

Unitary discriminants of characters

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In memory of Richard Parker

ABSTRACT. Together with David Schlang we computed the discriminants of the invariant Hermitian forms for all indicator o even degree absolutely irreducible characters of the ATLAS groups supplementing the tables of orthogonal determinants computed in collaboration with Richard Parker, Tobias Braun and Thomas Breuer. The methods that are used in the unitary case are described in this paper. A character has a well defined unitary discriminant, if and only if it is unitary stable, i.e. all irreducible unitary constituents have even degree. Computations for large degree characters are only possible because of a new method called *unitary condensation*. A suitable automorphism helps to single out a square class of the real subfield of the character field consisting of representatives of the discriminant of the invariant Hermitian forms. This square class can then be determined modulo enough primes.

MSC: 20C15; 11E12.

KEYWORDS: unitary representations of finite groups; unitary discriminants; invariants of Hermitian forms; Clifford invariants of quadratic forms

1 Introduction

Let K be a field, G a finite group and $n \in \mathbb{N}$. A group homomorphism $\rho : G \rightarrow \mathrm{GL}_n(K)$ is called a K -representation of G . Often $\rho(G)$ is contained in a smaller classical group, such as symplectic, unitary or orthogonal groups. This paper continues a long term project of the author with Richard Parker to specify these classical groups for finite fields and number fields. The fundamental methods for studying orthogonal representations are described in our joint paper [13]. Parker was mainly interested in finite fields K . For given even dimension $n = 2m$ there is a unique unitary and symplectic group over a finite field; however there are two non-isomorphic orthogonal groups, $O_{2m}^+(K)$ and $O_{2m}^-(K)$. So we were faced with the task: given a Brauer character χ of an irreducible orthogonal representation $\rho : G \rightarrow \mathrm{GL}_{2m}(K)$, decide whether $\rho(G)$ is contained in an orthogonal group $O_{2m}^+(K)$ or

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$O_{2m}^-(K)$, i.e. whether χ has orthogonal discriminant $O+$ or $O-$. The finitely many primes p dividing the group order $|G|$ can be handled case by case, the infinitely many other primes p are dealt with by computing the orthogonal discriminant of the corresponding indicator + ordinary character χ . This is possible because, for a given complex irreducible character χ of indicator + and even degree, there is a square class of the character field of χ containing determinants of the $\rho(G)$ -invariant quadratic forms, though, due to Schur indices, there might not be a representation ρ over the character field that affords the character χ (see [12]). Together with Breuer and Parker, the author compiled a list of these orthogonal discriminants for all indicator + even degree ordinary and Brauer characters for all groups in the ATLAS of finite groups of order smaller or equal to the order of the sporadic simple Harada-Norton group. The results are available in [1]. As the computations seem to be much more involved, we left out to determine the Clifford invariant for the indicator + characters over number fields and also did not handle unitary discriminants for ordinary complex irreducible characters of indicator o and even degree.

The present paper transfers the ideas of [13] to the unitary case. Many methods, such as restriction, induction, tensor products and symmetrizations, behave very similar to the ones for orthogonal characters. Also decomposition number techniques can be varied to obtain restrictions for the unitary discriminants (see Section 5.2), for instance showing that all primes dividing the unitary discriminant also do divide the group order (see Corollary 5.5).

However there is a great difference to the orthogonal case: The discriminant of a given L/K -Hermitian form H is only well defined modulo norms, $\text{disc}(H) \in K^\times/N_{L/K}(L^\times)$. For finite fields, the norm map is surjective, so it seems to be impossible to obtain any information about $\text{disc}(H)$ by reducing the representation modulo primes. The latter technique is important in [13] to handle large degree characters, for which we computed composition factors of a condensed module (see Section 8.1) modulo some well chosen primes (usually not dividing the group order) to get enough information on the square class containing the orthogonal discriminant. It is amazing that also this condensation method can be transferred to the unitary case: If there exists a suitable automorphism α of order 2 of G , interchanging the indicator o character χ with its complex conjugate $\bar{\chi} = \chi \circ \alpha$, then there is a square class of the real subfield of the character field of χ , called *the α -discriminant of χ* , that relates to the unitary discriminant of χ (see Theorem 7.6). In complete analogy to the orthogonal case, we can use skew adjoint units in the α -fixed algebra, to compute the α -discriminant of χ (see Section 8.2).

We start by recalling the basic notions for quadratic and Hermitian forms. As we are mainly interested in totally positive definite Hermitian forms over quadratic complex extensions L of totally real number fields K , Section 3 gives the important facts for this situation. In particular Hasse's norm theorem gives criteria for computing the norm subgroup $N_{L/K}(L^\times)$ of K^\times from local data. Invariant lattices provide a tool to use modular reduction to restrict the possible unitary discriminants. Therefore some facts about lattices in local Hermitian spaces are recalled in Section 3.2. The next section introduces unitary discriminants of characters. To appropriately use modular reduction techniques, we need

to have a finer notion of indicator for absolutely irreducible Brauer characters: There are two different sorts of indicator o Brauer characters; those, for which the corresponding simple module carries a non-zero invariant Hermitian form, and those for which there is no such invariant form. The former characters are called *unitary*, as the corresponding representation embeds into the unitary group. To simplify notation we also say that all real Brauer characters and all ordinary characters are unitary. Then a character is called *unitary stable* (see Definition 5.1) if and only if all its unitary constituents have even degree. Unitary stable characters are exactly those that have a well defined unitary discriminant. In particular indicator $-$ characters are unitary stable and Section 4.3 shows how to determine their unitary discriminant from their local Schur indices. Section 5 collects important character theoretic methods that can be automatised to compute and check unitary discriminants. For orthogonal characters there was no need to deal with covering groups, as there the orthogonal discriminants are predicted by theoretical methods. For unitary characters the Q_8 -trick described in Section 6 is a shortcut to obtain unitary discriminants of faithful characters of certain even covering groups.

The next section lays the ground for applying condensation techniques to compute unitary discriminants. The determinant of a symmetric bilinear form can be computed as the determinant of its adjoint involution, and hence as the square class of the determinant of any skew-adjoint unit. The adjoint involution of an Hermitian form H is an involution of second kind. Also here we can obtain the determinant of H intrinsically from the algebra $\text{End}(H)$ with involution (see Remark 4.8). There are two obstacles, however: The discriminant algebra from Remark 4.8 is hard to determine in general, and, the determinant of H is only defined modulo norms. We overcome these obstacles by defining an orthogonal subalgebra (Definition 7.4), $A \leq \text{End}(H)$, to which the adjoint involution of H restricts as an orthogonal involution. Orthogonal subalgebras can be determined as α -fixed algebras (Definition 7.2) for certain outer automorphisms α . Section 8 shows how this can be applied to compute unitary discriminants using modular condensation techniques. This is illustrated in two examples, a computational one for the Harada-Norton group and a smaller example for $U_3(7)$, that illustrates techniques that might generalise to compute all unitary discriminants of the groups $U_3(q)$ for odd prime powers q and other finite groups of Lie type. The last section illustrates the computation for the group $3.ON$, the Schur cover of the sporadic simple O’Nan group. Besides giving examples for the various methods described in this paper, the results are interesting, as this is one of the rare cases, where some unitary discriminants are not rational.

Special thanks go to David Schlang, who compiled a large list of unitary discriminants during his master thesis [18] and his fellowship awarded by the SFB TRR 195, Computeralgebra (Project-ID 286237555). With his help we obtained the unitary discriminants of all indicator o even degree ordinary characters for all groups in the ATLAS of finite groups [5] of order smaller or equal to the order of the Harada Norton group, the current status of the database of orthogonal discriminants. I also thank Thomas Breuer, who is permanently updating the OSCAR database of orthogonal and unitary discriminants of characters [1].

2 Quadratic and Hermitian forms

Let L be a field of characteristic $\neq 2$ and σ an automorphism of L of order 1 or 2. Put $K := \text{Fix}_\sigma(L)$ to be the fixed field of σ in L . Then either $\sigma = \text{id}$ and $L = K$ and we talk about quadratic forms or σ has order 2 and $[L : K] = 2$. A Hermitian space, or, more precisely, an L/K -Hermitian space is a finite dimensional vector space V over L with a non-degenerate Hermitian form $H : V \times V \rightarrow L$, i.e. a K -bilinear map such that $H(av, w) = aH(v, w)$ and $H(v, w) = \sigma(H(w, v))$ for all $v, w \in V$ and $a \in L$. Let

$$N := N_{L/K}(L^\times) = \{a\sigma(a) \mid a \in L^\times\}$$

denote the norm subgroup of K^\times . Then $(K^\times)^2 \leq N \leq K^\times$ and $N = (K^\times)^2$ if $L = K$.

Definition 2.1. *The discriminant of H is the signed determinant*

$$\text{disc}(H) := (-1)^{\binom{n}{2}} \det(H_B)N \in K^\times/N$$

where $n := \dim_L(V)$ is the dimension of V and $H_B := (H(b_i, b_j))_{i,j=1}^n \in L^{n \times n}$ is the Gram matrix of H with respect to any L -basis $B = (b_1, \dots, b_n)$ of V .

For quadratic forms H , i.e. $L = K$, the square class $\text{disc}(H) = d(K^\times)^2$ defines a unique étale K -algebra $\mathcal{D}(H) = K[X]/(X^2 - d)$ of degree 2 over K , the *discriminant algebra* of the quadratic form (see [13, Definition 3.1, Remark 3.2] also for the correct definition for fields of characteristic 2). For $L \neq K$ the class of the discriminant defines a unique quaternion algebra over K which we call the discriminant algebra of H :

Definition 2.2. *For $a, b \in K^\times$ let $(a, b)_K$ denote the central simple K -algebra with K -basis $(1, i, j, ij)$ such that $i^2 = a$, $j^2 = b$, $ij = -ji$. If $L = K[\sqrt{\delta}]$ is a quadratic extension of K we also put*

$$(L, b)_K := (\delta, b)_K.$$

For $\sigma \neq \text{id}$ the discriminant algebra of the Hermitian form H with $\text{disc}(H) = dN_{L/K}(L^\times)$ is defined as the class

$$\Delta(H) := [(L, d)_K] \in \text{Br}_2(L, K)$$

where $\text{Br}_2(L, K)$ is the exponent 2-subgroup of the Brauer group of central simple K -algebras that are split by L .

For $a, b \in K^\times$ it holds that $(L, a)_K \cong (L, b)_K$ if and only if $aN_{L/K}(L^\times) = bN_{L/K}(L^\times)$.

Definition 2.3. *For $[D] \in \text{Br}_2(L, K)$ the L -discriminant of $[D]$ is defined as*

$$\text{disc}_L([D]) = bN_{L/K}(L^\times) \in K^\times/N_{L/K}(L^\times)$$

where $b \in K^\times$ is such that $[D] = [(L, b)_K]$.

For an L/K -Hermitian form the L -discriminant of $\Delta(H)$ is just the discriminant of H .

3 Hermitian forms over number fields

We now assume that L is a complex number field with totally real subfield $K := L^+$, in particular $[L : K] = 2$ and $\text{id} \neq \sigma \in \text{Aut}_K(L)$ is the non-trivial K -linear field automorphism of L . Let (V, H) be a totally positive definite Hermitian space of dimension, say, $n := \dim_L(V)$.

Restriction of scalars turns V into a vector space V_K of dimension $2n$ over K . The Hermitian form H defines a quadratic form Q_H on V_K by

$$Q_H(v) := H(v, v) \text{ for all } v \in V_K.$$

The following proposition is given in [17] for general global fields. Note that in [17, Remark (10.1.4)] the determinant of the Hermitian form has to be replaced by its discriminant to obtain a correct statement.

Proposition 3.1. *(see [17, Remark (10.1.4), Theorem (10.1.7)]) Two Hermitian spaces (V, H) and (V', H') are isometric if and only if the quadratic spaces (V_K, Q_H) and $(V'_K, Q_{H'})$ are isometric. If $L = K[\sqrt{\delta}]$ then $\text{disc}(Q_H) = \delta^n$ and the Clifford invariant of Q_H is $\Delta(H)$.*

As Clifford invariant, discriminant, dimension, and signatures at all real places of K determine the isometry class of a quadratic form over K , we get the following corollary.

Corollary 3.2. *Two totally positive definite L/K Hermitian forms are isometric, if and only if they have the same dimension and the same discriminant algebra.*

An important feature for number fields is the so called Hasse principle or local-global principle:

- Two L/K Hermitian forms are isometric over K if and only if they are isometric over all its completions.
- Two quaternion algebras over K are isomorphic, if and only if they ramify at the same places of K . Moreover the number of ramified places is always even.
- Hasse Norm Theorem for quadratic extensions: For $a \in K$ it holds that $a \in N_{L/K}(L)$ if and only if $a \in N_{L_\varphi/K_\varphi}(L_\varphi)$ for all places φ of K .

The finite completions of K are in bijection to the maximal ideals of its ring of integers \mathbb{Z}_K . Let $\varphi \trianglelefteq \mathbb{Z}_K$ be such a non-zero prime ideal. Then there are three cases:

- a) $\varphi\mathbb{Z}_L$ is a maximal ideal of \mathbb{Z}_L , i.e. φ is inert in L/K and L_φ is the unique unramified quadratic extension of K_φ .
- b) $\varphi\mathbb{Z}_L = P_1P_2$ is a product of two distinct prime ideals $P_1 = \sigma(P_2)$ of \mathbb{Z}_L , i.e. φ is split in L/K and $L_\varphi \cong K_\varphi \oplus K_\varphi$.
- c) $\varphi\mathbb{Z}_L = P^2$ is the square of a prime ideal of \mathbb{Z}_L . Then φ is ramified in L/K .

The following local criterion for being a norm is important for the computations.

Lemma 3.3. *An element $0 \neq a \in K$ is a norm, $a \in N_{L/K}(L^\times)$ if and only if the following three criteria are satisfied:*

- (i) *a is totally positive*
- (ii) *$\nu_\varphi(a)$ is even for all prime ideals φ of K that are inert in L/K*
- (iii) *$a \in N_{L_\varphi/K_\varphi}(L_\varphi)$ for all ramified primes φ of K .*

Proof. The first statement guarantees that a is a norm for all infinite places of K , the second one that a is a local norm for all inert places and the third one for the ramified places. For the split places $L_\varphi = K_\varphi \oplus K_\varphi$ and all elements of K_φ are norms. \square

For the infinite completions the complex signature parametrizes Hermitian forms. As H is totally positive definite, this specifies H for the infinite completions of K .

Remark 3.4. The infinite places of K are ramified in the discriminant algebra $\Delta(H)$ if and only if $\text{disc}(H)$ consists of totally negative elements, if and only if $\dim_L(V) \equiv 2$ or $3 \pmod{4}$.

Let φ be a prime ideal of K that is split in L/K . Then the completion $\Delta(H)_\varphi$ contains the completion $L_\varphi \cong K_\varphi \oplus K_\varphi$ which contains zero divisors. So φ is not ramified in $\Delta(H)$.

Remark 3.5. No split prime ideal ramifies in the discriminant algebra of H .

Proposition 3.6. *Let φ be a prime ideal of K that is inert in L/K . Then φ is ramified in $\Delta(H)$ if and only if the φ -adic valuation $\nu_\varphi(d)$ is odd for all $d \in \text{disc}(H)$.*

Proof. Then L_φ/K_φ is the unique unramified extension of degree 2 of K_φ . In particular the norm group

$$N_{L_\varphi/K_\varphi} = \{x \in K_\varphi \mid \nu_\varphi(x) \in 2\mathbb{Z}\}.$$

Now φ is ramified in $\Delta(H) = (L, d)_K$ if and only if d is not a norm in L_φ/K_φ , i.e. if and only if $\nu_\varphi(d)$ is odd. \square

Remark 3.7. For a ramified place $\varphi = P^2$ we can always find a representative $d \in \mathbb{Z}_K$ of $\text{disc}(H)$ such that $d \notin \varphi$ (see Proposition 3.10 (c)). Assume that φ is not a dyadic prime, i.e. $2 \notin \varphi$. Then φ ramifies in the discriminant algebra of H if and only if d is not a square modulo φ .

3.1 Lattices

Assume that L/K is a quadratic extension of local or global number fields and (V, H) a finite dimensional non-degenerate L/K -Hermitian space.

Definition 3.8. A \mathbb{Z}_L -lattice Λ in V is a finitely generated \mathbb{Z}_L submodule of V that contains a basis of V . The dual lattice is

$$\Lambda^* := \{v \in V \mid H(v, \Lambda) \subseteq \mathbb{Z}_L\}$$

and again a \mathbb{Z}_L -lattice in V . We say that Λ is unimodular if $\Lambda = \Lambda^*$.

Example 3.9. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}[\sqrt{-10}]$. Then $P_5 := (5, \sqrt{-10}) \trianglelefteq \mathbb{Z}_L$ is not a principal ideal. Let (V, H) be an n -dimensional L/K -Hermitian space with orthonormal basis $B = (b_1, \dots, b_n)$, so $H_B = I_n$ and (V, H') such a space with $H'_B = \text{diag}(I_{n-1}, 1/5)$. Then

$$\Lambda = \langle B \rangle_{\mathbb{Z}_L} \text{ and } \Lambda' := \bigoplus_{i=1}^{n-1} \mathbb{Z}_L b_i \oplus P_5 b_n$$

are unimodular lattices in (V, H) and (V, H') respectively. If n is even, then

$$\Delta(H) = \begin{cases} (-1, -1)_{\mathbb{Q}} & n \equiv 2 \pmod{4} \text{ is ramified at } \infty, 2 \\ \mathbb{Q} & n \in 4\mathbb{Z} \end{cases}$$

whereas

$$\Delta(H') = \begin{cases} (-10, -5)_{\mathbb{Q}} & n \equiv 2 \pmod{4} \text{ is ramified at } \infty, 5 \\ (-10, 5)_{\mathbb{Q}} & n \in 4\mathbb{Z} \text{ is ramified at } 2, 5. \end{cases}$$

3.2 Lattices, the local picture

We now assume that we work over local fields of characteristic 0. So let K be a complete discrete valued field with valuation ring \mathbb{Z}_K and let L be a quadratic extension of K with valuation ring \mathbb{Z}_L , and let (V, H) be an L/K Hermitian space. Denote by $\pi \in \mathbb{Z}_L$ a generator of the maximal ideal of \mathbb{Z}_L , if L/K is inert, then we choose $\pi \in \mathbb{Z}_K$ and if L/K is non-dyadic ramified we choose π such that $\pi^2 \in \mathbb{Z}_K$, i.e. $\sigma(\pi) = -\pi$.

The next proposition is certainly well known, we indicate a proof as we need similar ideas for lattices invariant under a group.

Proposition 3.10. (a) *There is a \mathbb{Z}_L -lattice Λ in V such that $\pi\Lambda^* \subseteq \Lambda \subseteq \Lambda^*$. Such lattices are called square free.*

(b) *For any square free lattice Λ the Hermitian form induces a non-degenerate form*

$$\overline{H} : \Lambda/\pi\Lambda^* \times \Lambda/\pi\Lambda^* \rightarrow \mathbb{Z}_L/\pi\mathbb{Z}_L, \overline{H}(\ell + \pi\Lambda^*, m + \pi\Lambda^*) := H(\ell, m) + \pi\mathbb{Z}_L$$

which is Hermitian in the case that L/K is unramified and symmetric if L/K is ramified.

(c) *If L/K is ramified, then there is a \mathbb{Z}_L -lattice Λ in V such that $\Lambda = \Lambda^*$. For such a lattice the discriminant of the symmetric bilinear space $(\Lambda/\pi\Lambda^*, \overline{H})$ from (b) is congruent to the discriminant of a Gram matrix of H with respect to a basis of Λ .*

(d) Let L/K be inert and let Λ be a square free lattice. Then the valuation of $\text{disc}(H)$ is congruent modulo 2 to the dimension of the $\mathbb{Z}_L/\pi\mathbb{Z}_L$ -space Λ^*/Λ .

(e) For any square free lattice Λ

$$\tilde{H} : \Lambda^*/\Lambda \times \Lambda^*/\Lambda \rightarrow \mathbb{Z}_L/\pi\mathbb{Z}_L, \tilde{H}(\ell + \Lambda, m + \Lambda) := \pi H(\ell, m) + \pi\mathbb{Z}_L$$

is a Hermitian form in the case that L/K is unramified and a skew-symmetric form if L/K is non-dyadic and ramified.

Proof. (a) Let Λ be an integral lattice, i.e. $H(\Lambda, \Lambda) \subseteq \mathbb{Z}_L$. Then $\Lambda \subseteq \Lambda^*$ and Λ^*/Λ is a finitely generated torsion \mathbb{Z}_L -module. Let $\ell \in \Lambda^*$ be such that $\pi^2\ell \in \Lambda$. As $\sigma(\pi)$ is again a uniformizer of \mathbb{Z}_L there is a unit $u \in \mathbb{Z}_L^\times$ such that

$$H(\pi\ell, \pi\ell) = uH(\ell, \pi^2\ell) \in \mathbb{Z}_L.$$

In particular $\Lambda + \mathbb{Z}_L(\pi\ell)$ is again an integral lattice. Continuing like this, we eventually arrive at a lattice as in (a).

(b) follows from explicit computations.

(c) We continue with the computations in (a) and assume that $\ell \in \Lambda^*$ such that $\pi\ell \in \Lambda$. Then

$$\pi H(\ell, \ell) = H(\pi\ell, \ell) \in \mathbb{Z}_L$$

so $\nu_\pi(H(\ell, \ell)) \geq -1$. As $H(\ell, \ell) \in K$ the π -adic valuation $\nu_\pi(H(\ell, \ell))$ is even, and therefore $H(\ell, \ell) \in \mathbb{Z}_L$. So $\Lambda + \mathbb{Z}_L\ell$ is again an integral lattice and we can continue enlarging Λ until $\Lambda = \Lambda^*$. For a \mathbb{Z}_L -basis B of such a unimodular lattice Λ , the Gram matrix H_B is congruent mod π to the Gram matrix of the bilinear space $\Lambda/\pi\Lambda$ and so is its determinant.

(d) is clear and (e) follows again by direct computations. \square

4 Discriminants of characters

4.1 Characters of finite groups

Let G be a finite group and K be a field. A K -representation ρ of G is a group homomorphism $\rho : G \rightarrow \text{GL}_n(K)$ into the group of invertible $n \times n$ -matrices over K . The natural number n is then called the degree of the representation. The representation ρ is called *irreducible*, if the KG -module K^n has no non-trivial KG -submodules. The associated Frobenius character is the map

$$\chi_\rho : G \rightarrow K, g \mapsto \text{trace}(g).$$

For fields K of characteristic zero, Frobenius characters are in bijection to isomorphism classes of K -representations of G . As $\rho(g)$ is diagonalizable over an algebraic closure of K , the character values are sums of $|G|$ -th roots of unity. In particular the *character field* $\mathbb{Q}(\chi_\rho)$ of χ_ρ , the field extension $\mathbb{Q}(\chi_\rho(g) : g \in G)$ generated by the character values, is

an abelian number field. The *ordinary character table* displays the character values of the complex irreducible characters of G on representatives of the conjugacy classes of G . For more details see for instance [7].

To relate the representation theory over fields of characteristic 0 and positive characteristic, p , Brauer characters are introduced. An element of G is called a p' -element, if its order is not divisible by p . The set $G_{p'}$ denotes the set of all p' -elements in G ; it is a union of conjugacy classes, the p' -classes. A p -Brauer character is a certain class function $\varphi : G_{p'} \rightarrow \mathbb{C}$. Loosely speaking, the restriction $\chi \pmod{p}$ of a complex Frobenius character χ to the p' -classes of G is a p -Brauer character of G . If p does not divide the group order, then this restriction is a bijection between the irreducible p -Brauer characters and the irreducible complex characters. For primes p dividing the group order the p -Brauer character $\chi \pmod{p}$ might be reducible. The irreducible constituents of $\chi \pmod{p}$ are in bijection to the composition factors of $L/\varphi L$, for any lattice L invariant under the representation ρ affording χ and a suitably chosen prime ideal φ . To specify the ideal we also denote this modular reduction by $\chi \pmod{\varphi}$. See [13, Sections 6 and 7.1] and [8, Introduction] for the subtleties that might arise here and Section 9 for an example. For a more sophisticated introduction to Brauer characters see [7, Chapter 15] or, for a summary, [3, Section 2.1] and [8, Introduction]. We just note the following.

Remark 4.1. Let F be a sufficiently large finite field of characteristic p . Then the irreducible p -Brauer characters are in bijection to the simple FG -modules. We then also say that the corresponding F -representation of G affords the respective irreducible Brauer character.

Definition 4.2. (see for instance [8, Introduction, Section 9]) *Let χ be an absolutely irreducible ordinary or Brauer character of the finite group G and let ρ be a representation affording χ . The Frobenius-Schur indicator of χ is*

- + if χ is real valued and there is a non-degenerate $\rho(G)$ -invariant quadratic form or χ is the trivial 2-Brauer character.
- if χ is real valued and there is a non-degenerate $\rho(G)$ -invariant symplectic form but no invariant non-degenerate quadratic form.
- o in all other cases.

The Frobenius-Schur indicator of an ordinary character χ is easily calculated from its character values. If the Frobenius-Schur indicator of χ is o then there is a positive-definite $\rho(G)$ -invariant Hermitian form for any representation ρ affording χ . This is not automatically true for Brauer characters, as there are irreducible Brauer characters of indicator o for which the associated representation is not contained in a unitary group, for instance all indicator o irreducible 3-Brauer characters of $L_3(3)$ are not unitary, as all irreducible 3-modular representations of $L_3(3)$ are realisable over \mathbb{F}_3 .

Definition 4.3. *An irreducible p -Brauer character χ is called unitary, if there is a p -power q such that $\chi(g^{-1}) = \chi(g^q)$ for all $g \in G_{p'}$.*

Note that real valued Brauer characters (indicator $+$ or $-$) are always unitary as $\chi(g^{-1}) = \overline{\chi}(g)$ so $q = p^0 = 1$ is a possible choice. The unifying philosophy here is to consider symplectic and quadratic forms as “unitary” invariant forms over a suitable field extension. Any such non-degenerate form gives rise to an isomorphism between the representation and its (unitary) dual.

Lemma 4.4. (see [4, Lemma 4.4.1] and also [8, Introduction, Section 10] for a slightly too simplified version) *An indicator o absolutely irreducible Brauer character χ is unitary if and only if the representation ρ affording χ admits a non-degenerate unitary form.*

To simplify the exposition we also say that all ordinary characters are unitary.

4.2 Discriminants of characters

Let χ be an absolutely irreducible ordinary character of a finite group G of even degree $\chi(1) \in 2\mathbb{Z}$. Let $K := \mathbb{Q}(\chi)$ denote the character field of χ .

If the indicator of χ is $+$ then [12] shows that there is a unique square class $\text{disc}(\chi) \in K^\times / (K^\times)^2$ such that for any extension field L of K and any LG -module V affording the character χ the discriminants of all non-degenerate G -invariant quadratic forms Q on V are

$$\text{disc}(Q) = \text{disc}(\chi)(L^\times)^2.$$

This square class of the character field is called the *orthogonal discriminant* of χ and the *discriminant algebra* of χ is

$$\mathcal{D}(\chi) := K[X]/(X^2 - d) \text{ where } \text{disc}(\chi) = d(K^\times)^2.$$

Now assume that χ is an ordinary irreducible character of indicator o and of even degree. Then the character field $L = \mathbb{Q}(\chi)$ is a complex number field. Let $K := L^+$ denote the maximal real subfield of L and put $N := N_{L/K}(L^\times)$ to be the norm subgroup of K^\times . Then

$$(K^\times)^2 \leq N \leq K^\times.$$

If there is a L -representation V of G affording the character χ , then there is a 1-dimensional K -space of G -invariant L/K -Hermitian forms spanned by some totally positive definite Hermitian form, say, H . In particular all non-zero G -invariant L/K -Hermitian forms have the same discriminant, $\text{disc}(H)$.

Definition 4.5. *For a finite group G define*

$$\text{Irr}^o(G) := \{\chi \in \text{Irr}_{\mathbb{C}}(G) \mid \text{ind}(\chi) = o, \chi(1) \in 2\mathbb{Z}\}$$

the set of absolutely irreducible ordinary characters of G of even degree and indicator o . Let $\chi \in \text{Irr}^o(G)$, $L := \mathbb{Q}(\chi)$ and $K := L^+$. Assume that there is an L -representation V of G affording the character χ . Then the unitary discriminant of χ is defined as

$$\text{disc}(\chi) := \text{disc}(H) = dN_{L/K}(L^\times) \in K^\times / N_{L/K}(L^\times)$$

for any non-degenerate G -invariant Hermitian form H on V .

The discriminant algebra of χ is defined as

$$\Delta(\chi) := [(L, d)_K] = \Delta(H).$$

As $\chi(1)$ is assumed to be even, Remark 3.4 translates into the following remark.

Remark 4.6. If \wp is an infinite place of K , then \wp is ramified in $\Delta(\chi)$ if and only if $\chi(1) \equiv 2 \pmod{4}$.

This accounts for the fact that $\det(\chi)$ consists of totally positive elements of K .

Remark 3.5 gives us

Remark 4.7. Let $\chi \in \text{Irr}^o(G)$. If \wp is a finite place of $\mathbb{Q}(\chi)^+$ that is split in the extension $\mathbb{Q}(\chi)/\mathbb{Q}(\chi)^+$ then \wp is not ramified in $\Delta(\chi)$.

Nontrivial Schur indices are very rare for indicator o characters of the groups in [5] (see [6], [20]). In these rare cases we found ways to get around the direct computations by restricting the character to some appropriate subgroup. For the well-definedness of this discriminant we rely on [9, Definition (10.28), Corollary (10.35)]:

Remark 4.8. Let \mathcal{A} be a central simple L -algebra of even degree $2m$ and with involution ι of the second kind, i.e. the restriction of ι to the center L of \mathcal{A} is a field automorphism of order 2. Let K denote its fixed field. Then there is a central simple K -algebra with involution that is Brauer equivalent to $(L, \text{disc}(H))_K$ if $(\mathcal{A}, \iota) = (L^{2m \times 2m}, \iota_H)$ for some non-degenerate L/K -Hermitian form H on L^{2m} .

4.3 Indicator –

Let χ be an absolutely irreducible complex character with Frobenius-Schur indicator $-$. Then the character field K is a totally real number field and there is a KG -module V affording the orthogonal character $\psi := 2\chi$. Let $\mathcal{Q} := \text{End}_{KG}(V)$ denote the endomorphism algebra of V . This is a totally definite quaternion algebra with center K . Choose a maximal subfield $L = K[\sqrt{-\delta}]$ of \mathcal{Q} . Then $\delta \in K$ is totally positive and the L -discriminant of $[\mathcal{Q}]$ is represented by some totally positive $\gamma \in K$ such that

$$\mathcal{Q} = (-\delta, -\gamma)_K.$$

Remark 4.9. Considering \mathcal{Q} as an L -vector space of dimension 2, the norm form on \mathcal{Q} is a totally positive Hermitian form of discriminant $\text{disc}(\mathcal{Q}) = -\gamma$.

Proof. Let $L = K1 \oplus Ki \leq \mathcal{Q}$ and choose $j \in \mathcal{Q}$ such that $j^2 = -\gamma$ and $ij = -ji$. Then $(1, j)$ is an orthogonal L -basis of \mathcal{Q} with respect to the norm form. As the norm of 1 is 1 and the norm of j is γ the discriminant of this 2-dimensional Hermitian L -space is represented by $-\gamma$. \square

Corollary 4.10. *Let H be a non-zero G -invariant L/K -Hermitian form H on V . Then*

$$\text{disc}(H) = \text{disc}_L(\mathcal{Q})^{\chi(1)/2} \text{ and } \Delta(H) = [\mathcal{Q}]^{\chi(1)/2} \in \text{Br}_2(L, K).$$

5 Elementary character theoretic methods

This section describes elementary character theoretic methods that are used to compute the discriminants of unitary ordinary characters of G .

5.1 Unitary stability

In [13] the most important notion is the one of orthogonal stability. An ordinary or Brauer character χ is called *orthogonally stable*, if all its indicator + constituents have even degree. Orthogonally stable characters are exactly those that have a well defined orthogonal discriminant. For unitary characters we need a slightly stronger condition:

Definition 5.1. *An ordinary or Brauer character χ of a finite group G is called unitary stable, if all irreducible unitary constituents of χ have even degree.*

Similarly as in [13, Theorem 5.13] one has the following proposition.

Proposition 5.2. *An ordinary character is unitary stable, if and only if it has a well defined unitary discriminant algebra.*

The unitary discriminant of a sum of unitary stable characters is essentially the product of the unitary discriminants of the unitary summands. If these summands have a larger character field, then also their Galois conjugates appear and one needs to replace the product by the appropriate norm as in [13, Proposition 5.17], Summands of indicator $-$ are handled in Corollary 4.10. Similarly Schur indices for summands χ of indicator $+$ give rise to an additional correction factor, $\text{disc}_L(\mathcal{Q})^{\chi(1)/2}$ (see Theorem 7.4).

Remark 5.3. In our computations we often get equations for discriminants of unitary stable characters that *hold over* the complex field L . This means that the equations hold in $K^\times/N_{L/K}(L^\times)$ where $K = L^+$ is the totally real subfield of L . If there is no field mentioned we always mean the character field.

5.2 Modular reduction

Theorem 5.4. *Let $\chi \in \text{Irr}^o(G)$ and put $L := \mathbb{Q}(\chi)$. Let \wp be a finite place of L^+ that is inert in L/L^+ . If $\chi \pmod{\wp}$ is unitary stable then \wp is not ramified in the discriminant algebra $\Delta(\chi)$.*

Proof. The proof of this theorem is almost the same as the one of [13, Theorem 6.4], showing the result for real characters χ . Put $K = L^+$ to denote the real subfield of L . As \wp is inert, the completion L_\wp of L at the finite place \wp of K is the unique unramified extension of degree 2 of K_\wp . Denote by P the maximal ideal of \mathbb{Z}_{L_\wp} and by F the residue field \mathbb{Z}_{L_\wp}/P . After possibly passing to a suitable unramified extension of K_\wp we may assume that there is an $L_\wp G$ -module (V, H) affording the character χ . Let $\Lambda \subseteq \Lambda^*$ be a maximal integral G -invariant lattice in V . Then

$$P\Lambda^* \subseteq \Lambda \subseteq \Lambda^*$$

and $(\Lambda^*/\Lambda, \tilde{H})$ is a Hermitian FG -module by Proposition 3.10 (e). Now Λ is maximal integral, so this FG -module is the orthogonal direct sum of simple unitary modules. By assumption all these modules have even dimension, so $\dim_F(\Lambda^*/\Lambda)$ is even and Proposition 3.10 (d) shows that the φ -adic valuation of the discriminant of H is even. \square

Corollary 5.5. *Let $\chi \in \text{Irr}^o(G)$. Then all prime ideals of $\mathbb{Q}(\chi)^+$ that ramify in the discriminant algebra $\Delta(\chi)$ divide $2|G|$.*

For primes that are ramified in L/L^+ the module Λ^*/Λ carries a symplectic G -invariant form. So Proposition 3.10 (c) implies the following result.

Corollary 5.6. *Let $\chi \in \text{Irr}^o(G)$. If a non-dyadic prime φ is ramified in $\mathbb{Q}(\chi)/\mathbb{Q}(\chi)^+$ and $\chi \pmod{\varphi}$ is unitary stable, then $\text{disc}(\chi) \pmod{\varphi}$ is the orthogonal discriminant of $\chi \pmod{\varphi}$. This implies that φ is ramified in the discriminant algebra $\Delta(\chi)$ if and only if the orthogonal discriminant of $\chi \pmod{\varphi}$ is not a square in the residue field.*

For blocks of defect 1, also the converse of Theorem 5.4 is true. For general cyclic defect an analogous statement as [13, Theorem 6.12] is also true for unitary stable characters.

Proposition 5.7. *Let $\chi \in \text{Irr}^o(G)$ be in a p -block of defect 1. Put $L := \mathbb{Q}(\chi)$ and let φ be a finite place of L^+ that contains the rational prime p and is inert in L/L^+ . Then $\chi \pmod{\varphi}$ is unitary stable if and only if φ does not ramify in the discriminant algebra of χ .*

The proof is almost the same as for orthogonally stable characters in [13, Theorem 6.10]. In particular this shows that if φ is inert and χ in a block of defect 1, then the reduction $\chi \pmod{\varphi}$ has at most two unitary constituents.

Example 5.8. As an example let χ be one of the two complex conjugate indicator o irreducible ordinary faithful character of degree 116622 of the group $3.ON$, the Schur cover of the sporadic simple O’Nan group of order $2^9 \cdot 3^5 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$. Then $\mathbb{Q}(\chi) = \mathbb{Q}[\sqrt{-3}]$ and 7, 19 and 31 are norms in $\mathbb{Q}[\sqrt{-3}]/\mathbb{Q}$. The 11-modular reduction of χ is irreducible. Modulo 5 the character has defect 1 and $\chi \pmod{5} = 5643ab + 52668ab$. Whereas $5643a$ and $5643b$ are unitary, the other two 5-modular constituents do not admit a non-zero invariant form. So Proposition 5.7 implies that 5 divides the discriminant of χ and hence $\text{disc}(\chi) \in \{-5, -10\}$. Also the 3-modular reduction $\chi \pmod{3} = 6138 + 104346$ is unitary stable. Both 3-modular constituents have indicator $+$, $\text{disc}(6138) = O-$ and $\text{disc}(104346) = O+$. So the unitary discriminant of χ is not a square modulo 3, and hence $\text{disc}(\chi) = -10$.

5.3 The unitary discriminants for $O_{10}^+(2)$.

The group $O_{10}^+(2)$ is one example where modular reduction is enough to obtain all unitary discriminants.

Theorem 5.9. *The unitary discriminants of the characters χ in $\text{Irr}^\circ(O_{10}^+(2))$ are as follows:*

χ	$\chi(1)$	$\mathbb{Q}(\chi)$	$\text{disc}(\chi)$	$\Delta(\chi)$
33, 34	110670	$\mathbb{Q}[\sqrt{-15}]$	-1	$[(-1, -3)_{\mathbb{Q}}]$
51, 52	332010	$\mathbb{Q}[\sqrt{-15}]$	-2	$[(-2, -5)_{\mathbb{Q}}]$
68, 69	442680	$\mathbb{Q}[\sqrt{-15}]$	1	$[\mathbb{Q}]$
79, 80	711450	$\mathbb{Q}[\sqrt{-7}]$	-3	$[(-1, -3)_{\mathbb{Q}}]$
81, 82	711450	$\mathbb{Q}[\sqrt{-7}]$	-3	$[(-1, -3)_{\mathbb{Q}}]$

Proof. The first three pairs of characters χ with character field $L := \mathbb{Q}[\sqrt{-15}]$ have a unitary stable reduction modulo all primes that are inert in L/\mathbb{Q} . So by Theorem 5.4 no inert prime ramifies in the discriminant algebra of χ . Remark 4.7 tells us that no split prime ramifies in $\Delta(\chi)$ and Remark 4.6 shows that the infinite place of \mathbb{Q} ramifies in $\Delta(\chi)$ for the first two but not for the third of the three pairs of characters. For all three characters the 5-modular reduction is absolutely irreducible. From the database of orthogonal discriminants in [1] we get that this 5-modular reduction has orthogonal discriminant $O+$ for $\chi_{33/34}$ and $\chi_{68/69}$ and $O-$ for $\chi_{51/52}$. So Corollary 5.6 shows that 5 is ramified in $\Delta(\chi)$ only for $\chi = \chi_{51/52}$. As the number of ramified places in $\Delta(\chi)$ is even, this yields the behaviour of the prime 3. So $\Delta(\chi_{33})$ is ramified at ∞ and 3, $\Delta(\chi_{51})$ is ramified at ∞ and 5, and $\Delta(\chi_{61}) = [\mathbb{Q}]$.

The four characters of degree 711450 are absolutely irreducible modulo 5, 7, 17, and 31. Modulo the ramified prime, 7, their orthogonal discriminant is $O+$ according to [1]. As 2 is a norm in $\mathbb{Q}[\sqrt{-7}]/\mathbb{Q}$ the possible discriminants are -1 and -3. But -1 is not a square modulo 7 and hence $\text{disc}(\chi) = -3$ for these four characters χ . \square

5.4 Restriction and induction

Let G be a finite group and $\chi \in \text{Irr}^\circ(G)$. Denote by $L = \mathbb{Q}(\chi)$ the character field of χ . If one can construct an L -representation ρ affording the character χ , then finding a non-degenerate $\rho(G)$ -invariant Hermitian form boils down to solving a system of linear equations. A more sophisticated method, given in [19] to obtain the commuting algebra, can also be applied for finding invariant Hermitian forms.

Often the degree of χ is too big so that we cannot construct such a representation ρ , however sometimes the restriction of χ to a subgroup remains unitary stable and all its summands that occur in $\chi|_U$ with odd multiplicity have manageable degree.

Remark 5.10. If there is a subgroup $U \leq G$ such that the restriction $\chi|_U$ of χ to U is unitary stable, then $\text{disc}(\chi) = \text{disc}(\chi|_U)$.

Note that the parity of the degrees of the unitary constituents of the restriction matters, but their character fields are not relevant. For induction it is the opposite: If ψ is a character of some subgroup U and F a U -invariant form in a representation affording the character ψ , then the orthogonal sum of $[G : U]$ copies of F is a G -invariant form in the induced representation.

Remark 5.11. Let $U \leq G$ and let ψ be some character of U . If the induced character ψ^G is unitary stable then

$$\text{disc}(\psi^G) = \begin{cases} \text{disc}(\psi) & \text{if } [G : U] \text{ odd} \\ 1 & \text{if } [G : U] \text{ even} \end{cases}$$

This equation holds over the character field of ψ , which might be larger than the one of ψ^G .

5.5 Tensor products and symmetrizations

Tensor products and symmetrizations often give rise to equations relating discriminants of irreducible characters:

Remark 5.12. Let χ, ψ be two absolutely irreducible ordinary characters of G such that $\chi(1)$ is even. If the tensor product $\chi \cdot \psi$ is unitary stable, then

$$\text{disc}(\chi \cdot \psi) = \text{disc}(\chi)^{\psi(1)}$$

over any field containing $\mathbb{Q}(\chi)$ and $\mathbb{Q}(\psi)$.

Symmetrizations have been dealt with in [14] and illustrated with the example of the group $6.Suz$, where the symmetrizations of the complex character of degree 12 allow to conclude that all faithful $\chi \in \text{Irr}^o(6.Suz)$ have unitary discriminant $(-1)^{\chi(1)/2}$ (see [14, Section 6]). Symmetrizations are quite helpful for groups G for which one of its covering groups has a faithful character χ of fairly small degree. As for induced characters it is not important that $\chi(1)$ is even to use unitary stable symmetrizations of χ for getting equations for unitary discriminants over the character field of χ .

As an example consider the group $U_4(2)$. It has three pairs of complex conjugate indicator o irreducible characters of even degree, $\chi_{5,6}$ of degree 10, $\chi_{12,13}$ of degree 30, $\chi_{14,15}$ of degree 40. Symmetrizing the indicator o characters of degree 5, we obtain $\chi_{5,6}$ as the exterior square, i.e. partition $[1, 1]$, and $\chi_{14,15}$ as the symmetrization of degree 3 corresponding to the partition $[2, 1]$. The sum $\chi_{13} + \chi_{14}$ is the symmetrization of degree 6 for the partition $[4, 1, 1]$ of a suitable character of degree 4 of $2.U_4(2)$. Combining the results we obtain $\text{disc}(\chi) = (-1)^{\chi(1)/2}$ for all $\chi \in \text{Irr}^o(U_4(2))$.

6 Perfect groups with even center

Let G be a perfect group with a center of even order and $\chi \in \text{Irr}^o(G)$ be a faithful character of G . Put $L := \mathbb{Q}(\chi)$ and $K := L^+$ the maximal real subfield of the character field L .

Proposition 6.1. *Let \wp be a non-dyadic prime of K .*

(a) *If \wp is unramified in L/K then \wp is unramified in $\Delta(\chi)$.*

(b) If \wp is ramified in L/K and d is the sum of the dimensions of the orthogonal constituents of $\chi \pmod{\wp}$ then \wp is ramified in $\Delta(\chi)$ if and only if $(-1)^{d/2}$ is not a square modulo \wp .

Proof. (a) For any non-dyadic prime \wp of K the central element $z \in G$ of order 2 acts as $-id$ on all \wp -modular constituents of χ . As G is perfect, $\det(-id) = 1$ and thus all these constituents have even degree. Hence $\chi \pmod{\wp}$ is unitary stable. Now (a) follows from Remark 4.7 and Theorem 5.4.

(b) As in (a) all \wp -modular constituents of χ have even degree. Let P be the prime of L such that $P^2 = \wp\mathbb{Z}_L$. By Proposition 3.10 there is a $\mathbb{Z}_{L_P}G$ -lattice Λ such that $P\Lambda^* \subset \Lambda \subset \Lambda^*$. Moreover $(\Lambda/P\Lambda^*, \overline{H})$ is a non-degenerate symmetric bilinear space over the residue field $\mathbb{Z}_L/P \cong \mathbb{Z}_K/\wp$ and $(\Lambda^*/\Lambda, \tilde{H})$ is a symplectic $\mathbb{Z}_L/P\mathbb{Z}_L$ -module. In particular all orthogonal \wp -modular constituents occur with even multiplicity in Λ^*/Λ , so

$$d \equiv \dim(\Lambda/P\Lambda^*) \pmod{4}.$$

Clearly z acts as $-id$ on $\Lambda/P\Lambda^*$. Since G is perfect, the Spinor norm of $-id$ is trivial. But the Spinor norm of $-id$ is the determinant of \overline{H} (see for instance [10, Section 3.1.2]) and so also this determinant is trivial and $\text{disc}(\overline{H}) = (-1)^{d/2} \in \{1, -1\}$. Now Corollary 5.6 tells us that \wp is ramified in $\Delta(\chi)$, if and only if $(-1)^{d/2}$ is not a square modulo \wp . \square

Corollary 6.2. *Let G be a perfect group G whose center has an order divisible by 4 and let $\chi \in \text{Irr}^o(G)$ be faithful. Then only dyadic primes of K might ramify in $\Delta(\chi)$.*

Proof. Let z be a central element of order 4. Then z acts as a primitive fourth root of unity on all irreducible representations of G . Again, the determinant of such a representation is 1, so all faithful irreducible characters have a degree that is a multiple of 4. As this also holds for $\chi \in \text{Irr}^o(G)$, no infinite place of K ramifies in $\Delta(\chi)$. Proposition 6.1 shows that the only primes of K that might possibly ramify in $\Delta(\chi)$ are the dyadic primes of K . \square

6.1 The Q_8 -trick

Remark 6.3. Let G be a group with cyclic center of even order and let $z \in G$ denote the central element of order 2 in G . Assume that G contains a subgroup $U \cong Q_8$ with $z \in U$. Let χ be the unique faithful irreducible character of U . Then the restriction of any faithful character \mathfrak{X} of G to U is a multiple of χ and hence unitary stable. The methods from Section 4.3 allow to read off the unitary discriminant of \mathfrak{X} just from its degree and its character field: Put $L := \mathbb{Q}(\mathfrak{X})$ and let K denote its maximal real subfield. Then $(-1, -1)_K = (L, -d)_K$ for $d = \text{disc}_L((-1, -1)_K)$ and

$$\begin{aligned} \text{disc}(\mathfrak{X}) &= 1 \text{ if } \mathfrak{X}(1) \equiv 0 \pmod{4} \\ \text{disc}(\mathfrak{X}) &= -d \text{ if } \mathfrak{X}(1) \equiv 2 \pmod{4}. \end{aligned}$$

In both cases the discriminant algebra of \mathfrak{X} is

$$\Delta(\mathfrak{X}) = [(-1, -1)_K]^{\mathfrak{X}(1)/2} \in \text{Br}_2(L, K).$$

The assumption that Q_8 is contained in such a group occurs quite frequently. For instance all covering groups $2.A_n$, for $n \geq 4$, contain such a group Q_8 (as $2.A_4$ does) and hence the Q_8 -trick gives all the unitary discriminants of their faithful unitary characters.

To illustrate the Q_8 -trick, let $G = 2.S_6(3)$. Then all faithful $\chi \in \text{Irr}^o(2.S_6(3))$ have character field $L = \mathbb{Q}(\sqrt{-3})$. As $(-1, -1)_{\mathbb{Q}} = (-3, -2)_{\mathbb{Q}}$ the remark shows that $\text{disc}(\chi) = (-2)^{\chi(1)/2}$.

7 Antiadjoint automorphisms

In this section we assume that the characteristic of the field K is not 2. For quadratic KG -modules (V, Q) the paper [12] works out a very useful method to compute the determinant of the associated G -invariant symmetric bilinear form

$$B : V \times V \rightarrow K, B(x, y) := Q(x + y) - Q(x) - Q(y).$$

If there is an invertible $X \in \text{End}_K(V)$ such that $X^{ad} = -X$ (adjoint with respect to B) then $\det(B) = \det(X)(K^\times)^2$. There is an elementary proof by choosing a basis of V and considering the matrices X and B with respect to this basis. Then

$$X^{ad} = BX^{tr}B^{-1} = -X \text{ and hence } (XB) = -(XB)^{tr}$$

is a skew symmetric matrix. As the determinant of a skew symmetric matrix is always a square we get $\det(X) = \det(B)$ modulo squares.

Note that the adjoint involution of B inverts the group elements so we can search for skew adjoint invertible elements in the subalgebra of $\text{End}_K(V)$ spanned by the group elements without having the G -invariant quadratic form Q .

For any central simple K -algebra A there is a field extension E , such that $E \otimes_K A \cong E^{n \times n}$. For $a \in A \subseteq E \otimes_K A \cong E^{n \times n}$ we define the *reduced norm* of a as the determinant of the matrix $a \in E^{n \times n}$, $N_{red}(a) = \det(a)$ (see [15, Section 9]). This definition depends neither on the chosen splitting field E nor on the identification with a matrix ring. Moreover for any $a \in A$ we have $N_{red}(a) \in K$. We briefly recall the notion of a discriminant of an orthogonal involution.

Remark 7.1. ([9, Definition 7.2]) Let K be a field of characteristic $\neq 2$ and A be a central simple K -algebra of even degree $2m$, so $\dim_K(A) = (2m)^2$. An *orthogonal involution* ι on A is a K -algebra anti-automorphism ι of order 2 such that the K -subspace

$$A_- := \{a \in A \mid \iota(a) = -a\}$$

has dimension $m(2m - 1)$. Then A_- contains invertible elements. The *discriminant* of ι is defined as the square class

$$\text{disc}(\iota) := (-1)^m N_{red}(a)(K^\times)^2$$

for any invertible $a \in A_-$. In particular for $A = K^{2m \times 2m}$ and a symmetric $B \in \text{GL}_{2m}(K)$ such that $\iota = \iota_B$ is the adjoint involution with respect to B , then

$$\text{disc}(\iota) = \text{disc}(\iota_B) = \text{disc}(B).$$

The adjoint involution of an L/K -Hermitian form H is an involution of second kind on the endomorphism ring, as it restricts non trivially to the center L . Here the discriminant algebra from Remark 4.8 is hard to compute in general. Also it only singles out a class of $K^\times/N_{L/K}(L^\times)$ which, for number fields K , cannot be determined by looking at reductions modulo prime ideals of K , as for finite fields norms are surjective. To apply similar methods as for quadratic forms to Hermitian LG -modules (V, H) , we need to find an orthogonal subalgebra of $\text{End}_L(V)$:

Definition 7.2. *Let \mathcal{A} be a central simple L -algebra of even degree $2m$ and with L/K -involution ι of second kind. Then a K -subalgebra A of \mathcal{A} is called an orthogonal subalgebra of (\mathcal{A}, ι) , if and only if*

- (a) A is a central simple K -algebra with $LA = \mathcal{A}$.
- (b) A is invariant under ι , i.e. $\iota(A) = A$.
- (c) The restriction of ι to A is an orthogonal involution of A .

Example 7.3. Let ι be the adjoint involution of a Hermitian form H on V and $\mathcal{A} := \text{End}_L(V) \cong L^{n \times n}$. Then for any orthogonal basis B of H the subalgebra

$$A := \{a \in \mathcal{A} \mid {}^B a^B \in K^{n \times n}\} \cong K^{n \times n}$$

is an orthogonal subalgebra of (\mathcal{A}, ι) .

Theorem 7.4. *(see also [9, Proposition (10.33)]) Let (V, H) be a Hermitian L -space of even dimension $2m$, $\mathcal{A} := \text{End}_L(V)$, and $\iota = \iota_H$ the adjoint involution of the non-degenerate Hermitian form H . Let A be an orthogonal subalgebra of \mathcal{A} .*

- (a) $[A] \in \text{Br}_2(L, K)$.
- (b) $\text{disc}(H) = \text{disc}_L([A])^m \text{disc}(\iota|_A) \in K^\times/N_{L/K}(L^\times)$.

Proof. (a) Let A be an orthogonal subalgebra of \mathcal{A} . Then A is a central simple K -algebra such that $A \otimes_K L \cong L^{2m \times 2m}$, i.e. $A \cong \mathcal{Q}^{m \times m}$ for some quaternion algebra \mathcal{Q} with center K that is split by $L = K[\sqrt{-\delta}]$.

(b) Choose $b \in \text{disc}_L([A])$. Then $\mathcal{Q} = (L, b)_K$. Consider the case $m = 1$. Then

$$\mathcal{Q} = \langle i := \text{diag}(\sqrt{-\delta}, -\sqrt{-\delta}), j := \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \rangle_K \subseteq L^{2 \times 2}$$

with is invariant under the adjoint involution ι of the L/K -Hermitian form $B := \text{diag}(1, b)$. This involution is orthogonal on \mathcal{Q} and i is a skew-adjoint element so

$$\text{disc}(\iota|_{\mathcal{Q}}) = -\det(i) = -\delta$$

whereas $\text{disc}(B) = -b$. As $\delta = N_{L/K}(\sqrt{-\delta})$ is a norm and $\text{disc}(B) = \text{disc}(\iota)$ we have

$$\text{disc}(\iota) = \text{disc}_L([A])^m \text{disc}(\iota|_A). \tag{1}$$

For general m choose a suitable L -basis of V on which A acts as block matrices with blocks in \mathcal{Q} as above. Then the Hermitian adjoint involution ι of the block diagonal matrix $\mathbf{B} := \text{diag}(B, \dots, B)$ restrict to an orthogonal involution of A . As A is a central simple K -algebra, Skolem-Noether theorem tells us that any other orthogonal involution on A is of the form $\kappa_u \iota$, where κ_u is the conjugation by some symmetric unit $u \in A$. This changes both, $\text{disc}(u\mathbf{B}) = \det(u) \text{disc}(\mathbf{B})$ and $\text{disc}(\iota|_A)$ by $\det(u)$, so equation (1) holds for all Hermitian involutions $\iota = \iota_H$. \square

Corollary 7.5. *In the situation of Theorem 7.4 the discriminant of the Hermitian form H is*

$$\text{disc}(H) = (-1)^m N_{\text{red}}(X) \text{disc}_L([A])^m$$

for any skew adjoint invertible $X \in A$, i.e. $X = -\iota_H(X) \in A^\times$.
If $X, X' \in A^\times$ are both skew adjoint, then

$$N_{\text{red}}(X)(K^\times)^2 = N_{\text{red}}(X')(K^\times)^2.$$

7.1 The fixed algebra under an automorphism

Let G be a finite group and $\chi \in \text{Irr}^o(G)$. Put $2m := \chi(1)$, $L = \mathbb{Q}(\chi)$, and assume that there is an L -representation $\rho : G \rightarrow \text{GL}(V)$ affording the character χ . Put $K := L^+$ to denote the real subfield of L and let $H : V \times V \rightarrow L$ be a $\rho(G)$ -invariant L/K -Hermitian form on V . Assume that there is $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$ such that $\chi \circ \alpha = \bar{\chi}$. Then $\chi + \bar{\chi}$ extends to an absolutely irreducible character

$$\mathfrak{X} = \text{Ind}_G^{\mathcal{G}}(\chi)$$

of the semidirect product $\mathcal{G} := G : \langle \alpha \rangle$.

Theorem 7.6. *Let $A := \text{Fix}_\alpha(\rho) = \langle \rho(g) + \rho(\alpha(g)) \mid g \in G \rangle_K$ denote the α -fixed algebra.*

- (a) *Let \mathcal{V} be a $K\mathcal{G}$ -module affording the character $2\mathfrak{X}$ and put $\mathcal{Q} := \text{End}_{K\mathcal{G}}(\mathcal{V})$.
Then $[A] = [\mathcal{Q}] \in \text{Br}_2(L, K)$.*
- (b) *The algebra A is invariant under the adjoint involution of H , i.e. $\iota_H(A) = A$.*
- (c) *Assume that the Frobenius-Schur indicator of \mathfrak{X} is $+$.
Then A is an orthogonal subalgebra of $(\text{End}_L(V), \iota_H)$.*
- (d) *Assume that the Frobenius-Schur indicator of \mathfrak{X} is $-$.
Then $\text{disc}(H) = \text{disc}_L(\mathcal{Q})^m$ and $\Delta(H) = [\mathcal{Q}]^m$.*

Proof. (a) Write $L = K[\sqrt{-\delta}]$, then $\text{End}_L(V) = A \oplus \sqrt{-\delta}A$, in particular $LA = \text{End}_L(V) \cong L^{2m \times 2m}$. So $\dim_K(A) = (2m)^2$ and A is central simple.

To identify the class of the central simple K -algebra A in the Brauer group, we relate the centraliser of A in $\text{End}_K(V)$ to the Schur indices of \mathfrak{X} , i.e. to the class $[\mathcal{Q}] \in \text{Br}_2(L, K)$.

We closely follow [2, Section 2]:

Let $\bar{}$ denote the non-trivial Galois automorphism of L/K . Identify $\text{End}_L(V)$ with $L^{2m \times 2m}$ by choosing an L -basis B of V and define the K -linear endomorphism $\sigma \in \text{End}_K(V)$ by

$$\sigma\left(\sum_{i=1}^{2m} a_i B_i\right) = \sum_{i=1}^{2m} \bar{a}_i B_i.$$

For $X \in L^{2m \times 2m}$ the Galois conjugate matrix $\bar{X} = (\bar{X}_{ij})$ is the matrix of the endomorphism $\sigma X \sigma$. As $\chi \circ \alpha = \bar{\chi}$, there is some $X \in \text{GL}_{2m}(L)$ unique up to scalars in L^\times such that

$$X \rho(g) X^{-1} = \overline{\rho(\alpha(g))} \text{ for all } g \in G.$$

Then $\bar{X}X$ commutes with all $\rho(g)$, so $\bar{X}X = \lambda I_{2m}$ for some $\lambda \in L^\times$. It is easy to see that $\lambda = \bar{\lambda} \in K^\times$ and is well defined up to norms. Now the proof of [2, Proposition 4.2] shows that $[\mathcal{Q}] = [(L, \lambda)_K]$.

As $A = \{\rho(x) \mid x \in KG, \alpha(x) = x\}$ we see that $Xa = \bar{a}X$ for all $a \in A$. As endomorphisms of V this equation translates into $X \cdot a = \sigma \cdot a \cdot \sigma \cdot X$. So the composition $\beta := \sigma \cdot X$ is a K -linear map on V that commutes with all matrices in A and satisfies

$$\beta^2 = (\sigma X)^2 = (\sigma X \sigma)X = \bar{X}X = \lambda.$$

Also scalar multiplication γ by $\sqrt{-\delta} \in L$ commutes with all matrices in A . These two K -linear endomorphisms of V generate a 4-dimensional subalgebra of $\text{End}_K(V)$ and satisfy the relations

$$\gamma^2 = -\delta, \beta^2 = \lambda, \beta\gamma = -\gamma\beta$$

so $\langle \gamma, \beta \rangle_K \cong \mathcal{Q}$.

(b) To see that the adjoint involution ι_H restricts to a K -linear involution on $\text{Fix}_\alpha(\rho)$ note that $\iota_H(\rho(g)) = \rho(g^{-1})$ for all $g \in G$, so ι_H maps the generator $\rho(g) + \rho(\alpha(g))$ to the generator $\rho(g^{-1}) + \rho(\alpha(g^{-1}))$, in particular $\iota_H(A) = A$.

(c) It remains to show that the restriction of ι_H to A is an orthogonal involution on A , i.e. that the dimension of skew-adjoint elements in A is $m(2m - 1)$. This follows from the assumption on the indicator of \mathfrak{X} . As dimensions remain unchanged under field extensions we may replace K by the field \mathbb{R} of real numbers and L by \mathbb{C} . The assumption on the Frobenius-Schur indicator of \mathfrak{X} shows that the Schur index of \mathfrak{X} over the real numbers \mathbb{R} is 1. So the main result of [2] shows that there is a basis of $V \otimes_L \mathbb{C}$ such that the matrices in $\rho(G)$ with respect to this basis satisfy

$$\overline{\rho(g)} = \rho(\alpha(g)) \text{ for all } g \in G.$$

In particular the set $\rho(G)$ is invariant under complex conjugation and

$$A \otimes_K \mathbb{R} \cong \mathbb{R}^{2m \times 2m} \leq \text{End}_{\mathbb{C}}(V \otimes_L \mathbb{C}) = \mathbb{C}^{2m \times 2m}.$$

Also the one dimensional \mathbb{R} -space of invariant Hermitian forms is invariant under complex conjugation and hence all these forms are real and symmetric, inducing an orthogonal

involution on $A \otimes_K \mathbb{R}$.

(d) Now assume that the indicator of χ is $-$. Then by (a) the algebra $A \cong \mathbb{Q}^{m \times m}$ and the restriction of ι to A is a symplectic involution. As in Corollary 4.10 one gets that $\Delta(H) = [\mathbb{Q}]^m$. \square

Definition 7.7. *In the notation of Theorem 7.6 the algebra A is called the α -fixed algebra of $\chi \in \text{Irr}^o(G)$, the square class*

$$\text{disc}(\iota|_A) =: \text{disc}^\alpha(\chi) = d(K^\times)^2 \in K^\times / (K^\times)^2$$

is called the α -discriminant of χ and the étale K -algebra

$$\mathcal{D}^\alpha(\chi) := K[X]/(X^2 - d)$$

is called the α -discriminant algebra of χ .

Proposition 7.8. *In the notation of Theorem 7.6 all prime ideals of K that ramify in the α -discriminant algebra of χ divide $2|G|$.*

Proof. Let \wp be a prime ideal of K that does not divide $2|G|$. Then \wp is not ramified in L/K . Passing to the completions at \wp put

$$\Gamma_\wp := \langle \rho(g) \mid g \in G \rangle_{\mathbb{Z}_{K_\wp}} \leq \mathcal{A} \text{ and } \Delta_\wp := \langle \rho(g) + \rho(\alpha(g)) \mid g \in G \rangle_{\mathbb{Z}_{K_\wp}} \leq A.$$

Because \wp does not divide $|G|$ the order Γ_\wp is a maximal order in \mathcal{A} . As \wp does not divide $2 \text{disc}(L/K)$ we have $\mathbb{Z}_L \Delta_\wp = \Gamma_\wp$ and hence Δ_\wp is a maximal order in A . Both orders are invariant under the adjoint involution ι_H . In particular for any Δ_\wp -lattice $\Lambda \leq V_{K,\wp}$ also its H -dual lattice Λ^* is Δ_\wp -invariant. As Δ_\wp is a maximal order, there is $n \in \mathbb{Z}$ such that $\Lambda^* = \wp^n \Lambda$ and hence \wp is not ramified in the discriminant algebra of H . \square

8 Orthogonal and unitary condensation

8.1 Orthogonal condensation

Condensation methods play an important role in the construction of (modular) character tables and decomposition matrices for large finite groups (see for instance [16] for a brief description of the general idea and one of the many papers citing this article for further applications).

Fixed point condensation is an important tool to compute orthogonal discriminants [3, Section 3.3.2] for large degree characters due to the availability of very sophisticated programs to compute in large permutation representations (in particular in [21]). Let $W \cong \mathbb{Z}^{[G:U]}$ and $\rho_W : G \rightarrow \text{GL}(W)$ be such a permutation representation of the finite group G on the cosets of the subgroup U . For $S \leq G$ put

$$e_S := \frac{1}{|S|} \sum_{h \in S} h \in \mathbb{Q}G$$

to denote the projection on the S -fixed points and

$$N_S := N_G(S)$$

to be the normaliser of S in G . Then e_S is an idempotent in $\mathbb{Z}[\frac{1}{|S|}]G$ that is invariant under the natural involution ι of the group algebra with $\iota(g) = g^{-1}$ for all $g \in G$.

Remark 8.1. Let K be a field such that $|S| \neq 0$ in K . Put $W' := KW\rho_W(e_S)$ and $A := e_S KGe_S$. Then the natural involution ι on KG endows A with an orthogonal involution that we again denote by ι . Moreover W' is an A -module whose composition factors C are in bijection to the KG -composition factors of KW that contain a non-trivial S -fixed space.

In practice we never can be sure to have found enough generators for the condensed algebra A . So we need to work with a subalgebra A' for which we know the involution ι . Character theory predicts the dimensions of the composition factors of W' , so by computing an A' -composition series of W' we can be sure that the A -composition factors are isomorphic to the A' -composition factors of W' . For such a composition factor C , we can compute the A' -action, and thus the A -action on C .

Now let V be an orthogonal composition factor of KW and assume that Q is a non-degenerate G -invariant quadratic form on V . As $e_S = \iota(e_S)$ we have an orthogonal decomposition $(V, Q) = (V_1, Q_1) \perp (V_2, Q_2)$ where

$$V_1 = V\rho_W(e_S) \text{ and } V_2 = V\rho_W(1 - e_S)$$

and $\text{disc}(Q) = \text{disc}(Q_1) \text{disc}(Q_2)$.

Moreover V_2 is a KN_S -module and Q_2 is an N_S -invariant quadratic form on V_2 . If this module V_2 is orthogonally stable as N_S -module, then we can predict the orthogonal discriminant of (V_2, Q_2) and obtain the discriminant of (V, Q) by computing the one of the condensed module (V_1, Q_1) . If the restriction of V to N_S is not orthogonally stable then also (V_1, Q_1) is not an orthogonally stable N_S -module and there is no group that we can use to construct the invariant quadratic form Q_1 . Thanks to [12, Proposition 2.2] (see Section 7) we can use the involution ι on A' to compute the discriminant of the polarisation of Q_1 :

Remark 8.2. (see Remark 7.1) Assume that there is $a \in \rho_{V_1}(A)$ such that

- (a) $\iota(a) = -a$
- (b) $a \in \text{GL}(V_1)$

then the determinant of Q_1 is $\det(a)(K^\times)^2$.

Remark 8.3. If K is a number field, then the computation of the composition factors of the condensed module Ve_S is in general unfeasible as meat-axe methods only work well over finite fields. However, given an a-priori list of possible determinants of the composition factor (V_1, Q_1) (e.g. all square classes $d(K^\times)^2$ for which $K[\sqrt{d}]/K$ is only ramified at primes dividing the group order), then it is possible to deduce the correct square class by computing the determinant of (V_1, Q_1) modulo enough well chosen prime ideals (that usually do not divide the group order).

8.2 Unitary condensation

Thanks to Theorem 7.6 and Proposition 7.8 we can apply orthogonal condensation to determine the α -discriminant of an indicator o character and thus the discriminants of the invariant Hermitian forms.

We aim to compute the unitary discriminant of some $\chi \in \text{Irr}^o(G)$. So assume that we are in the situation of Theorem 7.6, in particular there is some automorphism α of order 2 of G such that $\bar{\chi} = \chi \circ \alpha$. Let V be an LG -module affording the character χ .

As before we have to choose two subgroups of G :

- (a) A subgroup $U \leq G$ such that χ is a constituent of the permutation character χ_U of G on the cosets of U .
- (b) A subgroup $S \leq G$ with normalizer $N_S := N_G(S)$ such that
 - (b1) $|S|$ is invertible in L ,
 - (b2) $V(1 - e_S)$ is a unitary stable LN_S -module
 - (b3) $\alpha(S) = S$

Because of condition (b3) the automorphism α satisfies $\alpha(e_S) = e_S$ and hence induces an algebra automorphism α on the condensed algebra $A = e_S \rho_W(LG) e_S$.

To compute the α -discriminant of the composition factor $V e_S$ of the natural A -module that corresponds to V we compute the determinant on this composition factor of a skew-adjoint unit X in the α -fixed algebra in A . As this singles out a square class of K^\times , for which there are only finitely many possibilities thanks to Proposition 7.8, we can compute composition factors and also the determinant of X modulo enough primes (not dividing the group order) to deduce the α -discriminant of $V e_S$. The unitary discriminant of $V(1 - e_S)$ can be computed in $N_G(S)$.

8.3 An example: The group HN .

The sporadic simple Harada-Norton group HN has order $2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 13$ and two pairs of complex conjugate indicator o absolutely irreducible characters of even degree: $\chi_{25} = \overline{\chi_{26}}$ of degree 656250 and character field $\mathbb{Q}(\chi_{25}) = \mathbb{Q}[\sqrt{-19}]$ and $\chi_{35} = \overline{\chi_{36}}$ of degree 1361920 and character field $\mathbb{Q}(\chi_{35}) = \mathbb{Q}[\sqrt{-10}]$. All Schur indices of $HN : 2$ are 1.

χ_{25} is irreducible modulo 5, 7, 19, and has two odd degree constituents modulo 11. As 19 is ramified in the character field, its 19-modular reduction is orthogonal, and from [1] we obtain $\text{disc}(\chi_{25} \pmod{19}) = O+$. As 5, 7, and 11 are norms the possible unitary discriminants of χ_{25} are -2 and -3 .

Similarly χ_{35} is irreducible modulo 2, 7, 19 and has two odd degree constituents modulo 11. As 11 is a norm in $\mathbb{Q}[\sqrt{-10}]/\mathbb{Q}$ the possible discriminants are 1, 3, 5, 15.

The missing information is obtained using unitary condensation. The outer elements in $HN.2$ interchange χ_{25} and χ_{26} and also χ_{35} with χ_{36} . The normaliser in $HN.2$ of the Sylow 5-subgroup does not contain an outer element of order 2. So we take a subgroup S of order 5^5 whose normaliser $N_S := N_{HN}(S)$ is the maximal subgroup of order 2000000 of HN .

Here we find an element $\alpha \in HN.2 \backslash HN$ of order 2 that normalises S . The group S has a 210-dimensional fixed space on the module with character χ_{25} and a 416-dimensional fixed space for χ_{35} . We start with a permutation representation on the 16500000 cosets of the maximal subgroup $U_3(8).3$ of HN . All four characters, $\chi_{25}, \chi_{26}, \chi_{35}$, and χ_{36} occur with multiplicity one in the corresponding permutation character. The group S has 5280 orbits, so we need to compute the action of a skew adjoint α -fixed element on the composition factors of dimension 210 respectively 416 of a 5280-dimensional module. We reduce modulo primes where -19 (resp. -10) are squares. It turns out that the S -fixed part of the α -determinant of both, χ_{25} and χ_{35} , is 33. On the orthogonal complement, the normaliser N_S acts unitary stably and we compute the discriminant here as 1. As 11 is a norm in both imaginary quadratic extensions we get

Theorem 8.4. *The unitary discriminants of the characters in $\text{Irr}^o(HN)$ are $\text{disc}(\chi_{25}) = -3$ and $\text{disc}(\chi_{35}) = 3$.*

8.4 An example: The group $SU(3, 7)$.

For a smaller example consider the group

$$G := SU(3, 7) := \{X \in \text{SL}_3(\mathbb{F}_{49}) \mid X \begin{pmatrix} 001 \\ 010 \\ 100 \end{pmatrix} \overline{X}^{tr} = \begin{pmatrix} 001 \\ 010 \\ 100 \end{pmatrix}\}$$

of order $2^7 \cdot 3 \cdot 7^3 \cdot 43$. The characters $\mathfrak{X} \in \text{Irr}^o(G)$ together with their character fields and unitary discriminants are given in the following table:

\mathfrak{X}	$\mathfrak{X}(1)$	$\mathbb{Q}(\mathfrak{X})$	$\text{disc}(\mathfrak{X})$
13, 14	258	$\mathbb{Q}[\sqrt{-1}]$	-7
15, 16	258	$\mathbb{Q}[\sqrt{-2}]$	-7
27, 28	344	$\mathbb{Q}[\sqrt{-1}]$	21
29 - 32	344	$\mathbb{Q}[\zeta_8]$	7
33 - 36	344	$\mathbb{Q}[\zeta_8]$	1
37 - 44	344	$\mathbb{Q}[y''_{48}]$	1
45 - 58	384	$\mathbb{Q}[u_{43}, \sqrt{-43}]$	1

A Sylow 7-subgroup S of G consists of the upper triangular matrices with 1 on the diagonal; the normaliser B of S is the group of all upper triangular matrices $B \cong 7^{1+2} : C_{48}$. The group B has 48 linear characters, the ones that restrict trivially to S , one character, ψ , of degree 48, restricting to the sum of the other 48 degree one characters of S , and 8 characters χ_1, \dots, χ_8 of degree 42. χ_1 has indicator $-$, χ_2 indicator $+$ and rational character field, $\chi_3 = \overline{\chi_4}$ have character field $\mathbb{Q}(\sqrt{-1})$ and χ_5, \dots, χ_8 have character field $\mathbb{Q}(\zeta_8)$. By Section 4.3 the unitary discriminant of χ_1 over the relevant character fields is -7 . The characters ψ and χ_2 is orthogonal, so [11, Corollary 4.4] implies that $\text{disc}(\chi_2) = -7$ and

$\text{disc}(\psi) = 1$. We compute the remaining unitary discriminants by explicitly constructing the representation and an invariant Hermitian form over the character field. We obtain $\text{disc}(\chi_3) = \text{disc}(\chi_4) = -7$ and $\text{disc}(\chi_5) = \dots = \text{disc}(\chi_8) = -1$.

The characters $\mathfrak{X}_{13}, \dots, \mathfrak{X}_{16}$ have a unitary stable restriction to B of discriminant -7 . The restriction to B of $\mathfrak{X}_{45}, \dots, \mathfrak{X}_{58}$ is the sum $\psi + \chi_1 + \dots + \chi_8$ and hence unitary stable of discriminant 1.

The other characters do not restrict unitary stably to B ; the non-unitary stable part consists of the respective S -fixed points. Write $B = S : \langle t \rangle$ where t is some diagonal matrix of order 48. One computes with GAP that the unitary stable part $V(1 - e_S)$ restricts to B with character as in the following table:

\mathfrak{X}_V	$\mathfrak{X}_{V(1-e_S)}$	$\text{disc}(V(1 - e_S))$	$\mathfrak{R}_{Ve_S}(t)$	$\text{disc}(Ve_S)$
27	$\chi_1 + \chi_{3,4} + \chi_{5-8} + \psi$	-7	$(\zeta_{12}^{-1}, \zeta_{12}^7)$	-3
28	$\chi_1 + \chi_{3,4} + \chi_{5-8} + \psi$	-7	$(-\zeta_{12}^{-1}, -\zeta_{12}^7)$	-3
29	$\chi_1 + \chi_2 + \chi_4 + \chi_{5-8} + \psi$	-7	$(-\zeta_{24}^{-5}, -\zeta_{24}^{11})$	-3
30	$\chi_1 + \chi_2 + \chi_3 + \chi_{5-8} + \psi$	-7	$(\zeta_{24}, \zeta_{24}^{-7})$	-3
31	$\chi_1 + \chi_2 + \chi_4 + \chi_{5-8} + \psi$	-7	$(\zeta_{24}^{-5}, \zeta_{24}^{11})$	-3
32	$\chi_1 + \chi_2 + \chi_3 + \chi_{5-8} + \psi$	-7	$(-\zeta_{24}, -\zeta_{24}^{-7})$	-3
33	$\chi_{1-4} + \chi_6 + \chi_7 + \chi_8 + \psi$	-1	$(-\zeta_{16}, \zeta_{16})$	$-2 - \sqrt{2}$
34	$\chi_{1-4} + \chi_5 + \chi_6 + \chi_7 + \psi$	-1	$(-\zeta_{16}^7, \zeta_{16}^7)$	$-2 - \sqrt{2}$
35	$\chi_{1-4} + \chi_5 + \chi_7 + \chi_8 + \psi$	-1	$(-\zeta_{16}^5, \zeta_{16}^5)$	$-2 + \sqrt{2}$
36	$\chi_{1-4} + \chi_5 + \chi_6 + \chi_8 + \psi$	-1	$(-\zeta_{16}^3, \zeta_{16}^3)$	$-2 + \sqrt{2}$
37	$\chi_{1-4} + \chi_5 + \chi_6 + \chi_8 + \psi$	-1	$(\zeta_{48}^{-7}, \zeta_{48})$	-3

All the characters \mathfrak{X}_{37-44} restrict to B as the sum $\chi_1 + \chi_2 + \chi_3 + \chi_4 + \psi +$ three characters of $\{\chi_5, \dots, \chi_8\}$ and the sum of two faithful characters of B/S . This yields that the discriminant of $V(1 - e_S)$ is -1 for \mathfrak{X}_{37-44} .

To obtain the discriminant of Ve_S just from the action of t , we consider the automorphism α of $G = U_3(7)$ that is given by applying the Frobenius automorphism to the entries of the matrices. Then $\alpha(t) = t^7$ and t^8 is in the fixed space of α .

For \mathfrak{X}_i with $27 \leq i \leq 32$ or $37 \leq i \leq 44$ the element t^{16} acts as

$$\rho_{Ve_S}(t^{16}) = \text{diag}(\zeta_3, \zeta_3^{-1})$$

on Ve_S so $\det(\rho_{Ve_S}(t^{16} - t^{-16})) = -3$ in all these cases. Note that 3 is a norm in $\mathbb{Q}(\mathfrak{X}_i)/\mathbb{Q}(\mathfrak{X}_i)^+$ for $i \in \{29, \dots, 32, 37, \dots, 44\}$.

For the four characters \mathfrak{X}_{33-36} we compute the action of the skew adjoint α -fixed endomorphism given by $t + t^7 - t^{-1} - t^{-7}$ as

$$a := \rho_{Ve_S}(t + t^7 - t^{-1} - t^{-7}) = \text{diag}(\zeta + \zeta^7 - \zeta^{-1} - \zeta^{-7}, -\zeta - \zeta^7 + \zeta^{-1} + \zeta^{-7})$$

where ζ is some primitive 16th root of unity of determinant $\det(a) = 8 \pm 4\sqrt{2}$, which is a norm in $\mathbb{Q}(\mathfrak{X}_i)/\mathbb{Q}(\mathfrak{X}_i)^+$ for $i \in \{29, \dots, 32\}$.

9 An example: The sporadic simple O’Nan group

Theorem 9.1. *The character fields, unitary discriminants, and discriminant algebras of the characters $\chi \in \text{Irr}^o(3.ON)$ are given in the following table.*

χ	$\chi(1)$	$\mathbb{Q}(\chi)$	$\text{disc}(\chi)$	$\Delta(\chi)$
3, 4	13376	$\mathbb{Q}[\sqrt{-31}]$	1	$[\mathbb{Q}]$
5, 6	25916	$\mathbb{Q}[\sqrt{-5}]$	1	$[\mathbb{Q}]$
31 – 34	342	$\mathbb{Q}[\sqrt{2}, \sqrt{-3}]$	-1	$[(-1, -1)_{\mathbb{Q}[\sqrt{2}]}]$
45 – 48	52668	$\mathbb{Q}[\sqrt{7}, \sqrt{-3}]$	$8 + 3\sqrt{7}$	$[(-3, 8 + 3\sqrt{7})_{\mathbb{Q}[\sqrt{7}]}]$
53, 54	63612	$\mathbb{Q}[\sqrt{-3}]$	55	$[(-3, 55)_{\mathbb{Q}}]$
57, 58	116622	$\mathbb{Q}[\sqrt{-3}]$	-10	$[(-3, -10)_{\mathbb{Q}}]$
59, 60	122760	$\mathbb{Q}[\sqrt{-3}]$	1	$[\mathbb{Q}]$
61 – 64	169290	$\mathbb{Q}[\sqrt{2}, \sqrt{-3}]$	-1	$[(-1, -1)_{\mathbb{Q}[\sqrt{2}]}]$
65, 68	169632	$\mathbb{Q}[\sqrt{15}, \sqrt{-3}]$	$-2\sqrt{15} + 15$	$[(-3, -2\sqrt{15} + 15)_{\mathbb{Q}[\sqrt{15}]}]$
66, 67	169632	$\mathbb{Q}[\sqrt{15}, \sqrt{-3}]$	$2\sqrt{15} + 15$	$[(-3, 2\sqrt{15} + 15)_{\mathbb{Q}[\sqrt{15}]}]$
69, 70	175770	$\mathbb{Q}[\sqrt{-3}]$	-11	$[(-3, -11)_{\mathbb{Q}}]$
71, 72	207360	$\mathbb{Q}[c_{19}, \sqrt{-3}]$	$3c_{19}^2 + 10c_{19} + 9$	$[(-3, 3c_{19}^2 + 10c_{19} + 9)_{\mathbb{Q}[c_{19}]}]$
75, 76	207360	$\mathbb{Q}[c_{19}, \sqrt{-3}]$	$-13c_{19}^2 + 3c_{19} + 76$	$[(-3, -13c_{19}^2 + 3c_{19} + 76)_{\mathbb{Q}[c_{19}]}]$
73, 74	207360	$\mathbb{Q}[c_{19}, \sqrt{-3}]$	$10c_{19}^2 - 13c_{19} - 29$	$[(-3, 10c_{19}^2 - 13c_{19} - 29)_{\mathbb{Q}[c_{19}]}]$
77 – 80	253440	$\mathbb{Q}[\sqrt{93}, \sqrt{-3}]$	1	$[\mathbb{Q}[\sqrt{93}]]$

Proof. Note that we only need to treat one of each pair of complex conjugate characters, as $\Delta(\chi) = \Delta(\bar{\chi})$ is the Clifford invariant of the orthogonal character $\chi + \bar{\chi}$.

- The characters $\chi_3, \chi_4 = \bar{\chi}_3$ are non-faithful. Their character field, $L = \mathbb{Q}[\sqrt{-31}]$, has class number 3 and is only ramified at the rational prime 31. The p -modular reduction of χ_3 is unitary stable for all primes but $p = 7$. As 7 is decomposed in L/\mathbb{Q} , Theorem 5.4 yields that 31 is the only prime that possibly ramifies in the discriminant algebra $\Delta(\chi_3)$. Modulo the ramified prime, 31, the orthogonal discriminant is a square. So Corollary 5.6 shows that 31 is not ramified in $\Delta(\chi_3)$ and hence $\Delta(\chi_3) = [\mathbb{Q}]$.
- Also $\chi_5, \chi_6 = \bar{\chi}_5$ are non-faithful. Their character field is $L = \mathbb{Q}[\sqrt{-5}]$, has class number 2, and is ramified at 2 and 5. The p -modular reduction of these two characters is unitary stable for all $p \neq 3, 7$. As 3 and 7 are decomposed in L/\mathbb{Q} , Theorem 5.4 yields that the only primes that are possibly ramified in the discriminant algebra are 2 and 5. The orthogonal discriminant of the 5-modular reduction of χ_5 is a square mod 5, Corollary 5.6 implies that the prime 5 is not ramified in $\Delta(\chi_5)$. As the number of ramified primes is even, the prime 2 cannot be the only ramified prime, so $\Delta(\chi_5) = [\mathbb{Q}]$.

All other characters in the table are faithful.

- The characters number 31–34 restrict absolutely irreducible to the maximal subgroup $C_3 \times L_3(7).2$. Their unitary discriminant has been computed by David Schlang by restricting further to the normaliser of a 7-Sylow subgroup of $L_3(7)$.
- Number 45-48 have character field $L := \mathbb{Q}[\sqrt{7}, \sqrt{-3}]$ which is ramified at 3 over its maximal real subfield $K = \mathbb{Q}[\sqrt{7}]$. Whereas L has class number 2, all ideals in K are principal ideals. K has a totally positive fundamental unit $u = 8 + 3\sqrt{7}$, which is not a norm in L/K . The only inert prime in L/K that divides the group order is the one dividing 2, here $\chi_{45} \pmod{2}$ is unitary stable, so no inert prime ramifies in the discriminant algebra. So it remains to decide whether the two prime ideals \wp_3, \wp'_3 of K ramify in $\Delta(\chi_{45})$, i.e. $\text{disc}(\chi_{45}) = u$, or $\text{disc}(\chi_{45}) = 1$ and $\Delta(\chi_{45}) = [K]$. The induced character $\text{Ind}_{3.ON:2}^{3.ON:2}(\chi_{45})$ has character field K and Schur index 2 at the two primes \wp_3, \wp'_3 of K that divide 3. If α is an outer automorphism of order 2 in $3.ON : 2$, then the α -fixed algebra A is equivalent to $(-3, u)_K$ by Theorem 7.6. However, as $\chi_{45}(1)$ is a multiple of 4, Theorem 7.4 shows that there is no correction factor because of the Schur indices of the induced character. So $\text{disc}(\chi_{45})$ is represented by $\text{disc}^\alpha(\chi_{45})$. To compute this α -discriminant we use condensation with the derived subgroup S of the normaliser in $3.ON$ of the 7-Sylow subgroup of order $2 \cdot 7^3$. χ_{45} occurs as a constituent in the permutation representation on the 41,938,920 cosets of a maximal subgroup of index 57 in the subgroup $L_3(7)$ of $3.ON$. The group S has 61704 orbits in this permutation character. After multiplication with a nontrivial central idempotent we compute composition factors in a 20568 condensed module. The determinant of a skew-adjoint element in the α -fixed algebra of the relevant composition factors is either a square (y) or not (n) modulo suitable primes:

χ	dim	37	43	61	67	73	79	97	103	109
45	72	yn	n	y	y	y	n	n	yn	yy
57	162	yy	yy	nn	yy	nn	yy	nn	nn	yy

Note that 7 is a square mod 37, 103, and 109 but not modulo the other primes, so one obtains for χ_{45} two constituents of dimension 72 modulo 37, 103, and 109 and one constituent of dimension 144 for the other primes. The results allow to conclude that the α -discriminant on the 72-dimensional S -fixed space of χ_{45} is $u \cdot 7 \cdot 31$.

- Theoretical arguments showing that $\text{disc}(\chi_{57}) = -10$ are given in Example 5.8. However, during the condensation of χ_{45} , we also computed the α -discriminant of χ_{57} on the 162 dimensional fixed space of the S as $\text{disc}^\alpha(Ve_S) = -21$. This is in accordance with Theorem 7.4. The character field of χ_{57} is $\mathbb{Q}[\sqrt{-3}]$. The induced character $\text{Ind}_{ON}^{ON:2}(\chi_{57})$ has character field \mathbb{Q} and Schur index 2 at the primes 2 and 5. By Theorem 7.6 the α -fixed algebra is Brauer equivalent to $(-3, 10)_{\mathbb{Q}}$ and of $\mathbb{Q}[\sqrt{-3}]$ -discriminant 10. As 3 and 7 are both norms in $\mathbb{Q}[\sqrt{-3}]/\mathbb{Q}$ Theorem 7.4 yields $\text{disc}(\chi_{57}) = -10$.
- The character field of the characters number 53,54,57-60 is $\mathbb{Q}[\sqrt{-3}]$ and of class number 1. The prime divisors 7, 19, 31 of the group order are split in this extension and hence norms.
- For the primes 5 and 11 the modular reduction of χ_{53} is not unitary stable. As both primes

divide the group order only with multiplicity 1, Proposition 5.7 yields that 55 divides the unitary discriminant of χ_{53} . So $\text{disc}(\chi_{53}) \in \{55, 110\}$. Now $\chi_{53} \pmod{3}$ is orthogonally stable and has square discriminant, so $\text{disc}(\chi_{53})$ is a square modulo 3, allowing to conclude that $\text{disc}(\chi_{53}) = 55$.

- The character χ_{59} reduces absolutely irreducible modulo 5 and 11 leaving 1 and 2 as possible unitary discriminants. Its reduction modulo 3 is not orthogonally stable, so we cannot obtain any more information about this character from the decomposition matrix. So we use the unitary condensation method described in Section 8.2. We start with the permutation module P of dimension 736560 on the cosets of the subgroup $L_3(7) \leq 3.ON$ and use fixed point condensation with the 7-Sylow subgroup H of $3.ON$. The group H has 2196 orbits on the 736560 cosets, falling into 732 orbits under the center $\langle z \rangle$ of $3.ON$. Computing modulo primes $p \equiv 1 \pmod{3}$ we specify the action of z as some primitive third root of unity in \mathbb{F}_p^\times and hence reduce the computations to find composition factors of a 732 dimensional \mathbb{F}_p -module for in total 25 suitable primes. These composition factors have dimension 3, 171, 192, 366, where the latter two correspond to characters number 53/54 resp. 59/60. For some $\alpha \in N_{3.ON,2}(H)$ of order 2 that is not contained in $3.ON$ we compute the determinants of an α -fixed skew-adjoint element in the condensed algebra as $3 \cdot 5 \cdot 11$ on the composition factor of dimension 192 and 7 for the one of dimension 366. On the orthogonal complement of the fixed space the relevant discriminants of the restriction to the normaliser of H are powers of 7, so one concludes that $\text{disc}(\chi_{59}) = 1$.
- The characters number 61-64 are tensor products, $169632 = 342 \cdot 495$ of one of the characters number 31-34 with a character of degree 495. So they have the same unitary discriminant as χ_{31} .
- Number 65-68 have character field $L := \mathbb{Q}[\sqrt{15}, \sqrt{-3}]$ an unramified extension of its maximal real subfield $K = \mathbb{Q}[\sqrt{15}]$. Both fields, L and K have class number 2. The fundamental unit of K is a norm. The prime ideals of K that are inert in L/K and divide the group order are the divisors of 2, 5, 11, and 19. The reduction modulo both prime ideals of K that divide 19 is absolutely irreducible. For 11, one prime ideal yields an absolutely irreducible reduction, the other one a non unitary stable reduction. By Proposition 5.7 exactly one of these two prime ideals, say \wp_{11} , ramifies in $\Delta(\chi_{65})$. Also the reduction modulo the unique prime ideal \wp_5 of K that divides 5 is not unitary stable. The ideal $\wp_3 \wp_5 \wp_{11} = (2\sqrt{15} + 15)$ has a totally positive generator, but no such combination involving the prime ideal dividing 2 has such a totally positive generator. So we conclude that the discriminant of χ_{65} is represented by $\pm 2\sqrt{15} + 15$.
- The character $\chi_{69} = \overline{\chi_{70}}$ has character field $\mathbb{Q}[\sqrt{-3}]$, is orthogonally stable modulo 3 with square discriminant. Its reduction modulo 5 is absolutely irreducible and modulo 11 we obtain $175770 = 42687 + 133083$. So Proposition 5.7, Theorem 5.4 and Corollary 5.6 together show that $\text{disc}(\chi_{69}) = -11$.
- The characters number 71 – 76 are algebraic conjugate and so are their unitary discriminants. It is enough to treat $\chi_{71} = \overline{\chi_{72}}$. The character field is $L = K[\sqrt{-3}]$ where $K = \mathbb{Q}[c_{19}]$ is the real subfield of L . The extension L/K is only ramified at 3, L has class number 3, K has class number 1 and all totally positive units in K are squares. The prime divisors of

the group order that are inert in L/K are 2, 5 and the three prime ideals of K that divide 11. Whereas $\chi_{71} \pmod{2}$ and $\pmod{5}$ is absolutely irreducible and hence unitary stable, its 11-modular reduction is unitary stable at the prime ideal of K that contains $c_{19} - 6$ and not unitary stable at the other two prime ideals. So the unitary discriminant is a totally positive generator of the product of these two prime ideals dividing 11 in K . One checks that this number is indeed a square modulo 3, as predicted by the orthogonal discriminant of $\chi_{71} \pmod{3}$.

- The last characters, $\chi_{77}, \dots, \chi_{80}$, have character field $L := \mathbb{Q}[\sqrt{93}, \sqrt{-3}]$ again of class number 3. The fundamental unit of $K = \mathbb{Q}[\sqrt{93}]$ is totally positive and a norm in L/K . The extension L/K is totally unramified. The inert primes dividing the group order are 3 and 11. As both reductions χ_{77} modulo the two prime ideals of K that divide 11 remain absolutely irreducible, the only possible ramified prime in $\Delta(\chi_{77})$ is the one that divides 3. As the number of ramified places is even this prime is also not ramified and hence the unitary discriminant of χ_{77} is 1. \square

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