm} + h_{ijk}^\beta R_{\beta\alpha lm})_m \\ &= h_{ijkmlm}^\alpha + (h_{pj}^\alpha R_{pilm} + h_{ipk}^\alpha R_{pjlm} + h_{ijp}^\alpha R_{pklm} + h_{ijk}^\beta R_{\beta\alpha lm})_m \\ &= h_{ijkmml}^\alpha + h_{pjkm}^\alpha R_{pilm} + h_{ipkm}^\alpha R_{pjlm} + h_{ijpm}^\alpha R_{pklm} + h_{ijkp}^\alpha R_{pmlm} + h_{ijkm}^\beta R_{\beta\alpha lm} \\ &\quad + (h_{pj}^\alpha R_{pilm} + h_{ipk}^\alpha R_{pjlm} + h_{ijp}^\alpha R_{pklm} + h_{ijk}^\beta R_{\beta\alpha lm})_m \\ &= (\Delta h_{ijjk}^\alpha)_l + 2h_{pjkm}^\alpha R_{pilm} + 2h_{ipkm}^\alpha R_{pjlm} + 2h_{ijpm}^\alpha R_{pklm} + h_{ijkp}^\alpha R_{pmlm} \\ &\quad + h_{pj}^\alpha R_{pilm} + h_{ipk}^\alpha R_{pjlm} + h_{ijp}^\alpha R_{pklm} + h_{ijk}^\beta R_{\beta\alpha lmm} + 2h_{ijkm}^\beta R_{\beta\alpha lm}. \end{aligned}$$

It follows that

$$\begin{aligned}
(4.4) \quad h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha &= h_{ijkl}^\alpha (\Delta h_{ijk}^\alpha)_l + 6h_{ijkl}^\alpha h_{pjkm}^\alpha R_{pilm} + h_{ijkl}^\alpha h_{ijkp}^\alpha R_{pmlm} \\
&\quad + 3h_{ijkl}^\alpha h_{pjkl}^\alpha R_{pilmm} + 2h_{ijkl}^\alpha h_{ijkm}^\delta R_{\delta\alpha lm} + h_{ijkl}^\alpha h_{ijk}^\delta R_{\delta\alpha lmm} \\
&= (h_{ijkl}^\alpha \Delta h_{ijk}^\alpha)_l - \sum (\Delta h_{ijk}^\alpha)^2 + 6h_{ijkl}^\alpha h_{pjkm}^\alpha R_{pilm} + h_{ijkl}^\alpha h_{ijkp}^\alpha R_{pmlm} \\
&\quad + 3h_{ijkl}^\alpha h_{pjkl}^\alpha R_{pilmm} + h_{ijkl}^\alpha h_{ijk}^\delta R_{\delta\alpha lmm} + 2h_{ijkl}^\alpha h_{ijkm}^\delta R_{\delta\alpha lm}.
\end{aligned}$$

By (4.1), the second term is equal to

$$\begin{aligned}
(4.5) \quad - \sum (\Delta h_{ijk}^\alpha)^2 &= -4 \sum (\Delta a_1^\alpha)^2 + (\Delta a_2^\alpha)^2 \\
&= -\frac{1}{4}(9S - 14)^2 \mathcal{B}_1 - \frac{7}{8}(25S - 28)|\nabla S|^2.
\end{aligned}$$

By (4.2), (4.3) and the minimality, the third term and the fourth term are

$$(4.6) \quad 6h_{ijkl}^\alpha h_{pjkm}^\alpha R_{pilm} + h_{ijkl}^\alpha h_{ijkp}^\alpha R_{pmlm} = 7\left(1 - \frac{S}{2}\right)\mathcal{B}_2 - 3S\left(1 - \frac{S}{2}\right)(3S - 4)^2.$$

By Lemma 1, the fifth term is equal to

$$\begin{aligned}
(4.7) \quad 3h_{ijkl}^\alpha h_{pjkl}^\alpha R_{pilmm} &= 3h_{ijkl}^\alpha h_{ijk}^\alpha R_{lilii} + 3h_{ijkl}^\alpha h_{mjk}^\alpha R_{miiimm} \\
&= -12(h_{111}^\alpha h_{1121}^\alpha S_2 + h_{112}^\alpha h_{1112}^\alpha S_1) \\
&= -\frac{3}{4}((\mathcal{B}_1)_2 + S_2(3S - 4))S_2 - \frac{3}{4}((\mathcal{B}_1)_1 + S_1(3S - 4))S_1 \\
&= -\frac{3}{4}\langle \nabla \mathcal{B}_1, \nabla S \rangle - \frac{3}{4}(3S - 4)|\nabla S|^2.
\end{aligned}$$

By Lemma 1 and (2.2), the sixth term is

$$\begin{aligned}
(4.8) \quad h_{ijkl}^\alpha h_{ijk}^\delta R_{\delta\alpha lmm} &= h_{ijkl}^\alpha h_{ijk}^\delta R_{\delta\alpha 122} + h_{ijk2}^\alpha h_{ijk}^\delta R_{\delta\alpha 211} \\
&= 2h_{ijk}^\delta (h_{ijk1}^\alpha (h_{12}^\alpha h_{112}^\delta - h_{111}^\alpha h_{11}^\delta - h_{112}^\alpha h_{12}^\delta + h_{11}^\alpha h_{111}^\delta)) \\
&\quad - h_{ijk2}^\alpha (h_{12}^\alpha h_{111}^\delta + h_{112}^\alpha h_{11}^\delta - h_{111}^\alpha h_{12}^\delta - h_{11}^\alpha h_{112}^\delta) \\
&= -\frac{1}{4}\langle \nabla \mathcal{B}_1, \nabla S \rangle - \frac{1}{8}(3S - 4)|\nabla S|^2 - \frac{1}{4}S(3S - 4)\mathcal{B}_1.
\end{aligned}$$

The final term is

$$\begin{aligned}
(4.9) \quad 2h_{ijkl}^\alpha h_{ijkm}^\delta R_{\delta\alpha lm} &= 64(\langle a, a_{11} \rangle \langle a, a_{22} \rangle + \langle b, a_{11} \rangle \langle b, a_{22} \rangle) \\
&= (S_{11} - \mathcal{B}_1)(S_{22} - \mathcal{B}_1) - (S_{12})^2 \\
&= (\mathcal{B}_1)^2 - \frac{1}{2}S(3S - 4)\Delta S - \frac{1}{2}|\text{Hess } S|^2 \\
&= \frac{1}{4}S^2(3S - 4)^2 + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2.
\end{aligned}$$

Combining (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9) gives that

$$\begin{aligned}
h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha &= (h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha)_l - \frac{1}{4}(9S - 14)^2 \mathcal{B}_1 - \frac{7}{8}(25S - 28)|\nabla S|^2 + 7(1 - \frac{S}{2})\mathcal{B}_2 \\
&\quad - 3S(1 - \frac{S}{2})(3S - 4)^2 - \frac{3}{4}\langle \nabla \mathcal{B}_1, \nabla S \rangle - \frac{3}{4}(3S - 4)|\nabla S|^2 - \frac{1}{4}\langle \nabla \mathcal{B}_1, \nabla S \rangle \\
&\quad - \frac{1}{8}(3S - 4)|\nabla S|^2 - \frac{1}{4}S(3S - 4)\mathcal{B}_1 + \frac{1}{4}S^2(3S - 4)^2 + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2 \\
&= (h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha)_l - (21S^2 - 64S + 49)\mathcal{B}_1 + 7(1 - \frac{S}{2})\mathcal{B}_2 + \frac{1}{4}S(3S - 4)^2(7S - 12) \\
&\quad - \frac{7}{2}(7S - 8)|\nabla S|^2 - \langle \nabla \mathcal{B}_1, \nabla S \rangle + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\frac{1}{2}\Delta \mathcal{B}_2 &= h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha + \mathcal{B}_3 \\
&= (h_{ijkl}^\alpha \Delta h_{ijkl}^\alpha)_l - (21S^2 - 64S + 49)\mathcal{B}_1 + 7(1 - \frac{S}{2})\mathcal{B}_2 + \frac{1}{4}S(3S - 4)^2(7S - 12) \\
&\quad - \frac{7}{2}(7S - 8)|\nabla S|^2 - \langle \nabla \mathcal{B}_1, \nabla S \rangle + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2 + \mathcal{B}_3,
\end{aligned}$$

which completes the proof.  $\square$

**4.3. The third Simons-type identity.** The following integral equality is the key to proving Theorem A (iii).

**Theorem 8.** *Let  $M$  be a closed minimal surface immersed in  $\mathbb{S}^N(1)$  with positive Gaussian curvature. Then*

$$\begin{aligned}
\int_M S(3S - 4)(3S - 5)(5S - 9) &= \int_M \left[ \frac{3}{2}(11S - 21)|\nabla S|^2 - \frac{5}{4}(\Delta S)^2 + 2\mathcal{C}_2 + 2\mathcal{C}_3 \right] \\
&= 2 \int_M \left[ \mathcal{B}_3 - \frac{1}{8}S(3S - 4)(45S^2 - 144S + 116) \right. \\
&\quad \left. + \frac{1}{8}(65S - 166)|\nabla S|^2 - \frac{5}{8}(\Delta S)^2 \right],
\end{aligned}$$

where  $\mathcal{C}_2 = 2 \sum [(a_{111}^\alpha - a_{122}^\alpha)^2 + (a_{211}^\alpha - a_{222}^\alpha)^2]$  and  $\mathcal{C}_3 = 2 \sum [(a_{112}^\alpha + a_{121}^\alpha)^2 + (a_{212}^\alpha + a_{221}^\alpha)^2]$ .

*Proof.* By (2.12) we obtain

$$(4.10) \quad \frac{1}{2}\Delta \mathcal{B}_1 = \frac{1}{4}\Delta^2 S + \frac{3}{4}\Delta S^2 - \Delta S.$$

Combining (3.12) we obtain

$$(4.11) \quad \mathcal{B}_2 = \frac{1}{4}S(3S - 4)(9S - 14) - \frac{1}{2}|\nabla S|^2 + \frac{15}{8}\Delta S^2 - \frac{9}{2}\Delta S + \frac{1}{4}\Delta^2 S.$$



First, by (2.12) and Lemma 2, we obtain

$$\begin{aligned}
& - (21S^2 - 64S + 49)\mathcal{B}_1 + \mathcal{B}_3 \\
& = \frac{1}{4}(-39S^2 + 112S - 80)\mathcal{B}_1 + \frac{13}{8}(7S - 8)|\nabla S|^2 + \mathcal{C}_2 + \mathcal{C}_3 \\
(4.12) \quad & = \frac{1}{8}S(3S - 4)(-39S^2 + 112S - 80) + \frac{13}{8}(7S - 8)|\nabla S|^2 \\
& \quad - \frac{39}{8}S^2\Delta S + 14S\Delta S - 10\Delta S + \mathcal{C}_2 + \mathcal{C}_3.
\end{aligned}$$

Second, by (4.11) we obtain

$$\begin{aligned}
(4.13) \quad 7\left(1 - \frac{S}{2}\right)\mathcal{B}_2 & = \frac{7}{8}S(2 - S)(3S - 4)(9S - 14) + \frac{7}{4}(S - 2)|\nabla S|^2 + \frac{105}{8}\Delta S^2 \\
& \quad - \frac{105}{16}S\Delta S^2 - \frac{63}{2}\Delta S + \frac{63}{4}S\Delta S + \frac{7}{4}\Delta^2 S - \frac{7}{8}S\Delta^2 S.
\end{aligned}$$

Also, trivial calculations give that

$$\begin{aligned}
(4.14) \quad & \frac{1}{8}S(3S - 4)(-39S^2 + 112S - 80) + \frac{7}{8}S(2 - S)(3S - 4)(9S - 14) \\
& \quad + \frac{1}{4}S(3S - 4)^2(7S - 12) \\
& = -\frac{1}{2}S(3S - 4)(3S - 5)(5S - 9).
\end{aligned}$$

Therefore, combining Theorem 7, (4.12), (4.13) and (4.14), we obtain

$$\begin{aligned}
(4.15) \quad \frac{1}{2}\Delta\mathcal{B}_2 & = (h_{ijkl}^\alpha\Delta h_{ijk}^\alpha)_l - \frac{1}{2}S(3S - 4)(3S - 5)(5S - 9) - \frac{1}{8}(91S - 92)|\nabla S|^2 \\
& \quad - \frac{39}{8}S^2\Delta S + \frac{119}{4}S\Delta S + \frac{105}{8}\Delta S^2 - \frac{105}{16}S\Delta S^2 - \frac{83}{2}\Delta S \\
& \quad + \frac{7}{4}\Delta^2 S - \frac{7}{8}S\Delta^2 S - \langle\nabla\mathcal{B}_1, \nabla S\rangle + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\text{Hess } S|^2 + \mathcal{C}_2 + \mathcal{C}_3.
\end{aligned}$$

To continue, we first have

$$\begin{aligned}
S\Delta S & = \frac{1}{2}\Delta S^2 - |\nabla S|^2, \\
S\Delta S^2 & = \frac{2}{3}\Delta S^3 - 2S|\nabla S|^2,
\end{aligned}$$

and

$$S^2\Delta S = \frac{1}{3}\Delta S^3 - 2S|\nabla S|^2,$$

which can be obtained by easy calculations. Next, since we have

$$\text{div}(\Delta S\nabla S) = (\Delta S)^2 + \langle\nabla S, \nabla(\Delta S)\rangle$$

and

$$\Delta(S\Delta S) = (\Delta S)^2 + 2\langle\nabla S, \nabla(\Delta S)\rangle + S\Delta^2 S,$$

we obtain

$$S\Delta^2 S = \Delta(S\Delta S) - 2\text{div}(\Delta S\nabla S) + (\Delta S)^2.$$

Also, since we have

$$\mathcal{B}_1 \Delta S = \frac{1}{2}(\Delta S)^2 + \frac{3}{2}S^2 \Delta S - 2S \Delta S$$

and

$$\operatorname{div}(\mathcal{B}_1 \nabla S) = \mathcal{B}_1 \Delta S + \langle \nabla \mathcal{B}_1, \nabla S \rangle,$$

we obtain

$$-\langle \nabla \mathcal{B}_1, \nabla S \rangle = -\operatorname{div}(\mathcal{B}_1 \nabla S) + \frac{1}{2}(\Delta S)^2 + \frac{3}{2}S^2 \Delta S - 2S \Delta S.$$

Hence, we have

$$\begin{aligned} -\frac{39}{8}S^2 \Delta S &= -\frac{13}{8}\Delta S^3 + \frac{39}{4}S|\nabla S|^2, \\ \frac{119}{4}S \Delta S &= \frac{119}{8}\Delta S^2 - \frac{119}{4}|\nabla S|^2, \\ -\frac{105}{16}S \Delta S^2 &= -\frac{35}{8}\Delta S^3 + \frac{105}{8}S|\nabla S|^2, \\ -\frac{7}{8}S \Delta^2 S &= -\frac{7}{8}\Delta(S \Delta S) + \frac{7}{4}\operatorname{div}(\Delta S \nabla S) - \frac{7}{8}(\Delta S)^2, \text{ and} \\ -\langle \nabla \mathcal{B}_1, \nabla S \rangle &= -\operatorname{div}(\mathcal{B}_1 \nabla S) + \frac{1}{2}(\Delta S)^2 + \frac{1}{2}\Delta S^3 - 3S|\nabla S|^2 - \Delta S^2 + 2|\nabla S|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\Delta \mathcal{B}_2 &= (h_{ijkl}^\alpha \Delta h_{ijk}^\alpha)_l - \frac{1}{2}S(3S-4)(3S-5)(5S-9) - \frac{1}{8}(91S-92)|\nabla S|^2 \\ &\quad - \frac{39}{8}S^2 \Delta S + \frac{119}{4}S \Delta S + \frac{105}{8}\Delta S^2 - \frac{105}{16}S \Delta S^2 - \frac{83}{2}\Delta S \\ &\quad + \frac{7}{4}\Delta^2 S - \frac{7}{8}S \Delta^2 S - \langle \nabla \mathcal{B}_1, \nabla S \rangle + \frac{1}{4}(\Delta S)^2 - \frac{1}{2}|\operatorname{Hess} S|^2 + \mathcal{C}_2 + \mathcal{C}_3 \\ &= (h_{ijkl}^\alpha \Delta h_{ijk}^\alpha)_l - \frac{1}{2}S(3S-4)(3S-5)(5S-9) - \frac{11}{2}\Delta S^3 + 27\Delta S^2 - \frac{83}{2}\Delta S \\ &\quad + \frac{7}{4}\Delta^2 S - \frac{7}{8}\Delta(S \Delta S) + \frac{7}{4}\operatorname{div}(\Delta S \nabla S) - \operatorname{div}(\mathcal{B}_1 \nabla S) + \frac{1}{4}(34S-65)|\nabla S|^2 \\ &\quad - \frac{1}{8}(\Delta S)^2 - \frac{1}{2}|\operatorname{Hess} S|^2 + \mathcal{C}_2 + \mathcal{C}_3. \end{aligned}$$

Integrating on both sides, we obtain

$$(4.16) \quad \begin{aligned} &\int_M \frac{1}{2}S(3S-4)(3S-5)(5S-9) \\ &= \int_M \left[ \frac{1}{4}(34S-65)|\nabla S|^2 - \frac{1}{8}(\Delta S)^2 - \frac{1}{2}|\operatorname{Hess} S|^2 + \mathcal{C}_2 + \mathcal{C}_3 \right]. \end{aligned}$$

By Reilly's formula (cf. [22]), we obtain

$$\int_M [2(\Delta S)^2 - 2|\operatorname{Hess} S|^2 + (S-2)|\nabla S|^2] = 0.$$

Therefore, (4.16) becomes

$$\int_M \frac{1}{2} S(3S-4)(3S-5)(5S-9) = \int_M \left[ \frac{3}{4} (11S-21) |\nabla S|^2 - \frac{5}{8} (\Delta S)^2 + C_2 + C_3 \right].$$

Combining Lemma 2, we obtain

$$\begin{aligned} \int_M S(3S-4)(3S-5)(5S-9) &= 2 \int_M \left[ \mathcal{B}_3 - \frac{1}{8} S(3S-4)(45S^2 - 144S + 116) \right. \\ &\quad \left. + \frac{1}{8} (65S - 166) |\nabla S|^2 - \frac{5}{8} (\Delta S)^2 \right], \end{aligned}$$

which completes the proof.  $\square$

**4.4. Proof of Theorem A.** First, we give an upper bound of the integral of  $|\nabla S|^2$ .

**Lemma 4.** *Let  $M$  be a closed surface minimally immersed into  $\mathbb{S}^N(1)$ . If  $\frac{5}{3} \leq S \leq \frac{9}{5}$  and  $S_{\min} = \inf_{p \in M} S(p)$ , then*

$$\int_M |\nabla S|^2 \leq \int_M S(3S-4)(S-S_{\min}).$$

*Proof.* Since  $\operatorname{div}(S\nabla S) = S\Delta S + \langle \nabla S, \nabla S \rangle$ , then using (2.12) we obtain

$$\begin{aligned} \int_M |\nabla S|^2 &= \int_M -S\Delta S \\ &= \int_M -S(2\mathcal{B}_1 - S(3S-4)) \\ &= \int_M (-2S\mathcal{B}_1 + S^2(3S-4)) \\ &\leq -2S_{\min} \int_M \mathcal{B}_1 + \int_M S^2(3S-4) \\ &= \int_M S(3S-4)(S-S_{\min}), \end{aligned}$$

which proves the lemma.  $\square$

By Theorems 5 and 6, Theorem A (i)-(ii) is true. It remains to prove Theorem A (iii). Notice that

$$\begin{aligned} \int_M (\Delta S)^2 &= 2 \int_M \mathcal{B}_1 \Delta S - \int_M S(3S-4) \Delta S \\ &= 2 \int_M \mathcal{B}_1 \Delta S - 3 \int_M S^2 \Delta S + 4 \int_M S \Delta S \\ &= 2 \int_M \mathcal{B}_1 \Delta S + \int_M (6S-4) |\nabla S|^2. \end{aligned}$$

By the divergence theorem and (3.12) we obtain

$$\begin{aligned} \frac{1}{2} \int_M \mathcal{B}_1 \Delta S &= \frac{1}{2} \int_M S \Delta \mathcal{B}_1 \\ &= \int_M \left( -\frac{7}{2} |\nabla S|^2 + \frac{11}{4} S |\nabla S|^2 - \frac{1}{4} S^2 (3S - 4)(9S - 14) + S \mathcal{B}_2 \right) \\ &\leq \int_M \left( \left( \frac{11}{4} S - \frac{7}{2} \right) |\nabla S|^2 - \frac{1}{4} S^2 (3S - 4)(9S - 14) + S_{\max} \left( \frac{1}{4} S (3S - 4)(9S - 14) - \frac{1}{2} |\nabla S|^2 \right) \right). \end{aligned}$$

Then

$$2 \int_M \mathcal{B}_1 \Delta S \leq \int_M (11S - 14 - 2S_{\max}) |\nabla S|^2 - \int_M S(3S - 4)(9S - 14)(S - S_{\max}).$$

Combining these calculations gives that

$$(4.17) \quad \int_M (\Delta S)^2 \leq \int_M (17S - 18 - 2S_{\max}) |\nabla S|^2 - \int_M S(3S - 4)(9S - 14)(S - S_{\max}).$$

Inserting (4.17) into Theorem 8 and using Lemma 4 give that

$$\begin{aligned} 0 &\geq \int_M S(3S - 4) \left[ -\frac{1}{2}(3S - 5)(5S - 9) + \frac{5}{8}(9S - 14)(S - S_{\max}) \right] \\ &\quad + \int_M \left( -\frac{19}{8}S - \frac{9}{2} + \frac{5}{4}S_{\max} \right) |\nabla S|^2 \\ &\geq \int_M S(3S - 4) \left[ -\frac{1}{2}(3S - 5)(5S - 9) + \frac{5}{8}(9S - 14)(S - S_{\max}) \right] \\ &\quad + \int_M \left( -\frac{19}{8}S - \frac{9}{2} + \frac{5}{4}S_{\max} \right) S(3S - 4)(S - S_{\min}) \\ &\geq \int_M S(3S - 4) \left[ -\frac{1}{2}(3S - 5)(5S - 9) + \frac{5}{8}(9S - 14)(S - S_{\max}) \right] \\ &\quad + \int_M \left( -\frac{19}{8}S_{\max} - \frac{9}{2} + \frac{5}{4}S_{\max} \right) S(3S - 4)(S - S_{\min}) \\ &= \int_M S(3S - 4) \left[ \frac{1}{2}(3S - 5)(9 - 5S) + \frac{5}{8}(9S - 14)(S - S_{\max}) + (S - S_{\min}) \left( -\frac{9}{8}S_{\max} - \frac{9}{2} \right) \right] \\ &=: \int_M S(3S - 4) \frac{\mathcal{A}}{8}, \end{aligned}$$

where  $\mathcal{A} = 4(3S - 5)(9 - 5S) + 5(9S - 14)(S - S_{\max}) + (S - S_{\min})(-9S_{\max} - 36)$ . Here, with the condition  $\frac{5}{3} \leq S \leq \frac{9}{5}$ , we used the relation

$$-\frac{19}{8}S - \frac{9}{2} + \frac{5}{4}S_{\max} < 0.$$

Assume that  $\mathcal{A} > 0$ . Then, with  $S > \frac{4}{3}$ , we have

$$0 \geq \int_M S(3S - 4) \mathcal{A} > 0,$$

which creates a contradiction. Hence,  $\mathcal{A}$  cannot be identically positive. To estimate  $\mathcal{A}$ , we have

$$\begin{aligned}\mathcal{A} &= 4(3S - 5)(9 - 5S) + 5(S - S_{\max})(9S - 14) - (S - S_{\min})(9S_{\max} + 36) \\ &\geq 4(3S - 5)(9 - 5S) + 5(S_{\min} - S_{\max})(9S_{\max} - 14) - (S_{\max} - S_{\min})(9S_{\max} + 36) \\ &= 4(3S - 5)(9 - 5S) - (S_{\max} - S_{\min})(54S_{\max} - 34).\end{aligned}$$

Suppose that  $S_{\max} - S_{\min} < \varepsilon$ , where  $\frac{5}{3} \leq S \leq \frac{9}{5}$ . Then

$$\begin{aligned}\mathcal{A} &> 4(3S - 5)(9 - 5S) - \varepsilon(54S_{\max} - 34) \\ &> 4(3S - 5)(9 - 5S) - \varepsilon(54(S_{\min} + \varepsilon) - 34) \\ &> 4(3S_{\min} - 5)(9 - 5S_{\max}) - \varepsilon(54S_{\min} - 34) - 54\varepsilon^2 \\ &> 4(3S_{\min} - 5)(9 - 5(S_{\min} + \varepsilon)) - \varepsilon(54S_{\min} - 34) - 54\varepsilon^2 \\ &= 4(3S_{\min} - 5)(9 - 5S_{\min}) + \varepsilon(134 - 114S_{\min}) - 54\varepsilon^2.\end{aligned}$$

Then we obtain that

$$\mathcal{A}_1 := 4(3S_{\min} - 5)(9 - 5S_{\min}) + \varepsilon(134 - 114S_{\min}) - 54\varepsilon^2$$

cannot be identically nonnegative. Let

$$\mathcal{F} = (134 - 114S_{\min})^2 + 864(3S_{\min} - 5)(9 - 5S_{\min}).$$

If

$$\frac{(134 - 114S_{\min}) - \sqrt{\mathcal{F}}}{108} \leq \varepsilon \leq \frac{(134 - 114S_{\min}) + \sqrt{\mathcal{F}}}{108} =: \varepsilon_0,$$

then  $\mathcal{A} > \mathcal{A}_1 \geq 0$ , which creates a contradiction. Choosing  $\varepsilon = \varepsilon_0$ , one has

$$S_{\max} - S_{\min} \geq \varepsilon_0.$$

Therefore, we proved Theorem A. □

**Acknowledgments.** *The authors are grateful to the reviewers for their insightful comments that significantly enhanced this work. The authors would like to thank Professor Zizhou Tang (Nankai University) for his encouragements and support. Finally, the authors want to thank Haiyang Wang for his discussions when he was a graduate student in Beijing Normal University.*

## REFERENCES

- [1] Benko K, Kothe M, Semmler K D, et al. Eigenvalues of the Laplacian and curvature. *Colloq Math*, 1979, 42: 19–31
- [2] Bolton J, Woodward L M. On the Simon conjecture for minimal immersions with  $S^1$ -symmetry. *Math Z*, 1988, 200: 111–121
- [3] Calabi E. Minimal immersions of surfaces in euclidean spheres. *J Differential Geom*, 1967, 1: 111–125
- [4] Chang S P. On minimal hypersurfaces with constant scalar curvatures in  $\mathbb{S}^4$ . *J Differential Geom*, 1993, 37(3): 523–534.

- [5] Cheng Q M. The rigidity of Clifford torus  $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$ . *Comment Math Helv*, 1996, 71(1): 60–69
- [6] Chern S S, do Carmo M, Kobayashi S. Minimal submanifolds of a sphere with second fundamental form of constant length. In: *Functional Analysis and Related Fields: Proceedings of a Conference in honor of Professor Marshall Stone, held at the University of Chicago, May 1968*. Berlin: Springer, 1970, 59–75
- [7] Dillen F. Minimal immersions of surfaces into spheres. *Arch Math (Basel)*, 1987, 49(1): 94–96
- [8] Ding Q, Xin Y L. On Chern’s problem for rigidity of minimal hypersurfaces in the spheres. *Adv Math*, 2011, 227(1): 131–145
- [9] do Carmo M, Wallach N. Minimal immersions of spheres into spheres. *Ann of Math (2)*, 1971, 93: 43–62
- [10] Gu J R, Xu H W, Xu Z Y, et al. A survey on rigidity problems in geometry and topology of submanifolds. In: *Proceedings of the 6th International Congress of Chinese Mathematicians. Advanced Lectures in Mathematics*, vol. 37. Beijing-Boston: Higher Education Press-International Press, 2016, 79–99
- [11] Guadalupe I, Rodriguez L. Normal curvature of surfaces in space forms. *Pacific J Math*, 1983, 106(1): 95–103
- [12] Itoh T. A characterization of the generalized Veronese surfaces. *Proc Amer Math Soc*, 1988, 104(2): 571–576
- [13] Kozłowski M, Simon U. Minimal immersion of 2-manifolds into spheres. *Math Z*, 1984, 186(3): 377–382
- [14] Lawson H B. Local rigidity theorems for minimal hypersurfaces. *Ann of Math (2)*, 1969, 89: 187–197
- [15] Lei L, Xu H W, Xu Z Y. On the generalized Chern conjecture for hypersurfaces with constant mean curvature in a sphere. *Sci China Math*, 2021, 64(7): 1493–1504
- [16] Li A M, Li J M. An intrinsic rigidity theorem for minimal submanifolds in a sphere. *Arch Math (Basel)*, 1991, 58(6): 582–594
- [17] Li H Z, Simon U. Quantization of curvature for compact surfaces in  $S^n$ . *Math Z*, 2003, 245(2): 201–216
- [18] Lu Z Q. Normal scalar curvature conjecture and its applications. *J Funct Anal*, 2011, 261(5): 1284–1308
- [19] Okayasu T. Minimal immersions of curvature pinched 2-manifolds into spheres. *Kodai Math J*, 1987, 10(1): 116–126
- [20] Pavlista T. Geometrische Abschätzungen kleiner Eigenwerte des Laplaceoperators. PhD Dissertation. Berlin: Technische Universität Berlin, 1984
- [21] Peng C K, Terng C L. The scalar curvature of minimal hypersurfaces in spheres. *Math Ann*, 1983, 266(1): 105–113
- [22] Reilly R C. Applications of the Hessian operator in a Riemannian manifold. *Indiana Univ Math J*, 1977, 26(3): 459–472
- [23] Sakamoto K. On the curvature of minimal 2-spheres in spheres. *Math Z*, 1998, 228(4): 605–627
- [24] Scherfner M, Weiss S, Yau S T. A review of the Chern conjecture for isoparametric hypersurfaces in spheres. In: *Advances in Geometric Analysis, Advanced Lectures in Mathematics*, vol. 21. Beijing-Boston: Higher Education Press-International Press, 2012, 175–187
- [25] Simon U. Eigenvalues of the Laplacian and minimal immersions into spheres. In: *Differential Geometry*, Montreal: Pitman, Boston, MA, 1985, 131: 115–120
- [26] Simons J. Minimal varieties in Riemannian manifolds. *Ann of Math (2)*, 1968, 88: 62–105

- [27] Tang Z Z, Wei D Y, Yan W J. A sufficient condition for a hypersurface to be isoparametric. *Tohoku Math J (2)*, 2020, 72(4): 493–50
- [28] Tang Z Z, Yan W J. On the Chern conjecture for isoparametric hypersurfaces. *Sci China Math*, 2023, 66(1): 143–162
- [29] Xu H W, Xu Z Y. On Chern’s conjecture for minimal hypersurfaces and rigidity of self-shrinkers. *J Funct Anal*, 2017, 273(11): 3406–3425
- [30] Yang D. Minimal surfaces in a unit sphere pinched by intrinsic curvature and normal curvature. *Sci China Math*, 2019, 62(9): 1779–1792
- [31] Yang H C, Cheng Q M. A note on the pinching constant of minimal hypersurfaces with constant scalar curvature in the unit sphere. *Chinese Sci Bull*, 1991, 36(1): 1–6
- [32] Yang H C, Cheng Q M. An estimate of the pinching constant of minimal hypersurfaces with constant scalar curvature in the unit sphere. *Manuscripta Math*, 1994, 84(1): 89–100
- [33] Yang H C, Cheng Q M. Chern’s conjecture on minimal hypersurfaces. *Math Z*, 1998, 227(3): 377–390

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