
Stochastic Optimization Using Ricci Flow

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Abstract

This paper proposes a theoretical framework for modeling and optimizing the bounded functions based on the Fourier series approximation and Ricci flow. Specifically, the initial manifold, \mathcal{M}_0 is approximated using Fourier series approximation in conjunction with the center and boundary sampling procedure introduced in the paper. The manifold is iteratively evolved using an algorithm that involves sampling along circles defined by the Riemannian metric tensor. Furthermore, the function is optimized by applying inverse Ricci flow i.e. instead of regularizing the manifold, flow allows for the high curvature regions to be accentuated. This allows for the singularities to occur at potential global optima assuming the deviation of the manifold at any point is smaller than the optimum.

1 Introduction

Due to intrinsic uncertainty, stochastic processes are far more challenging to model and optimize in comparison to the corresponding deterministic processes[1]. From its inception [2,3], a significant body of work dedicated to modeling and optimization of these processes has since been developed [4,5].

The modeling approach includes approximating the function to be log-concave [6,7]. Therefore, the results obtained using these methods are limited in their ability to predict the local optimum of highly non-convex functions.

Moreover, a large amount of the work skews towards using unidirectional randomized search methods such as random walk [8,9] and hit-and-run [10], and simulated annealing [11]. In addition, zeroth order optimization [12] is a favored method to reduce computational complexity [13].

Monte Carlo-based sampling methods for optimizing stochastic optimization find applications in a variety of problems such as Machine learning, supply chain networking, engineering design, and scheduling [14]. However, these methods are effective only when a small sample size is needed. Owing to the curse of dimensionality, stochastic optimization suffers from large computation complexity as the sample size increases drastically [15].

To address the problem of optimizing highly non-convex functions with reasonable accuracy, we propose a theoretical framework outlined in the paper. The motivation of this work is to exploit the topological structure embedded in the data and use appropriate methods to deform this structure strategically toward obtaining optimum solutions.

The paper is broadly divided into two sections. In section 2, wave superposition is used to approximate the function manifold. A detailed sampling procedure is outlined to approximate the initial manifold and its evolution. In section 3, the manifold is optimized using a strategic combination of different types of Ricci flows. The flow allows for singularities to happen at optimum locations. These locations are filtered based on the sign of the objective function (maxima or minima). The filtered points are checked and compared for the best feasible solution.

The method outlined in the paper is a theoretical framework that can serve as a basis for solving highly non-convex problems. However, the method can be tailored for specific problems depending on the problem specifications.

2 Wave superposition for topology approximation

The motivation for modeling the stochastic process is to extract the topology embedded in the sampled dataset. The superposition of sine waves or Fourier series approximation can approximate a manifold using a set of samples. The manifold approximation allows for the sampling function to become continuous. In other words, the information is extrapolated in the regions beyond the sampling point in a smooth continuous space. This allows us to use sophisticated optimization techniques that allow for the search of the optimum using the curvature information.

Assumptions: The following assumptions are made in order to approximate and optimize the function using this method:

- The function can be approximated reasonably accurately such that deviation at no point is larger than the optimum value of the function.
- The manifold is smooth.
- The function is bounded and continuous.

2.1 Manifold approximation

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded sampling function such that

$$f : \prod_{d \in \mathcal{D}} [d_L, d_U] \rightarrow \mathbb{R}, \quad \mathcal{D} = \{1, 2, 3, \dots, n\}$$

The function can be approximated at any point \mathbf{x} in the sampled domain using the Fourier series transformation as follows:

$$f(\mathbf{x}) \approx \sum_{\mathbf{k} \in K} a_{\mathbf{k}} e^{j\mathbf{k}\omega\mathbf{x}} \quad (1)$$

where ω represents fundamental frequency in each dimension and K is a set of indices of all the frequency components. We will denote the R.H.S. in the above equation as $F(x)$.

The coefficients a_k can be approximated using N samples as follows:

$$a_{\mathbf{k}} = \frac{1}{N} \sum_{i=1}^N f(\mathbf{x}_i) e^{-j\mathbf{k}\omega\mathbf{x}_i} \quad (2)$$

If we define the error function as following:

$$\epsilon(N) = |f(x) - F(x)|$$

Then the error function approaches zero as the number of samples reaches to infinity, i.e.

$$\text{As } N \rightarrow \infty, \quad \epsilon \rightarrow 0$$

Please note that due to the superposition of sine waves, the manifold is smooth in nature. The number of waves superimposed in the same direction favoring the optima provides local information about the distribution of optimum points. However, in order to minimize sampling, we adopt a more detailed procedure for approximating the manifold as outlined below.

2.2 Sampling procedure and manifold evolution

We begin by approximating the manifold using a minimal number of sampled data points that allow for the manifold to best approximate the functions at the corner points and center point.

Therefore, for the first iteration, we define sampling using the following two sets of samples:

1. **Boundary sampling:** These samples are ideally collected corresponding to the corner points in all the dimensions. These points can alternately be taken as the extreme points in the feasible space. However, if the feasible space is unknown, these points can be taken arbitrarily such that it allows for a large set of data points to lie in this region. These sets are chosen based on the specifics of the problem under consideration.

2. Midpoint sampling: The points are ideally evaluated at the midpoint such that each coordinate corresponds to the midpoint of the respective dimensions bounds. We denote this midpoint as \mathbf{p} .

Using these samples and by adopting the outlined procedure for approximating the manifold in section 2.1, we approximate the manifold. Let's call this manifold, \mathcal{M}_0^{n+1}

Next, we draw circles on this manifold as it evolves at constant intervals such that the radii of each of these circles, $C_{\mathbf{p},r}$ on the z-evolved manifold, \mathcal{M}_z^{n+1} are given by the Riemannian metric tensor from the midpoint to the surface of the sphere.

$$C_{\mathbf{p},r} = \{\mathbf{q} \in \mathcal{M}_z^{n+1} \mid d(\mathbf{p}, \mathbf{q}) = r\}$$

where g is geodesic distance from \mathbf{p} to \mathbf{q} .

For each circle, we randomly pick a point and evaluate the function, $F(x)$ value at this point.

if $|f(x) - F(x)| \leq \alpha$, we reject the sample

else, we accept the sample.

At each iteration, we collect the samples and recreate the function by using the Fourier series approximation method in section 2.1.

We perform these iterations until the circle $C_{\mathbf{p},r}$ crosses the boundary of the manifold. The algorithm is written as follows.

Algorithm 1 Manifold Approximation and Sampling

- 1: Initialize the number of dimensions, n .
 - 2: Define the bounds for each dimension.
 - 3: **Boundary Sampling:**
 - 4: Collect samples at the corner points in all dimensions.
 - 5: For unknown feasible space is unknown, arbitrarily choose extreme points.
 - 6: **Midpoint Sampling:**
 - 7: Sample the midpoint, \mathbf{p} .
 - 8: **Approximate Initial Manifold:**
 - 9: Approximate the initial manifold, denoted as \mathcal{M}_0^{n+1} .
 - 10: **Circular sampling and Manifold evolution**
 - 11: **for** each iteration **do**
 - 12: **Sampling on Circles:**
 - 13: Draw circles centered at \mathbf{p} with radii determined by the Riemannian metric tensor.
 - 14: **for** each circle $C_{\mathbf{p},r}$ defined by $d(\mathbf{p}, \mathbf{q}) = r$ **do**
 - 15: Randomly select a point \mathbf{q} on the circle.
 - 16: Evaluate the function $F(\mathbf{q})$ at point \mathbf{q} .
 - 17: **end for**
 - 18: **Sample Acceptance Criteria:**
 - 19: **if** $|f(\mathbf{q}) - F(\mathbf{q})| \leq \alpha$ **then**
 - 20: Reject the sample.
 - 21: **else**
 - 22: Accept the sample.
 - 23: **end if**
 - 24: Evolve the manifold \mathcal{M}_z^{n+1} at constant intervals using accepted samples.
 - 25: **end for**
 - 26: **Check Termination:**
 - 27: Continue the iterations until the circle $C_{\mathbf{p},r}$ crosses the boundary of the manifold.
 - 28: **Output:**
 - 29: The final optimized manifold.
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The final manifold is then optimized using appropriate flow conditions on the manifold. The flow should be such that it allows the detection of extreme points on the manifold.

3 Ricci Flow Informed Optimization

To find the optimum on the manifold, we apply an iterative process using a combination of Ricci flow and inverse Ricci flow as defined below.

In the first iteration, for the Riemannian metric, $g(t)$, the manifold, \mathcal{M}_z^{n+1} for $t \in (0, T]$ is deformed using inverse Ricci flow which is defined as follows:

$$\frac{\partial g}{\partial t} = 2 * Ric(g)$$

This inverse Ricci flow causes the space to deform such that the curvature of the point of largest curvature increases until the singularity formation at this point.

In the next iteration, we apply the Ricci flow on \mathcal{M}_z^{n+1} with the exception that at the point of singularity in the previous iteration:

$$\frac{\partial g}{\partial t} = -2 * Ric(g)$$

This process continues until the desired number of candidate points are gathered for optima.

Algorithm 2 Finding the Optimum on the Manifold

- Initialize the manifold \mathcal{M}_z^{n+1} and the Riemannian metric $g(t)$ for $t \in (0, T]$.
- 2: **First Iteration: Inverse Ricci Flow**
for each $t \in (0, T]$ **do**
 - 4: Deform the manifold \mathcal{M}_z^{n+1} using the inverse Ricci flow:

$$\frac{\partial g}{\partial t} = 2 \cdot Ric(g)$$

end for
 - 6: **Next Iteration: Ricci Flow**
if point of singularity from the previous iteration **then**
 - 8: Deform the manifold \mathcal{M}_z^{n+1} using the Ricci flow:

$$\frac{\partial g}{\partial t} = -2 \cdot Ric(g)$$

else
 - 10: Deform the manifold \mathcal{M}_z^{n+1} using the inverse Ricci flow:

$$\frac{\partial g}{\partial t} = 2 \cdot Ric(g)$$

end if
 - 12: Continue until the desired number of candidate points for optima are reached.
 - Filtering Candidate Points**
 - 14: Filter the candidate points based on the objective function (minima or maxima).
 Choose the best solution among the filtered candidate points as the local optimum.
 - 16: **Output:**
 The local optimum on the manifold.
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Since we approximated the manifold using Fourier series approximation, the filtered candidate points depending on the objective function (minima or maxima) obtained using the above procedure are checked, and the best solution is chosen as the local optimum.

4 Results and Conclusion

In this paper, we proposed a novel method for optimizing the stochastic processes. Using a defined sampling procedure that allows for boundary and center point sampling, a manifold is approximated using Fourier series approximation. The manifold is iteratively evolved using samples on the circles formed on the evolved matrix using the Riemannian metric tensor. We continue the procedure until the circle crosses the boundary of the manifold. For optimization, we de-regularize the manifold using a combination of Ricci flow and inverse Ricci flow such that the regions of large curvature in the appropriate direction depend on maxima or minima form singularities. These points form a set of candidates for optima. These points are checked and compared for the best solution. Thus obtained solution is the optimum point.

5 Data Availability Statement

The manuscript has no associated data.

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