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SOME RIGIDITY RESULTS ON SHRINKING GRADIENT RICCI SOLITON

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ABSTRACT. Suppose (M^n, g, f) is a complete shrinking gradient Ricci soliton. We give several rigidity results under some natural conditions, generalizing the results in [\[25,](#page-12-0) [14\]](#page-12-1). Using maximum principle, we prove that shrinking gradient Ricci soliton with constant scalar curvature $R = 1$ is isometric to a finite quotient of $\mathbb{R}^2 \times \mathbb{S}^2$, giving a new proof of the main results of Cheng-Zhou [\[9\]](#page-12-2).

1. INTRODUCTION

For an *n*-dimensional complete Riemannian manifold (M, g) and a smooth potential function f on (M, g) , the triple (M, g, f) is called a gradient shrinking Ricci soliton or shrinker if

(1.1)
$$
Ric + \text{Hess } f = \frac{1}{2}g,
$$

where Ric is the Ricci curvature of (M, g) and Hess f is the Hessian of f. Shrinkers are viewed as a natural extension of Einstein manifolds. More importantly, shrinkers play an important role in the Ricci flow as they correspond to some self-similar solutions and arise as limits of dilations of Type I singularities in the Ricci flow. Shrinkers can also be regarded as critical points of the Perelman's entropy functional and play a significant role in Perelman's resolution of the Poincaré conjecture [\[22,](#page-12-3) [23,](#page-12-4) [24\]](#page-12-5).

The study of solitons has become increasingly important in both the study of the Ricci flow and metric measure space. Solitons play a direct role as singularity dilations in the Ricci flow proof of uniformiza-tion. In [\[22\]](#page-12-3), Perelman introduced the ancient κ -solutions, which play an important role in the singularity analysis, and he also proved that suitable blow down limit of ancient κ -solutions must be a shrinking gradient Ricci soliton. In [\[23\]](#page-12-4), Perelman proved that any two dimensional non-flat ancient κ -soluition must be the standard S^2 , and he

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also classified three dimensional shrinking gradient Ricci soliton under the assumption of nonnegative curvature and κ -noncollapseness. Due to the work of Perelman [\[23\]](#page-12-4), Ni-Wallach [\[21\]](#page-12-6), Cao-Chen-Zhu [\[5\]](#page-12-7), the classification of three dimensional shrinking gradient Ricci soliton is complete. For more work on the classification of gradient Ricci soliton under various curvature condition, see [\[1\]](#page-11-0), [\[3\]](#page-11-1), [\[5\]](#page-12-7), [\[4\]](#page-12-8), [\[7\]](#page-12-9), [\[8\]](#page-12-10), [\[11\]](#page-12-11), [\[20\]](#page-12-12), [\[25\]](#page-12-0), [\[27\]](#page-13-0), [\[30\]](#page-13-1), [\[31\]](#page-13-2).

In this paper, we study some rigidity problem about the shrinking gradient Ricci soliton.

In section 2, we provide some preliminary knowledge which will be used throughout the paper.

In section 3, we impose the additional assumption which is called condition A as follows:

$$
|R(u, v, u, v)| \le A \cdot Ric(u, u)
$$

for any $|u| = |v| = 1$ and $u \perp v$, where A is a positive constant. There are so many examples satisfying condition A but having mixed sectional curvature. An explicit example is the Kähler shrinking gradient Ricci soliton on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ constructed by Cao and Koiso independently. Next we can state the following splitting result.

Theorem 1.1. Let (M^n, g, f) be a shrinking gradient Ricci soliton satisfying condition A , then the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where $\mathbb N$ is an $n-1$ dimensional shrinking gradient Ricci soliton with positive Ricci curvature.

Remark Petersen-Wylie [\[25\]](#page-12-0) and Guan-Lu-Xu [\[14\]](#page-12-1) proved the above theorem independently if (M^n, g, f) has nonnegative sectional curvature.

 $\sum_{i,j=1}^n R(u,e_i,v,e_j)Ric(e_i,e_j)$, where $\{e_i\}_{i=1}^n$ are local orthonormal ba-In section 4, at first we define a symmetric two tensor h by $h(u, v) =$ sis, then we state our main results as follows.

Theorem 1.2. Let (M^n, g, f) be a shrinking gradient Ricci soliton, if $Ric \geq 0$ and $h \leq \frac{1}{2}Ric$, then the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where $\mathbb{\bar{N}}$ is a compact Einstein manifold.

Based on the above theorem, we give a new proof of Theorem 1.4 in [\[26\]](#page-13-3).

Corollary 1.3. [\[26\]](#page-13-3) If (M^n, g, f) is a shrinking gradient Ricci soliton with nonnegative sectional curvature and $R \leq 1$, then the universal cover of M is isometric to either \mathbb{R}^n or $\mathbb{S}^2 \times \mathbb{R}^{n-2}$

In section 5, we focus our attention on 4-dimensional gradient shrinking Ricci solitons with constant scalar curvature. Recall that in Pe-tersen and Wylie's paper [\[25\]](#page-12-0), a gradient Ricci soliton (M, q) is said to be rigid if it is isometric to a quotient $N \times \mathbb{R}^k$, the product soliton of an Einstein manifold N of positive scalar curvature with the Gaussian soliton \mathbb{R}^k . Therefore, a gradient Ricci soliton has constant scalar curvature if it is rigid. Conversely, for the complete shrinking case, Prof. Huai-Dong Cao raised the following

Conjecture: Let (M^n, g, f) , $n \geq 4$, be a complete *n*-dimensional gradient shrinking Ricci soliton. If (M, q) has constant scalar curvature, then it must be rigid, i.e., a finite quotient of $\mathbb{N}^k \times \mathbb{R}^{n-k}$ for some Einstein manifold N of positive scalar curvature.

About this conjecture, Petersen-Wylie [\[26\]](#page-13-3) proved the following interesting result.

Theorem 1.4 ([\[26\]](#page-13-3)). A Ricci shrinker is rigid iff it has constant scalar curvature and is radially flat, that is, the sectional curvautre

$$
K(\nabla f, \cdot) = 0.
$$

Later, Fernández-López and García-Río [\[12\]](#page-12-13) characterize the rigidity using the rank of Ricci curvature.

Theorem 1.5 ([\[12\]](#page-12-13)). A Ricci shrinker is rigid iff it has constant scalar curvature and the Ricci curvature has constant rank.

In the same paper, they also proved that the possible value of R is $\{0, 1, ..., \frac{n-1}{2}\}$ $\frac{-1}{2}, \frac{n}{2}$ $\frac{n}{2}$. In dimension 4, If $R = 0$, (M^4, g) is isometric to \mathbb{R}^4 ; if $R=\frac{3}{2}$ $\frac{3}{2}$, then (M^4, g) is isometric to $\mathbb{R} \times \mathbb{S}^3$; if $R = 2$, then (M^4, g) is isometric to a compact Einstein manifold with $Ric = \frac{1}{2}$ $rac{1}{2}g$.

Recently, Cheng-Zhou [\[9\]](#page-12-2) proved a four dimensional Ricci shrinker with $R = 1$ is rigid. They applied Δ_f to the quantity

$$
tr(Ric^3) - \frac{1}{4},
$$

and they got the following nice inequality

$$
\Delta_f\left(f(tr(Ric^3) - \frac{1}{4})\right) \ge 9f(tr(Ric^3) - \frac{1}{4}),
$$

at last they used the integration by parts to derive that

$$
tr(Ric^3) - \frac{1}{4} = 0
$$

over M, and this implies that $\lambda_1 + \lambda_2 = 0$, so the Ricci curvature has rank 2, finally they obtain the rigidity by Theorem [3.2.](#page-5-0)

We want to point remark that (3.11) in Cheng-Zhou [\[9\]](#page-12-2) gives that

$$
\frac{1}{3}tr(Ric^3) - \frac{1}{12} = \lambda_2 \lambda_3 \lambda_4,
$$

this implies that the quantity they used is $\sigma_3(Ric)$, since $\lambda_1 = 0$ in this situation.

We restate the main theorem in [\[9\]](#page-12-2) as follows.

Theorem 1.6 ([\[9\]](#page-12-2)). Let (M, g, f) be a 4-dimensional complete noncompact shrinking gradient Ricci soliton. If M has constant scalar curvature $R = 1$, then it must be isometric to a finite quotient of $\mathbb{R}^2 \times S^2$.

Our new proof is inspired by [\[25\]](#page-12-0), where they assumed the sectional curvature is nonnegative. Denote the eigenvalues of Ricci curvature by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. then it is easy to observe that $Rm * Ric \geq 0$. They applied Δ_f directly to the sum of the smallest k eigenvalues and obtained

$$
\Delta_f(\lambda_1 + \lambda_2 + \dots + \lambda_k) \leq (\lambda_1 + \lambda_2 + \dots + \lambda_k)
$$

holds in the barrier sense, at last they derived that the Ricci curvature has constant rank by standard maximum principle.

In the setting of $R = 1$, it is not that easy, and the most important thing is to derive the following inequality,

$$
\Delta_f(\lambda_1 + \lambda_2)
$$

\n
$$
\leq \frac{2\nabla f \cdot \nabla(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2) - 2(\lambda_1^2 + \lambda_2^2)}{f} (1 - 2\lambda_1 - 2\lambda_2)
$$

\n
$$
-2(\lambda_1 + \lambda_2) + 4(\lambda_1^2 + \lambda_2^2),
$$

then

$$
\frac{\lambda_1 + \lambda_2}{f}
$$

satisfies

$$
\Delta_f \frac{\lambda_1 + \lambda_2}{f} \leq -0.9 \cdot \frac{\lambda_1 + \lambda_2}{f}
$$

outside a compact set, next we can use similar trick as [\[18\]](#page-12-14) to obtain a uniform positive lower bound of $\lambda_1 + \lambda_2$, this is impossible unless $\lambda_1 + \lambda_1 \equiv 0.$

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2. Preliminary

Suppose (M^n, g, f) is a shrinking gradient Ricci soliton $\nabla^2 f + Ric =$ 1 $\frac{1}{2}g$. At first we recall some basic formulas which will be used during the paper.

$$
(2.2) \t dR = 2Ric(\nabla f),
$$

$$
(2.3) \t\t R + \Delta f = \frac{n}{2},
$$

$$
(2.4) \t\t R + |\nabla f|^2 = f,
$$

(2.5)
$$
\Delta_f R = R - 2|Ric|^2,
$$

(2.6)
$$
\Delta_f R_{ij} = R_{ij} - 2R_{ikjl}R_{kl},
$$

where $\Delta_f Ric = \Delta Ric - \nabla_{\nabla f} Ric$ in the last formula.

Next we state the estimate of potential function f in Cao-Zhou [\[6\]](#page-12-15).

Theorem 2.1 ([\[6\]](#page-12-15)). Suppose (M^n, g, f) is an noncompact shrinking gradient Ricci soliton, then there exist C_1 and C_2 such that

$$
\left(\frac{1}{2}d(x,p) - C_1\right)^2 \le f(x) \le \left(\frac{1}{2}d(x,p) + C_2\right)^2,
$$

where p is the minimal point of f .

Remark . Chen [\[2\]](#page-11-2) proved that any shrinking gradient Ricci soliton has $R \geq 0$, so due to $R + |\nabla f|^2 = f$ we derive that $|\nabla f|(x) \leq \frac{1}{2}$ $\frac{1}{2}d(x,p)+$ C_2 .

The following splitting theorem for shrinking gradient Ricci soliton will be important for us.

Theorem 2.2 (Naber, [\[20\]](#page-12-12)). For any n-dimensional shrinking gradient Ricci soliton (M^n, g, f) with bounded curvature and a sequence of points $x_i \in M$ going to infinity along an integral curve of ∇f , by choosing a subsequence if necessary, (M, g, x_i) converges smoothly to a product manifold $\mathbb{R} \times \mathbb{N}^{n-1}$, where $\mathbb N$ is a shrinking gradient Ricci soliton.

3. Structure of shrinking gradient Ricci soliton satisfying condition A

There is one explicit curvature condition called 2-nonnegative flag curvature which is defined as follows.

Definition 3.1. (M^n, g) is said to have 2-nonnegative flag curvature if

 $R(e_1, e_2, e_1, e_2) + R(e_1, e_3, e_1, e_3) > 0$

for any othonormal basis e_1, e_2, e_3 .

Remark Qu-Wu [\[28\]](#page-13-4) proved that 2-nonnegative flag curvature implies condition A. 2-nonnegative flag curvature is also considered in Li-Ni [\[16\]](#page-12-16).

Proposition 3.2. Suppose (M^n, g, f) is a shrinking gradient Ricci soliton satisfying condition A, then the rank of Ricci curvature tenor is constant.

Proof: Denote the eigenvalues of Ricci curvature by $\lambda_1 \leq \lambda_2 \leq \cdots \leq$ λ_n .

Claim. $\Delta_f \lambda_1 \leq (1 + 2A(n-1))\lambda_1$ in the barrier sense.

Actually, at x, assume e_1 satisfies $Ric(x)(e_1, e_1) = \lambda_1(x)$, then extend e_1 to an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ such that $\{e_i\}_{i=1}^n$ are the eigenvectors of $Ric(x)$ corresponding to eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}.$ Take parallel transport of e_1 along all the geodesics from x , then in a neighborhood $B(x, \delta)$ we get a smooth function $Ric(y)(e_1(y), e_1(y))$ satisfying $Ric(y)(e_1(y), e_1(y)) \geq \lambda(y)$ and $Ric(x)(e_1(x), e_1(x)) = \lambda_1(x)$.

So at $x \in M$,

$$
\Delta_f Ric(e_1, e_1)
$$

= $(\Delta_f Ric)(e_1, e_1)$
= $Ric(e_1, e_1) - 2 \sum_{i=1}^n R(e_1, e_i, e_1, e_i) Ric(e_i, e_i)$
 $\leq Ric(e_1, e_1) + 2A(n - 1)Ric(e_1, e_1) |Ric|$
= $Ric(e_1, e_1) (1 + 2A(n - 1) |Ric|).$

 \Box

Suppose there exists $q \in M$ such that $\lambda_1(q) = 0$, then by the strong maximum principle, we get $\lambda_1 \equiv 0$ on M.

Similar argument implies that $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ satisfies

$$
\Delta_f(\lambda_1 + \lambda_2 + \dots + \lambda_k) \le (1 + 2A(n-1)|Ric|)(\lambda_1 + \lambda_2 + \dots + \lambda_k)
$$

in the barrier sense for any $2 \leq k \leq n$. So we can again use the maximum principle to derive that either $\lambda_1 + \lambda_2 + \cdots + \lambda_k > 0$ or $\lambda_1 + \lambda_2 + \cdots + \lambda_k \equiv 0$ on M.

Proposition 3.3. Under the same assumption with Proposition [3.2,](#page-5-0) the kernel of Ricci curvature is a parallel subbundle.

Proof: Given any section $V \in \text{ker}(Ric)$, choose local orthonormal basis $\{e_1, e_2, \dots, e_n\}$. Due to the nonnegativity of Ricci curvature, $Ric(V, V) = 0$ is the same as $Ric(V) = 0$. According to condition A,

$$
Ric(V, V) = 0
$$
 also implies $R(V, e_i, V, e_i) = 0$.
\n
$$
\Delta_f Ric(V, V)
$$

\n
$$
= \Delta Ric(V, V) - \nabla_{\nabla f} Ric(V, V)
$$

\n
$$
= \nabla_k \nabla_k (R_{ij} V^i V^j) - (\nabla_{\nabla f} Ric)(V, V) - 2Ric(\nabla_{\nabla f} V, V)
$$

\n
$$
= (\Delta Ric)(V, V) + 4\nabla_k R_{ij} \nabla_k V^i V^j + 2Ric(\Delta V, V) + 2R_{ij} \nabla_k V^i \nabla_k V^j
$$

\n
$$
- (\nabla_{\nabla f} Ric)(V, V) - 2Ric(\nabla_{\nabla f} V, V)
$$

\n
$$
= (\Delta_f Ric)(V, V) + 4\nabla_k R_{ij} \nabla_k V^i V^j + 2Ric(\Delta_f V, V) + 2R_{ij} \nabla_k V^i \nabla_k V^j
$$

\n
$$
= Ric(V, V) - 2R(V, e_i, V, e_i) Ric(e_i, e_i) + 4\nabla_k R_{ij} \nabla_k V^i V^j + 2R_{ij} \nabla_k V^i \nabla_k V^j
$$

\n
$$
= 4\nabla_k R_{ij} \nabla_k V^i V^j + 2R_{ij} \nabla_k V^i \nabla_k V^j.
$$

Since

$$
\nabla_k (R_{ij} \nabla_k V^i V^j) = \nabla_k R_{ij} \nabla_k V^i V^j + R_{ij} \Delta V^i V^j + R_{ij} \nabla_k V^i \nabla_k V^j
$$

= $\nabla_k R_{ij} \nabla_k V^i V^j + R_{ij} \nabla_k V^i \nabla_k V^j$,

so

$$
0 = \Delta_f Ric(V, V) = -2R_{ij}\nabla_k V^i \nabla_k V^j,
$$

hence $\nabla_k V \in \text{ker}(\text{Ric})$.

Combining Proposition [3.2,](#page-5-0) Proposition [3.3](#page-5-1) with De Rham's splitting Theorem, we can get the following structure result for shrinking gradient Ricci soliton.

Theorem 3.4. Let (M^n, g, f) be a shrinking gradient Ricci soliton satisfying condition A, then the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where $\mathbb N$ has positive Ricci curvature.

Assume (M^n, g, f) has nonnegative sectional curvature, Petersen-Wylie [\[25\]](#page-12-0) and Guan-Lu-Xu [\[14\]](#page-12-1) proved the above Theorem indepen-dently. Moreover, Munteanu-Wang [\[18\]](#page-12-14) obtained that (M^n, g, f) is compact if it has nonnegative sectional curvature and positive Ricci curvature. So N is compact in the above Theorem. Here our condition A is weaker than theirs. In [\[28,](#page-13-4) [29\]](#page-13-5), the authors derived the soliton (M^4, g, f) is also compact under condition A and other natural conditions.

4. Rigidity via nonnegative sectional curvature

Define a symmetric two tensor h by $h(u, v) = \sum_{i,j=1}^{n} R(u, e_i, v, e_j) Ric(e_i, e_j),$ where $\{e_i\}_{i=1}^n$ are local orthonormal basis.

Theorem 4.1. Let (M^n, g, f) be a shrinking gradient Ricci soliton, if $Ric \geq 0$ and $h \leq \frac{1}{2}Ric$, then the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where $\overline{\mathbb{N}}$ is a compact Einstein manifold.

Proof: Because $Ric \geq 0$ and $h \leq \frac{1}{2}Ric$,

$$
\langle h - \frac{1}{2}Ric, Ric \rangle = R_{ikjl}R_{ij}R_{kl} - \frac{1}{2}|Ric|^2 \le 0.
$$

Recall the formula [\(2.6\)](#page-4-0), direct calculation gives

$$
\frac{1}{2}\Delta_f |Ric|^2 = |\nabla Ric|^2 + |Ric|^2 - 2R_{ikjl}R_{ij}R_{kl}
$$

Claim. $|\nabla Ric| \equiv 0$ on M.

Actually, integrating the above identity over $B(p, r)$, the right hand is always nonnegative, so the goal is to prove

$$
\lim_{r \to \infty} \int_{B(p,r)} \Delta_f |Ric|^2 e^{-f} = 0.
$$

Thanks to the estimate $\int_M (|Ric|^2 + |\nabla Ric|^2) e^{-f} < C$ in [\[15\]](#page-12-17), it is easy to see that there exists a sequence $r_i \to \infty$ such that $\int_{\partial B(p,r_i)} (|Ric|^2 + |\nabla Ric|^2) e^{-f} \to$ 0. Hence

$$
\left| \int_{B(p,r_i)} \Delta_f |Ric|^2 e^{-f} \right|
$$

=
$$
\left| \int_{\partial B(p,r_i)} \langle \nabla |Ric|^2 e^{-f}, \nu \rangle \right|
$$

$$
\leq \int_{\partial B(p,r_i)} 2 |Ric| |\nabla Ric| e^{-f}
$$

$$
\leq \int_{\partial B(p,r_i)} (|Ric|^2 + |\nabla Ric|^2) e^{-f} \to 0.
$$

So $|\nabla Ric| \equiv 0$ on M, De Rham's splitting Theorem implies that the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where N is a compact Einstein manifold.

Corollary 4.2. If (M^n, g, f) is a shrinking gradient Ricci soliton with nonnegative sectional curvature and $R \leq 1$, then the universal cover of M is isometric to either \mathbb{R}^n or $\mathbb{S}^2 \times \mathbb{R}^{n-2}$

Proof: Choose local orthonormal basis $\{e_i\}_{i=1}^n$, due to the assumption, for any i ,

$$
2Ric(e_i, e_i) = Ric(e_i, e_i) + \sum_{j \neq i} R(e_i, e_j, e_i, e_j) \leq Ric(e_i, e_i) + \sum_{j \neq i} Ric(e_j, e_j) = R \leq 1,
$$

hence $Ric(e_i, e_i) \leq \frac{1}{2}$ $\frac{1}{2}$, i.e. $Ric \leq \frac{1}{2}$ $\frac{1}{2}g$. So

$$
h(u, u) = \sum_{i=1}^{n} R(u, e_i, u, e_i) Ric(e_i, e_i) \leq \frac{1}{2} \sum_{i=1}^{n} R(u, e_i, u, e_i) = \frac{1}{2} Ric(u, u).
$$

Then from the above Theorem we know that the universal cover of M is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$, where $\mathbb N$ is a compact Einstein manifold. Since $R \leq 1$, N has to be \mathbb{S}^2 . В последните последните последните последните и последните последните последните последните последните после
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Remark . Corollary 4.2 appeared in Petersen-Wylie's paper [\[26\]](#page-13-3), but our proof is different from theirs. Under the same assumption, they first got the scalar curvature is identically one using Naber's result [\[20\]](#page-12-12), then the conclusion follows from their main theorem that shrinking gradient soliton satisfying $0 \leq Ric \leq \frac{1}{2}$ $\frac{1}{2}g$ and constant scalar curvature condition must be rigid.

According to the above discussion, we can give "the third proof" using our structure Theorem [3.4,](#page-6-0)

Third proof of Corollary [4.2](#page-7-0)

Obviously the condition A holds, then the universal cover of $Mⁿ$ is isometric to $\mathbb{R}^k \times \mathbb{N}^{n-k}$ by Theorem [3.4,](#page-6-0) where $\mathbb N$ is a shrinking gradient Ricci soliton with positive Ricci curvature. Because Naber's theorem implies the scalar curvature is identically one, ∇f is identically zero using formula [\(2.3\)](#page-4-0), i.e. $Ric = \frac{1}{2}$ $\frac{1}{2}g$ on N, so N is \mathbb{S}^2 .

Theorem 4.3. Let (M^n, g, f) be a shrinking gradient Ricci soliton with bounded curvature, if $Ric \geq 0$ and the scalar curvature $R < 1 - \delta$ for some $0 < \delta < 1$, then M^n is flat.

Proof: Suppose on the contrary, then the strong maximum principle gives $R > 0$ on M.

Because the curvature is bounded, formula [\(2.5\)](#page-4-0) and quadratic growth of f from Theorem [2.1,](#page-4-1) it is easy to see that outside a compact set, f has no critical point. Formula (2.3) implies that the scalar curvature is always increasing along the integral curve of f , hence the scalar curvature has a positive lower bound. Next we can apply Theorem [2.2](#page-4-2) to obtain that (M, g, f) converge along the integral curve of ∇f to $\mathbb{R} \times \mathbb{N}^{n-1}$, where \mathbb{N}^{n-1} is a nonflat shrinking gradient Ricci soliton satisfying the same assumption, then we play the same game on N. When N is of dimension 4, According to the main Theorem in Munteanu-Wang [\[19\]](#page-12-18) we get the asymptotic limit of N is either $\mathbb{R} \times \mathbb{S}^3$ or $\mathbb{R}^2 \times \mathbb{S}^2$ or their quotients. In any case the scalar curvature of the asymptotic limit can't be smaller than $1 - \delta$. Contradiction. Remark In dimension 4, Munteanu-Wang [\[17\]](#page-12-19) proved that bounded scalar curvature implies bounded curvature operator, so the bounded curvature assumption can be removed.

5. constant scalar curvature

To prove the main theorem, it is necessary to derive the nonnegativity of Ricci curvature. Actually,

Lemma 5.1 ([\[12\]](#page-12-13),[\[9\]](#page-12-2)). Suppose (M^4, g, f) is a four dimensional Ricci shrinker with $R = 1$, then $Ric \geq 0$.

The following formula which is derived in plays an important role.

(5.7)
$$
\nabla_{\nabla f} Ric = Ric \circ (Ric - \frac{1}{2}g) + R(\nabla f, \cdot, \nabla f, \cdot).
$$

Next we state the main result proved in Cheng-Zhou [\[9\]](#page-12-2) and give a new proof.

Theorem 5.2 ([\[9\]](#page-12-2)). Let (M, g, f) be a 4-dimensional complete noncompact shrinking gradient Ricci soliton. If M has constant scalar curvature $R = 1$, then it must be isometric to a finite quotient of $\mathbb{R}^2 \times S^2$.

Proof: Denote the eigenvalues of Ricci curvature by $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq$ λ_4 .

Claim.

$$
\Delta_f(\lambda_1 + \lambda_2)
$$

\n
$$
\leq \frac{2\nabla f \cdot \nabla(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2) - 2(\lambda_1^2 + \lambda_2^2)}{f} (1 - 2\lambda_1 - 2\lambda_2)
$$

\n
$$
-2(\lambda_1 + \lambda_2) + 4(\lambda_1^2 + \lambda_2^2)
$$

in the barrier sense.

Actually, at x, because $R = 1$, $Ric(\nabla f) = 0$, so we choose $e_1 =$ ∇f $\frac{\nabla f}{|\nabla f|}$, then extend e_1 to an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that ${e_i}_{i=1}^4$ are the eigenvectors of $Ric(x)$ corresponding to eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Take parallel transport of $\{e_i\}_{i=1}^4$ along all the geodesics from x, then in a neighborhood $B(x, \delta)$ we get a smooth function $u(y) = Ric(y)(e_1(y), e_1(y)) + Ric(y)(e_2(y), e_2(y))$ satisfying $u(y) \ge$ $\lambda_1(y) + \lambda_2(y)$ and $u(x) = \lambda_1(x) + \lambda_2(x)$.

So, at x , by (5.7) ,

$$
-2R(\nabla f, e_i, \nabla f, e_i) = 2\nabla_k f_{ii} f_k + 2\lambda_i^2 - \lambda_i
$$

=
$$
-2(\nabla_{\nabla f} Ric)(e_i, e_i) + 2\lambda_i^2 - \lambda_i,
$$

Hence

$$
2R(e_1, e_2, e_1, e_2) = \frac{2\nabla f \cdot \nabla u + u - 2u^2}{f}.
$$

$$
\Delta_{f}u(x) = \Delta_{f} (Ric(y)(e_{1}(y), e_{1}(y)) + Ric(y)(e_{2}(y), e_{2}(y)))|_{y=x}
$$
\n
$$
= (\Delta_{f}Ric)(e_{1}, e_{1})(x) + (\Delta_{f}Ric)(e_{2}, e_{2})(x)
$$
\n
$$
= \lambda_{1} + \lambda_{2} - 2 \sum_{i=1}^{4} R_{1i1i}\lambda_{i}
$$
\n
$$
= (\lambda_{1} + \lambda_{2}) - 2R_{1212}(\lambda_{1} + \lambda_{2}) - 2\lambda_{3}(\lambda_{3} - R_{3434}) - 2\lambda_{4}(\lambda_{4} - R_{3434})
$$
\n
$$
= (\lambda_{1} + \lambda_{2}) - 2R_{1212}(\lambda_{1} + \lambda_{2}) - 2(\lambda_{3}^{2} + \lambda_{4}^{2}) + 2(\lambda_{3} + \lambda_{4})R_{3434}
$$
\n
$$
= (\lambda_{1} + \lambda_{2}) - 2R_{1212}(\lambda_{1} + \lambda_{2}) - 2(\lambda_{3}^{2} + \lambda_{4}^{2}) + (2R_{1212} + (\lambda_{3} + \lambda_{4}) - (\lambda_{1} + \lambda_{2}))(\lambda_{3} + \lambda_{4})
$$
\n
$$
= (\lambda_{1} + \lambda_{2}) - (\lambda_{1} + \lambda_{2})(\lambda_{3} + \lambda_{4}) + 2R_{1212}(\lambda_{3} + \lambda_{4} - \lambda_{1} - \lambda_{2}) + (\lambda_{3} + \lambda_{4})^{2} - 2(\lambda_{3}^{2} + \lambda_{4}^{2})
$$
\n
$$
= 2R_{1212}(1 - 2\lambda_{1} - 2\lambda_{2}) + (\lambda_{1} + \lambda_{2}) + (\lambda_{1} + \lambda_{2})(1 - \lambda_{3} - \lambda_{4}) + (\lambda_{3} + \lambda_{4})^{2} - 2(\lambda_{3}^{2} + \lambda_{4}^{2})
$$
\n
$$
= \frac{2\nabla f \cdot \nabla(\lambda_{1} + \lambda_{2}) + (\lambda_{1} + \lambda_{2}) - 2(\lambda_{1}^{2} + \lambda_{2}^{2})
$$
\n
$$
+ (\lambda_{1} + \lambda_{2})^{2} + (1 - \lambda_{1} - \lambda_{2})^{2} - 2(\frac{1}{
$$

this finish the proof of the Claim.

Next we consider the function $\frac{\lambda_1 + \lambda_2}{f}$,

$$
\Delta_f \frac{\lambda_1 + \lambda_2}{f} = \frac{1}{f} \Delta_f (\lambda_1 + \lambda_2) + \Delta_f \frac{1}{f} (\lambda_1 + \lambda_2) + 2 \nabla \frac{1}{f} \cdot \nabla (\lambda_1 + \lambda_2)
$$

\n
$$
\leq \frac{1}{f} \left\{ \frac{2 \nabla f \cdot \nabla (\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2) - 2(\lambda_1^2 + \lambda_2^2)}{f} (1 - 2\lambda_1 - 2\lambda_2) \right\}
$$

\n
$$
-2(\lambda_1 + \lambda_2) + 4(\lambda_1 + \lambda_2)^2 \right\} + \left(\frac{1}{f} + \frac{1}{f^2}\right) (\lambda_1 + \lambda_2) - \frac{2 \nabla f \cdot \nabla (\lambda_1 + \lambda_2)}{f^2}
$$

\n
$$
= \frac{\lambda_1 + \lambda_2}{f} \left\{ -2 + 4(\lambda_1 + \lambda_2) + \frac{1 - 2(\lambda_1 + \lambda_2)}{f} (1 - 2\lambda_1 - 2\lambda_2) \right\}
$$

\n
$$
- \frac{4 \nabla f \cdot \nabla (\lambda_1 + \lambda_2)}{f} + 1 + \frac{1}{f} \right\}
$$

By Theorem [2.2,](#page-4-2) we know the asymptotic limit of (M^4, g) along the integral curve of f is $\mathbb{R} \times \mathbb{N}^3$, where \mathbb{N}^3 is a three dimensional shrinking gradient Ricci soliton, hence is a finite quotient of $\mathbb{R} \times \mathbb{S}^2$ due to $R = 1$. So $\lambda_1 + \lambda_2$ tends to zero at infinity.

Because $R = 1$, Munteanu-Wang [\[17\]](#page-12-19) obtained that Riemannian curvature is bounded, hence its derivative is also bounded due to Shi's estimate. So $|\nabla f \cdot \nabla (\lambda_1 + \lambda_2)|$ is bouned by $C \cdot |\nabla f| \cdot |\nabla Ric|$, hence $\nabla f \cdot \nabla(\lambda_1 + \lambda_2)$ $\frac{\lambda_1 + \lambda_2}{f}$ is sufficiently small outside a compact set.

In all, we get

$$
\Delta_f \frac{\lambda_1 + \lambda_2}{f} \le -0.9 \cdot \frac{\lambda_1 + \lambda_2}{f}
$$

outside a compact set D.

Suppose $\lambda_1 + \lambda_2$ is not identically zero on $M \setminus D$, then we can apply similar argument as [\[10\]](#page-12-20) or [\[18\]](#page-12-14) to derive that

$$
\frac{\lambda_1 + \lambda_2}{f} \ge \frac{a}{f}
$$

for some small positive a outside a compact set. So $\lambda_1 + \lambda_2 \ge a$, which contradict with the fact that $\lambda_1 + \lambda_2$ tends to zero at infinity.

So $\lambda_1 + \lambda_2 \equiv 0$ on $M \setminus D$, hence the function

$$
G = tr(Ric3) - \frac{1}{2}|Ric|2,
$$

is 0 on $M \setminus D$.

Because G is an analytic function, has to be zero, we obtain that $G \equiv 0$ on M. Moreover, the equation $0 = \Delta_f R = R - 2|Ric|^2$ implies that

$$
G = tr(Ric3) - |Ric|2 + \frac{1}{4}R
$$

= $\sum_{i=1}^{4} (\lambda_i - \frac{1}{2})^2 \lambda_i = 0.$

Finally we get $\lambda_1 = \lambda_2 \equiv 0$ and $\lambda_3 = \lambda_4 \equiv \frac{1}{2}$ $\frac{1}{2}$ due to $Ric \geq 0$. This implies the Ricci curvature has constant rank 2. Therefore, any 4 dimensional shrinking gradient Ricci soliton with $R = 1$ is isometric to a finite quotient of $\mathbb{R}^2 \times \mathbb{S}^2$ by [\[12\]](#page-12-13).

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