Several Examples of Application of the Simple Equations Method (SEsM) for Obtaining Exact Solutions of Nonlinear PDEs

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Abstract

We apply the Simple Equations Method (SEsM) for obtaining exact solutions of nonlinear differential equations. We discuss several examples with goal to illustrate the results from the use of derivatives of composite functions in the algorithm of SEsM. The discussed examples contain derivatives of functions which are composite functions of solutions of two simple equations.

1 Introduction

The complex systems are in the most cases non-linear ones [\[1\]](#page-9-0)- [\[16](#page-10-0)]. Thus, large efforts are focused on the study of the effects of this nonlinearity. Such effects are studied by the methods of the time series analysis and by models based on differential or difference equations [\[17\]](#page-10-1) - [\[31](#page-10-2)]. Usually, the model equations are nonlinear partial differential or difference equations. Thus, the exact and approximate analytical solutions od such equation are of great interest. The methodology for obtaining such solutions is in development since several decades. At the beginning, researchers tried to transform the nonlinearity of the studied equation and even to remove it by means of appropriate transformation. One example is the Hopf-Cole transformation [\[32\]](#page-10-3), [\[33\]](#page-11-0). It transforms the nonlinear Burgers equation to the linear heat equation. Following this way, one arrived at the Method of Inverse Scattering Transform [\[34\]](#page-11-1) - [\[36\]](#page-11-2) the method of Hirota [\[37](#page-11-3)], [\[38](#page-11-4)]. Another line of research connected to the use of transformations is [\[39\]](#page-11-5).

Below, we discuss and apply the SEsM (Simple Equations Method) [\[40](#page-11-6)] -[\[42\]](#page-11-7). This method is a result of another branch of research on the methodology. Kudryashov and then Kudryashov and Loguinova developed the Method of Simplest Equation (MSE) [\[43\]](#page-11-8),[\[44](#page-11-9)] . This method is based on determination of singularity order n of the solved nonlinear partial differential equation and searching of particular solution of this equation as series containing powers of solutions of a simpler equation called simplest equation. The SEsM methodology has a long story until its recent formulation which was given in [\[45\]](#page-11-10) - [\[48\]](#page-11-11). SEsM was proposed by Vitanov after many years of research which started almost 35 years ago [\[49](#page-11-12)] - [\[53](#page-12-0)]. Then, in 2009 [\[54](#page-12-1)], [\[55\]](#page-12-2) and in 2010, Vitanov and co-authors used the ordinary differential equation of Bernoulli as simplest equation [\[56\]](#page-12-3) and applied the simplest version of of SEsM called Modified Method of Simplest Equation (MMSE) to ecology and population dynamics [\[57\]](#page-12-4). MMSE used a balance equation [\[58](#page-12-5)], [\[59\]](#page-12-6) to determine the kind of the simplest equation and truncation of the series of solutions of the simplest equation. MMSE is equivalent of the MSE mentioned above. Up to 2018 the contributions to the methodology and its application have been connected to the MMSE [\[60](#page-12-7)] - [\[64\]](#page-12-8). This research was based on single simplest equation and one balance equation. The construction of the solution of the solved equation was chosen to be as power series containing powers of the solutions of the simplest equation.

Recently Vitanov extended the capacity of the methodology by inclusion of the possibility of use of more than one simplest equation. This version is called SEsM - Simple Equations Method as the used simple equations are more simple than the solved nonlinear partial differential equation but these simple equations in fact can be quite complicated. We note that a variant of SEsM based on two simple equations was applied in [\[66\]](#page-12-9) and the first description of the methodology was made in $|45|$ and then in $|46|$ - $|48|$. For more applications of specific cases of the methodology see [\[67\]](#page-13-0), [\[68](#page-13-1)].

In this article we will show several examples of application of SEsM. We illustrate the use of composite function which is a function of two simple functions. Each simple functions can be a function of two independent variables. The structure of the article is as follows. We describe SEsM in Sect 2. In Sect. 3, we supply the information needed for the use of derivatives of composite functions in SEsM. Several examples are shown in Sect. 4 and Sect. 5 presents some concluding remarks.

2 The Simple Equations Method (SEsM)

We consider a a system of nonlinear partial differential equations

$$
\mathcal{E}_{i}[u_{1}(x, \ldots, t), \ldots, u_{n}(x, \ldots, t)]=0, i=1, \ldots, n.
$$
 (1)

 $\mathcal{E}_i[u_1(x,\ldots,t),\ldots,u_n(x,\ldots,t)]$ depends on the functions $u_1(x,\ldots,t),\ldots,u_n(x,\ldots,t)$ and some of their derivatives $(u_i \text{ can be a function of more than 1 spatial})$ coordinates). Step 1 of SEsM includes transformations

$$
u_i(x, ..., t) = T_i[F_i(x, ..., t), G_i(x, ..., t), ...].
$$
\n(2)

Here $T_i(F_i, G_i, \ldots)$ is some function of another functions F_i, G_i, \ldots Note that T_i are composite functions. In general $F_i(x, \ldots, t)$, $G_i(x, \ldots, t)$, ... can be functions of several spatial variables as well as of the time. The goal of the transformations is to transform the nonlinearity of the solved differential equations to more treatable kind of nonlinearity. In the best case, the transformation removes the nonlinearity and the solved nonlinear differential equation is reduced to a linear equation.

The nonlinearities in the solved equations can be different kinds. For a example, for the case of one solved equation the transformation $T(F, G, \ldots)$ can be the Painleve expansion. If the solved equation has polynomial nonlinearities one can skip this step.

Next, one makes Step 2. of SEsM where the functions $F_i(x, ..., t)$, $G_i(x, ..., t)$, ... are represented as a function of other functions $f_{i1},..., f_{iN}, g_{i1},..., g_{iM}$, ..., which are connected to solutions of some differential equations (these equations can be partial or ordinary differential equations) that are more simple than Eq.(2). The forms of the functions $F_i(f_1, \ldots, f_N)$, $G_i(x, \ldots, t)$, ... can be different. At Step 3. of SEsM, we choose the functions used in F_i, G_i, \ldots the functions $f_{i1}, \ldots, f_{iN}, g_{i1}, \ldots, g_{iM}$ are solutions of PDEs which are more simple than the solved nonlinear partial differential equation. These more simple equations usually are ordinary differential equations. In many cases the form of the simple equations is determined by a balance equations. Balance equations may be needed in order to ensure that the system of algebraic equations from Step 4. contains more than one term in any of the equations. This is needed for a non-trivial solution of the solved nonlinear partial differential equation.

At Step 4. of SEsM we apply the steps 1 - 3 to Eqs.(2) and this transforms the left-hand side of these equations. In the most cases the result of this transformation are functions which are sum of terms where each term contains some function multiplied by a coefficient. This coefficient contains some of the parameters of the solved equations and some of the parameters of

the solution. In the most cases a balance procedure must be applied in order to ensure that the above-mentioned relationships for the coefficients contain more than one term (e.g., if the result of the transformation is a polynomial then the balance procedure has to ensure that the coefficient of each term of the polynomial is a relationship that contains at least two terms). This balance procedure may lead to one or more additional relationships among the parameters of the solved equation and parameters of the solution. The last relationships are called balance equations.

Finally at Step 4. of SEsM We can obtain a nontrivial solution of Eq. (2) if all coefficients mentioned in Step 3 are set to 0. This condition leads to a system of nonlinear algebraic equations. Each nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation.

3 Composite functions in SEsM

Let us consider the function $h(x_1, \ldots, x_d)$. It is a function of d independent variables x_{1}, \ldots, x_{d} . We assume hat h is a composite function of m other functions $g_1^{(1)}$ $j^{(1)}_1, \ldots, g^{(m)}_n$

$$
h(x_1, \ldots, x_d) = f[g^{(1)}(x_1, \ldots, x_d), \ldots, g^{(m)}(x_1, \ldots, x_d)].
$$
 (3)

Following notations are introduced.

- 1. $\vec{\nu} = (\nu_1, \dots, \nu_d)$ is a d-dimensional index containing the integer nonnegative numbers ν_1, \ldots, ν_d .
- 2. $\vec{z} = (z_1, \ldots, z_d)$ is a d-dimensional object containing the real numbers z_1, \ldots, z_d .
- 3. $| \vec{\nu} | = \sum$ d $i=1$ ν_i is the sum of the elements of the d-dimensional index $\vec{\nu}$.
- 4. $\vec{\nu}! = \prod$ d $i=1$ $\nu_i!$ is the factorial of the multicomponent index $\vec{\nu}$.
- 5. $\vec{z}^{\vec{\nu}} = \prod$ d $\frac{i=1}{i}$ $z_i^{\nu_i}$ is the $\vec{\nu}$ -th power of the multicomponent variable \vec{z} .
- 6. $D^{\vec{\nu}}_{\vec{x}} = \frac{\partial^{|\vec{\nu}|}}{\partial x_1^{\nu_1}...\hat{\delta}}$ $\frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}...\partial x_d^{\nu_d}}$, $|\vec{\nu}| > 0$ is the $\vec{\nu}$ -th derivative with respect to the multicomponent variable \vec{x} . We note that in this notation $D_{\vec{x}}^{\vec{0}}$ is the identity operator.
- 7. $|| \vec{z} || = \text{max} | z_i |$ is the maximum value component of the multicomponent variable \vec{z} in the interval $1 \leq i \leq d$.
- 8. For the d-dimensional index $\vec{l} = (l_1, \ldots, l_d)$ $(l_1, \ldots, l_d$ are integers) we have $\vec{l} \leq \vec{\nu}$ when $l_i \leq \nu_i, i = 1, \ldots, d$. Then we define

$$
\begin{pmatrix} \vec{\nu} \\ \vec{l} \end{pmatrix} = \prod_{i=1}^d \begin{pmatrix} \nu_i \\ l_i \end{pmatrix} = \frac{\vec{\nu}!}{\vec{l}!(\vec{\nu}-\vec{l})!}.
$$

- 9. Ordering of vector indexes. For two vector indexes $\vec{\mu} = (\mu_1, \dots, \mu_d)$ and $\vec{\nu} = (\nu_1, \dots, \nu_d)$ we have $\vec{\mu} \prec \vec{\nu}$ when one of the following holds
	- (a) $|\vec{\mu}| < |\vec{\nu}|$.
	- (b) $| \vec{\mu} | = | \vec{\nu} |$ and $\mu_1 < \nu_1$.
	- (c) $| \vec{\mu} | = | \vec{\nu} |, \mu_1 = \nu_1, \ldots \mu_k = \nu_k$ and $\mu_{k+1} < \nu_{k+1}$ for some $1 \leq k < d$.

Below we use also the notation

$$
h_{(\vec{\nu})} = D_{\vec{x}}^{\vec{\nu}} h; \quad f_{(\vec{\lambda})} = D_{\vec{y}}^{\vec{\lambda}} f; \quad g_{(\vec{\mu})}^{(i)} = D_{\vec{x}}^{\vec{\mu}} g^{(i)}; \quad \vec{g}_{(\vec{\mu})} = (g_{(\vec{\mu})}^{(1)}, \dots, g_{(\vec{\mu})}^{(m)}).
$$
 (4)

The Faa di Bruno formula for the composite derivative of a function containing functions of many variables is [\[69](#page-13-2)]

$$
h_{(\vec{\nu})} = \sum_{1 \leq |\vec{\lambda}| \leq n} f_{(\vec{\lambda})} \sum_{s=1}^{n} \sum_{p_s(\vec{\nu}, \vec{\lambda})} (\vec{\nu}!) \prod_{j=1}^{s} \frac{[\vec{g}_{(\vec{l}_j)}]^{\vec{k}_j}}{(\vec{k}_j!) [\vec{l}_j!]^{|\vec{k}_j|}}.
$$
(5)

In [\(5\)](#page-4-0) $n = |\vec{v}|$. In addition,

$$
p_s(\vec{\nu}, \vec{\lambda}) = {\vec{k_1}, \dots, \vec{k_s}; \vec{l_1}, \dots, \vec{l_s}}, \quad |\vec{k_i}| > 0.
$$
 (6)

Moreover,

$$
0 \prec \vec{l}_1 \ldots \prec \vec{l}_s, \quad \sum_{i=1}^s \vec{k}_i = \vec{\lambda}, \quad \sum_{i=1}^s |\vec{k}_i| \vec{l}_i = \vec{\nu}.
$$
 (7)

[\(5\)](#page-4-0) can be simplified by a change of the notation [\[69\]](#page-13-2). We introduce

$$
p(\vec{\nu}, \vec{\lambda}) = {\vec{k_1}, \dots, \vec{k_n}; \vec{l_1}, \dots, \vec{l_n}}, \quad 1 \le s \le n,
$$
 (8)

and,

$$
\vec{k}_i = 0; \quad \vec{l}_i = 0, \quad 1 \le i \le n - s, \quad |\vec{k}_i| > 0, \quad n - s + 1 \le i \le n. \tag{9}
$$

Finally $0 \prec \vec{l}_{n-s+1} \ldots \prec \vec{l}_n$ are such that $\sum_{n=1}^{n}$ $i=1$ $\vec{k}_i = \vec{\lambda}_i$ and $\sum_{ }^{n}$ $i=1$ $|\vec{k}_i|\vec{l}_i = \vec{\nu}$. Then [\(5\)](#page-4-0) can be written as

$$
h_{(\vec{\nu})} = \sum_{1 \leq |\vec{\lambda}| \leq n} f_{(\vec{\lambda})} \sum_{p(\vec{\nu}, \vec{\lambda})} (\vec{\nu}) \prod_{j=1}^{n} \frac{[\vec{g}_{(\vec{l}_{j})}]^{\vec{k}_{j}}}{(\vec{k}_{j}!) [\vec{l}_{j}!]^{\vec{k}_{j}]}}.
$$
(10)

We discuss below the specific case when the composite function h is a function of two independent variables x_1 and x_2 . In addition we consider the case of composite function containing two functions of two independent variables. In this case the composite function is a function of the functions $g^{(1)}(x_1, x_2)$ and $g^{(2)}(x_1, x_2)$. The Faa di Bruno formula for composite function containing two functions which are functions of two variables is

$$
h_{(\vec{\nu})} = \frac{\partial^{\nu_1 + \nu_2} h}{\partial x_1^{\nu_1} \partial x_2^{\nu_2}} = \sum_{1 \le (\lambda_1 + \lambda_2) \le \nu_1 + \nu_2} \frac{\partial^{\lambda_1 + \lambda_2} f}{\partial g^{(1)} \lambda_1 \partial g^{(2)} \lambda_2} \left\{ \sum_{s=1}^{\nu_1 + \nu_2} \sum_{p_s(\vec{\nu}, \vec{\lambda})} (\nu_1! \nu_2!) \times \prod_{j=1}^s \left[\frac{1}{(k_{j,1}! k_{j,2}!)(l_{j,1}! + l_{j,2}!)^{k_{j,1} + k_{j,2}} \prod_{i=1}^2 \left(\frac{\partial^{l_{j,1} + l_{j,2}}}{\partial x_1^{l_{j,1}} \partial x_2^{l_{j,2}} g^{(i)}} \right)^{k_{j,i}} \right] \right\}.
$$
 (11)

The version of [\(11\)](#page-5-0) occurring from [\(10\)](#page-5-1) is

$$
h_{(\vec{\nu})} = \frac{\partial^{\nu_1 + \nu_2} h}{\partial x_1^{\nu_1} \partial x_2^{\nu_2}} = \sum_{1 \le (\lambda_1 + \lambda_2) \le \nu_1 + \nu_2} \frac{\partial^{\lambda_1 + \lambda_2} f}{\partial g^{(1) \lambda_1} \partial g^{(2) \lambda_2}} \left\{ \sum_{p(\vec{\nu}, \vec{\lambda})} (\nu_1! \nu_2!) \times \prod_{j=1}^n \left[\frac{1}{(k_{j,1}! k_{j,2}!)(l_{j,1}! + l_{j,2}!)^{k_{j,1} + k_{j,2}} \prod_{i=1}^2 \left(\frac{\partial^{l_{j,1} + l_{j,2}}}{\partial x_1^{l_{j,1}} \partial x_2^{l_{j,2}}} g^{(i)} \right)^{k_{j,i}} \right] \right\}.
$$
 (12)

4 An example of use of composite functions in SEsM

We are going to show how the methodology of SEsM works in presence of composite functions. We will use a specific form of the composite function: composite function of a function of 2 variables $h = f[g^{(1)}(x,t), g^{(2)}(x,t)]$

$$
h = \alpha + \beta_1 g^{(1)} + \beta_2 g^{(2)} + \gamma_1 g^{(1)^2} + \gamma_2 g^{(2)^2} + \gamma_3 g^{(1)} g^{(2)}.
$$
 (13)

The example is connected to the equation

$$
(1+h^2)\left(\frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial t^2}\right) - 2h\left[\left(\frac{\partial h}{\partial x}\right)^2 - \left(\frac{\partial h}{\partial t}\right)^2\right] = h(1-h^2). \tag{14}
$$

We apply SEsM and skip Step. 1 (no transformation of the nonlinearity) as the noninearity of the equation is polynomial one. We have $h =$ $f[g^{(1)}(x,t), g^{(2)}(x,t)]$. The needed derivatives are as follows

$$
\frac{\partial h}{\partial x} = \frac{\partial f}{\partial g^{(1)}} \frac{\partial g^{(1)}}{\partial x} + \frac{\partial f}{\partial g^{(2)}} \frac{\partial g^{(2)}}{\partial x},\tag{15}
$$

$$
\frac{\partial h}{\partial t} = \frac{\partial f}{\partial g^{(1)}} \frac{\partial g^{(1)}}{\partial t} + \frac{\partial f}{\partial g^{(2)}} \frac{\partial g^{(2)}}{\partial t}.
$$
 (16)

$$
\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 f}{\partial g^{(1)^2}} \left(\frac{\partial g^{(1)}}{\partial x}\right)^2 + 2 \frac{\partial^2 f}{\partial g^{(1)} \partial g^{(2)}} \frac{\partial g^{(1)}}{\partial x} \frac{\partial g^{(2)}}{\partial x} + \frac{\partial f}{\partial g^{(1)}} \frac{\partial^2 g^{(1)}}{\partial x^2} + \frac{\partial^2 f}{\partial g^{(2)^2}} \left(\frac{\partial g^{(2)}}{\partial x}\right)^2 + \frac{\partial f}{\partial g^{(2)}} \frac{\partial^2 g^{(2)}}{\partial x^2}.
$$
 (17)

$$
\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 f}{\partial g^{(1)^2}} \left(\frac{\partial g^{(1)}}{\partial t}\right)^2 + 2 \frac{\partial^2 f}{\partial g^{(1)} \partial g^{(2)}} \frac{\partial g^{(1)}}{\partial t} \frac{\partial g^{(2)}}{\partial t} + \frac{\partial f}{\partial g^{(1)}} \frac{\partial^2 g^{(1)}}{\partial t^2} + \frac{\partial^2 f}{\partial g^{(2)^2}} \left(\frac{\partial g^{(2)}}{\partial t}\right)^2 + \frac{\partial f}{\partial g^{(2)}} \frac{\partial^2 g^{(2)}}{\partial t^2}.
$$
 (18)

[\(14\)](#page-5-2) becomes

$$
(1+h^2)\left\{\frac{\partial^2 f}{\partial g^{(1)}}\left[\left(\frac{\partial g^{(1)}}{\partial x}\right)^2 - \left(\frac{\partial g^{(1)}}{\partial t}\right)^2\right] + 2\frac{\partial^2 f}{\partial g^{(1)}\partial g^{(2)}}\left[\frac{\partial g^{(1)}}{\partial x}\frac{\partial g^{(2)}}{\partial x} - \frac{\partial g^{(1)}}{\partial t}\frac{\partial g^{(2)}}{\partial t}\right] + \frac{\partial f}{\partial g^{(1)}}\left[\frac{\partial^2 g^{(1)}}{\partial x^2} - \frac{\partial^2 g^{(1)}}{\partial t^2}\right] + \frac{\partial^2 f}{\partial g^{(2)}}\left[\left(\frac{\partial g^{(2)}}{\partial x}\right)^2 - \left(\frac{\partial g^{(2)}}{\partial t}\right)^2\right] + \frac{\partial f}{\partial g^{(2)}}\left[\frac{\partial^2 g^{(2)}}{\partial x^2} - \frac{\partial^2 g^{(2)}}{\partial t^2}\right] - 2h\left\{\left(\frac{\partial f}{\partial g^{(1)}}\right)^2 \left[\left(\frac{\partial g^{(1)}}{\partial x}\right)^2 - \left(\frac{\partial g^{(1)}}{\partial t}\right)^2\right] + \left(\frac{\partial f}{\partial g^{(2)}}\right)^2 \left[\left(\frac{\partial g^{(2)}}{\partial x}\right)^2 - \left(\frac{\partial g^{(2)}}{\partial t}\right)^2\right] + \frac{\partial f}{\partial g^{(2)}}\left[\frac{\partial g^{(1)}}{\partial g^{(2)}}\frac{\partial g^{(2)}}{\partial g^{(2)}} - \frac{\partial g^{(1)}}{\partial t}\frac{\partial g^{(2)}}{\partial t}\right] \right\} = h(1-h^2)(19)
$$

The composite function h is of the kind [\(13\)](#page-5-3) where $\alpha = 0, \beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 0$. In addition $g^{(1)}$ does not depend on t and $g^{(2)}$ does not depend on x. Let $\gamma_3 = A$. The composite function becomes

$$
h(x,t) = Ag^{(1)}(\alpha x)g^{(2)}(\delta \gamma t), \quad \delta = \pm 1.
$$
 (20)

The composite function [\(20\)](#page-6-0) allows for complicated simple equations for $g^{(1)}$ and $g^{(2)}$. These equations can be of the kind of equations for the elliptic functions of Jacobi:

$$
\left(\frac{dg^{(1)}}{dx}\right)^2 = \alpha (a_1 g^{(1)^4} + b_1 g^{(1)^2} + c_1)
$$

$$
\left(\frac{dg^{(2)}}{dx}\right)^2 = \beta (a_2 g^{(2)^4} + b_2 g^{(2)^2} + c_2).
$$
 (21)

Because of all above, [\(14\)](#page-5-2) is reduced to a system of algebraic equations

$$
\alpha^{2}b_{1} - \gamma^{2}b_{2} = 1 \n\alpha^{2}a_{1} + \gamma^{2}A^{2}c_{2} = 0 \n\gamma^{2}a_{2} + \alpha^{2}A^{2}c_{1} = 0.
$$
\n(22)

[\(22\)](#page-7-0) has various non-trivial solutions (Step 7 of SEsM). For an example, one of these solutions is when $\alpha^2 - \gamma^2 < 1$. We can consider A as a free parameter. Then $\alpha^2 = \gamma^2 + \frac{A^2 - 1}{A^2 + 1}$. Thus,

$$
h(x,t) = A \text{cn} \left\{ \alpha x; \frac{A^2[\alpha^2(A^2+1)+1]}{\alpha^2(A^2+1)^2} \right\} \text{cn} \left\{ \delta \gamma t; \frac{A^2[\gamma^2(A^2+1)-1]}{\gamma^2(A^2+1)^2} \right\}. (23)
$$

In [\(23\)](#page-7-1) cn(αx ; k₁) and cn(γt ; k₂) are corresponding Jacobi elliptic functions of modulus $0 \leq k_1 \leq 1$ and $0 \leq k_2 \leq 1$ respectively.

[\(22\)](#page-7-0) has an interesting specific case when $k_1 = 1$ and $k_1 = 0$. Then cn($\alpha x; k_1$) = sech(αx) and cn($\delta \gamma t$) = cos($\delta \gamma t$). Then,

$$
h(x,t) = \frac{\cos\left[\frac{\delta}{(A^2+1)^{1/2}}\right]t}{\cosh\left[\frac{A^2}{A^2+1}\right]^{1/2}x}.
$$
 (24)

[\(24\)](#page-7-2) can be obtained also straightforward on the basis of the composite func-tion [\(20\)](#page-6-0) if one takes for $g^{(1)}$ and $g^{(2)}$ the corresponding simple equations for the trigonometric and hyperbolic function respectively.

There are many other possible solutions. Several other examples are as follows. Let $k_1 = k_2 = 1$ and $\alpha^2 = 1/(1 - A)$; $\gamma^2 = A^2/(1 - A^2)$. Then,

$$
h(x,t) = a \frac{\sinh\left[(1/(1-A^2))^{1/2} x \right]}{\cosh\left[(A^2/(1-A^2))^{1/2} t \right]}
$$
(25)

We note that this solution is specific case of the more complicated solution

$$
h(x,t) = A \frac{\operatorname{sn}(\alpha x; k_1)}{\operatorname{cn}(\alpha x; k_1)} \operatorname{dn}(\delta \gamma t; k_2),\tag{26}
$$

where

$$
k_1^2 = \frac{\alpha^2 (1 - A^2)^2 + A^2}{\alpha^2 (1 - A^2)};
$$
 $k_2^2 = \frac{A^2 - \gamma^2 (1 - A^2)^2}{\gamma^2 A^2 (1 - A^2)};$ $\gamma^2 = \alpha^2 A^2$

Another example is when

$$
k_1^2 = 1 - \frac{1 - \alpha^2 (A^2 + 1)/A^2}{\alpha^2 (A^2 + 1)}; \quad k_2^2 = \frac{A^2 [1 - \gamma^2 (\alpha^2 + 1)]}{\gamma^2 (A^2 + 1)}; \quad \alpha^2 = A^2 \gamma^2 \tag{27}
$$

The corresponding solution is

$$
h(x,t) = A \mathrm{dn}(\alpha x, k_1) \mathrm{sn}(\delta \gamma t; k_2)
$$
\n(28)

[\(28\)](#page-8-0) has a specific case when $k_1 = \left[1 - 1/A^4\right]^{1/2}$, and $k_2 = 1$. In this case $\alpha^2 = A^2/(A^2 + 1)$, and $\gamma^2 = A^2/(A^2 + 1)$. The solution becomes

$$
h(x,t) = A \mathrm{dn} \left[\frac{A^2 x}{A^2 + 1}; \left(1 - \frac{1}{A^4} \right)^{1/2} \right] \tanh \left[\delta \left(\frac{At}{A^2 + 1} \right) \right] \tag{29}
$$

Another solution of the system of nonlinear algebraic equations is

$$
k_1^2 = 1 - \frac{\alpha^2 (A^2 - 1)/A^2 - 1}{\alpha^2 (A^2 - 1)^2}; \quad k_2^2 = 1 - \frac{A^2 [\gamma^2 (A^2 - 1) - 1]}{\gamma^2 (A^2 - 1)}; \quad \alpha^2 = A^2 \gamma^2
$$
\n(30)

The corresponding solution is

$$
h(x,t) = A \mathrm{dn}(\alpha x; k_1) \frac{\mathrm{sn}(\delta \gamma t; k_2)}{\mathrm{cn}(\delta \gamma t; k_2)}
$$
(31)

Here we have again the specific solution $k_1 = (1 - 1/A^4)^{1/2}$, $k_2 = 0$. Then $\alpha^2 = A^2/(A^2 - 1)$ and $\gamma^2 = 1/(A^2 - 1)$. The solution is

$$
h(x,t) = A \text{dn} \left[\frac{A^2 x}{A^2 - 1}; \left(1 - \frac{1}{A^4} \right)^{1/2} \right] \tan \left[\left(\frac{A^2}{A^2 - 1} \right)^{1/2} t \right] \tag{32}
$$

5 Concluding remarks

Yhis paper is devoted to an aspect of the application of the Simple Equations Method (SEsM) for obtaining excat soluions of nonliner differential equations. This aspect is the use of composite function in the process of the application of the methodology. The need of use of composite functions in SEsM occurs in a natural way. The reason is that the searched solution of the solved equation has to be constructed as a composite function of functions which are solutions of more simple differential equations. This leads to the need of use of the Faa di Bruno relationship for the derivatives of a composite functions. This use of composite functions in the methodology of SEsM opens a possibility for obtaining additional results on the methodology as well as specific solutions of many nonlinear differential equations.

References

- [1] R. Axelrod, M. Cohen. Harnessing Complexity (Basic Books, New York, 2001)
- [2] P. G. Drazin. Nonlinear Systems (Cambridge University Press, Cambridge, UK, 1992).
- [3] R. Lambiotte, M. Ausloos. Journal of Statistical Mechanics: Theory and Experiment P08026 (2007)
- [4] M. Mitchell. Artificial Intelligence **170**, 1194 1212 (2006).
- [5] R. Kutner, M. Ausloos, D. Grech, T. Di Matteo, C. Schinckus, H. E. Stanley. Physica A 516, 240 - 253 (2019).
- [6] N.K. Vitanov, F. H. Busse. ZAMP 48, 310 324 (1997).
- [7] R. M. May, S. A. Levin, G. Sugihara. Nature 451, 893 895 (2008).
- [8] S. A. Sheard, A. Mostashari. Systems Engineering, 12, 295 311 (2009).
- [9] M. Bahrami, N. Chinichian, A. Hosseiny, G. Jafari, M. Ausloos. Physica A 540, 123203 (2020).
- [10] N.K. Vitanov. Physics Letters A, 248, 338-346, (1998)
- [11] R. J. Lempert. PNAS USA, 99, Suppl. 3, 7309 7313.
- [12] N. K. Vitanov, M. Ausloos, G. Rotundo. Advances in Complex Systems 15, Supp. 01, 1250049 (2012)
- [13] J. Foster. Cambridge Journal of Economics 29, 873 892 (2005).
- [14] N. K.Vitanov, K. N. Vitanov. Mathematical Social Sciences, 80, 108 114 (2016).
- [15] L. Cameron, D. Larsen-Freeman. Journal of Applied Linguistics, 17, 226 - 239 (2007).
- [16] L. A. N. Amaral, A. Scala, M. Barthelemy, H. E. Stanley. Proceedings of the National Academy of Sciences, $97, 11149 - 11152$ (2000).
- [17] J. K. Hale. Oscillations in Nonlinear Systems (Dover, New York, 1991).
- [18] A. S. Pikovsky, D. L. Shepelyansky. Phys. Rev. Lett. 100, 094101 (2008).
- [19] I. P. Jordanov. Comp. rend. Acad. Sci. Bulg, 61, 307 314 (2008).
- [20] Y. Niu, S. Gong. Phys. Rev. A 73, 053811 (2006).
- [21] H. Kantz, T. Schreiber. Nonlinear Time Series Analysis (Cambridge University Press, Cambridge, UK, 2004).
- [22] R. Struble. Nonlinear Differential Equations. (Dover, New York, 2018).
- [23] P. J. Brockwell, R. A. Davis, M. V. Calder. Introduction to Time Series and Forecasting. (Springer, New York, 2002).
- [24] N. K. Vitanov. Physica D 136 , $322 339$ (2000)
- [25] A. Fuchs. Nonlinear Dynamics in Complex Systems. (Springer, Berlin, 2014)
- [26] S. Goldstein. Phychological Inquiry, 8 , 125 128 (1997).
- [27] N. K. Vitanov, Z. Dimitrova, H. Kantz. Physics Letters A, 346 350-355 (2006)
- [28] K. Kawasaki, T. Ohta. Kink dynamics in one-dimensional nonlinear systems. Physica A, 116, 573 – 593 (1982).
- [29] N. K. Vitanov, K. N. Vitanov. Physica A, 527, 121174 (2019).
- [30] N. K. Vitanov, R. Borisov, K. N. Vitanov. Physica A, 581, 126207 (2021).
- [31] H. Sedaghat. Nonlinear Difference Equations: Theory with Applications to Social Science Models (Springer Science & Business Media, Dordrecht, 2013)
- [32] E. Hopf. Communications on Pure and Applied Mathematics, 3, 201 – 230 (1950).
- [33] J. D. Cole. Quarterly of Applied Mathematics 9, 225 – 236 (1951).
- [34] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur. Studies in Applied Mathematics, $53, 249 - 315 (1974)$.
- [35] M. J. Ablowitz, P. A. Clarkson. Solitons, Nonlinear Evolution Equations and Inverse Scattering. (Cambridge University Press, Cambridge, UK, 1991).
- [36] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. R. Miura. Phys. Rev. Lett. **19**, $1095 - 1097$ (1967).
- [37] R. Hirota. Phys. Rev. Lett. 27, 1192 – 1194 (1971).
- [38] R. Hirota. The Direct Method in Soliton Theory. (Cambridge University Press, Cambridge, UK, 2004).
- [39] N. K. Vitanov. Axioms, 12, No. 12, 1106 (2023).
- [40] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. Entropy, 23, No.1, 10 (2020).
- [41] N. K. Vitanov, *Entropy*, **24**, No. 11, 1653, (2022).
- [42] N. K. Vitanov. *AIP Conference Proceedings* **2459**, 020003 (2022).
- [43] N. A. Kudryshov. Chaos, Solitons & Fractals 24, 1217 – 1231 (2005).
- [44] N. A. Kudryashov, N. B. Loguinova. Applied Mathematics and Computation **205**, $361 - 365$ (2008).
- [45] N. K. Vitanov. Pliska Studia Mathematica Bulgarica, $30, 29 - 42$ (2019).
- [46] N. K. Vitanov. Journal of Theoretical and Applied Mechanics, 49, 107 -122 (2019).
- [47] N. K. Vitanov. AIP Conference Proceedings 2159, 030038 (2019).
- [48] N. K. Vitanov, Z. I. Dimitrova. AIP Conference Proceedings 2159, 030039 (2019).
- [49] N. Martinov, N. Vitanov. Journal of Physics A: Mathematical and General 25 , $L51 - L56$ (1992).
- [50] N. Martinov, N. Vitanov. J. Phys A: Math. Gen. 25, 3609 3613 (1992)
- [51] N. K. Martinov, N. K. Vitanov. Canadian Journal of Physics, 72, 618 624 (1994).
- [52] N. Martinov, N. Vitanov. Zeitschrift für Physik B. 100 , $129 135$ (1996).
- [53] N. K. Vitanov. Journal of Physics A: Mathematical and General, 29, 5195 – 5207 (1996).
- [54] N. K. Vitanov, I. P. Joranov, Z. I. Dimitrova. Communications in Nonlinear Science and Numerical Simulation 14, 2379 – 2388 (2009).
- [55] N. K. Vitanov, I. P. Jordanov, Z. I. Dimitrova. Applied Mathematics and Computation 215, 2950– 2964 (2009).
- [56] N. K. Vitanov. Communications in Nonlinear Science and Numerical Simulation 15, 2050 – 2060 (2010).
- [57] N. K. Vitanov, Z. I. Dimitrova. Communications in Nonlinear Science and Numerical Simulation 15, 2836 – 2845 (2010).
- [58] N. K. Vitanov, Z. I. Dimitrova, H. Kantz. Applied Mathematics and Computation, 216, 2587 – 2595 (2010).
- [59] N. K. Vitanov. Communications in Nonlinear Science and Numerical Simulation, 16, 1176 – 1185 (2011).
- [60] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. Communications in Nonlinear Science and Numerical Simulation, 16, 3033 – 3044 (2011).
- [61] N. K. Vitanov. Communications in Nonlinear Science and Numerical Simulation, 16, 4215 – 4231 (2011).
- [62] N. K. Vitanov. Pliska Studia Mathematica Bulgarica 21, 257 266 (2012).
- [63] N. K. Vitanov, Z. I. Dimitrova. Applied Mathematics and Computation, 247, $213 - 217$ (2014).
- [64] N. K. Vitanov, Z. I. Dimitrova, T. I. Ivanova. Applied Mathematics and Computation, 315, 372 – 380 (2017).
- [65] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. Applied Mathematics and Computation, 269, 363 – 378 (2015).
- [66] N. K. Vitanov,Z. I. Dimitrova. Journal of Theoretical and Applied Mechanics, Sofia, 48, No. 1, 59 – 68 (2018).
- [67] N. K. Vitanov. AIP Conference Porceedings, 2321, 030035 (2021).
- [68] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. Entropy, 23, 10 (2021).
- [69] G. M. Constantine, T. H. Savits. Transactions of the American Mathematical Society, 348, 503 – 520 (1996).