A CHENG-YAU TYPE ESTIMATE FOR THE SYMPLECTIC CALABI-YAU EQUATION

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ABSTRACT. In the setting of Donaldson's conjecture on the Calabi-Yau equation on symplectic 4-manifolds, we prove an a priori estimate which in the Kähler case resembles a classical estimate of Cheng-Yau.

1. INTRODUCTION

The Calabi Conjecture, solved by Yau in 1976 [21], says that given a smooth positive volume form σ on a compact Kähler manifold (M, ω) with total mass $\int_M \sigma = \int_M \omega^n$, we can find a unique Kähler metric $\tilde{\omega}$ in the same cohomology class as ω whose volume element is pointwise equal to σ , i.e $\tilde{\omega}^n = \sigma$. The uniqueness statement was proved by Calabi in 1954 [3], and the existence follows via a continuity method from *a priori* C^{∞} estimates for a Kähler metric which only depend on its volume form, its cohomology class, and the underlying complex manifold.

In 2006, Donaldson [6] proposed a conjectural generalization of this result to symplectic 4-manifolds. More precisely, suppose (M^4, J) is a closed almost-complex manifold and ω is a symplectic form taming J, which means that

 $\omega(X, JX) > 0$, for all $X \neq 0$.

The (1, 1)-part of ω with respect to J is denoted by $\omega^{(1,1)}$. Let $\tilde{\omega}$ be another symplectic form, cohomologous to ω and compatible with J, which means that it tames J and furthermore

$$\tilde{\omega}(X,Y) = \tilde{\omega}(JX,JY).$$

There are associated Hermitian metrics g, \tilde{g} which are defined by

$$g(X,Y) = \frac{1}{2} \left(\omega(X,JY) + \omega(Y,JX) \right), \quad \tilde{g}(X,Y) = \tilde{\omega}(X,JY)$$

with corresponding volume forms $(\omega^{(1,1)})^2$ and $\tilde{\omega}^2$. Donaldson then conjectured the following:

Conjecture 1.1. Let (M^4, J) be a closed almost-complex 4-manifold, ω a symplectic form taming J, $\tilde{\omega}$ a cohomologous symplectic form compatible with J, and σ a smooth positive volume form with $\int_M \sigma = \int_M \omega^2$. If $\tilde{\omega}$ satisfies the Calabi-Yau equation

1

(1.1)
$$\tilde{\omega}^2 = \sigma,$$

VALENTINO TOSATTI

then for any $k \ge 0$, we can bound $\|\tilde{\omega}\|_{C^k(M,g)}$ by a constant that depends only on k and on bounds on $\sigma, \omega, (M, J)$.

When J is integrable and ω is compatible with J, we are on a Kähler surface and Conjecture 1.1 follows from the aforementioned theorem of Yau [21]. If solved, Donaldson's conjecture would have striking applications in symplectic geometry, see [6, 15]. Donaldson's conjecture was first investigated by Weinkove [20] in the case when ω is also compatible with J, where he showed that it holds provided J is close to being integrable. In [18], Tosatti-Weinkove-Yau showed that Conjecture 1.1 in general would follow if one could prove a bound

(1.2)
$$\operatorname{tr}_{q}\tilde{g} \leqslant C,$$

where C depends only on $\sigma, \omega, (M, J)$ (we will call such constants uniform), and that (1.2) can indeed be established when g has nonnegative curvature in a suitable sense. Further progress on Donaldson's conjecture has since proceeded in two main directions: proving it on explicit examples [14, 7, 1, 2, 17, 19], and for general manifolds reducing the bound (1.2) to bounding an "almost-Kähler potential" function. For this, following [20, 18], one defines a function $\varphi \in C^{\infty}(M, \mathbb{R})$ by

(1.3)
$$\Delta_{\tilde{g}}\varphi = 2 - \operatorname{tr}_{\tilde{g}}g, \quad \sup_{M}\varphi = 0,$$

which is uniquely determined and in the Kähler case would satisfy $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$, i.e. φ would be a familiar Kähler potential. Estimate (1.2) was then successively reduced to proving uniform bounds for $\int_M e^{-\alpha\varphi}\omega^2$ (for some $\alpha > 0$) in [18] in 2007, for $\int_M |\varphi|\omega^2$ in [13, Rmk 3.1] in 2009 (with a very recent new proof in [8]), for *any* integral bound for φ in [17] in 2016, and lastly to a uniform positive lower bound for the Lebesgue measure of $\{\varphi \ge -C\}$ for any given uniform C in [17].

Nevertheless, none of these results use the crucial assumption that $\tilde{\omega}$ and ω are cohomologous: if this is not the case then one has to replace the constant 2 in (1.3) with another suitable constant, namely

(1.4)
$$\Delta_{\tilde{g}}\varphi = \frac{2\int_{M}\omega\wedge\tilde{\omega}}{\int_{M}\tilde{\omega}^{2}} - \operatorname{tr}_{\tilde{g}}g, \quad \sup_{M}\varphi = 0,$$

but there is no essential difference in all the above results, which still show that to obtain uniform C^{∞} bounds for $\tilde{\omega}$ (and a uniform lower bound $\tilde{\omega} \ge C^{-1}\omega$) it suffices to have for example any uniform integral bound for φ (or just a uniform lower bound for the measure of a superlevel set), provided the constant $\frac{2\int_M \omega \wedge \tilde{\omega}}{\int_M \tilde{\omega}^2}$ is uniformly bounded above. Consider the Kähler case, where J is integrable and ω is compatible with it, and degenerate $[\tilde{\omega}]$ to a limiting class $[\alpha]$ on the boundary of the Kähler cone with $\int_M \alpha^2 > 0$ (there are many such examples, see e.g. [11]). In this case the constant in (1.4) remains uniformly bounded, but the corresponding solution metrics $\tilde{\omega}$ cannot converge smoothly since $[\alpha]$ does not contain any Kähler metric, hence in this case no uniform integral/measure bound for φ solving (1.4) can hold. In other words, if one wants to use the almost-Kähler potential φ to prove (1.2), then some new idea has to be used which makes crucial use of the cohomological assumption.

In this note we take a different approach. Because of the cohomological assumption, we can write

(1.5)
$$\tilde{\omega} = \omega + da,$$

for some real 1-form a. We are free to modify a by adding df for any smooth function f, and if we choose f solving the elliptic equation $d_{\tilde{g}}^* df = -d_{\tilde{g}}^* a$, we can then assume without loss that

(1.6)
$$d^*_{\tilde{q}}a = 0.$$

The 1-form *a* satisfying (1.5) and (1.6) is then uniquely determined modulo the addition of a \tilde{g} -harmonic 1-form, and we can fix this ambiguity for example by requiring that *a* be $L^2(\tilde{g})$ -orthogonal to the space of such forms. The Calabi-Yau equation (1.1) is then a nonlinear elliptic system for the 1-form *a* (when dim M = 4). Our main result is then the following:

Theorem 1.2. In the above setting, we have the estimate

(1.7)
$$\operatorname{tr}_{g}\tilde{g} \leqslant C(1+\sup_{M}|a|_{\tilde{g}}^{2}),$$

where C is a uniform constant.

We make a few remarks about this result.

1. It is important for our arguments that the gauge-fixing condition (1.6) and the norm on the RHS of (1.7) are both with respect to \tilde{g} as opposed to g. Changing (1.6) to $d_g^* a = 0$ does not seem completely out of the question, but replacing $|a|_{\tilde{g}}^2$ with $|a|_g^2$ in (1.7) does not seem feasible.

2. In the Kähler case, i.e. when J is integrable and ω is compatible with J, we actually have that $a = d^c \varphi$ where φ is the usual Kähler potential so that $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$. Indeed $dd^c \varphi = i\partial\overline{\partial}\varphi$, and

$$d^*_{\tilde{q}}d^c\varphi = -*_{\tilde{g}}d*_{\tilde{g}}d^c\varphi = *_{\tilde{g}}d(d\varphi \wedge \tilde{\omega}) = 0,$$

and for any \tilde{g} -harmonic 1-form α ,

$$\int_M d^c \varphi \wedge *_{\tilde{g}} \alpha = \int_M d^c \varphi \wedge J(\alpha) \wedge \tilde{\omega} = \int_M d\varphi \wedge \alpha \wedge \tilde{\omega} = 0.$$

Thus, in the Kähler case, estimate (1.7) becomes

(1.8)
$$\operatorname{tr}_{g}\tilde{g} \leqslant C(1 + \sup_{M} |\nabla \varphi|_{\tilde{g}}^{2})$$

which in the local setting of pseudoconvex domains in \mathbb{C}^n was proved by Cheng-Yau [4, Proposition 7.1]. Tracing through our arguments easily shows that in the Kähler case (1.8) holds in all dimensions, and it does not use any

VALENTINO TOSATTI

knowledge about L^{∞} a priori bounds for φ (which were famously proved by Yau [21] using Moser iteration).

3. An estimate of the different but related form

(1.9)
$$\operatorname{tr}_{g}\tilde{g} \leqslant C(1 + \sup_{M} |\nabla\varphi|_{g}^{2}),$$

is known for different geometric PDEs on Kähler manifolds, including for example the complex Hessian equation [9] and the Monge-Ampère equation for (n-1)-psh functions [16], see also [10]. In these settings, (1.9) is then used together with a blowup argument and a Liouville theorem [5, 16, 10] to show that $\sup_M |\nabla \varphi|_g^2 \leq C$ and hence $\operatorname{tr}_g \tilde{g} \leq C$. However, the fact that \tilde{g} appears on the RHS of (1.7) instead of g makes it ill-suited to blowup arguments.

The proof of Theorem 1.2 is done by a maximum principle argument, and interestingly it uses crucially that the dimension of M is 4 (while the aforementioned results in [18, 17] apply in all even dimensions $2n \ge 4$), except in the Kähler case where our argument works in all dimensions.

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2. Proof of Theorem 1.2

We work in the setting described in the Introduction. Define a smooth function F on M by

$$\sigma = e^F(\omega^{(1,1)})^2,$$

so that in any local chart the Calabi-Yau equation (1.1) can be written as

(2.10)
$$\det(\tilde{g}) = e^{F} \det(g)$$

Since g, \tilde{g} are both Hermitian with respect to J, and dim M = 4, it follows from (2.10) that we have

(2.11)
$$\operatorname{tr}_{\tilde{q}}g = e^{-F}\operatorname{tr}_{q}\tilde{g},$$

which we will use repeatedly, often without mention.

As in [18], we will use covariant derivatives with respect the Chern connection ∇ of g (also known as "canonical connection", see [18, §2]). We recall two estimates proved in [18]. The first one is the differential inequality from [18] (which can be extracted from the proof of Lemma 3.2 there)

(2.12)
$$\Delta_{\tilde{g}} \operatorname{tr}_{g} \tilde{g} \ge |\nabla \tilde{g}|_{g,\tilde{g}}^{2} - C \operatorname{tr}_{g} \tilde{g} \operatorname{tr}_{\tilde{g}} g - C \ge |\nabla \tilde{g}|_{g,\tilde{g}}^{2} - C (\operatorname{tr}_{g} \tilde{g})^{2} - C,$$

using (2.11), where the norm $|\cdot|_{g,\tilde{g}}^2$ uses both g and \tilde{g} , see (2.31) below for the definition. The second one is the following Cauchy-Schwarz type inequality

[18, (3.20)]

(2.13)
$$|\nabla \tilde{g}|_{g,\tilde{g}}^2 \ge \frac{|\nabla \mathrm{tr}_g \tilde{g}|_{\tilde{g}}^2}{\mathrm{tr}_g \tilde{g}}$$

Given these preliminaries, the main claim is then the following:

Proposition 2.1. There is a uniform C > 0 such that for any small $\varepsilon > 0$ we have the differential inequality

$$(2.14) \qquad \Delta_{\tilde{g}}|a|_{\tilde{g}}^2 \geqslant \frac{1}{2}|\tilde{\nabla}a|_{\tilde{g}}^2 + \frac{1}{2}|\overline{\tilde{\nabla}}a|_{\tilde{g}}^2 + \frac{1}{C}(\operatorname{tr}_g \tilde{g})^2 - \varepsilon|\nabla \tilde{g}|_{g,\tilde{g}}^2 - \frac{C}{\varepsilon^2}|a|_{\tilde{g}}^4 - C.$$

In fact, as follows from the proof, the constants $\frac{1}{2}$ on the RHS of (2.14) can be taken to be $1 - \delta$ for any given $\delta > 0$, at the expense of making *C* larger.

Proof of Theorem 1.2 assuming Proposition 2.1. Let us first assume that Proposition 2.1 holds, and use it to prove (1.7). For this, combining (2.12) and (2.14), and throwing away two positive terms, we have

$$\Delta_{\tilde{g}}(\operatorname{tr}_{g}\tilde{g}+A|a|_{\tilde{g}}^{2}) \geqslant |\nabla \tilde{g}|_{g,\tilde{g}}^{2} - C_{0}(\operatorname{tr}_{g}\tilde{g})^{2} - C + \frac{A}{C_{1}}(\operatorname{tr}_{g}\tilde{g})^{2} - \varepsilon A|\nabla \tilde{g}|_{g,\tilde{g}}^{2} - \frac{CA}{\varepsilon^{2}}|a|_{\tilde{g}}^{4} - CA|\nabla \tilde{g}|_{g,\tilde{g}}^{2} - \frac{CA}{\varepsilon^{2}}|a|_{\tilde{g}}^{4} - CA|\nabla \tilde{g}|_{g,\tilde{g}}^{2} - \varepsilon A|\nabla \tilde{g}|_{g,\tilde{g}}^{$$

where C_0, C_1 are uniform constants, and now if we choose $A = C_1(C_0 + 1)$ and $\varepsilon = \frac{1}{A}$, we obtain

$$\Delta_{\tilde{g}}(\mathrm{tr}_{g}\tilde{g}+A|a|_{\tilde{g}}^{2}) \geqslant (\mathrm{tr}_{g}\tilde{g})^{2}-C|a|_{\tilde{g}}^{4}-C,$$

and the maximum principle concludes the proof of (1.7).

Beginning of proof of Proposition 2.1. We now proceed to prove Proposition 2.1. As in [18] we also use the Chern connection $\tilde{\nabla}$ of \tilde{g} , and the formalism of moving frames. Thus, we work in a local \tilde{g} -unitary frame $\tilde{\theta}^i, i = 1, 2$, which are (1, 0)-forms, and we have the first structure equations

$$d\tilde{\theta}^i = -\tilde{\theta}^i_j \wedge \tilde{\theta}^j + \tilde{\Theta}^i,$$

where $\tilde{\theta}_j^i$ are the connection 1-forms and $\tilde{\Theta}^i$ are the torsion forms, which are of type (0,2) and equal to

$$\tilde{\Theta}^i = \tilde{N}^i_{\overline{jk}} \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^k},$$

where \tilde{N}_{jk}^i are the components of the Nijenhuis tensor of J (and are skew-symmetric in j, k). The second structure equations read

$$d\tilde{\theta}^i_j = -\tilde{\theta}^i_k \wedge \tilde{\theta}^k_j + \tilde{\Omega}^i_j$$

where $\tilde{\Omega}_{i}^{i}$ are the curvature 2-forms. They can be written as

$$\tilde{\Omega}^i_j = \tilde{R}^i_{jk\overline{\ell}} \tilde{\theta}^k \wedge \overline{\tilde{\theta}^\ell} + \tilde{K}^i_{jk\ell} \tilde{\theta}^k \wedge \tilde{\theta}^\ell + \tilde{K}^i_{j\overline{k\overline{\ell}}} \overline{\tilde{\theta}^k} \wedge \overline{\tilde{\theta}^\ell},$$

and the Ricci curvature form is defined to be

$$\operatorname{Ric}(\tilde{g}) = \frac{\sqrt{-1}}{2\pi} \tilde{\Omega}_i^i,$$

where here and in the following we will mostly omit summation signs. It is well-known that $\operatorname{Ric}(\tilde{g})$ is a closed real 2-form which represents the first Chern class $c_1(M, J)$ of the complex vector bundle $(T^{\mathbb{R}}M, J)$. Its (1, 1)-part is given by

$$\frac{\sqrt{-1}}{2\pi}\tilde{R}_{k\overline{\ell}}\tilde{\theta}^k\wedge\overline{\tilde{\theta}^\ell},\quad \tilde{R}_{k\overline{\ell}}=\tilde{R}^i_{ik\overline{\ell}}.$$

Its relation with the Calabi-Yau equation is this [18, (3.16)]: if g, \hat{g} are two *J*-Hermitian metrics then we have

(2.15)
$$\frac{1}{2\pi} dd^c \log \frac{\det(g)}{\det(\hat{g})} = \operatorname{Ric}(\hat{g}) - \operatorname{Ric}(g),$$

where for a real-valued function f we define (1,0) and (0,1) forms $\partial f, \overline{\partial} f$ by $df = \partial f + \overline{\partial} f$, and define $d^c f = \frac{\sqrt{-1}}{2}(\overline{\partial} f - \partial f)$. A different sign convention was used in [18], so the term $dd^c f$ here is equal to the term $-\frac{1}{2}d(Jdf)$ in [18]. The trace of the (1,1)-part of $dd^c f$ with respect to \tilde{g} equals its Laplacian $\Delta_{\tilde{g}}f$, see [18, Lemma 2.5], which is equal to the usual Laplace-Beltrami operator of \tilde{g} (up to a factor of $\frac{1}{2}$). It can be written as

$$\Delta_{\tilde{g}}f = 2\frac{\tilde{\omega} \wedge dd^c f}{\tilde{\omega}^2}.$$

If we have a tensor T then we can consider its covariant derivatives ∇T and $\overline{\nabla}T$. For example, if in our frame T has components $\tilde{T}^i_{j\overline{k}}$ (as an example) then its covariant derivatives $\overline{\nabla}T$ and $\overline{\nabla}T$ have components $\tilde{T}^i_{j\overline{k},p}$ and $\tilde{T}^i_{j\overline{k},\overline{p}}$ respectively, which can be obtained as follows:

$$\tilde{T}^{i}_{j\overline{k},p}\tilde{\theta}^{p}+\tilde{T}^{i}_{j\overline{k},\overline{p}}\overline{\tilde{\theta}^{p}}=d\tilde{T}^{i}_{j\overline{k}}+\tilde{T}^{q}_{j\overline{k}}\tilde{\theta}^{i}_{q}-\tilde{T}^{i}_{q\overline{k}}\tilde{\theta}^{q}_{j}-\tilde{T}^{i}_{j\overline{q}}\overline{\tilde{\theta}^{q}_{k}},$$

and similarly for tensors of other types. We can easily compute [18, (2.28)] that if f is a real-valued function then we have

(2.16)
$$dd^{c}f = \tilde{f}_{\overline{i}}\overline{\tilde{\Theta}^{i}} + \tilde{f}_{i\overline{j}}\overline{\tilde{\theta}^{i}} \wedge \overline{\tilde{\theta}^{j}} + \tilde{f}_{i}\tilde{\Theta}^{i}$$

To simplify computations, denote by $\alpha = a^{(1,0)}$, so that

$$a = \alpha + \overline{\alpha}$$

and we have

$$|a|_{\tilde{g}}^2 = 2|\alpha|_{\tilde{g}}^2,$$

where we are using \tilde{g} here as a Riemannian metric on $T^{\mathbb{R}}M$ on the LHS and as a Hermitian metric on $T^{1,0}M$ on the RHS, and the equality follows from the fact that the Riemannian metric is *J*-invariant. In our frame we can write $\alpha = \tilde{\alpha}_i \tilde{\theta}^i$, so that

$$|\alpha|_{\tilde{q}}^2 = |\tilde{\alpha}_i|^2.$$

We then compute

$$d|\alpha|_{\tilde{g}}^{2} = \tilde{\alpha}_{i,j}\overline{\tilde{\alpha}_{i}}\overline{\tilde{\theta}}^{j} + \tilde{\alpha}_{i,\overline{j}}\overline{\tilde{\alpha}_{i}}\overline{\tilde{\theta}}^{j} + \tilde{\alpha}_{i}\overline{\tilde{\alpha}_{i,\overline{j}}}\overline{\tilde{\theta}}^{j} + \tilde{\alpha}_{i}\overline{\tilde{\alpha}_{i,j}}\overline{\tilde{\theta}}^{j},$$

hence

$$\partial |\alpha|_{\tilde{g}}^2 = \tilde{\alpha}_{i,j} \overline{\tilde{\alpha}_i} \tilde{\theta}^j + \tilde{\alpha}_i \overline{\tilde{\alpha}_{i,\overline{j}}} \tilde{\theta}^j,$$

$$(d\partial |\alpha|_{\tilde{g}}^2)^{(1,1)} = (\tilde{\alpha}_{i,j\overline{k}}\overline{\tilde{\alpha}_i} + \tilde{\alpha}_{i,j}\overline{\tilde{\alpha}_{i,k}} + \tilde{\alpha}_{i,\overline{k}}\overline{\tilde{\alpha}_{i,\overline{j}}} + \tilde{\alpha}_i\overline{\tilde{\alpha}_{i,\overline{j}k}})\overline{\tilde{\theta}^k} \wedge \tilde{\theta}^j$$

and since $dd^c |\alpha|_{\tilde{g}}^2 = -\sqrt{-1}d\partial |\alpha|_{\tilde{g}}^2$, this implies that

(2.17)
$$\Delta_{\tilde{g}} |\alpha|_{\tilde{g}}^2 = |\tilde{\alpha}_{i,j}|^2 + |\tilde{\alpha}_{i,\overline{j}}|^2 + \tilde{\alpha}_{i,k\overline{k}}\overline{\tilde{\alpha}_i} + \tilde{\alpha}_i\overline{\tilde{\alpha}_{i,\overline{k}k}}$$

Next, we need a commutation relation for covariant derivatives of α . We start with the definition

$$d\tilde{\alpha}_i = \tilde{\alpha}_j \tilde{\theta}_i^j + \tilde{\alpha}_{i,j} \tilde{\theta}^j + \tilde{\alpha}_{i,\overline{j}} \overline{\tilde{\theta}^j},$$

and applying d again

 $\begin{array}{l} (2.18) \\ 0 = dd\tilde{\alpha}_i = \tilde{\alpha}_j \tilde{\Omega}_i^j + \tilde{\alpha}_{i,jp} \tilde{\theta}^p \wedge \tilde{\theta}^j + \tilde{\alpha}_{i,j\overline{p}} \overline{\tilde{\theta}^p} \wedge \tilde{\theta}^j + \tilde{\alpha}_{i,j\overline{p}} \tilde{\Theta}^j + \tilde{\alpha}_{i,\overline{j}p} \tilde{\theta}^p \wedge \overline{\tilde{\theta}^j} + \tilde{\alpha}_{i,\overline{j}\overline{p}} \overline{\tilde{\theta}^p} \wedge \overline{\tilde{\theta}^j} + \tilde{\alpha}_{i,\overline{j}\overline{p}} \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^j} + \tilde{\alpha}_{i,\overline{j}\overline{p}} \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^j} + \tilde{\alpha}_{i,\overline{j}\overline{p}} \overline{\tilde{$

$$\tilde{\alpha}_{j}\tilde{R}^{j}_{ik\overline{\ell}}\tilde{\theta}^{k}\wedge\overline{\tilde{\theta}^{\ell}}-\tilde{\alpha}_{i,k\overline{\ell}}\tilde{\theta}^{k}\wedge\overline{\tilde{\theta}^{\ell}}+\tilde{\alpha}_{i,\overline{\ell}k}\tilde{\theta}^{k}\wedge\overline{\tilde{\theta}^{\ell}}=0,$$

and so we obtain the commutation relation

(2.19)
$$\tilde{\alpha}_{i,k\overline{\ell}} = \tilde{\alpha}_{i,\overline{\ell}k} + \tilde{\alpha}_j \tilde{R}^j_{ik\overline{\ell}},$$

exactly like in the Kähler case. Taking the (0, 2)-part of (2.18) gives

$$\tilde{\alpha}_{j}\tilde{K}_{i\overline{k}\overline{\ell}}^{j}\overline{\tilde{\theta}^{k}}\wedge\overline{\tilde{\theta}^{\ell}}+\tilde{\alpha}_{i,j}\tilde{N}_{\overline{k}\overline{\ell}}^{j}\overline{\tilde{\theta}^{k}}\wedge\overline{\tilde{\theta}^{\ell}}+\tilde{\alpha}_{i,\overline{\ell}\overline{k}}\overline{\tilde{\theta}^{k}}\wedge\overline{\tilde{\theta}^{\ell}}=0,$$

which, after skew-symmetrizing $\tilde{\alpha}_{i,\overline{\ell k}}$ in k,ℓ , gives the commutation relation

(2.20)
$$\tilde{\alpha}_{i,\overline{\ell k}} = \tilde{\alpha}_{i,\overline{k\ell}} - 2\tilde{\alpha}_{j}\tilde{K}^{j}_{i\overline{k\ell}} - 2\tilde{\alpha}_{i,j}\tilde{N}^{j}_{\overline{k\ell}}.$$

Using (2.19) in (2.17) gives

$$\Delta_{\tilde{g}} |\alpha|_{\tilde{g}}^2 = |\tilde{\alpha}_{i,j}|^2 + |\tilde{\alpha}_{i,\overline{j}}|^2 + 2\operatorname{Re}(\tilde{\alpha}_{i,\overline{k}k}\overline{\tilde{\alpha}_i}) + \tilde{\alpha}_j\overline{\tilde{\alpha}_i}\tilde{R}_{ik\overline{k}}^j,$$

and recalling from [18, (2.21)] that

(2.21)
$$\tilde{R}^{j}_{ik\overline{k}} = \tilde{R}_{i\overline{j}} - 4\tilde{N}^{q}_{\overline{p}\overline{j}}\overline{\tilde{N}^{p}_{\overline{q}\overline{i}}} - 4\tilde{N}^{p}_{\overline{q}\overline{j}}\overline{\tilde{N}^{i}_{\overline{p}\overline{q}}}$$

we obtain

$$\Delta_{\tilde{g}}|\alpha|_{\tilde{g}}^{2} = |\tilde{\alpha}_{i,j}|^{2} + |\tilde{\alpha}_{i,\overline{j}}|^{2} + 2\operatorname{Re}(\tilde{\alpha}_{i,\overline{k}k}\overline{\tilde{\alpha}_{i}}) + \tilde{R}_{i\overline{j}}\overline{\tilde{\alpha}_{j}}\overline{\tilde{\alpha}_{i}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{j}}^{q}\overline{\tilde{N}_{\overline{q}\overline{i}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{q}}^{p}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{q}}^{p}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{q}}^{p}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{q}}^{p}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{N}_{\overline{p}\overline{q}}^{p}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}}\widetilde{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\widetilde{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}}\widetilde{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}}\widetilde{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}\overline{q}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p}}}} - 4\tilde{\alpha}_{j}\overline{\tilde{N}_{\overline{p$$

To deal with the term $\tilde{R}_{i\overline{j}}\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}$, observe that differentiating the PDE (2.10) and using (2.15) we have that $\operatorname{Ric}(\tilde{g}) = \operatorname{Ric}(g) - \frac{1}{2\pi}dd^{c}F$, which is a fixed background tensor (independent of \tilde{g}), hence we can write $\tilde{R}_{i\overline{j}} = \tilde{T}_{i\overline{j}}$ where $T = \sqrt{-1}\tilde{T}_{i\overline{j}}\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{j}}$ is some fixed tensor. This implies that

$$\tilde{R}_{i\overline{j}}\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}} \geqslant -|T|_{\tilde{g}}|\alpha|_{\tilde{g}}^{2} \geqslant -C\mathrm{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2}.$$

Likewise, if we let now $\tilde{T}_{i\overline{j}} = \tilde{N}_{\overline{pj}}^q \overline{\tilde{N}_{q\overline{i}}^p}$, then $T = \sqrt{-1}\tilde{T}_{i\overline{j}}\tilde{\theta}^i \wedge \overline{\tilde{\theta}^j}$ is some fixed tensor (independent of \tilde{g}), and so by the same logic we can bound

$$-4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\tilde{N}^{q}_{\overline{pj}}\overline{\tilde{N}^{p}_{\overline{q}i}} \geqslant -C\mathrm{tr}_{g}\tilde{g}|\alpha|^{2}_{\tilde{g}}.$$

Unfortunately this doesn't work directly for the last term

(2.22)
$$-4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\tilde{N}^{p}_{\overline{qj}}\overline{\tilde{N}^{i}_{pq}},$$

since the corresponding tensor $T = \sqrt{-1}\tilde{N}_{\bar{q}\bar{j}}^{\bar{p}}\overline{\tilde{N}_{pq}^{i}}\tilde{\theta}^{i}\wedge \overline{\tilde{\theta}^{j}}$ depends also on \tilde{g} . To understand more precisely how to bound this, denote by θ^{i} a local unitary frame for the background metric g, so that we can write

$$\tilde{\theta}^i = a^i_j \theta^j, \quad \theta^i = b^i_j \tilde{\theta}^j,$$

where the local matrices (a_j^i) and (b_j^i) are inverses of each other, i.e. $a_j^i b_i^k = \delta_{jk}$. We can express the components of the Nijenhuis tensor with respect to the frame θ^i as

$$N^{i}_{\overline{j}\overline{k}} = \tilde{N}^{p}_{\overline{qr}}\overline{a^{j}_{j}}\overline{a^{r}_{k}}b^{i}_{p}, \quad \text{hence } \tilde{N}^{i}_{\overline{j}\overline{k}} = N^{p}_{\overline{qr}}\overline{b^{j}_{j}}\overline{b^{r}_{k}}a^{i}_{p},$$

and the components N_{jk}^i are all uniformly bounded. Similarly the components of (1, 0)-form α are given by $\alpha_i = a_i^j \tilde{\alpha}_j$, and so we can write

$$4\tilde{\alpha}_{j}\overline{\tilde{\alpha}_{i}}\tilde{N}^{p}_{\overline{qj}}\overline{\tilde{N}^{i}_{\overline{pq}}} = 4\alpha_{w}\overline{\alpha_{h}}b^{w}_{j}\overline{b^{h}_{i}}N^{k}_{\overline{\ell\tau}}\overline{N^{t}_{\overline{uv}}}b^{\ell}_{q}\overline{b^{r}_{j}}a^{p}_{k}b^{w}_{p}b^{v}_{q}\overline{a^{i}_{t}} = 4\alpha_{w}\overline{\alpha_{h}}b^{w}_{j}N^{k}_{\overline{\ell\tau}}\overline{N^{h}_{\overline{kv}}}\overline{b^{\ell}_{q}}\overline{b^{r}_{j}}b^{v}_{q}.$$

and working at an arbitrary point we may choose our unitary frames so that at this point we have

$$a_j^i = \sqrt{\lambda_i} \delta_{ij}$$

where $\lambda_1, \lambda_2 > 0$ are the eigenvalues of the Hermitian metric \tilde{g} with respect to g. This implies that

$$b_j^i = \frac{1}{\sqrt{\lambda_j}} \delta_{ij},$$

and so at our point our term (2.22) simplifies to

$$-4\sum_{j,q=1}^{2}\lambda_{j}^{-1}\lambda_{q}^{-1}\alpha_{j}\overline{\alpha_{h}}N^{k}_{\overline{qj}}\overline{N^{h}_{\overline{kq}}},$$

and since $N_{\overline{qj}}^k$ is skew-symmetric in q, j, only the terms in the sum with $j \neq q$ survive, and so this equals

$$-4\lambda_1^{-1}\lambda_2^{-1}\alpha_1\overline{\alpha_h}N_{21}^k\overline{N_{k2}^h} - 4\lambda_2^{-1}\lambda_1^{-1}\alpha_2\overline{\alpha_h}N_{12}^k\overline{N_{k1}^h}$$

but from the Calabi-Yau equation (1.1) we have that

(2.23)
$$\lambda_1 \lambda_2 = e^F,$$

which is uniformly bounded, and so our term (2.22) can be bounded below by

$$-C\sum_{h} |\alpha_{h}|^{2} = -C|\alpha|_{g}^{2} \ge -C\mathrm{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2},$$

and combining all of the above gives

$$\Delta_{\tilde{g}} |\alpha|_{\tilde{g}}^2 \geqslant |\tilde{\alpha}_{i,j}|^2 + |\tilde{\alpha}_{i,\overline{j}}|^2 + 2\text{Re}(\tilde{\alpha}_{i,\overline{k}k}\overline{\tilde{\alpha}_i}) - C\text{tr}_g \tilde{g} |\alpha|_{\tilde{g}}^2.$$

Observe that here we have used crucially that $\dim M = 4$ so that there are only two eigenvalues. This fact will be also used further below.

The main claim (2.24). If we denote by $\Delta_{\tilde{g}}^{H}a = -(dd_{\tilde{g}}^{*}a + d_{\tilde{g}}^{*}da)$ the \tilde{g} -Hodge Laplacian of $a = \alpha + \overline{\alpha}$, then the main claim is that

$$(2.24) \quad 2\operatorname{Re}(\tilde{\alpha}_{i,\overline{k}k}\overline{\tilde{\alpha}_{i}}) \geqslant \frac{1}{2}\tilde{g}(\Delta_{\tilde{g}}^{H}a,a) - \frac{1}{2}|\tilde{\alpha}_{i,j}|^{2} - C\operatorname{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2} - C|\nabla\tilde{g}|_{g,\tilde{g}}|\alpha|_{\tilde{g}}^{2}.$$

End of proof of Proposition 2.1 assuming (2.24). Assuming that (2.24) holds, let us complete the proof of Proposition 2.1. Plugging in, we get

$$\Delta_{\tilde{g}}|\alpha|_{\tilde{g}}^2 \ge \frac{1}{2}|\tilde{\alpha}_{i,j}|^2 + |\tilde{\alpha}_{i,\overline{j}}|^2 + \frac{1}{2}\tilde{g}(\Delta_{\tilde{g}}^H a, a) - C\mathrm{tr}_g\tilde{g}|\alpha|_{\tilde{g}}^2 - C|\nabla\tilde{g}|_{g,\tilde{g}}|\alpha|_{\tilde{g}}^2.$$

We compute

$$d\alpha = d(\tilde{\alpha}_i \tilde{\theta}^i) = \tilde{\alpha}_{i,j} \tilde{\theta}^j \wedge \tilde{\theta}^i + \tilde{\alpha}_{i,\overline{j}} \overline{\tilde{\theta}^j} \wedge \tilde{\theta}^i + \tilde{\alpha}_i \tilde{\Theta}^i,$$

and recall that

$$da = d\alpha + \overline{d\alpha} = \tilde{\omega} - \omega,$$

and taking the (1, 1)-part

$$(da)^{(1,1)} = \tilde{\omega} - \omega^{(1,1)},$$

hence

$$|(da)^{(1,1)}|_{\tilde{g}}^2 = 2 - 2\mathrm{tr}_{\tilde{g}}g + |g|_{\tilde{g}}^2 \ge 2 - 2\mathrm{tr}_{\tilde{g}}g + \frac{(\mathrm{tr}_{\tilde{g}}g)^2}{2} \ge \frac{(\mathrm{tr}_{\tilde{g}}g)^2}{4} - 2,$$

but we also have

$$(da)^{(1,1)} = (\tilde{\alpha}_{i,\overline{j}} - \overline{\tilde{\alpha}_{j,\overline{i}}})\tilde{\theta}^{j} \wedge \tilde{\theta}^{i},$$
$$|(da)^{(1,1)}|_{\tilde{g}}^{2} = |\tilde{\alpha}_{i,\overline{j}} - \overline{\tilde{\alpha}_{j,\overline{i}}}|^{2} \leqslant 4|\tilde{\alpha}_{i,\overline{j}}|^{2},$$

and so

$$|\tilde{\alpha}_{i,\overline{j}}|^2 \geqslant \frac{(\mathrm{tr}_{\tilde{g}}g)^2}{16} - \frac{1}{2} \geqslant \frac{(\mathrm{tr}_g\tilde{g})^2}{C} - \frac{1}{2}.$$

This gives

$$\Delta_{\tilde{g}}|\alpha|_{\tilde{g}}^2 \ge \frac{1}{2}|\tilde{\alpha}_{i,j}|^2 + \frac{1}{2}|\tilde{\alpha}_{i,\overline{j}}|^2 + \frac{(\operatorname{tr}_g \tilde{g})^2}{C} + \frac{1}{2}\tilde{g}(\Delta_{\tilde{g}}^H a, a) - C\operatorname{tr}_g \tilde{g}|\alpha|_{\tilde{g}}^2 - C|\nabla \tilde{g}|_{g,\tilde{g}}|\alpha|_{\tilde{g}}^2 - C,$$

and we also have

$$\frac{1}{2}|\tilde{\alpha}_{i,j}|^2 + \frac{1}{2}|\tilde{\alpha}_{i,\overline{j}}|^2 \geqslant \frac{1}{4}|\tilde{\nabla}a|_{\tilde{g}}^2 + \frac{1}{4}|\overline{\tilde{\nabla}}a|_{\tilde{g}}^2,$$

and recalling that $|a|_{\tilde{g}}^2 = 2|\alpha|_{\tilde{g}}^2$, we obtain (2.25)

$$\Delta_{\tilde{g}}|a|_{\tilde{g}}^2 \ge \frac{1}{2}|\tilde{\nabla}a|_{\tilde{g}}^2 + \frac{1}{2}|\overline{\tilde{\nabla}}a|_{\tilde{g}}^2 + \frac{(\mathrm{tr}_g\tilde{g})^2}{C} + \tilde{g}(\Delta_{\tilde{g}}^Ha, a) - C\mathrm{tr}_g\tilde{g}|a|_{\tilde{g}}^2 - C|\nabla\tilde{g}|_{g,\tilde{g}}|a|_{\tilde{g}}^2 - C.$$

We now deal with the term with the Hodge Laplacian. Using the gauge-fixing condition (1.6) we have

$$\Delta_{\tilde{g}}^{H}a = -dd_{\tilde{g}}^{*}a - d_{\tilde{g}}^{*}da = -d_{\tilde{g}}^{*}da = d_{\tilde{g}}^{*}\omega - d_{\tilde{g}}^{*}\tilde{\omega} = d_{\tilde{g}}^{*}\omega,$$

since $*_{\tilde{g}}\tilde{\omega} = \tilde{\omega}$ and so $d^*_{\tilde{g}}\tilde{\omega} = 0$. We also have the well-known formula

$$*_{\tilde{g}}\omega = \frac{2\omega \wedge \tilde{\omega}}{\tilde{\omega}^2}\tilde{\omega} - \omega + 2\omega^{(2,0)+(0,2)} = \operatorname{tr}_{\tilde{g}}g\,\tilde{\omega} - \omega + 2\omega^{(2,0)+(0,2)},$$
$$d *_{\tilde{g}}\omega = d\operatorname{tr}_{\tilde{g}}g \wedge \tilde{\omega} + 2d(\omega^{(2,0)+(0,2)}),$$
$$*_{\tilde{g}}d *_{\tilde{g}}\omega = d^c\operatorname{tr}_{\tilde{g}}g + 2 *_{\tilde{g}}d(\omega^{(2,0)+(0,2)}),$$

and so

$$\begin{aligned} \Delta_{\tilde{g}}^{H}a &= d_{\tilde{g}}^{*}\omega = -d^{c}\mathrm{tr}_{\tilde{g}}g - 2*_{\tilde{g}}d(\omega^{(2,0)+(0,2)}), \\ |\tilde{g}(\Delta_{\tilde{g}}^{H}a,a)| \leqslant |\tilde{g}(d^{c}\mathrm{tr}_{\tilde{g}}g,a)| + 2|\tilde{g}(*_{\tilde{g}}d(\omega^{(2,0)+(0,2)}),a)| \\ &\leqslant C|\nabla\mathrm{tr}_{\tilde{g}}g|_{\tilde{g}}|a|_{\tilde{g}} + 2|*_{\tilde{g}}d(\omega^{(2,0)+(0,2)})|_{\tilde{g}}|a|_{\tilde{g}} \\ &= C|\nabla(\mathrm{tr}_{g}\tilde{g}e^{-F})|_{\tilde{g}}|a|_{\tilde{g}} + 2|d(\omega^{(2,0)+(0,2)})|_{\tilde{g}}|a|_{\tilde{g}} \\ &\leqslant C|\nabla\mathrm{tr}_{g}\tilde{g}|_{\tilde{g}}|a|_{\tilde{g}} + C\mathrm{tr}_{g}\tilde{g}|\nabla(e^{-F})|_{\tilde{g}}|a|_{\tilde{g}} + C(\mathrm{tr}_{g}\tilde{g})^{\frac{3}{2}}|a|_{\tilde{g}} \\ &\leqslant C|\nabla\mathrm{tr}_{g}\tilde{g}|_{\tilde{g}}|a|_{\tilde{g}} + C(\mathrm{tr}_{g}\tilde{g})^{\frac{3}{2}}|a|_{\tilde{g}} \\ &\leqslant \varepsilon \frac{|\nabla\mathrm{tr}_{g}\tilde{g}|_{\tilde{g}}^{2}}{\mathrm{tr}_{g}\tilde{g}} + \frac{C}{\varepsilon}\mathrm{tr}_{g}\tilde{g}|a|_{\tilde{g}}^{2} + \varepsilon(\mathrm{tr}_{g}\tilde{g})^{2} + \frac{C}{\varepsilon}|a|_{\tilde{g}}^{4} \\ &\leqslant \varepsilon \frac{|\nabla\mathrm{tr}_{g}\tilde{g}|_{\tilde{g}}^{2}}{\mathrm{tr}_{g}\tilde{g}} + 2\varepsilon(\mathrm{tr}_{g}\tilde{g})^{2} + \frac{C}{\varepsilon^{2}}|a|_{\tilde{g}}^{4} \\ &\leqslant \varepsilon |\nabla\tilde{g}|_{g,\tilde{g}}^{2} + 2\varepsilon(\mathrm{tr}_{g}\tilde{g})^{2} + \frac{C}{\varepsilon^{2}}|a|_{\tilde{g}}^{4}, \end{aligned}$$

using (2.13), and so assuming without loss that ε is small, and substituting in (2.25) gives

$$\Delta_{\tilde{g}}|a|_{\tilde{g}}^2 \ge \frac{1}{2}|\tilde{\nabla}a|_{\tilde{g}}^2 + \frac{1}{2}|\overline{\tilde{\nabla}}a|_{\tilde{g}}^2 + \frac{(\operatorname{tr}_g \tilde{g})^2}{C} - \varepsilon|\nabla\tilde{g}|_{g,\tilde{g}}^2 - \frac{C}{\varepsilon^2}|a|_{\tilde{g}}^4 - C,$$

which is exactly (2.14).

Proof of the main claim (2.24) modulo (2.27) and (2.28). We now need to prove the main claim (2.24). Since the Hodge Laplacian is a real operator, we have

$$\Delta_{\tilde{g}}^{H}a = \Delta_{\tilde{g}}^{H}\alpha + \Delta_{\tilde{g}}^{H}\overline{\alpha} = \Delta_{\tilde{g}}^{H}\alpha + \overline{\Delta_{\tilde{g}}^{H}\alpha},$$

and

$$(\Delta_{\tilde{g}}^{H}\overline{\alpha})^{(0,1)} = \overline{(\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)}},$$

10

and so
(2.26)

$$\tilde{g}(\Delta_{\tilde{g}}^{H}a,a) = \tilde{g}(\Delta_{\tilde{g}}^{H}\alpha + \Delta_{\tilde{g}}^{H}\overline{\alpha}, \alpha + \overline{\alpha})$$

$$= \tilde{g}(\Delta_{\tilde{g}}^{H}\alpha, \overline{\alpha}) + \tilde{g}(\Delta_{\tilde{g}}^{H}\alpha, \alpha) + \tilde{g}(\Delta_{\tilde{g}}^{H}\overline{\alpha}, \alpha) + \tilde{g}(\Delta_{\tilde{g}}^{H}\overline{\alpha}, \overline{\alpha})$$

$$= \tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)}, \overline{\alpha}) + \tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)}, \alpha) + \tilde{g}((\Delta_{\tilde{g}}^{H}\overline{\alpha})^{(0,1)}, \alpha) + \tilde{g}((\Delta_{\tilde{g}}^{H}\overline{\alpha})^{(1,0)}, \overline{\alpha})$$

$$= \tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)}, \overline{\alpha}) + \tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)}, \alpha) + \tilde{g}(\overline{(\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)}}, \alpha) + \tilde{g}(\overline{(\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)}}, \alpha) + \tilde{g}(\overline{(\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)}}, \overline{\alpha})$$

$$= 2\text{Re}\tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)}, \overline{\alpha}) + 2\text{Re}\tilde{g}((\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)}, \alpha).$$

Next, we claim that

(2.27)
$$(\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)} = 2\tilde{\alpha}_{k,\tilde{i}\tilde{i}}\tilde{\theta}^{k},$$

(2.28)
$$(\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)} = \left(-4\tilde{\alpha}_{i,j}\tilde{N}_{k\bar{i}}^{j} - 2\tilde{\alpha}_{j}\tilde{K}_{i\bar{k}\bar{i}}^{j} + 2\tilde{\alpha}_{i}\tilde{N}_{\bar{j}\bar{k},j}^{j}\right)\overline{\theta^{k}}.$$

We will prove these by long direct computations, but first let us assume we have these and complete the proof of the main claim (2.24). Combining (2.26), (2.27) and (2.28) gives

$$(2.29) \frac{1}{2}\tilde{g}(\Delta_{\tilde{g}}^{H}a,a) = 2\operatorname{Re}(\tilde{\alpha}_{k,\tilde{i}i}\overline{\tilde{\alpha}_{k}}) + \operatorname{Re}\left(-4\tilde{\alpha}_{i,j}\tilde{\alpha}_{k}\tilde{N}_{\tilde{k}\tilde{i}}^{j} - 2\tilde{\alpha}_{j}\tilde{\alpha}_{k}\tilde{K}_{\tilde{i}\tilde{k}\tilde{i}}^{j} + 2\tilde{\alpha}_{i}\tilde{\alpha}_{k}\tilde{N}_{\tilde{j}\tilde{k},j}^{i}\right),$$

and we can bound the last 3 terms in (2.29) using the same method as before. For the first term, using the same notation as earlier, we can write

$$\tilde{\alpha}_{i,j}\tilde{\alpha}_k\tilde{N}_{\overline{k}\overline{i}}^j = \sum_{i,k=1}^2 \lambda_i^{-1}\lambda_k^{-1}\alpha_{i,j}\alpha_k N_{\overline{k}\overline{i}}^j = \sum_{i\neq k}\lambda_i^{-1}\lambda_k^{-1}\alpha_{i,j}\alpha_k N_{\overline{k}\overline{i}}^j = e^{-F}\sum_{i\neq k}\alpha_{i,j}\alpha_k N_{\overline{k}\overline{i}}^j,$$

using that N_{ki}^{j} is skew-symmetric in i, k as well as the Calabi-Yau equation (2.23). Using the Calabi-Yau equation again, we can bound

$$-\operatorname{Re}(4\tilde{\alpha}_{i,j}\tilde{\alpha}_{k}\tilde{N}_{\overline{k}\tilde{i}}^{j}) \leqslant C \sum_{j} \sum_{i \neq k} |\alpha_{i,j}| |\alpha_{k}| \leqslant C \sum_{j} \sum_{i \neq k} \lambda_{i}^{-\frac{1}{2}} \lambda_{k}^{-\frac{1}{2}} |\alpha_{i,j}| |\alpha_{k}| \lambda_{j}^{-\frac{1}{2}} \lambda_{j}^{\frac{1}{2}}$$
$$= C \sum_{j} \sum_{i \neq k} |\tilde{\alpha}_{i,j}| |\tilde{\alpha}_{k}| \lambda_{j}^{\frac{1}{2}} \leqslant C |\alpha|_{\tilde{g}} \sqrt{\operatorname{tr}_{g}\tilde{g}} \sum_{i,j} |\tilde{\alpha}_{i,j}|$$
$$\leqslant \frac{1}{2} |\tilde{\alpha}_{i,j}|^{2} + C \operatorname{tr}_{g}\tilde{g} |\alpha|_{\tilde{g}}^{2}.$$

The second terms in the parenthesis in (2.29) is bounded by $C \operatorname{tr}_{g} \tilde{g} |\alpha|_{\tilde{g}}^{2}$ arguing exactly as we did to bound (2.22): using the skew-symmetry of $K_{i\bar{k}i}^{j}$ in k, i and the Calabi-Yau equation (2.23) we have

$$\tilde{\alpha}_j \tilde{\alpha}_k \tilde{K}^j_{i\overline{k}i} = \sum_{i,k=1}^2 \lambda_i^{-1} \lambda_k^{-1} \alpha_j \alpha_k K^j_{i\overline{k}i} = \sum_{i \neq k} \lambda_i^{-1} \lambda_k^{-1} \alpha_j \alpha_k K^j_{i\overline{k}i} = e^{-F} \sum_{i \neq k} \alpha_j \alpha_k K^j_{i\overline{k}i}$$

and so

$$-\operatorname{Re}(2\tilde{\alpha}_{j}\tilde{\alpha}_{k}\tilde{K}_{\overline{iki}}^{j}) \leqslant C|\alpha|_{g}^{2} \leqslant C\operatorname{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2}$$

The last term in the parenthesis in (2.29)

(2.30)
$$\tilde{\alpha}_i \tilde{\alpha}_k \tilde{N}^i_{j\overline{k},j}$$

requires more work. First, we explain in detail the term $|\nabla \tilde{g}|_{g,\tilde{g}}^2$ in (2.12). As in [18, (3.9)] there are functions $a_{k\ell}^i$ defined by

$$da_m^i - a_j^i \theta_m^j + a_m^k \tilde{\theta}_k^i = a_{k\ell}^i a_m^k \tilde{\theta}^\ell,$$

which are the components of $\nabla \tilde{g}$ in the frame $\{\tilde{\theta}^i\}$, so that [18, Lemma 4.2]

$$|\nabla \tilde{g}|^2_{\tilde{g}} = |a^i_{k\ell}|^2$$

The mixed norm $|\nabla \tilde{g}|_{g,\tilde{g}}^2$ that appears in (2.12) is given by

(2.31)
$$|\nabla \tilde{g}|_{g,\tilde{g}}^2 = |a_{p\ell}^i a_k^p|^2$$

We can now go back to the term in (2.30) and using [18, Lemma 4.4 (i)] we can write it as

$$\tilde{\alpha}_i \tilde{\alpha}_k \tilde{N}_{j\overline{k},j}^i = \alpha_\ell \alpha_m b_k^m \overline{b_k^s} b_j^u \overline{b_j^r} N_{\overline{rs},u}^\ell + \alpha_\ell \alpha_m b_i^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_{uj}^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_{uj}^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_{uj}^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_{uj}^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_{uj}^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^u + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} N_{\overline{rs}}^t a_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} A_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} A_u^i a_u^i + \alpha_\ell \alpha_m b_j^\ell b_k^m \overline{b_k^s} \overline{b_j^r} A_u^i + \alpha_\ell \alpha_m$$

and using the skew-symmetry of $N^i_{\overline{jk},j}$ in j,k and the Calabi-Yau equation we have

$$\alpha_{\ell}\alpha_{m}b_{k}^{m}\overline{b_{k}^{s}}b_{j}^{u}\overline{b_{j}^{r}}N_{\overline{rs},u}^{\ell} = \sum_{j,k=1}^{2}\lambda_{j}^{-1}\lambda_{k}^{-1}\alpha_{i}\alpha_{k}N_{\overline{jk},j}^{i} = \sum_{j\neq k}^{2}\lambda_{j}^{-1}\lambda_{k}^{-1}\alpha_{i}\alpha_{k}N_{\overline{jk},j}^{i} = e^{-F}\sum_{j\neq k}\alpha_{i}\alpha_{k}N_{\overline{jk},j}^{i},$$

whose absolute value is bounded by $C|\alpha|_g^2 \leqslant C \mathrm{tr}_g \tilde{g} |\alpha|_{\tilde{g}}^2$. Similarly, using the skew-symmetry of N_{jk}^t in j,k and the Calabi-Yau equation we have

$$\begin{aligned} \alpha_{\ell} \alpha_{m} b_{i}^{\ell} b_{k}^{m} \overline{b_{k}^{s} b_{j}^{r}} N_{\overline{rs}}^{t} a_{uj}^{i} a_{t}^{u} &= \sum_{i,j,k=1}^{2} \alpha_{i} \alpha_{k} \lambda_{i}^{-\frac{1}{2}} \lambda_{k}^{-1} \lambda_{j}^{-\frac{1}{2}} N_{jk}^{t} a_{uj}^{i} a_{t}^{u} \\ &= \sum_{i} \sum_{j \neq k} \alpha_{i} \alpha_{k} \lambda_{i}^{-\frac{1}{2}} \lambda_{k}^{-1} \lambda_{j}^{-\frac{1}{2}} N_{j\overline{k}}^{t} a_{uj}^{i} a_{t}^{u} \\ &= e^{-\frac{F}{2}} \sum_{i} \sum_{j \neq k} \alpha_{i} \alpha_{k} \lambda_{i}^{-\frac{1}{2}} \lambda_{k}^{-\frac{1}{2}} N_{j\overline{k}}^{t} a_{uj}^{i} a_{t}^{u} \\ &= e^{-\frac{F}{2}} \sum_{i} \sum_{j \neq k} \tilde{\alpha}_{i} \tilde{\alpha}_{k} N_{j\overline{k}}^{t} a_{uj}^{i} a_{t}^{u}, \end{aligned}$$

whose absolute value is bounded by

$$C|\alpha|^2_{\tilde{g}}|a^i_{uj}a^u_t| = C|\alpha|^2_{\tilde{g}}|\nabla \tilde{g}|_{g,\tilde{g}},$$

and so

$$-\operatorname{Re}(2\tilde{\alpha}_{i}\tilde{\alpha}_{k}\tilde{N}^{i}_{\overline{j}\overline{k},j}) \leqslant C\operatorname{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2} + C|\alpha|_{\tilde{g}}^{2}|\nabla\tilde{g}|_{g,\tilde{g}}$$

12

Putting all these together we obtain

$$\frac{1}{2}\tilde{g}(\Delta_{\tilde{g}}^{H}a,a) \leqslant 2\operatorname{Re}(\tilde{\alpha}_{k,\tilde{i}i}\overline{\tilde{\alpha}_{k}}) + C\operatorname{tr}_{g}\tilde{g}|\alpha|_{\tilde{g}}^{2} + C|\alpha|_{\tilde{g}}^{2}|\nabla\tilde{g}|_{g,\tilde{g}} + \frac{1}{2}|\tilde{\alpha}_{i,j}|^{2},$$

which is exactly the main claim (2.24).

Proof of (2.27) and (2.28). To complete the proof of the main claim (2.24), we are left with showing (2.27) and (2.28), which are Bochner-Kodaira-Weitzenböck type formulas. For these, recall that by definition of the Hodge star $*_{\tilde{g}}$, for any two (p, q)-forms β, γ we have

$$\beta \wedge \overline{*}_{\tilde{g}} \gamma = \tilde{g}(\beta, \gamma) \tilde{\omega}^2 = \tilde{g}(\beta, \gamma) \sqrt{-1} \tilde{\theta}^1 \wedge \overline{\tilde{\theta}^1} \wedge \sqrt{-1} \tilde{\theta}^2 \wedge \overline{\tilde{\theta}^2} = -\tilde{g}(\beta, \gamma) \tilde{\theta}^1 \wedge \overline{\tilde{\theta}^1} \wedge \tilde{\theta}^2 \wedge \overline{\tilde{\theta}^2}$$

and also $*_{\tilde{g}}\overline{\alpha} = \overline{*_{\tilde{g}}\alpha}$ (i.e. $*_{\tilde{g}}$ is a real operator), from which we can compute the Hodge star on basic combinations of our frame elements, using the following notation: for $i \in \{1, 2\}$ we let $\hat{i} \in \{1, 2\}$ be such that $\{i, \hat{i}\} = \{1, 2\}$ (unordered), i.e. $\hat{1} = 2, \hat{2} = 1$. Then from the definition we have

$$\begin{split} *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} \wedge \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{\hat{i}}}) &= -1, \\ *_{\tilde{g}}\tilde{\theta}^{i} &= \tilde{\theta}^{i} \wedge \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{\hat{i}}}, \quad *_{\tilde{g}}\overline{\tilde{\theta}^{i}} &= -\overline{\tilde{\theta}^{i}} \wedge \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{\hat{i}}} \\ *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{\hat{i}}}) &= -\tilde{\theta}^{i}, \quad *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{i}}) = \tilde{\theta}^{\hat{i}}, \\ *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{\hat{i}}} \wedge \overline{\tilde{\theta}^{\hat{i}}}) &= \overline{\tilde{\theta}^{i}}, \quad *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{\hat{i}}} \wedge \overline{\tilde{\theta}^{\hat{i}}}) = -\overline{\tilde{\theta}^{\hat{i}}}, \\ *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \tilde{\theta}^{j}) &= \tilde{\theta}^{i} \wedge \tilde{\theta}^{j}, \quad *_{\tilde{g}}(\overline{\tilde{\theta}^{i}} \wedge \overline{\tilde{\theta}^{j}}) = \overline{\tilde{\theta}^{i}} \wedge \overline{\tilde{\theta}^{j}}, \\ *_{\tilde{g}}(\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{j}}) &= \left\{ \begin{array}{l} \tilde{\theta}^{\hat{i}} \wedge \overline{\tilde{\theta}^{\hat{i}}} & \text{if } i = j, \\ -\tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{j}} & \text{if } i = j. \end{array} \right. \end{split}$$

With these, we can now start the computation of $\Delta_{\tilde{g}}^{H}\alpha = -dd_{\tilde{g}}^{*}\alpha - d_{\tilde{g}}^{*}d\alpha$, recalling that

$$d^*_{\tilde{g}}\alpha = -*_{\tilde{g}} d*_{\tilde{g}} \alpha,$$

we compute

$$\begin{split} *_{\tilde{g}}\alpha &= \tilde{\alpha}_{i}\theta^{i} \wedge \theta^{i} \wedge \theta^{i}, \\ d *_{\tilde{g}} \alpha &= \tilde{\alpha}_{i,\overline{j}}\overline{\tilde{\theta}^{j}} \wedge \tilde{\theta}^{i} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} = \tilde{\alpha}_{i,\overline{i}}\overline{\tilde{\theta}^{i}} \wedge \tilde{\theta}^{i} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}}, \\ d_{\tilde{g}}^{*}\alpha &= -\tilde{\alpha}_{i,\overline{i}}, \end{split}$$

$$(2.32) dd_{\tilde{g}}^* \alpha = -\tilde{\alpha}_{i,\bar{i}k} \tilde{\theta}^k - \tilde{\alpha}_{i,\bar{i}k} \overline{\tilde{\theta}^k}.$$

$$d\alpha = \tilde{\alpha}_{i,j} \tilde{\theta}^j \wedge \tilde{\theta}^i + \tilde{\alpha}_{i,\bar{j}} \overline{\tilde{\theta}^j} \wedge \tilde{\theta}^i + \tilde{\alpha}_i \tilde{N}_{\bar{j}k}^i \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^k}$$

$$(2.33) *_{\tilde{g}} d\alpha = \tilde{\alpha}_{i,j} \tilde{\theta}^j \wedge \tilde{\theta}^i - \tilde{\alpha}_{i,\bar{i}} \tilde{\theta}^i \wedge \overline{\tilde{\theta}^i} + \tilde{\alpha}_{i,\bar{i}} \tilde{\theta}^i \wedge \overline{\tilde{\theta}^i} + \tilde{\alpha}_i \tilde{N}_{\bar{j}k}^i \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^k}.$$

We first prove (2.27). We are thus interested in the (1,0)-part of $d_{\tilde{g}}^* d\alpha$, so we first take the exterior derivative of (2.33) and take its (2,1)-part (which becomes (1,0) after taking the Hodge star) (2.34)

$$(d*_{\tilde{g}}d\alpha)^{(2,1)} = \tilde{\alpha}_{i,j\overline{k}}\overline{\tilde{\theta}^k} \wedge \tilde{\theta}^j \wedge \tilde{\theta}^i - \tilde{\alpha}_{i,\overline{i}k}\tilde{\theta}^k \wedge \tilde{\theta}^i \wedge \overline{\tilde{\theta}^i} + \tilde{\alpha}_{i,\overline{i}k}\tilde{\theta}^k \wedge \tilde{\theta}^i \wedge \overline{\tilde{\theta}^i} + 2\tilde{\alpha}_i \tilde{N}_{\overline{j}k}^i \overline{\tilde{N}_{pq}^j} \tilde{\theta}^p \wedge \tilde{\theta}^q \wedge \overline{\tilde{\theta}^k},$$

and in the first term in (2.34) we must have $j = \hat{i}$ and we can write it as

$$\tilde{\alpha}_{i,i\overline{i}}\overline{\overline{\theta}{}^{i}}\wedge\tilde{\theta}{}^{i}\wedge\tilde{\theta}{}^{i}+\tilde{\alpha}_{i,i\overline{i}}\overline{\overline{\theta}{}^{i}}\wedge\tilde{\theta}{}^{i}\wedge\tilde{\theta}{}^{i},$$

and its Hodge star equals

$$-\tilde{\alpha}_{i,i\overline{i}}\tilde{\theta}^{\hat{i}}+\tilde{\alpha}_{i,i\overline{i}}\tilde{\theta}^{i}=(-\tilde{\alpha}_{\hat{k},k\overline{\hat{k}}}+\tilde{\alpha}_{k,k\overline{\hat{k}}})\tilde{\theta}^{k}=(-\tilde{\alpha}_{i,k\overline{i}}+\tilde{\alpha}_{k,i\overline{i}})\tilde{\theta}^{k}.$$

In the second term we must have k = i and in the third term $k = \hat{i}$ so we can write them as

$$-\tilde{\alpha}_{i,\overline{i}i}\tilde{\theta}^i\wedge\tilde{\theta}^i\wedge\overline{\tilde{\theta}^i}+\tilde{\alpha}_{i,\overline{i}\hat{i}}\tilde{\theta}^{\hat{i}}\wedge\tilde{\theta}^i\wedge\overline{\tilde{\theta}^i},$$

and their Hodge star equals

$$(\tilde{\alpha}_{i,\overline{i}i} + \tilde{\alpha}_{i,\overline{i}\widehat{i}})\tilde{\theta}^i = \tilde{\alpha}_{k,\overline{i}i}\tilde{\theta}^k$$

In the last term in (2.34) we must have $q = \hat{p}$ so it equals

$$2\tilde{\alpha}_i \tilde{N}_{j\overline{p}}^{\underline{i}} \overline{\tilde{N}_{p\overline{p}}^j} \tilde{\theta}^p \wedge \tilde{\theta}^{\hat{p}} \wedge \overline{\tilde{\theta}^p} + 2\tilde{\alpha}_i \tilde{N}_{j\overline{p}}^{\underline{j}} \overline{\tilde{N}_{p\overline{p}}^j} \tilde{\theta}^p \wedge \tilde{\theta}^{\hat{p}} \wedge \overline{\tilde{\theta}^p},$$

and its Hodge star equals

$$\begin{split} 2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{p}}^{\underline{i}}\overline{\tilde{N}_{\overline{p}\overline{p}}^{j}}\tilde{\theta}^{\hat{p}} - 2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{p}}^{\underline{i}}\overline{\tilde{N}_{\overline{p}\overline{p}}^{j}}\tilde{\theta}^{p} &= \left(2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{k}}^{\underline{i}}\overline{\tilde{N}_{\overline{k}\overline{k}}^{j}} - 2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{k}}^{\underline{i}}\overline{\tilde{N}_{\overline{k}\overline{k}}^{j}}\right)\tilde{\theta}^{k} \\ &= \left(2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{\ell}}^{\underline{i}}\overline{\tilde{N}_{\ell k}^{j}} - 2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{\ell}}^{\underline{i}}\overline{\tilde{N}_{\overline{k}\overline{\ell}}^{j}}\right)\tilde{\theta}^{k} \\ &= -4\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{\ell}}^{\underline{i}}\overline{\tilde{N}_{\overline{k}\overline{\ell}}^{j}}\tilde{\theta}^{k}, \end{split}$$

using the skew-symmetry of $\tilde{N}_{\overline{k\ell}}^{j}$ in k, ℓ . Putting these together gives

$$-(d_{\tilde{g}}^*d\alpha)^{(1,0)} = (*_{\tilde{g}}d *_{\tilde{g}} d\alpha)^{(1,0)} = *_{\tilde{g}} \left((d *_{\tilde{g}} d\alpha)^{(2,1)} \right)$$
$$= \left(-\tilde{\alpha}_{i,k\bar{i}} + \tilde{\alpha}_{k,i\bar{i}} + \tilde{\alpha}_{k,\bar{i}i} - 4\tilde{\alpha}_i \tilde{N}_{\bar{j}\bar{\ell}}^{\underline{i}} \overline{\tilde{N}_{k\bar{\ell}}^j} \right) \tilde{\theta}^k$$

and combining this with (2.32) gives

$$(\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)} = \left(\tilde{\alpha}_{i,\bar{i}k} - \tilde{\alpha}_{i,k\bar{i}} + \tilde{\alpha}_{k,i\bar{i}} + \tilde{\alpha}_{k,\bar{i}i} - 4\tilde{\alpha}_{i}\tilde{N}_{\bar{j}\ell}\overline{\tilde{N}_{\bar{j}\ell}}\right)\tilde{\theta}^{k}$$

From the commutation relation (2.19) we obtain

$$\begin{split} \tilde{\alpha}_{i,\overline{i}k} &- \tilde{\alpha}_{i,k\overline{i}} = - \tilde{\alpha}_j \tilde{R}^j_{ik\overline{i}}, \\ \tilde{\alpha}_{k,i\overline{i}} &- \tilde{\alpha}_{k,\overline{i}i} = \tilde{\alpha}_j \tilde{R}^j_{ki\overline{i}}, \end{split}$$

while [18, (2.16)] gives

$$\tilde{R}^{j}_{ki\bar{i}} - \tilde{R}^{j}_{ik\bar{i}} = 4\tilde{N}^{j}_{\overline{p}\overline{i}}\overline{\tilde{N}^{p}_{k\bar{i}}},$$

and so

$$(\Delta_{\tilde{g}}^{H}\alpha)^{(1,0)} = \left(2\tilde{\alpha}_{k,\tilde{i}i} + 4\tilde{\alpha}_{j}\tilde{N}_{\tilde{p}\tilde{i}}^{j}\overline{\tilde{N}_{ki}^{p}} - 4\tilde{\alpha}_{i}\tilde{N}_{\tilde{j}\tilde{\ell}}^{j}\overline{\tilde{N}_{k\ell}^{j}}\right)\tilde{\theta}^{k} = 2\tilde{\alpha}_{k,\tilde{i}i}\tilde{\theta}^{k}$$

which proves (2.27). The proof of (2.28) is similar. First, from (2.32) we have

(2.35)
$$(dd_{\tilde{g}}^*\alpha)^{(0,1)} = -\tilde{\alpha}_{i,\bar{i}\bar{k}}\overline{\tilde{\theta}^k}.$$

We are then interested in the (0,1)-part of $d_{\hat{g}}^* d\alpha$, so we take the exterior derivative of (2.33) and then take the (1,2)-part to obtain

$$\begin{split} (d*_{\tilde{g}} d\alpha)^{(1,2)} &= \tilde{\alpha}_{i,j} \tilde{N}^{j}_{\overline{pq}} \overline{\tilde{\theta}^{p}} \wedge \overline{\tilde{\theta}^{q}} \wedge \tilde{\theta}^{i} - \tilde{\alpha}_{i,j} \tilde{N}^{i}_{\overline{pq}} \overline{\tilde{\theta}^{j}} \wedge \overline{\tilde{\theta}^{q}} - \tilde{\alpha}_{i,\overline{ik}} \overline{\tilde{\theta}^{k}} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} \\ &+ \tilde{\alpha}_{i,\overline{ik}} \overline{\tilde{\theta}^{k}} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} + \tilde{\alpha}_{i,p} \tilde{N}^{i}_{\overline{jk}} \overline{\tilde{\theta}^{p}} \wedge \overline{\tilde{\theta}^{j}} \wedge \overline{\tilde{\theta}^{k}} + \tilde{\alpha}_{i} \tilde{N}^{i}_{\overline{jk},p} \tilde{\theta}^{p} \wedge \overline{\tilde{\theta}^{j}} \wedge \overline{\tilde{\theta}^{k}} \\ &= \tilde{\alpha}_{i,j} \tilde{N}^{j}_{\overline{pq}} \overline{\tilde{\theta}^{p}} \wedge \overline{\tilde{\theta}^{q}} \wedge \tilde{\theta}^{i} - \tilde{\alpha}_{i,\overline{ik}} \overline{\tilde{\theta}^{k}} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} + \tilde{\alpha}_{i,\overline{ik}} \overline{\tilde{\theta}^{k}} \wedge \tilde{\theta}^{i} \wedge \overline{\tilde{\theta}^{i}} \\ &+ \tilde{\alpha}_{i} \tilde{N}^{i}_{\overline{jk},p} \tilde{\theta}^{p} \wedge \overline{\tilde{\theta}^{j}} \wedge \overline{\tilde{\theta}^{k}}, \end{split}$$

and in the first term in (2.36) we must have $q = \hat{p}$ so we can write it as

$$\tilde{\alpha}_{p,j}\tilde{N}^{j}_{\overline{p}\overline{p}}\overline{\tilde{\theta}^{p}}\wedge\overline{\tilde{\theta}^{p}}\wedge\tilde{\theta}^{p}+\tilde{\alpha}_{\hat{p},j}\tilde{N}^{j}_{\overline{p}\overline{p}}\overline{\tilde{\theta}^{p}}\wedge\overline{\tilde{\theta}^{\hat{p}}}\wedge\tilde{\theta}^{\hat{p}},$$

and its Hodge star equals

$$\tilde{\alpha}_{p,j}\tilde{N}^{j}_{\overline{p}\overline{p}}\overline{\overline{\theta}}^{\hat{p}} - \tilde{\alpha}_{\hat{p},j}\tilde{N}^{j}_{\overline{p}\overline{p}}\overline{\overline{\theta}}^{p} = \tilde{\alpha}_{p,j}\tilde{N}^{j}_{\overline{p}\overline{i}}\overline{\overline{\theta}}^{i} - \tilde{\alpha}_{i,j}\tilde{N}^{j}_{\overline{p}\overline{i}}\overline{\overline{\theta}}^{p} = -2\tilde{\alpha}_{i,j}\tilde{N}^{j}_{\overline{p}\overline{i}}\overline{\overline{\theta}}^{p}$$

In the second and third terms in (2.36) we must have k = i so we can write them as

$$-\tilde{\alpha}_{i,\overline{ii}}\overline{\tilde{\theta}^{i}}\wedge\tilde{\theta}^{i}\wedge\overline{\tilde{\theta}^{i}}+\tilde{\alpha}_{i,\overline{ii}}\overline{\tilde{\theta}^{i}}\wedge\tilde{\theta}^{i}\wedge\overline{\tilde{\theta}^{i}},$$

and their Hodge star equals

$$-\tilde{\alpha}_{i,\overline{ii}}\overline{\tilde{\theta}^{i}} - \tilde{\alpha}_{i,\overline{\tilde{i}i}}\overline{\tilde{\theta}^{\hat{i}}} = -\tilde{\alpha}_{i,\overline{ki}}\overline{\tilde{\theta}^{k}}$$

In the last term in (2.36) we must have $k = \hat{j}$ and so we can write it as

$$\tilde{\alpha}_i \tilde{N}^i_{j\bar{j},j} \tilde{\theta}^j \wedge \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^j} + \tilde{\alpha}_i \tilde{N}^i_{j\bar{j},j} \tilde{\theta}^j \wedge \overline{\tilde{\theta}^j} \wedge \overline{\tilde{\theta}^j},$$

and its Hodge star equals

$$\tilde{\alpha}_i \tilde{N}^i_{\overline{j}\overline{j},j} \overline{\overline{\theta}^j} - \tilde{\alpha}_i \tilde{N}^i_{\overline{j}\overline{j},j} \overline{\overline{\theta}^j} = \tilde{\alpha}_i \tilde{N}^i_{\overline{j}\overline{k},j} \overline{\overline{\theta}^k} - \tilde{\alpha}_i \tilde{N}^i_{\overline{k}\overline{j},j} \overline{\overline{\theta}^k} = 2\tilde{\alpha}_i \tilde{N}^i_{\overline{j}\overline{k},j} \overline{\overline{\theta}^k}.$$

Putting these together gives

$$-(d_{\tilde{g}}^*d\alpha)^{(0,1)} = (*_{\tilde{g}}d *_{\tilde{g}} d\alpha)^{(0,1)} = *_{\tilde{g}} \left((d *_{\tilde{g}} d\alpha)^{(1,2)} \right)$$
$$= \left(-2\tilde{\alpha}_{i,j}\tilde{N}_{ki}^j - \tilde{\alpha}_{i,\overline{k}i} + 2\tilde{\alpha}_i\tilde{N}_{j\overline{k},j}^i \right) \overline{\tilde{\theta}^k},$$

and combining this with (2.35) gives

$$(\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)} = \left(\tilde{\alpha}_{i,\overline{ik}} - 2\tilde{\alpha}_{i,j}\tilde{N}_{\overline{ki}}^{j} - \tilde{\alpha}_{i,\overline{ki}} + 2\tilde{\alpha}_{i}\tilde{N}_{\overline{jk},j}^{i}\right)\overline{\tilde{\theta}^{k}}.$$

From the commutation relation (2.20) we obtain

$$\tilde{\alpha}_{i,\overline{ik}} - \tilde{\alpha}_{i,\overline{ki}} = -2\tilde{\alpha}_j \tilde{K}^j_{\overline{iki}} - 2\tilde{\alpha}_{i,j} \tilde{N}^j_{\overline{ki}},$$

and so

$$(\Delta_{\tilde{g}}^{H}\alpha)^{(0,1)} = \left(-4\tilde{\alpha}_{i,j}\tilde{N}_{\overline{k}i}^{j} - 2\tilde{\alpha}_{j}\tilde{K}_{i\overline{k}i}^{j} + 2\tilde{\alpha}_{i}\tilde{N}_{\overline{j}\overline{k},j}^{i}\right)\overline{\tilde{\theta}^{k}},$$

which proves (2.28) and concludes the proof of Proposition 2.1, and hence of Theorem 1.2.

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