CONVERGENCE RATES OF LANDWEBER-TYPE METHODS FOR INVERSE PROBLEMS IN BANACH SPACES

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ABSTRACT. Landweber-type methods are prominent for solving ill-posed inverse problems in Banach spaces and their convergence has been well-understood. However, how to derive their convergence rates remains a challenging open question. In this paper, we tackle the challenge of deriving convergence rates for Landweber-type methods applied to ill-posed inverse problems, where forward operators map from a Banach space to a Hilbert space. Under a benchmark source condition, we introduce a novel strategy to derive convergence rates when the method is terminated by either an *a priori* stopping rule or the discrepancy principle. Our results offer substantial flexibility regarding step sizes, by allowing the use of variable step sizes. By extending the strategy to deal with the stochastic mirror descent method for solving nonlinear ill-posed systems with exact data, under a benchmark source condition we also obtain an almost sure convergence rate in terms of the number of iterations.

1. Introduction

Due to their simplicity of implementation and low computational complexity per iteration, Landweber-type methods are well-recognized for solving ill-posed inverse problems in Banach spaces. Although their convergence has been well-understood, deriving their convergence rates remains a challenging open question. In this paper, we will consider Landweber-type methods for solving ill-posed inverse problems, where the forward operators map from a Banach space to a Hilbert space, and develop novel strategies to tackle the challenge of deriving convergence rates under a benchmark source condition on sought solutions.

We consider ill-posed inverse problems governed by the operator equation

$$F(x) = y, \tag{1}$$

where $F : \text{dom}(F) \subset X \to Y$ is a Fréchet differentiable operator mapping from a Banach spaces X to a Hilbert space Y, with domain $\text{dom}(F)$. The Fréchet derivative of F at $x \in \text{dom}(F)$ is denoted by $F'(x)$. For numerous examples of inverse problems that take the form of equation (1), such as integral equations of the first kind, tomography problems, and parameter estimation in partial differential equations, see [9, 11, 20, 28].

We assume (1) has a solution, i.e. $y \in \text{Ran}(F)$, the range of F. In practical applications, prior feature information about the sought solution is often available, and it is important to incorporate this information into the reconstruction process. Let $\mathcal{R} : X \to (-\infty, \infty]$ be a proper, lower semi-continuous, strong convex function that takes into account the available feature information. To reconstruct a solution of (1) that aligns with the features described

by $\mathcal{R}$, we start with an initial guess $x_0 \in X$ with $\partial\mathcal{R}(x_0) \neq \emptyset$ and take $\xi_0 \in \partial\mathcal{R}(x_0)$, where $\partial\mathcal{R}(x_0)$ denotes the subdifferential of $\mathcal{R}$ at x_0. We want to determine a solution $x^\dagger$ of (1) such that

$$D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) = \min\left\{ D_{\mathcal{R}}^{\xi_0}(x, x_0) : F(x) = y \right\}, \tag{2}$$

where $D_{\mathcal{R}}^{\xi_0}(x, x_0)$ denotes the Bregman distance induced by $\mathcal{R}$ at x_0 in the direction ξ_0; for the definition of subdifferential, Bregman distance and other relevant concepts from convex analysis, see [31]. A brief review of these concepts is provided in Section 2. We are mainly interested in the situation where the problem (2) is ill-posed in the sense that its solution does not depend continuously on the data.

In applications, data acquired through experiments inevitably contains noise. Assume, instead of the exact data y, we only have a noisy data y^δ satisfying

$$\|y^\delta - y\| \leq \delta$$

with a small noise level $\delta > 0$. Directly substituting y with y^δ in the minimization problem (2) may render the problem ill-defined, even if well-defined, the solution obtained from y^δ may not depend continuously on the data. To effectively utilize noisy data y^δ to approximate the solution of (2), regularization techniques must be employed to mitigate this instability.

Although many regularization methods have been developed for solving (2) with noisy data, in this paper we will focus on the following Landweber-type method

$$\begin{aligned}
\xi_{k+1}^\delta &= \xi_k^\delta - \gamma_k^\delta F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta), \\
x_{k+1}^\delta &= \arg\min_{x \in X}\left\{ \mathcal{R}(x) - \langle \xi_{k+1}^\delta, x \rangle \right\},
\end{aligned} \tag{3}$$

where $x_0^\delta := x_0$ and $\xi_0^\delta := \xi_0$. This method builds on various efforts to extend the classical Landweber iteration ([5, 16, 19, 22, 23, 24, 26, 29]) and can be interpreted as a mirror descent method ([18, 21]). Each iteration involves two steps: the first performs a gradient step in the dual space X^* of X, followed by the second step, which uses the mirror map defined by $\mathcal{R}$ to pull the element back to the original primal space X. Under the tangential cone condition

$$\|F(\bar{x}) - F(x) - F'(x)(\bar{x} - x)\| \leq \eta\|F(\bar{x}) - F(x)\|$$

around a sought solution with $\eta \in [0, 1)$, the convergence has been proved when the iteration is terminated by *a priori* or *a posteriori* stopping rules, see [13, 16, 19, 26] for instance.

To assess how fast the iteration converges towards the desired solution, understanding the convergence rates is crucial and has significant theoretical interest. For ill-posed inverse problems, deriving such rates typically relies on assuming appropriate source conditions are satisfied by the sought solution. In Hilbert spaces, the linear Landweber iteration is well-studied, where it is known to be an order-optimal regularization method ([9]). For the more general method (3) in Hilbert spaces with a nonlinear forward operator F and a regularization functional $\mathcal{R}(x) = \frac{1}{2}\|x\|^2$, convergence rate analyses have been undertaken in [13, 22, 23]. Specifically, in the seminal work by [13], the nonlinear Landweber iteration

$$x_{k+1}^\delta = x_k^\delta - \gamma F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \tag{4}$$

in Hilbert spaces with a constant step size has been considered. Under the range invariance condition

$$F'(x) = Q_x F'(x^\dagger) \quad \text{and} \quad \|I - Q_x\| \le \kappa_0 \|x - x^\dagger\| \tag{5}$$

in a neighborhood of the sought solution $x^\dagger$, it has been shown that, if the method is terminated by the discrepancy principle

$$\|F(x_{k_\delta}^\delta) - y^\delta\| \le \tau\delta < \|F(x_k^\delta) - y^\delta\|, \quad 0 \le k < k_\delta \tag{6}$$

with suitably large $\tau > 1$, then the convergence rate

$$\|x_{k_\delta}^\delta - x^\dagger\| = O(\delta^{2\mu/(1+2\mu)})$$

holds, provided the source condition

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^\mu \omega$$

is satisfied for some $0 < \mu \le 1/2$ with sufficiently small $\|\omega\|$.

Deriving convergence rates for the method (3) in its full generality is a very challenging task. The main difficulties arise from several aspects: the possible nonlinearity of the forward operator F, the non-quadratic nature of the regularization functional $\mathcal{R}$, the non-Hilbertian structure of the underlying space X, and the possible use of variable step-sizes γ_k^δ. When F is a bounded linear operator and the step-size is constant, the convergence rate analysis on the corresponding method has been rigorously studied in [17]. In that work, through the interpretation of the method as a dual gradient method, the convergence rates have been derived successfully under varational source conditions ([14]) when the iteration is terminated by either an *a priori* stopping rule or the discrepancy principle. The approach developed in [17] heavily relies on the linearity of the forward operator and the constancy of the step-size. However, it is unclear how to extend the strategy presented in [17] to address the equation (1) when the forward operator is nonlinear or when variable step sizes are employed.

In this paper, we will develop entirely novel ideas to tackle the challenge of deriving convergence rate of the method (3) under the benchmark source condition

$$\xi^\dagger := \xi_0 + F'(x^\dagger)^* \lambda^\dagger \in \partial\mathcal{R}(x^\dagger) \text{ for some } \lambda^\dagger \in Y \tag{7}$$

on the sought solution $x^\dagger$, where $\partial\mathcal{R}$ denotes the subdifferential of $\mathcal{R}$. Note that, under the tangential cone condition, (2) can be equivalently stated as

$$\min\left\{ D_{\mathcal{R}}^{\xi_0}(x, x_0) : F'(x^\dagger)(x - x^\dagger) = 0 \right\}.$$

The Karush-Kuhn-Tucker (KKT) condition for this problem is exactly (7). In the context of ill-posed problems in infinite-dimensional spaces, the mere existence of a solution does not guarantee the satisfaction of the KKT condition. Therefore, (7) is commonly referred to as a source condition. This source condition has been extensively utilized in deriving convergence rates for regularization methods of ill-posed inverse problems ([6, 14, 17]). Under the range invariance condition (5) on F and the source condition (7) on $x^\dagger$, we will show that the convergence rate

$$\|x_{k_\delta}^\delta - x^\dagger\| = O(\delta^{1/2})$$

holds if k_δ is chosen by a suitable *a priori* stopping rule or the discrepancy principle (6). Our results hold for any strongly convex regularization functional $\mathcal{R}$ with great flexibility on step-sizes by allowing the use of variable ones. Thus constant and adaptive choices of

step-sizes are covered by our results. The key point for deriving our convergence rate is the observation that, under the condition (5), $\xi_k^\delta - \xi_0 \in \mathrm{Ran}(F'(x^\dagger)^*)$ for every k. This motivates us to reformulate the method (3) in an equivalent manner which defines an auxiliary sequence $\{\lambda_k^\delta\}$ such that $\xi_k^\delta = \xi_0 + F'(x^\dagger)^* \lambda_k^\delta$ for all k. Based on this sequence $\{\lambda_k^\delta\}$, the source condition (7), and the sequence $\{x_k^\delta\}$, we can construct a sequence of quantities which obey a nice recursive relation. Through the investigation of these quantities leads us to obtain the desired convergence rate under an *a priori* stopping rule. Based on this *a priori* result, after a careful comparison we obtain the convergence rate when the method is terminated by the discrepancy principle. Furthermore, we extend the idea to deal with the stochastic mirror descent method, which is a stochastic version of the method (3) for solving nonlinear ill-posed systems; we consider the exact data case and obtain an almost sure convergence rate result under a benchmark source condition.

This paper is organized as follows. In Section 2 we collect some basic facts of convex analysis in Banach spaces. In Section 3 we devote to deriving convergence rates of the Landweber-type method (3) when terminated by either *a priori* stopping rules or the discrepancy principle. In Section 4 we extend the idea to derive convergence rate for the stochastic mirror descent method for nonlinear ill-posed systems with exact data. Finally, in Section 5 we provide some numerical results to validate the theoretical results on convergence rates.

2. Preliminaries

In this section, we collect some basic facts on convex analysis in Banach spaces; for more details one may refer to [31].

Throughout the paper, the same notation $\langle \cdot, \cdot \rangle$ will be used to denote either the inner product in a Hilbert space or the duality pairing in a Banach space. Let X be a Banach space with norm $\|\cdot\|$, we use X^* to denote its dual space. For a convex function $f : X \to (-\infty, \infty]$, the set

$$\mathrm{dom}(f) := \{x \in X : f(x) < \infty\}$$

is called the effective domain of f. If $\mathrm{dom}(f) \neq \emptyset$, f is called proper. The subdifferential of f is the set-valued mapping $\partial f : X \rightrightarrows X^*$ defined by

$$\partial f(x) := \{\xi \in X^* : f(\bar{x}) - f(x) - \langle \xi, \bar{x} - x \rangle \geq 0 \text{ for all } \bar{x} \in X\}$$

for each $x \in X$. Note that $\partial f(x)$ could be empty for some x. The domain and graph of ∂f are defined respectively as

$$\mathrm{dom}(\partial f) := \{x \in \mathrm{dom}(f) : \partial f(x) \neq \emptyset\}$$

and

$$\mathrm{graph}(\partial f) := \{(x, \xi) \in X \times X^* : x \in \mathrm{dom}(\partial f) \text{ and } \xi \in \partial f(x)\}.$$

Given $x \in \mathrm{dom}(f)$, an element $\xi \in \partial f(x)$ is called a subgradient of f at x. The Bregman distance induced by f at x in the direction $\xi \in \partial f(x)$ is defined by

$$D_f^\xi(\bar{x}, x) := f(\bar{x}) - f(x) - \langle \xi, \bar{x} - x \rangle, \quad \forall \bar{x} \in X$$

which is always nonnegative and satisfies the identity

$$D_f^{\xi_2}(x, x_2) - D_f^{\xi_1}(x, x_1) = D_f^{\xi_2}(x_1, x_2) + \langle \xi_2 - \xi_1, x_1 - x \rangle \tag{1}$$

for all $x \in \mathrm{dom}(f)$ and $(x_1, \xi_1), (x_2, \xi_2) \in \mathrm{graph}(\partial f)$.

For a proper function $f : X \to (-\infty, \infty]$, its convex conjugate is defined by

$$f^*(\xi) := \sup_{x \in X} \{\langle \xi, x \rangle - f(x)\}, \quad \xi \in X^*$$

which is a convex function taking values in $(-\infty, \infty]$. If $f : X \to (-\infty, \infty]$ is proper, lower semi-continuous and convex, f^* is also proper and

$$\xi \in \partial f(x) \iff x \in \partial f^*(\xi) \iff f(x) + f^*(\xi) = \langle \xi, x \rangle. \tag{2}$$

A proper function $f : X \to (-\infty, \infty]$ is called strongly convex if there exists a constant $\sigma > 0$ such that

$$f(t\bar{x} + (1-t)x) + \sigma t(1-t)\|\bar{x} - x\|^2 \leq tf(\bar{x}) + (1-t)f(x) \tag{3}$$

for all $\bar{x}, x \in \text{dom}(f)$ and $t \in [0, 1]$. It is easy to see that if $f : X \to (-\infty, \infty]$ is strongly convex in the sense of (3), then

$$D_f^\xi(\bar{x}, x) \geq \sigma\|x - \bar{x}\|^2 \tag{4}$$

for all $\bar{x} \in \text{dom}(f)$, $x \in \text{dom}(\partial f)$ and $\xi \in \partial f(x)$. Furthermore, for a proper, lower semi-continuous, strongly convex function $f : X \to (-\infty, \infty]$ satisfying (3), it is known from [31, Corollary 3.5.11] that $\text{dom}(f^*) = X^*$, f^* is Fréchet differentiable and its gradient ∇f^* maps X^* to X with

$$\|\nabla f^*(\xi) - \nabla f^*(\eta)\| \leq \frac{\|\xi - \eta\|}{2\sigma} \tag{5}$$

for all $\xi, \eta \in X^*$. Consequently, it follows from (2) and (5) that

$$D_f^\xi(\bar{x}, x) = f^*(\xi) - f^*(\bar{\xi}) - \langle \xi - \bar{\xi}, \nabla f^*(\bar{\xi}) \rangle \leq \frac{1}{4\sigma}\|\xi - \bar{\xi}\|^2 \tag{6}$$

for any $(x, \xi), (\bar{x}, \bar{\xi}) \in \text{graph}(\partial f)$.

3. Convergence rates of Landweber-type methods

The convergence of general Landweber-type methods in Banach spaces, including (3) as a specific case, has been studied in [16, 19], particularly when the iteration is terminated using the discrepancy principle (6). For the method (3), the convergence analysis relies on the following standard conditions on $\mathcal{R}$ and F.

Assumption 1. $\mathcal{R} : X \to (-\infty, \infty]$ *is a proper, lower semi-continuous, strongly convex function in the sense that there is a constant $\sigma > 0$ such that*

$$\mathcal{R}(t\bar{x} + (1-t)x) + \sigma t(1-t)\|\bar{x} - x\|^2 \leq t\mathcal{R}(\bar{x}) + (1-t)\mathcal{R}(x)$$

for all $\bar{x}, x \in \text{dom}(\mathcal{R})$ and $0 \leq t \leq 1$.

Assumption 2.   (i) *X is a Banach space and Y is a Hilbert space.*
  (ii) *There exists $\rho > 0$ such that $B_{2\rho}(x_0) \subset \text{dom}(F)$ and (2) has a solution $x^\dagger$ such that $D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) \leq \frac{1}{2}\sigma\rho^2$.*
  (iii) *There is a constant L such that $\|F'(x)\| \leq L$ for all $x \in B_{2\rho}(x_0)$ and there exists $0 \leq \eta < 1$ such that*

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leq \eta\|F(\tilde{x}) - F(x)\|$$

for all $\tilde{x}, x \in B_{2\rho}(x_0)$.

Under Assumption 1 and Assumption 2. it has been shown in [19, Lemma 3.2] that $x^\dagger$ is the unique solution of (2), and by using (4) we can conclude that $\|x^\dagger - x_0\|^2 \le \rho^2/2 < \rho^2$, i.e. $x^\dagger \in B_\rho(x_0)$. Furthermore, we have the following convergence result.

Theorem 3.1. *Let Assumption 1 and Assumption 2 hold. Consider the Landweber-type method (3). Let $\tau > (1+\eta)/(1-\eta)$ be a given number and let $\{\gamma_k^\delta\}$ be chosen by one of the following rules:*

(i) $\gamma_k^\delta = \gamma/L^2$ *with* $0 < \gamma < 4\sigma(1 - \eta - (1+\eta)/\tau)$;

(ii) $\gamma_k^\delta = \min\left\{\frac{\gamma\|F(x_k^\delta) - y^\delta\|^2}{\|F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)\|^2}, \bar\gamma\right\}$ *for some* $\bar\gamma > 0$ *and* $0 < \gamma < 4\sigma(1 - \eta - (1+\eta)/\tau)$;

(iii) $\gamma_k^\delta = \min\left\{\frac{\gamma((1-\eta)\|F(x_k^\delta) - y^\delta\| - (1+\eta)\delta)\|F(x_k^\delta) - y^\delta\|}{\|F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)\|^2}, \bar\gamma\right\}$ *for some* $\bar\gamma > 0$ *and* $0 < \gamma < 4\sigma$.

Then the discrepancy principle (6) outputs a finite integer k_δ and there is a solution x^* of (1) in $B_{2\rho}(x_0) \cap \mathrm{dom}(\mathcal{R})$ such that

$$\lim_{\delta \to 0} \|x_{k_\delta}^\delta - x^*\| = 0 \quad \text{and} \quad \lim_{\delta \to 0} D_{\mathcal{R}}^{\xi_{k_\delta}^\delta}(x^*, x_{k_\delta}^\delta) = 0.$$

If in addition, $\mathrm{Ran}(F'(x)^*) \subset \overline{\mathrm{Ran}(F'(x^\dagger)^*)}$ *for all* $x \in B_{2\rho}(x_0)$, *then* $x^* = x^\dagger$.

Theorem 3.1 with γ_k^δ chosen by either (i) or (ii) has been proved in [16, 19], and the similar arguments there can be applied straightforwardly to establish the convergence result when γ_k^δ is chosen by (iii). In the method (3), the regularization functional $\mathcal{R}$ can be chosen in various ways to detect the features of sought solutions.

Example 3.2. Here are some choices of the regularization functional $\mathcal{R}$ that are useful in detecting features of sought solutions in applications.

(a) When X is a Hilbert space and the sought solution lies in a closed convex set C, we may take

$$\mathcal{R}(x) := \frac{1}{2}\|x\|^2 + \iota_C(x),$$

where ι_C denote the indicator function of C, i.e. $\iota_C(x) = 0$ if $x \in C$ and $\iota_C(x) = \infty$ otherwise. It is clear that this $\mathcal{R}$ satisfies Assumption 1 with $\sigma = 1/2$.

(b) When the sought solution lies in $L^2(\Omega)$ with sparsity feature, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, we may take

$$\mathcal{R}(x) := \frac{1}{2}\int_\Omega |x(\omega)|^2 d\omega + \beta \int_\Omega |x(\omega)| d\omega, \quad x \in L^2(\Omega),$$

where $\beta > 0$ is a large number. Clearly, this $\mathcal{R}$ satisfies Assumption 1 with $\sigma = 1/2$.

(c) When the sought solution is piece-wise constant on some bounded domain $\Omega \subset \mathbb{R}^d$, we may use

$$\mathcal{R}(x) := \frac{1}{2}\int_\Omega |x(\omega)|^2 d\omega + \beta \int_\Omega |Dx|$$

for some large number $\beta > 0$, where $\int_\Omega |Dx|$ denotes the total variation of x over Ω defined by ([1])

$$\int_\Omega |Dx| := \sup\left\{\int_\Omega x \,\mathrm{div}\mathbf{u}\,dx : \mathbf{u} \in C_0^1(\Omega, \mathbb{R}^d) \text{ and } \|\mathbf{u}\|_{L^\infty(\Omega)} \le 1\right\}.$$

It is easy to show that this $\mathcal{R}$ satisfies Assumption 1 with $\sigma = 1/2$.

(d) When the sought solution $x^\dagger$ is a probability density function over a bounded domain $\Omega \subset \mathbb{R}^d$, i.e. $x^\dagger \geq 0$ a.e. on Ω and $\int_\Omega x = 1$, we may take

$$\mathcal{R}(x) := f(x) + \iota_\Delta(x),$$

where ι_Δ denotes the indicator function of

$$\Delta := \left\{ x \in L^1_+(\Omega) : \int_\Omega x^\dagger = 1 \right\}$$

and

$$f(x) = \begin{cases} \int_\Omega x \log x, & \text{if } x \in L^1_+(\Omega) \text{ and } x \log x \in L^1(\Omega), \\ +\infty, & \text{otherwise} \end{cases}$$

is the negative of the Boltzmann-Shannon entropy. Here $L^1_+(\Omega) := \{x \in L^1(\Omega) : x \geq 0 \text{ a.e. on } \Omega\}$. It is known that $\mathcal{R}$ satisfies Assumption 1 with $\sigma = 1/2$; see [4, 7, 10, 17].

The performance of the method (3) has been demonstrated in [16, 19] through various numerical experiments. These results indicate that the method (3) not only captures the features of the sought solution with a suitable choice of $\mathcal{R}$, but also enjoys a certain convergence speed when the sought solutions exhibit desirable properties. This raises the natural question of whether it is possible to establish convergence rates for the method (3) when the sought solutions satisfy specific source conditions. In this section, we will derive the convergence rate for the method (3) under the assumption that the sought solution satisfies the benchmark source condition (7), provided that F meets the following additional condition.

Assumption 3. *There exists $\kappa_0 \geq 0$ such that for any $x \in B_{2\rho}(x_0)$ there is a bounded linear operator $Q_x : Y \to Y$ such that*

$$F'(x) = Q_x F'(x^\dagger) \quad \text{and} \quad \|I - Q_x\| \leq \kappa_0 \|x - x^\dagger\|.$$

Assumption 3 is actually stronger than condition (iii) in Assumption 2. Indeed, using similar arguments as in [13, 22] we can conclude that, if $8\kappa_0\rho < 1$, then

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leq \eta \|F(\tilde{x}) - F(x)\|$$

for all $\tilde{x}, x \in B_{2\rho}(x_0)$, where $\eta := 5\kappa_0\rho/(1 - 8\kappa_0\rho)$. This implies that condition (iii) in Assumption 2 holds with a small η, provided that $\kappa_0\rho$ is sufficiently small.

It should be noted that when F is a bounded linear operator, Assumption 3 holds automatically with $\kappa_0 = 0$. Moreover, [13, 27] provide examples of nonlinear ill-posed inverse problems that satisfy Assumption 3, including those arising from nonlinear integral equations of the first kind and parameter identification in partial differential equations. Below we will provide one more example that satisfies Assumption 3.

Example 3.3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the Neumann boundary value problem of the semi-linear elliptic equation

$$-\triangle y + g(y) = x \quad \text{in } \Omega, \qquad \partial_\nu y = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\partial_\nu y$ denotes the normal derivative of y in the direction of the unit outward normal ν to $\partial\Omega$ and $g(y)$ is a given function defined on $\mathbb{R}$ satisfying the following properties:

- $g \in C^1(\mathbb{R})$ and there is a constant $\mu_0 > 0$ such that $g'(y) \geq \mu_0$ for all $y \in \mathbb{R}$.

- g' is locally Lipschitz continuous, i.e. for any $M > 0$ there is $L_M \geq 0$ such that

$$|g'(y_1) - g'(y_2)| \leq L_M |y_1 - y_2|$$

  for all $y_1, y_2 \in \mathbb{R}$ satisfying $|y_1|, |y_2| \leq M$.

Let $r > d/2$ be a fixed number. We consider the inverse problem of determining the source term $x \in L^r(\Omega)$ from an $L^2(\Omega)$-measurement of y. According to [30, Theorem 4.7 & Theorem 4.16], for each $x \in L^r(\Omega)$ problem (1) has a unique weak solution $y_x \in H^1(\Omega) \cap C(\overline{\Omega})$ and for any $\tilde{x}, x \in L^r(\Omega)$ there holds

$$\|y_{\tilde{x}} - y_x\|_{H^1(\Omega)} + \|y_{\tilde{x}} - y_x\|_{C(\overline{\Omega})} \leq C\|\tilde{x} - x\|_{L^r(\Omega)}, \tag{2}$$

where C denotes a generic constant depending only on μ_0 and Ω. Since $H^1(\Omega) \hookrightarrow L^2(\Omega)$, it makes sense to define the operator $F : L^r(\Omega) \to L^2(\Omega)$ by $F(x) := y_x$ for any $x \in L^r(\Omega)$ and thus our inverse problem reduces to the form (1).

According to [30, Theorem 4.17], this F is Fréchet differentiable, and for any $x \in L^r(\Omega)$, its Fréchet derivative $F'(x)$ is an operator from $L^r(\Omega)$ to $H^1(\Omega) \subset L^2(\Omega)$. For each $h \in L^r(\Omega)$, $v := F'(x)h$ is the unique weak solution of

$$-\triangle v + g'(y_x)v = h \quad \text{in } \Omega, \qquad \partial_\nu v = 0 \quad \text{on } \partial\Omega.$$

Next, we show that this operator F satisfies Assumption 3. To see this, we consider the linear subspace

$$S := \{u \in H^1(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$$

of $H^1(\Omega)$. Let $x_0 \in L^r(\Omega)$ be fixed and let $\rho > 0$ be a fixed number. For any $\tilde{x}, x \in B_\rho(x_0) := \{z \in L^r(\Omega) : \|z - x_0\|_{L^r(\Omega)} < \rho\}$, we define the linear operator $Q_{\tilde{x},x} : S \subset H^1(\Omega) \to H^1(\Omega) \subset L^2(\Omega)$, where, for each $v \in S$, $w := Q_{\tilde{x},x}v \in H^1(\Omega)$ is the unique weak solution of the linear elliptic problem

$$-\triangle w + g'(y_{\tilde{x}})w = -\triangle v + g'(y_x)v \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Since $\partial_\nu v = 0$ on $\partial\Omega$, we can write

$$\begin{cases} -\triangle(w-v) + g'(y_{\tilde{x}})(w-v) = (g'(y_x) - g'(y_{\tilde{x}}))v & \text{in } \Omega, \\ \partial_\nu(w-v) = 0 & \text{on } \partial\Omega. \end{cases}$$

By the theory of linear elliptic equations, we conclude that

$$\|w - v\|_{H^1(\Omega)} \leq C\|(g'(y_x) - g'(y_{\tilde{x}}))v\|_{L^2(\Omega)} \leq \|g'(y_x) - g'(y_{\tilde{x}})\|_{L^\infty(\Omega)}\|v\|_{L^2(\Omega)}.$$

By virtue of (2) and the local Lipschitz continuity of g', there exists a constant C_ρ depending on Ω, μ_0, x_0 and ρ such that

$$\|w - v\|_{L^2(\Omega)} \leq \|w - v\|_{H^1(\Omega)} \leq C_\rho\|\tilde{x} - x\|_{L^r(\Omega)}\|v\|_{L^2(\Omega)}, \quad \forall v \in S.$$

Since $C_0^\infty(\Omega) \subset S$, S is dense in $L^2(\Omega)$. Thus, the above inequality implies that $Q_{\tilde{x},x}$ can be extended to a bounded linear operator from $L^2(\Omega)$ to itself and

$$\|I - Q_{\tilde{x},x}\|_{L^2(\Omega) \to L^2(\Omega)} \leq C_\rho\|\tilde{x} - x\|_{L^r(\Omega)}. \tag{4}$$

Note that, for any $h \in L^r(\Omega)$ we have $v := F'(x)h \in S$, and for the function w defined by (3) we have $w = F'(\tilde{x})h$. Consequently

$$F'(\tilde{x})h = w = Q_{\tilde{x},x}v = Q_{\tilde{x},x}F'(x)h, \quad \forall h \in L^r(\Omega)$$

which shows that $F'(\tilde{x}) = Q_{\tilde{x},x}F'(x)$. This, together with (4), shows that F satisfies Assumption 3.

3.1. **Some key estimates.** Based on Assumptions 1–3, our aim is to derive convergence rates of the Landweber-type method (3) under the benchmark source condition (7) on the sought solution $x^\dagger$, when the method is terminated by either an *a priori* stopping rule or the discrepancy principle.

In this subsection we will establish some key estimates that will be used in the subsequent analysis. Note from the definition of x_k^δ in (3) that $\xi_k^\delta \in \partial\mathcal{R}(x_k^\delta)$. We first show that, up to a preassigned stopping index, the iterates always stay in $B_{2\rho}(x_0)$ and the Bregman distances $\{D_{\mathcal{R}}^{\xi_k^\delta}(x^\dagger, x_k^\delta)\}$ obey a useful recursive inequality.

Lemma 3.4. *Let Assumption 1 and Assumption 2 hold. Suppose γ_k^δ is chosen such that*

$$\underline{\gamma} \le \gamma_k^\delta \le \min\left\{ \frac{\gamma\|F(x_k^\delta) - y^\delta\|^2}{\|F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)\|^2}, \bar{\gamma} \right\} \tag{5}$$

for some positive constants $\underline{\gamma}$, γ and $\bar{\gamma}$ with

$$c_0 := \frac{1}{2}\left(1 - \eta - \frac{\gamma}{4\sigma}\right) > 0. \tag{6}$$

Let $\hat{k}_\delta$ be an integer such that $c_1\delta^2\hat{k}_\delta < \frac{1}{2}\sigma\rho^2$, where $c_1 := (1+\eta)^2\bar{\gamma}/(4c_0)$. Then $x_k^\delta \in B_{2\rho}(x_0)$ for $0 \le k \le \hat{k}_\delta$ and

$$\Delta_{k+1}^\delta - \Delta_k^\delta \le -c_0\gamma_k^\delta\|F(x_k^\delta) - y^\delta\|^2 + c_1\delta^2 \tag{7}$$

for all integers $0 \le k < \hat{k}_\delta$, where $\Delta_k^\delta := D_{\mathcal{R}}^{\xi_k^\delta}(x^\dagger, x_k^\delta)$.

Proof. We first show by induction that

$$x_k^\delta \in B_{2\rho}(x_0) \quad \text{and} \quad \Delta_k^\delta \le D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) + c_1\delta^2 k \tag{8}$$

for all integers $0 \le k \le \hat{k}_\delta$. It is trivial for $k = 0$ as $\xi_0^\delta = \xi_0$ and $x_0^\delta = x_0$. Next we assume that (8) holds for all $0 \le k \le l$ for some $l < \hat{k}_\delta$ and show that (8) still holds for $k = l+1$. By the second equation in (3), we have $\xi_l^\delta \in \partial\mathcal{R}(x_l^\delta)$. Thus we may use (1) and (6) to obtain

$$\Delta_{l+1}^\delta - \Delta_l^\delta = D_{\mathcal{R}}^{\xi_{l+1}}(x_l, x_{l+1}) + \left\langle \xi_{l+1}^\delta - \xi_l^\delta, x_l^\delta - x^\dagger \right\rangle$$
$$\le \frac{1}{4\sigma}\left\|\xi_{l+1}^\delta - \xi_l^\delta\right\|^2 + \left\langle \xi_{l+1}^\delta - \xi_l^\delta, x_l^\delta - x^\dagger \right\rangle.$$

By virtue of the first equation in (3), we further obtain

$$\Delta_{l+1}^\delta - \Delta_l^\delta \le \frac{1}{4\sigma}(\gamma_l^\delta)^2 \left\|F'(x_l^\delta)^*(F(x_l^\delta) - y^\delta)\right\|^2 - \gamma_l^\delta \left\langle F(x_l^\delta) - y^\delta, F'(x_l^\delta)(x_l^\delta - x^\dagger) \right\rangle.$$

By using $\|y^\delta - y\| \le \delta$ and (iii) of Assumption 2, we then have

$$\Delta_{l+1}^\delta - \Delta_l^\delta \le \frac{1}{4\sigma}(\gamma_l^\delta)^2 \left\|F'(x_l^\delta)^*(F(x_l^\delta) - y^\delta)\right\|^2 - \gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2$$
$$- \gamma_l^\delta \left\langle F(x_l^\delta) - y^\delta, y^\delta - y + y - F(x_l^\delta) - F'(x_l^\delta)(x^\dagger - x_l^\delta) \right\rangle$$
$$\le \frac{1}{4\sigma}(\gamma_l^\delta)^2 \left\|F'(x_l^\delta)^*(F(x_l^\delta) - y^\delta)\right\|^2 - \gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2$$
$$+ \gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\| \left((1+\eta)\delta + \eta\|F(x_l^\delta) - y^\delta\|\right). \tag{9}$$

In view of (5), we can obtain

$$\Delta_{l+1}^\delta - \Delta_l^\delta \le -\left(1 - \eta - \frac{\gamma}{4\sigma}\right)\gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2 + (1+\eta)\gamma_l^\delta \delta \left\|F(x_l^\delta) - y^\delta\right\|$$
$$= -2c_0\gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2 + (1+\eta)\gamma_l^\delta \delta \left\|F(x_l^\delta) - y^\delta\right\|.$$

Combining this with the inequality

$$(1+\eta)\delta \left\|F(x_l) - y^\delta\right\| \le c_0 \left\|F(x_l^\delta) - y^\delta\right\|^2 + \frac{(1+\eta)^2}{4c_0}\delta^2$$

shows that

$$\Delta_{l+1}^\delta - \Delta_l^\delta \le -c_0\gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2 + \frac{(1+\eta)^2}{4c_0}\gamma_l^\delta \delta^2$$
$$\le -c_0\gamma_l^\delta \left\|F(x_l^\delta) - y^\delta\right\|^2 + c_1\delta^2, \tag{10}$$

where for the last step we used $\gamma_l^\delta \le \bar\gamma$. By virtue of this inequality and the induction hypothesis, we have

$$\Delta_{l+1}^\delta \le \Delta_l^\delta + c_1\delta^2 \le D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) + c_1\delta^2(l+1) \le \frac{1}{2}\sigma\rho^2 + c_1\delta^2\hat{k}_\delta < \sigma\rho^2$$

which together with Assumption 1 and (4) implies that $\sigma\|x_{l+1}^\delta - x^\dagger\|^2 \le \sigma\rho^2$ and hence $\|x_{l+1}^\delta - x^\dagger\| \le \rho$. Since Assumption 1 and (ii) in Assumption 2 imply $\|x^\dagger - x_0\| \le \rho$, we thus have $\|x_{l+1}^\delta - x_0\| \le 2\rho$, i.e. $x_{l+1}^\delta \in B_{2\rho}(x_0)$. We therefore complete the proof of (8). As a direct consequence, we can see that (10) holds for all $0 \le l < \hat{k}_\delta$ which shows the desired result. $\qquad\square$

In the following we will focus on deriving convergence rates under the benchmark source condition (7). Let $A := F'(x^\dagger)$. The key idea is the observation that, under Assumption 3, $\xi_k^\delta - \xi_0 \in \operatorname{Ran}(A^*)$ for each integer k. This leads us to introducing an auxiliary sequence which plays a crucial role in our argument. To be more precise, by using Assumption 3 we can write the first equation in (3) as

$$\xi_{k+1}^\delta = \xi_k^\delta - \gamma_k^\delta A^* Q_{x_k^\delta}^* \left(F(x_k^\delta) - y^\delta\right).$$

Thus, if we define $(\lambda_k^\delta, \xi_k^\delta, x_k^\delta)$ by setting $\lambda_0^\delta = 0$ and

$$\begin{aligned}
\xi_k^\delta &= \xi_0 + A^*\lambda_k^\delta, \\
x_k^\delta &= \arg\min_{x\in X}\left\{\mathcal{R}(x) - \langle\xi_k^\delta, x\rangle\right\}, \\
\lambda_{k+1}^\delta &= \lambda_k^\delta - \gamma_k^\delta Q_{x_k^\delta}^* \left(F(x_k^\delta) - y^\delta\right),
\end{aligned} \tag{11}$$

then these $(\xi_k^\delta, x_k^\delta)$ are the same as the ones produced by (3). Note that the definition of λ_{k+1}^δ relies on $Q_{x_k^\delta}$ which requires the information of $x^\dagger$. The reformulation (11) of (3) is not for computational purpose, instead it will be used only for theoretical analysis and a proper use of the additional sequence $\{\lambda_k^\delta\}$ will enable us to derive the convergence rate under the benchmark source condition (7).

Since Y is a Hilbert space and $\{\lambda_k^\delta\}$ is a sequence in Y, we may use the polarization identity and the definition of λ_{k+1}^δ to obtain

$$
\begin{aligned}
&\left\|\lambda_{k+1}^\delta - \lambda^\dagger\right\|^2 - \left\|\lambda_k^\delta - \lambda^\dagger\right\|^2 \\
&= \left\|\lambda_{k+1}^\delta - \lambda_k^\delta\right\|^2 + 2\left\langle \lambda_{k+1}^\delta - \lambda_k^\delta, \lambda_k^\delta - \lambda^\dagger \right\rangle \\
&= (\gamma_k^\delta)^2 \left\| Q_{x_k^\delta}^*(F(x_k^\delta) - y^\delta) \right\|^2 - 2\gamma_k^\delta \left\langle Q_{x_k^\delta}^*(F(x_k^\delta) - y^\delta), \lambda_k^\delta - \lambda^\dagger \right\rangle \\
&= (\gamma_k^\delta)^2 \left\| Q_{x_k^\delta}^*(F(x_k^\delta) - y^\delta) \right\|^2 - 2\gamma_k^\delta \left\langle (Q_{x_k^\delta}^* - I)(F(x_k^\delta) - y^\delta), \lambda_k^\delta - \lambda^\dagger \right\rangle \\
&\quad - 2\gamma_k^\delta \left\langle F(x_k^\delta) - y - A(x_k^\delta - x^\dagger) + y - y^\delta, \lambda_k^\delta - \lambda^\dagger \right\rangle \\
&\quad - 2\gamma_k^\delta \left\langle A(x_k^\delta - x^\dagger), \lambda_k^\delta - \lambda^\dagger \right\rangle.
\end{aligned}
$$

By the condition on Q_x, we have

$$
\|I - Q_{x_k^\delta}^*\| \leq \kappa_0 \|x_k^\delta - x^\dagger\|.
$$

According to Lemma 3.4 and the strong convexity of $\mathcal{R}$, $\|x_k^\delta - x^\dagger\|$ is bounded for $0 \leq k \leq \hat{k}_\delta$ and thus we can find a constant $c_2 > 0$ independent of k and δ such that

$$
\|Q_{x_k^\delta}^*\|^2 \leq (1 + \kappa_0 \|x_k^\delta - x^\dagger\|)^2 \leq c_2, \quad \forall 0 \leq k \leq \hat{k}_\delta.
$$

Therefore

$$
\begin{aligned}
&\left\|\lambda_{k+1}^\delta - \lambda^\dagger\right\|^2 - \left\|\lambda_k^\delta - \lambda^\dagger\right\|^2 \\
&\leq c_2 (\gamma_k^\delta)^2 \left\| F(x_k^\delta) - y^\delta \right\|^2 + 2\kappa_0 \gamma_k^\delta \left\| x_k^\delta - x^\dagger \right\| \left\| F(x_k^\delta) - y^\delta \right\| \left\| \lambda_k^\delta - \lambda^\dagger \right\| \\
&\quad + 2\gamma_k^\delta \delta \left\| \lambda_k^\delta - \lambda^\dagger \right\| + 2\gamma_k^\delta \left\| F(x_k^\delta) - y - A(x_k^\delta - x^\dagger) \right\| \left\| \lambda_k^\delta - \lambda^\dagger \right\| \\
&\quad - 2\gamma_k^\delta \left\langle A^*\lambda_k^\delta - A^*\lambda^\dagger, x_k^\delta - x^\dagger \right\rangle.
\end{aligned}
\tag{12}
$$

By using the source condition (7), the relation between ξ_k^δ and λ_k^δ, and the strong convexity of $\mathcal{R}$, we have

$$
\begin{aligned}
\left\langle A^*\lambda_k^\delta - A^*\lambda^\dagger, x_k^\delta - x^\dagger \right\rangle &= \left\langle \xi_k^\delta - \xi^\dagger, x_k^\delta - x^\dagger \right\rangle \\
&= D_{\mathcal{R}}^{\xi_k^\delta}(x^\dagger, x_k^\delta) + D_{\mathcal{R}}^{\xi^\dagger}(x_k^\delta, x^\dagger) \\
&\geq \Delta_k^\delta + \sigma \|x_k^\delta - x^\dagger\|^2.
\end{aligned}
$$

By virtue of Assumption 3 and (iii) in Assumption 2 we have

$$
\begin{aligned}
\left\| F(x_k^\delta) - y - A(x_k^\delta - x^\dagger) \right\| &= \left\| \int_0^1 \left(F'(x^\dagger + t(x_k^\delta - x^\dagger)) - A \right)(x_k^\delta - x^\dagger) dt \right\| \\
&= \left\| \int_0^1 \left(Q_{x^\dagger + t(x_k^\delta - x^\dagger)} - I \right) A(x_k^\delta - x^\dagger) dt \right\| \\
&\leq \int_0^1 \left\| Q_{x^\dagger + t(x_k^\delta - x^\dagger)} - I \right\| \left\| A(x_k^\delta - x^\dagger) \right\| dt \\
&\leq \frac{1}{2} \kappa_0 \left\| x_k^\delta - x^\dagger \right\| \left\| A(x_k^\delta - x^\dagger) \right\|
\end{aligned}
$$

$$\leq \frac{1}{2}(1+\eta)\kappa_0 \left\| x_k^\delta - x^\dagger \right\| \left\| F(x_k^\delta) - y \right\|$$

$$\leq \frac{1}{2}(1+\eta)\kappa_0 \left\| x_k^\delta - x^\dagger \right\| \left(\left\| F(x_k^\delta) - y^\delta \right\| + \delta \right).$$

Combining these estimates with (12) gives

$$\left\| \lambda_{k+1}^\delta - \lambda^\dagger \right\|^2 - \left\| \lambda_k^\delta - \lambda^\dagger \right\|^2$$
$$\leq c_2(\gamma_k^\delta)^2 \left\| F(x_k^\delta) - y^\delta \right\|^2 + (3+\eta)\kappa_0 \gamma_k^\delta \left\| x_k^\delta - x^\dagger \right\| \left\| F(x_k^\delta) - y^\delta \right\| \left\| \lambda_k^\delta - \lambda^\dagger \right\|$$
$$+ (1+\eta)\kappa_0 \gamma_k^\delta \delta \left\| x_k^\delta - x^\dagger \right\| \left\| \lambda_k^\delta - \lambda^\dagger \right\| + 2\gamma_k^\delta \delta \left\| \lambda_k^\delta - \lambda^\dagger \right\|$$
$$- 2\sigma \gamma_k^\delta \left\| x_k^\delta - x^\dagger \right\|^2 - 2\gamma_k^\delta \Delta_k^\delta.$$

By the boundedness of $\|x_k^\delta - x^\dagger\|$, we can find a constant c_3 independent of k and δ such that

$$\|\lambda_{k+1}^\delta - \lambda^\dagger\|^2 - \|\lambda_k^\delta - \lambda^\dagger\|^2$$
$$\leq c_2(\gamma_k^\delta)^2 \|F(x_k^\delta) - y^\delta\|^2 + (3+\eta)\kappa_0 \gamma_k^\delta \|x_k^\delta - x^\dagger\| \|F(x_k^\delta) - y^\delta\| \|\lambda_k^\delta - \lambda^\dagger\|$$
$$+ c_3 \gamma_k^\delta \delta \|\lambda_k^\delta - \lambda^\dagger\| - 2\sigma \gamma_k^\delta \|x_k^\delta - x^\dagger\|^2 - 2\gamma_k^\delta \Delta_k^\delta$$

for all $0 \leq k \leq \hat{k}_\delta$. By virtue of the Young's inequality we have

$$(3+\eta)\kappa_0 \|x_k^\delta - x^\dagger\| \|F(x_k^\delta) - y^\delta\| \|\lambda_k^\delta - \lambda^\dagger\|$$

$$\leq 2\sigma \|x_k^\delta - x^\dagger\|^2 + \frac{(3+\eta)^2 \kappa_0^2}{8\sigma} \|F(x_k^\delta) - y^\delta\|^2 \|\lambda_k^\delta - \lambda^\dagger\|^2,$$

we consequently obtain

$$\|\lambda_{k+1}^\delta - \lambda^\dagger\|^2 \leq \left(1 + \frac{(3+\eta)^2 \kappa_0^2}{8\sigma} \gamma_k^\delta \|F(x_k^\delta) - y^\delta\|^2 \right) \|\lambda_k^\delta - \lambda^\dagger\|^2$$
$$+ c_2(\gamma_k^\delta)^2 \|F(x_k^\delta) - y^\delta\|^2 + c_3 \gamma_k^\delta \delta \|\lambda_k^\delta - \lambda^\dagger\| - 2\gamma_k^\delta \Delta_k^\delta.$$

With the help of the condition $\gamma_k^\delta \leq \bar{\gamma}$, we thus obtain the following result.

Lemma 3.5. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Let the step-size γ_k be chosen to satisfy (5) and (6), and let $\hat{k}_\delta$ be chosen as in Lemma 3.4. If $x^\dagger$ satisfies the source condition (7), then*

$$\|\lambda_{k+1}^\delta - \lambda^\dagger\|^2 \leq \left(1 + \frac{(3+\eta)^2 \kappa_0^2}{8\sigma} \gamma_k^\delta \|F(x_k^\delta) - y^\delta\|^2 \right) \|\lambda_k^\delta - \lambda^\dagger\|^2$$
$$+ c_2 \bar{\gamma} \gamma_k^\delta \|F(x_k^\delta) - y^\delta\|^2 + c_3 \bar{\gamma} \delta \|\lambda_k^\delta - \lambda^\dagger\| - 2\gamma_k^\delta \Delta_k^\delta \qquad (13)$$

for $0 \leq k < \hat{k}_\delta$, where c_2 and c_3 are positive constants independent of k and δ.

Based on Lemma 3.5, we will show under the source condition (7) that λ_k^δ is bounded for all $0 \leq k \leq \hat{k}_\delta$ if $\hat{k}_\delta$ is chosen as $\hat{k}_\delta := [c\delta^{-1}]$ for some positive constant $c > 0$, where, for any given number t, $[t]$ denotes the largest integer $\leq t$. To this end, we need the following elementary result ([18]).

Lemma 3.6. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of nonnegative numbers such that*

$$a_k^2 \leq b_k^2 + c \sum_{j=0}^{k-1} a_j, \quad k = 0, 1, \cdots,$$

where $c \geq 0$ is a constant. If $\{b_k\}$ is non-decreasing, then

$$a_k \leq b_k + ck, \quad k = 0, 1, \cdots.$$

Lemma 3.7. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Let the step-size γ_k be chosen to satisfy (5) and (6). If $x^\dagger$ satisfies the source condition (7) and if $\hat{k}_\delta$ is chosen as $\hat{k}_\delta = [c\delta^{-1}]$ for some positive constant c, then there exist $C > 0$ and $\bar{\delta} > 0$ such that*

$$\|\lambda_k^\delta - \lambda^\dagger\| \leq C, \quad \forall 0 \leq k \leq \hat{k}_\delta \text{ and } 0 < \delta \leq \bar{\delta}.$$

Proof. Note that $c_1 \delta^2 \hat{k}_\delta \leq c_1 c\delta$. Thus there is $\bar{\delta} > 0$ such that $c_1 \delta^2 \hat{k}_\delta \leq \frac{1}{2}\sigma\rho^2$ for all $0 < \delta \leq \bar{\delta}$. Consequently, the result in Lemma 3.5 holds for $0 \leq k \leq \hat{k}_\delta$ and $0 < \delta \leq \bar{\delta}$. We set

$$
\begin{aligned}
a_k &:= \|\lambda_k^\delta - \lambda^\dagger\|, \\
b_k &:= c_2 \bar{\gamma}\gamma_k^\delta \|F(x_k^\delta) - y^\delta\|^2, \\
\beta_k &:= \frac{(3+\eta)^2 \kappa_0^2}{8\sigma} \gamma_k^\delta \|F(x_k^\delta) - y^\delta\|^2.
\end{aligned}
$$

Then from (13) in Lemma 3.5 it follows that

$$a_{k+1}^2 \leq (1+\beta_k)a_k^2 + b_k + c_3\bar{\gamma}\delta a_k, \quad \forall 0 \leq k < \hat{k}_\delta.$$

By recursively using this inequality we have

$$a_k^2 \leq a_0^2 \prod_{l=0}^{k-1}(1+\beta_l) + \sum_{i=0}^{k-1}(b_i + c_3\bar{\gamma}\delta a_i)\prod_{l=i+1}^{k-1}(1+\beta_l).$$

Note that

$$
\begin{aligned}
\prod_{l=i+1}^{k-1}(1+\beta_l) &= \exp\left(\log \prod_{l=i+1}^{k-1}(1+\beta_l)\right) \\
&= \exp\left(\sum_{l=i+1}^{k-1}\log(1+\beta_l)\right) \\
&\leq \exp\left(\sum_{l=i+1}^{k-1}\beta_l\right),
\end{aligned}
$$

where for the last step we used the inequality $\log(1+t) \leq t$ for $t \geq 0$. Therefore

$$a_k^2 \leq a_0^2 \exp\left(\sum_{l=0}^{k-1}\beta_l\right) + \sum_{i=0}^{k-1}(b_i + c_3\bar{\gamma}\delta a_i)\exp\left(\sum_{l=i+1}^{k-1}\beta_l\right).$$

According to Lemma 3.4, we have

$$c_0 \sum_{l=0}^{k-1}\gamma_l^\delta \|F(x_l^\delta) - y^\delta\|^2 \leq \Delta_0^\delta + c_1\delta^2 k \leq D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) + c_1\delta^2 \hat{k}_\delta. \tag{14}$$

Thus, there is a constant $\tilde{C}$ such that

$$\exp\left(\sum_{l=0}^{k-1}\beta_l\right) = \exp\left(\frac{(3+\eta)^2\kappa_0^2}{8\sigma}\sum_{l=0}^{k-1}\gamma_l^\delta\|F(x_l) - y^\delta\|^2\right) \leq \tilde{C}$$

for all $0 \le k \le \hat{k}_\delta$. Consequently

$$a_k^2 \le \tilde{C}\left(a_0^2 + \sum_{i=0}^{k-1} b_i\right) + \tilde{C}c_3\bar{\gamma}\delta\sum_{i=0}^{k-1} a_i, \quad \forall 0 \le k \le \hat{k}_\delta.$$

By virtue of Lemma 3.6 then we can conclude that

$$\|\lambda_k^\delta - \lambda^\dagger\| = a_k \le \sqrt{\tilde{C}\left(a_0^2 + \sum_{i=0}^{k-1} b_i\right) + \tilde{C}c_3\bar{\gamma}\delta k}$$

$$= \sqrt{\tilde{C}\left(a_0^2 + c_2\bar{\gamma}\sum_{i=0}^{k-1} \gamma_i^\delta\|F(x_i^\delta) - y^\delta\|^2\right) + \tilde{C}c_3\bar{\gamma}\delta k}.$$

With the help of (14), we can find a positive constant C' such that

$$\|\lambda_k^\delta - \lambda^\dagger\| \le C'(1 + k\delta), \quad \forall 0 \le k \le \hat{k}_\delta \text{ and } 0 < \delta \le \bar{\delta}.$$

Since $\hat{k}_\delta = [c\delta^{-1}]$, we therefore complete the proof. $\square$

3.2. **Convergence rate under** *a priori* **stopping rule.** The following result gives the convergence rate of the method (3) under an *a prior* stopping rule when the source condition (7) holds and the step-size γ_k^δ is chosen properly.

Theorem 3.8. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Let the step-size γ_k^δ be chosen to satisfy (5) and (6). If $x^\dagger$ satisfies the source condition (7) and if $\hat{k}_\delta$ is chosen as $\hat{k}_\delta = [c\delta^{-1}]$ for some positive constant c, then there exist $C > 0$ and $\bar{\delta} > 0$ such that*

$$D_{\mathcal{R}}^{\xi_{\hat{k}_\delta}^\delta}(x^\dagger, x_{\hat{k}_\delta}^\delta) \le C\delta, \quad \forall 0 < \delta \le \bar{\delta}.$$

Consequently $\|x_{\hat{k}_\delta}^\delta - x^\dagger\| = O(\delta^{1/2})$ by the strong convexity of $\mathcal{R}$.

Proof. By virtue of Lemma 3.5 and Lemma 3.7, there is a positive constant C such that

$$\|\lambda_{k+1}^\delta - \lambda^\dagger\|^2 \le \|\lambda_k^\delta - \lambda^\dagger\|^2 + C\gamma_k^\delta\|F(x_k^\delta) - y^\delta\|^2 + C\delta - 2\gamma_k^\delta\Delta_k^\delta. \tag{15}$$

for all integers $0 \le k < \hat{k}_\delta$ and $0 < \delta \le \bar{\delta}$. Multiplying this equation by c_0/C, adding to the equation (7) in Lemma 3.4, and using $\gamma_k^\delta \ge \underline{\gamma}$, we have

$$\Delta_{k+1}^\delta + \frac{c_0}{C}\|\lambda_{k+1}^\delta - \lambda^\dagger\|^2 \le \Delta_k^\delta + \frac{c_0}{C}\|\lambda_k^\delta - \lambda^\dagger\|^2 + c_0\delta + c_1\delta^2 - c_4\Delta_k^\delta$$

for all $0 \le k < \hat{k}_\delta$, where $c_4 := 2c_0\underline{\gamma}/C$. Recursively using this inequality gives

$$\Delta_{\hat{k}_\delta}^\delta + \frac{c_0}{C}\left\|\lambda_{\hat{k}_\delta}^\delta - \lambda^\dagger\right\|^2 + c_4\sum_{k=0}^{\hat{k}_\delta-1}\Delta_k^\delta \le \Delta_0 + \frac{c_0}{C}\|\lambda_0 - \lambda^\dagger\|^2 + (c_0 + c_1\delta)\delta\hat{k}_\delta, \tag{16}$$

where $\Delta_0 := \Delta_0^\delta = D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0)$. According to Lemma 3.4 we have

$$\Delta_k^\delta \ge \Delta_{\hat{k}_\delta}^\delta - c_1(\hat{k}_\delta - k)\delta^2, \quad 0 \le k \le \hat{k}_\delta$$

and thus

$$\sum_{k=0}^{\hat{k}_\delta-1}\Delta_k^\delta \ge \hat{k}_\delta\Delta_{\hat{k}_\delta}^\delta - c_1\delta^2\sum_{k=0}^{\hat{k}_\delta-1}(\hat{k}_\delta - k) = \hat{k}_\delta\Delta_{\hat{k}_\delta}^\delta - \frac{1}{2}c_1\hat{k}_\delta(\hat{k}_\delta + 1)\delta^2.$$

Combining this with (16) gives

$$\min\{1, c_4\}(\hat{k}_\delta + 1)\Delta^\delta_{\hat{k}_\delta} \leq \Delta_0 + \frac{c_0}{C}\|\lambda^\dagger\|^2 + (c_0 + c_1\delta)\delta\hat{k}_\delta + \frac{1}{2}c_1 c_4 \hat{k}_\delta(\hat{k}_\delta + 1)\delta^2.$$

Therefore, there is a constant C' independent of δ such that

$$\Delta^\delta_{\hat{k}_\delta} \leq C'\left(\frac{1}{\hat{k}_\delta + 1} + \hat{k}_\delta\delta^2 + \delta\right), \quad \forall 0 < \delta \leq \bar{\delta}.$$

According to the choice of $\hat{k}_\delta$, we thus obtain $\Delta^\delta_{\hat{k}_\delta} \leq C\delta$. Finally, by the strong convexity of $\mathcal{R}$, we have $\|x^\delta_{\hat{k}_\delta} - x^\dagger\|^2 \leq \Delta^\delta_{\hat{k}_\delta}/\sigma = O(\delta)$. The proof is complete. $\qquad\square$

Remark 3.9. Theorem 3.8 requires the step-size γ^δ_k to satisfy (5) and (6). Actually there are various choices of γ^δ_k satisfying this requirement:

(i) If γ^δ_k is taken to be a constant step-size $\gamma^\delta_k = \gamma/L^2$ with $0 < \gamma < 4\sigma(1 - \eta)$, where L is the finite constant appeared in (iii) of Assumption 2, then (5) and (6) are satisfied with $\underline{\gamma} = \bar{\gamma} = \gamma/L^2$.

(ii) If γ^δ_k is chosen as

$$\gamma^\delta_k = \min\left\{\frac{\gamma\|F(x^\delta_k) - y^\delta\|^2}{\|F'(x^\delta_k)^*(F(x^\delta_k) - y^\delta)\|^2}, \bar{\gamma}\right\}$$

for some constants $\bar{\gamma} > 0$ and $0 < \gamma < 4\sigma(1 - \eta)$, then (5) and (6) are satisfied with $\underline{\gamma} = \min\{\gamma/L^2, \bar{\gamma}\}$. This choice of γ^δ_k is related to the minimal error method.

(iii) Let $\tau > (1 + \eta)/(1 - \eta)$ be a given number and set $r^\delta_k := F(x^\delta_k) - y^\delta$. If γ^δ_k is chosen as

$$\gamma^\delta_k = \begin{cases} \min\left\{\frac{\gamma_0((1-\eta)\|r^\delta_k\| - (1+\eta)\delta)\|r^\delta_k\|}{\|F'(x^\delta_k)^* r^\delta_k\|^2}, \bar{\gamma}\right\} & \text{if } \|r^\delta_k\| > \tau\delta, \\ \min\left\{\frac{\gamma_0(1-\eta)}{L^2}, \bar{\gamma}\right\}, & \text{otherwise} \end{cases}$$

for some constant $0 < \gamma_0 < 4\sigma$, then by noting that

$$\frac{\gamma_0((1-\eta)\|r^\delta_k\| - (1+\eta)\delta)\|r^\delta_k\|}{\|F'(x^\delta_k)^* r^\delta_k\|^2} \geq \gamma_0\left(1 - \eta - \frac{1+\eta}{\tau}\right)\frac{\|r^\delta_k\|^2}{\|F'(x^\delta_k)^* r^\delta_k\|^2}$$

$$\geq \frac{\gamma_0}{L^2}\left(1 - \eta - \frac{1+\eta}{\tau}\right)$$

when $\|r^\delta_k\| > \tau\delta$, we can see that (5) and (6) are satisfied with

$$\gamma = \gamma_0(1 - \eta) \quad \text{and} \quad \underline{\gamma} = \min\left\{\frac{\gamma_0}{L^2}\left(1 - \eta - \frac{1+\eta}{\tau}\right), \bar{\gamma}\right\}.$$

Therefore, the result in Theorem 3.8 holds for all the above choices of step-sizes, i.e. if the source condition (7) holds and $\hat{k}_\delta = [c\delta^{-1}]$, then $\Delta^\delta_{\hat{k}_\delta} = O(\delta)$ and $\|x^\delta_{k_\delta} - x^\dagger\| = O(\delta^{1/2})$.

Remark 3.10. When F is a bounded linear operator, $\mathcal{R}(x) = \frac{1}{2}\|x\|^2$ and γ^δ_k is chosen as

$$\gamma^\delta_k = \frac{\gamma\|Fx^\delta_k - y^\delta\|^2}{\|F^*(Fx^\delta_k - y^\delta)\|^2},$$

the corresponding method of (3) becomes the minimal error method for solving linear ill-posed problems. It has been shown in [8] that minimal error method is not a regularization

method when terminated by an *a priori* stopping rule. Our result shows that if this step-size is modified into the one as listed in (ii) of Remark 3.9, convergence rate can be derived under an *a priori* stopping rule if the source condition (7) holds. Our result does not contradict the one in [8] because we have imposed an upper bound on γ_k^δ. Placing such an upper bound can enhance stability in numerical computation.

3.3. **Convergence rate under the discrepancy principle.** Next we turn to derive the convergence rate of the Landweber-type method (3) under the source condition (7) when the iteration is terminated by the discrepancy principle (6) for some number $\tau > 1$. We need the following result.

Lemma 3.11. *Let Assumption 1 and Assumption 2 hold. Let the step-size γ_k^δ be chosen to satisfy (5). If $\tau > 1$ and $\gamma > 0$ are chosen such that*

$$\tau > \frac{1+\eta}{1-\eta} \quad and \quad \gamma < 4\sigma\left(1 - \eta - \frac{1+\eta}{\tau}\right), \tag{17}$$

then the discrepancy principle (6) outputs a finite integer k_δ, $x_k^\delta \in B_{2\rho}(x_0)$ for all $0 \le k \le k_\delta$, and

$$\Delta_{k+1}^\delta - \Delta_k^\delta \le -c_5\gamma_k^\delta\|F(x_k^\delta) - y^\delta\|^2$$

for all $0 \le k < k_\delta$, where $\Delta_k^\delta := D_{\mathcal{R}}^{\xi_k^\delta}(x^\dagger, x_k^\delta)$ and $c_5 := 1 - \eta - \frac{1+\eta}{\tau} - \frac{\gamma}{4\sigma} > 0$.

Proof. This can be done by essentially following the proof in [19, Lemma 3.4] or [16, Lemma 3.1].   □

Theorem 3.12. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Let the step-size γ_k^δ be chosen to satisfy (5). Assume that $\tau > 1$ and $\gamma > 0$ are chosen such that (17) holds and let k_δ be the integer determined by the discrepancy principle (6). If the source condition (7) holds, then there is a positive constant C such that*

$$\Delta_{k_\delta}^\delta \le C\delta \quad and \quad \|x_{k_\delta}^\delta - x^\dagger\| \le C\delta^{1/2} \tag{18}$$

for all $\delta > 0$, where $\Delta_{k_\delta}^\delta := D_{\mathcal{R}}^{\xi_{k_\delta}^\delta}(x^\dagger, x_{k_\delta}^\delta)$.

Proof. By the strong convexity of $\mathcal{R}$, the second estimate in (18) follows from the first one. Thus we only need to show the first estimate in (18).

Since $\tau > 1$ and $\gamma > 0$ satisfy (17), the step-size γ_k^δ satisfies (5) and (6). Thus, by taking $\hat{k}_\delta := [\delta^{-1}]$, we may use Theorem 3.8 to conclude that there exists constants $C' > 0$ and $\bar\delta > 0$ such that

$$D_{\mathcal{R}}^{\xi_{\hat{k}_\delta}^\delta}(x^\dagger, x_{\hat{k}_\delta}^\delta) \le C'\delta, \quad \forall 0 < \delta \le \bar\delta. \tag{19}$$

We will use this result to show the first estimate in (18). According to Lemma 3.11, $\Delta_{k_\delta}^\delta \le D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) < \infty$. Thus, we can find a positive constant C such that $\Delta_{k_\delta}^\delta \le C\delta$ for $\delta > \bar\delta$. In the following we assume $0 < \delta \le \bar\delta$.

We consider k_δ according to two cases. If $k_\delta \ge \hat{k}_\delta$, then we may use Lemma 3.11 and (19) to obtain

$$\Delta_{k_\delta}^\delta \le D_{\mathcal{R}}^{\xi_{\hat{k}_\delta}^\delta}(x^\dagger, x_{\hat{k}_\delta}^\delta) \le C'\delta.$$

It remains only to consider the case $k_\delta < \hat{k}_\delta$. According to Lemma 3.7, there is a constant C independent of δ such that $\|\lambda_{k_\delta}^\delta - \lambda^\dagger\| \le C$. Thus, by using $\xi_{k_\delta}^\delta = \xi_0 + A^* \lambda_{k_\delta}^\delta$ and the source condition (7), we can obtain

$$\begin{aligned}
\Delta_{k_\delta}^\delta &\le D_{\mathcal{R}}^{\xi_{k_\delta}^\delta}(x^\dagger, x_{k_\delta}^\delta) + D_{\mathcal{R}}^{\xi^\dagger}(x_{k_\delta}^\delta, x^\dagger) = \langle \xi_{k_\delta}^\delta - \xi^\dagger, x_{k_\delta}^\delta - x^\dagger \rangle \\
&= \langle A^* \lambda_{k_\delta}^\delta - A^* \lambda^\dagger, x_{k_\delta}^\delta - x^\dagger \rangle = \langle \lambda_{k_\delta}^\delta - \lambda^\dagger, A(x_{k_\delta}^\delta - x^\dagger) \rangle \\
&\le \|\lambda_{k_\delta}^\delta - \lambda^\dagger\| \|A(x_{k_\delta}^\delta - x^\dagger)\| \\
&\le C \|A(x_{k_\delta}^\delta - x^\dagger)\|.
\end{aligned}$$

By Assumption 2 (iii), $\|y^\delta - y\| \le \delta$, and $\|F(x_{k_\delta}^\delta) - y^\delta\| \le \tau\delta$, we can conclude that

$$\begin{aligned}
\Delta_{k_\delta}^\delta &\le C(1+\eta)\|F(x_{k_\delta}^\delta) - y\| \\
&\le C(1+\eta)\left(\delta + \|F(x_{k_\delta}^\delta) - y^\delta\|\right) \\
&\le C(1+\eta)(1+\tau)\delta.
\end{aligned}$$

The proof is therefore complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

By applying the above result to the method (3) with the step-size γ_k^δ chosen by those rules listed as (i) and (ii) in Remark 3.9, we immediately obtain the following result.

Corollary 3.13. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Consider the method (3) with γ_k^δ chosen by one of the following rules:*

(i) $\gamma_k^\delta = \gamma/L^2$ *for some $\gamma > 0$;*

(ii) $\gamma_k^\delta = \min\left\{ \frac{\gamma\|F(x_k^\delta) - y^\delta\|^2}{\|F'(x_k^\delta)^*(F(x_k^\delta) - y^\delta)\|^2}, \bar{\gamma} \right\}$ *for some $\gamma > 0$ and $\bar{\gamma} > 0$.*

Assume $\tau > 1$ and $\gamma > 0$ are chosen such that (17) holds and let k_δ be the integer determined by the discrepancy principle (6). If the source condition (7) holds, then there is a constant $C > 0$ such that

$$D_{\mathcal{R}}^{\xi_{k_\delta}^\delta}(x^\dagger, x_{k_\delta}^\delta) \le C\delta \quad \text{and} \quad \|x_{k_\delta}^\delta - x^\dagger\| \le C\delta^{1/2}$$

for all $\delta > 0$.

In Theorem 3.12 and Corollary 3.13, $\tau > 1$ and $\gamma > 0$ are required to satisfy (17). In applications we may need to take τ close to $(1+\eta)/(1-\eta)$ in order to get more accurate reconstruction results. As a direct consequence, the theoretical result only allows to use very small γ. This may lead to increase the required number of iterations and thus consume more computational time. For the step-size γ_k^δ given by (iii) in Remark 3.9, fortunately we have the following result which allows γ_0 to be a much larger number.

Theorem 3.14. *Let Assumption 1, Assumption 2 and Assumption 3 hold. Let $\tau > (1+\eta)/(1-\eta)$ and let the step-size γ_k^δ be chosen as*

$$\gamma_k^\delta = \min\left\{ \frac{\gamma_0((1-\eta)\|r_k^\delta\| - (1+\eta)\delta)\|r_k^\delta\|}{\|F'(x_k^\delta)^* r_k^\delta\|^2}, \bar{\gamma} \right\} \tag{20}$$

with $\bar{\gamma} > 0$ and $0 < \gamma_0 < 4\sigma$ whenever $\|r_k^\delta\| > \tau\delta$, where $r_k^\delta := F(x_k^\delta) - y^\delta$. Then the discrepancy principle (6) outputs a finite integer k_δ. If the source condition (7) holds, then there is a constant $C > 0$ such that

$$D_{\mathcal{R}}^{\xi_{k_\delta}^\delta}(x^\dagger, x_{k_\delta}^\delta) \le C\delta \quad \text{and} \quad \|x_{k_\delta}^\delta - x^\dagger\| \le C\delta^{1/2}$$

for all $\delta > 0$.

Proof. The key point is to show that the same result in Lemma 3.11 holds under the relaxed requirement on γ_0. To see this, we note that, when $x_k^\delta \in B_{2\rho}(x_0)$ with $\|r_k^\delta\| > \tau\delta$ and γ_k^δ is chosen by (20), we may follow the derivation of (9) to obtain

$$
\begin{aligned}
\Delta_{k+1}^\delta - \Delta_k^\delta &\le \frac{1}{4\sigma}(\gamma_k^\delta)^2 \|F'(x_k^\delta)^* r_k^\delta\|^2 - \left((1-\eta)\|r_k^\delta\| - (1+\eta)\delta\right)\gamma_k^\delta \|r_k^\delta\| \\
&\le -\left(1 - \frac{\gamma_0}{4\sigma}\right)\left((1-\eta)\|r_k^\delta\| - (1+\eta)\delta\right)\gamma_k^\delta \|r_k^\delta\| \\
&\le -\left(1 - \frac{\gamma_0}{4\sigma}\right)\left(1 - \eta - \frac{1+\eta}{\tau}\right)\gamma_k^\delta \|r_k^\delta\|^2.
\end{aligned}
$$

Recall that $0 < \gamma_0 < 4\sigma$ and $\tau > (1+\eta)/(1-\eta)$. Based on the above inequality, we may use a similar induction argument in the proof of Lemma 3.4 to show that, if $n \ge 0$ is an integer such that $\|r_k^\delta\| > \tau\delta$ for all $0 \le k \le n$, then $x_k^\delta \in B_{2\rho}(x_0)$ for all $0 \le k \le n$ and

$$
\Delta_{k+1}^\delta \le \Delta_k^\delta - c_5 \|r_k^\delta\|^2, \quad \forall 0 \le k < n, \tag{21}
$$

where

$$
c_5 := \left(1 - \frac{\gamma_0}{4\sigma}\right)\left(1 - \eta - \frac{1+\eta}{\tau}\right)\min\left\{\frac{\gamma_0}{L^2}\left(1 - \eta - \frac{1+\eta}{\tau}\right), \bar\gamma\right\} > 0.
$$

If the discrepancy principle (6) does not output a finite integer, then $\|r_k^\delta\| > \tau\delta$ for all integers $k \ge 0$. Consequently, it follows from (21) that

$$
c_5 \tau^2 \delta^2 (n+1) \le c_5 \sum_{k=0}^{n} \|r_k^\delta\|^2 \le \Delta_0^\delta = D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) < \infty
$$

for any integer $n \ge 0$. Letting $n \to \infty$ gives a contradiction. Therefore, the discrepancy principle (6) determines a finite integer k_δ with

$$
x_k^\delta \in B_{2\rho}(x_0) \text{ for } 0 \le k \le k_\delta \quad \text{and} \quad \Delta_{k+1}^\delta \le \Delta_k^\delta \text{ for } 0 \le k < k_\delta. \tag{22}
$$

Finally let us derive the given convergence rate. To this end, we define γ_k^δ as in (iii) of Remark 3.9 for $k > k_\delta$. For this choice of step-size, we know that the result in Theorem 3.8 holds, i.e. for $\hat k_\delta = [\delta^{-1}]$ there exist positive constants $C > 0$ and $\bar\delta > 0$ such that

$$
D_{\mathcal{R}}^{\xi_{\hat k_\delta}^\delta}(x^\dagger, x_{\hat k_\delta}^\delta) \le C\delta, \quad \forall 0 < \delta \le \bar\delta.
$$

Based on this inequality and (22), we may use the same argument in the proof of Theorem 3.12 to derive that $\Delta_{k_\delta}^\delta \le C\delta$ for all $\delta > 0$. The proof is complete. $\qquad\square$

Remark 3.15. (i) When X is a Hilbert space, $\mathcal{R}(x) = \frac{1}{2}\|x\|^2$ and $\gamma_k^\delta = $ constant, the method (3) becomes the classical Landweber iteration (4) studied in [13] in which the convergence rate result under the source condition (7) requires $\|\lambda^\dagger\|$ to be sufficiently small. It is interesting to see that our result does not require this smallness condition.

(ii) For the method (3) with a constant step-size and F being a bounded linear operator, convergence rates have been established in [17] under general variational source conditions by interpreting the method as a dual gradient method. Our work in this paper, restricted to this special case, can even present new results by allowing the use of variable step-sizes.

4. **Convergence rate of stochastic mirror descent method**

In this section we will extend the idea to deal with a stochastic version of the method (3) for solving nonlinear ill-posed system

$$F_i(x) = y_i, \quad i = 1, \cdots, N \tag{1}$$

with exact data, where, for each $i = 1, \cdots, N$, the operator $F_i : X \to Y_i$ is a Fréchet differentiable operator from a common Banach space X to a Hilbert space Y_i. We assume (1) has a solution and look for a solution whose feature is described by a proper, lower semi-continuous, strongly convex function $\mathcal{R} : X \to (-\infty, \infty]$. Let $(x_0, \xi_0) \in X \times X^*$ be an initial guess with $x_0 \in X$ and $\xi_0 \in \partial\mathcal{R}(x_0)$, we intend to find a solution $x^\dagger$ of (1) such that

$$D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) = \min\left\{ D_{\mathcal{R}}^{\xi_0}(x, x_0) : F_i(x) = y_i, \ i = 1, \cdots, N \right\}. \tag{2}$$

Let $Y_1 \times \cdots \times Y_N$ be the product space of $Y_1, \cdots, Y_N$ with the natural inner product inherited from those of Y_i. By introducing the operator $F : X \to Y_1 \times \cdots \times Y_N$ as

$$F(x) := (F_1(x), \cdots, F_N(x))$$

and $y := (y_1, \cdots, y_N)$, we can write (1) and (2) into the form (1) and (2) respectively. In this way, we may use the method (3) to solve the problem. Correspondingly, the updating formula from ξ_k to ξ_{k+1} takes the form

$$\xi_{k+1} = \xi_k - \gamma_k \sum_{i=1}^{N} F_i'(x_k)^*(F_i(x_k) - y_i)$$

which requires calculating the terms $F_i'(x_k)^*(F_i(x_k) - y_i)$ for all $i = 1, \cdots, N$ at each iteration step and hence it can be time-consuming when N is large. In order to make the method more efficient, we may take a random term from the sum in the above equation to update ξ_{k+1}. This leads to the following stochastic mirror descent method.

Algorithm 1. *Take $(x_0, \xi_0) \in X \times X^*$ with $x_0 \in \mathrm{dom}(F)$ and $\xi_0 \in \partial\mathcal{R}(x_0)$. Let $\{\gamma_k\}$ be a given sequence of positive numbers. For $k \geq 0$ do the following:*

(i) *Pick $i_k \in \{1, \cdots, N\}$ randomly via the uniform distribution;*
(ii) *Update $\xi_{k+1} = \xi_k - \gamma_k F_{i_k}'(x_k)^*(F_{i_k}(x_k) - y_{i_k})$;*
(iii) *Calculate $x_{k+1} = \arg\min_{x \in X} \{\mathcal{R}(x) - \langle \xi_{k+1}, x \rangle\}$.*

When every F_i is a bounded linear operator, Algorithm 1 has been proposed and analyzed in [18]; under Assumption 1 on $\mathcal{R}$ the convergence has been established and, when the sought solution satisfies a benchmark source condition, a convergence rate has been derived. In order to carry out the analysis of Algorithm 1 for nonlinear ill-posed system, beside Assumption 1 on $\mathcal{R}$, we also assume the following standard conditions on F_i for $i = 1, \cdots, N$.

Assumption 4. (i) *X is a Banach space, Y_i is a Hilbert space for each i.*
(ii) *There exists $\rho > 0$ such that $B_{2\rho}(x_0) \subset \mathrm{dom}(F)$ and (2) has a solution $x^\dagger$ such that $D_{\mathcal{R}}^{\xi_0}(x^\dagger, x_0) \leq \sigma\rho^2$.*
(iii) *There exists $0 \leq \eta < 1$ such that*

$$\|F_i(\tilde{x}) - F_i(x) - F_i'(x)(\tilde{x} - x)\| \leq \eta\|F_i(\tilde{x}) - F_i(x)\|$$

for all $\tilde{x}, x \in B_{2\rho}(x_0)$ and $i = 1, \cdots, N$.

(iv) *There is $\kappa_0 \geq 0$ such that for each $i \in \{1, \cdots, N\}$ and each $x \in B_{2\rho}(x_0)$, there exists a bounded linear operator $Q_x^i : Y_i \to Y_i$ such that*

$$F_i'(x) = Q_x^i F_i'(x^\dagger) \quad and \quad \|I - Q_x^i\| \leq \kappa_0 \|x - x^\dagger\|.$$

Note that when $N = 1$, Assumption 4 reduces to Assumption 2 and Assumption 3. Let

$$L := \max_{i=1,\cdots,N} \sup_{x \in B_{2\rho}(x_0)} \|F_i'(x)\|.$$

From (iv) of Assumption 4 it is easy to see $L < \infty$. From simplicity of exposition, throughout this section we set $A_i := F_i'(x^\dagger)$ and define $A : X \to Y_1 \times \cdots \times Y_N$ by

$$Ax := (A_1 x, \cdots, A_N x), \quad x \in X.$$

It is easy to see that the adjoint A^* of A is given by

$$A^* z = \sum_{i=1}^N A_i^* z_i, \quad \forall z := (z_1, \cdots, z_N) \in Y_1 \times \cdots \times Y_N.$$

Assumption 4 will enable us to derive a convergence rate of Algorithm 1 under the source condition (7) of the sought solution $x^\dagger$. i.e. there exists $\lambda^\dagger = (\lambda_1^\dagger, \cdots, \lambda_N^\dagger) \in Y_1 \times \cdots \times Y_N$ such that

$$\xi^\dagger := \xi_0 + A^* \lambda^\dagger = \xi_0 + \sum_{i=1}^N A_i^* \lambda_i^\dagger \in \partial \mathcal{R}(x^\dagger). \tag{3}$$

In the following result we first show that Algorithm 1 is well-defined and the Bregman distance $D_{\mathcal{R}}^{\xi_k}(x^\dagger, x_k)$ is monotonically decreasing.

Lemma 4.1. *Let Assumption 1 and (i)-(iii) of Assumption 4 hold with $L < \infty$. Consider Algorithm 1. Let $\bar{\gamma} := \sup_k \gamma_k$ and assume*

$$c_6 := 1 - \eta - \frac{\bar{\gamma} L^2}{4\sigma} > 0. \tag{4}$$

Then $x_k \in B_{2\rho}(x_0)$ and

$$\Delta_{k+1} - \Delta_k \leq -c_6 \gamma_k \|F_{i_k}(x_k) - y_{i_k}\|^2$$

for all integers $k \geq 0$, where $\Delta_k := D_{\mathcal{R}}^{\xi_k}(x^\dagger, x_k)$.

Proof. When $x_k \in B_{2\rho}(x_0)$, we may use the similar argument in the proof of Lemma 3.4 to obtain

$$\Delta_{k+1} - \Delta_k \leq \frac{1}{4\sigma} \|\xi_{k+1} - \xi_k\|^2 + \langle \xi_{k+1} - \xi_k, x_k - x^\dagger \rangle$$

$$= \frac{\gamma_k^2}{4\sigma} \|F_{i_k}'(x_k)^*(F_{i_k}(x_k) - y_{i_k})\|^2 - \gamma_k \langle F_{i_k}(x_k) - y_{i_k}, F_{i_k}'(x_k)(x_k - x^\dagger) \rangle$$

$$\leq \frac{\gamma_k^2}{4\sigma} \|F_{i_k}'(x_k)^*(F_{i_k}(x_k) - y_{i_k})\|^2 - (1 - \eta)\gamma_k \|F_{i_k}(x_k) - y_{i_k}\|^2.$$

By using $\|F_{i_k}'(x_k)\| \leq L$, we further obtain

$$\Delta_{k+1} - \Delta_k \leq - \left(1 - \eta - \frac{\gamma_k L^2}{4\sigma}\right) \gamma_k \|F_{i_k}(x_k) - y_{i_k}\|^2 \leq -c_6 \gamma_k \|F_{i_k}(x_k) - y_{i_k}\|^2.$$

Based on this, we may use an induction argument, as done in the proof of Lemma 3.4, to conclude the proof. $\square$

Note that, once $(x_0, \xi_0) \in X \times X^*$ and $\{\gamma_k\}$ are fixed, the sequence $\{(x_k, \xi_k)\}$ in Algorithm 1 is completely determined by the sample path $\{i_0, i_1, \cdots\}$; changing the sample path can result in a different iterative sequence and thus $\{(x_k, \xi_k)\}$ is a random sequence. We need to analyze Algorithm 1 using tools from stochastic calculus. Let $\mathcal{F}_0 = \emptyset$ and, for each integer $k \geq 0$, let $\mathcal{F}_k$ denote the σ-algebra generated by the random variables i_l for $0 \leq l < k$. Then $\{\mathcal{F}_k : k \geq 0\}$ form a filtration which is natural to Algorithm 1. Let $\mathbb{E}$ denote the expectation associated with this filtration, see [3]. There holds the tower property $\mathbb{E}[\mathbb{E}[\Phi|\mathcal{F}_k]] = \mathbb{E}[\Phi]$ for any random variable Φ.

Based on Lemma 4.1, it is easy to derive the convergence of $\{x_k\}$ to a solution of (2) in expectation and almost surely by the arguments in [18] with minor modifications. Our focus here is on deriving convergence rate of Algorithm 1 under the source condition (7). To this end, we need to interpret Algorithm 1 from an alternative perspective. Note that, under (iv) of Assumption 4, the updating formula for ξ_{k+1} can be written as

$$\xi_{k+1} = \xi_k - \gamma_k A_{i_k}^* (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k}).$$

This motivates us to consider the following algorithm.

Algorithm 2. *Take $(x_0, \xi_0) \in X \times X^*$ with $x_0 \in \mathrm{dom}(F)$ and $\xi_0 \in \partial \mathcal{R}(x_0)$. Set $\lambda_0 = (0, \cdots, 0) \in Y_1 \times \cdots \times Y_N$. For $k \geq 0$ do the following:*

(i) *Pick $i_k \in \{1, \cdots, N\}$ randomly via the uniform distribution;*

(ii) *Update $\lambda_{k+1} = (\lambda_{k+1,1}, \cdots, \lambda_{k+1,N}) \in Y_1 \times \cdots \times Y_N$ by $\lambda_{k+1,i} = \lambda_{k,i}$ for $i \neq i_k$ and*

$$\lambda_{k+1,i_k} = \lambda_{k,i_k} - \gamma_k (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k});$$

(iii) *Calculate $\xi_{k+1} = \xi_0 + A^* \lambda_{k+1}$ and $x_{k+1} = \arg\min_{x \in X} \{\mathcal{R}(x) - \langle \xi_{k+1}, x \rangle\}$.*

Note that Algorithm 2 is not an implementable one, it is only used to produce an extra sequence $\{\lambda_k\}$ in $Y_1 \times \cdots \times Y_N$ which is crucial for the forthcoming analysis. Note that for ξ_{k+1} defined by algorithm 2 we have

$$\begin{aligned}
\xi_{k+1} = \xi_0 + A^* \lambda_{k+1} &= \xi_0 + \sum_{i \neq i_k} A_i^* \lambda_{k+1,i} + A_{i_k}^* \lambda_{k+1,i_k} \\
&= \xi_0 + \sum_{i \neq i_k} A_i^* \lambda_{k,i} + A_{i_k}^* \left(\lambda_{k,i_k} - \gamma_k (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k}) \right) \\
&= \xi_k - \gamma_k A_{i_k}^* (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k}) \\
&= \xi_k - \gamma_k F_{i_k}'(x_k)^* (F_{i_k}(x_k) - y_{i_k}).
\end{aligned}$$

Therefore both Algorithm 1 and Algorithm 2 can produce the same sequence $\{x_k, \xi_k\}$ if they start from the same initial guess, follow the same sample path and use the same step-size.

Lemma 4.2. *Let Assumption 1 and Assumption 4 hold. Let $\{\gamma_k\}$ satisfy (4). Consider Algorithm 2 and assume the source condition (3) holds. Then*

$$\begin{aligned}
\mathbb{E}[\|\lambda_{k+1} - \lambda^\dagger\|^2 | \mathcal{F}_k] \leq{}& \left(1 + \frac{(3+\eta)^2 \kappa_0^2}{8\sigma N} \gamma_k \|F(x_k) - y\|^2 \right) \|\lambda_k - \lambda^\dagger\|^2 \\
&+ \frac{C\gamma_k^2}{N} \|F(x_k) - y\|^2 - \frac{2\gamma_k}{N} \Delta_k
\end{aligned}$$

for all integers $k \geq 0$, where $\Delta_k := D_{\mathcal{R}}^{\xi_k}(x^\dagger, x_k)$ and C is a positive constant independent of k.

Proof. By the definition of λ_{k+1} and the polarization identity we have

$$
\begin{aligned}
\|\lambda_{k+1} - \lambda^\dagger\|^2 &= \sum_{i \neq i_k} \|\lambda_{k+1,i} - \lambda_i^\dagger\|^2 + \|\lambda_{k+1,i_k} - \lambda_{i_k}^\dagger\|^2 \\
&= \sum_{i \neq i_k} \|\lambda_{k,i} - \lambda_i^\dagger\|^2 + \|\lambda_{k,i_k} - \lambda_{i_k}^\dagger - \gamma_k (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k})\|^2 \\
&= \|\lambda_k - \lambda^\dagger\|^2 + \gamma_k^2 \left\| (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k}) \right\|^2 \\
&\quad - 2\gamma_k \left\langle (Q_{x_k}^{i_k})^* (F_{i_k}(x_k) - y_{i_k}), \lambda_{k,i_k} - \lambda_{i_k}^\dagger \right\rangle.
\end{aligned}
$$

By using Assumption 4 and following the similar argument in establishing Lemma 3.5 we can obtain

$$
\begin{aligned}
\|\lambda_{k+1} - \lambda^\dagger\|^2 &\leq \|\lambda_k - \lambda^\dagger\|^2 + \gamma_k^2 (1 + \kappa_0 \|x_k - x^\dagger\|)^2 \|F_{i_k}(x_k) - y_{i_k}\|^2 \\
&\quad + 2\kappa_0 \gamma_k \|x_k - x^\dagger\| \|F_{i_k}(x_k) - y_{i_k}\| \|\lambda_{k,i_k} - \lambda_{i_k}^\dagger\| \\
&\quad - 2\gamma_k \left\langle F_{i_k}(x_k) - y_{i_k} - A_{i_k}(x_k - x^\dagger), \lambda_{k,i_k} - \lambda_{i_k}^\dagger \right\rangle \\
&\quad - 2\gamma_k \left\langle A_{i_k}(x_k - x^\dagger), \lambda_{k,i_k} - \lambda_{i_k}^\dagger \right\rangle \\
&\leq \|\lambda_k - \lambda^\dagger\|^2 + \gamma_k^2 (1 + \kappa_0 \|x_k - x^\dagger\|)^2 \|F_{i_k}(x_k) - y_{i_k}\|^2 \\
&\quad + (3 + \eta)\kappa_0 \gamma_k \|x_k - x^\dagger\| \|F_{i_k}(x_k) - y_{i_k}\| \|\lambda_{k,i_k} - \lambda_{i_k}^\dagger\| \\
&\quad - 2\gamma_k \left\langle A_{i_k}^* (\lambda_{k,i_k} - \lambda_{i_k}^\dagger), x_k - x^\dagger \right\rangle.
\end{aligned}
$$

According to Lemma 4.1 and the strong convexity of $\mathcal{R}$, $\{x_k\}$ is bounded. Thus, we can find a positive constant C such that

$$
\begin{aligned}
\|\lambda_{k+1} - \lambda^\dagger\|^2 &\leq \|\lambda_k - \lambda^\dagger\|^2 + C\gamma_k^2 \|F_{i_k}(x_k) - y_{i_k}\|^2 \\
&\quad + (3 + \eta)\kappa_0 \gamma_k \|x_k - x^\dagger\| \|F_{i_k}(x_k) - y_{i_k}\| \|\lambda_{k,i_k} - \lambda_{i_k}^\dagger\| \\
&\quad - 2\gamma_k \left\langle A_{i_k}^* (\lambda_{k,i_k} - \lambda_{i_k}^\dagger), x_k - x^\dagger \right\rangle.
\end{aligned}
$$

Taking the conditional expectation on $\mathcal{F}_k$ gives

$$
\begin{aligned}
\mathbb{E}[\|\lambda_{k+1} - \lambda^\dagger\|^2 | \mathcal{F}_k] &\leq \|\lambda_k - \lambda^\dagger\|^2 + \frac{C\gamma_k^2}{N} \sum_{i=1}^N \|F_i(x_k) - y_i\|^2 \\
&\quad + \frac{(3 + \eta)\kappa_0}{N} \gamma_k \|x_k - x^\dagger\| \sum_{i=1}^N \|F_i(x_k) - y_i\| \|\lambda_{k,i} - \lambda_i^\dagger\| \\
&\quad - \frac{2\gamma_k}{N} \left\langle \sum_{i=1}^N A_i^* (\lambda_{k,i} - \lambda_i^\dagger), x_k - x^\dagger \right\rangle.
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\sum_{i=1}^N \|F_i(x_k) - y_i\| \|\lambda_{k,i} - \lambda_i^\dagger\| &\leq \left(\sum_{i=1}^N \|F_i(x_k) - y_i\|^2 \right)^{1/2} \left(\sum_{i=1}^N \|\lambda_{k,i} - \lambda_i^\dagger\|^2 \right)^{1/2} \\
&= \|F(x_k) - y\| \|\lambda_k - \lambda^\dagger\|.
\end{aligned}
$$

Combining this with the above inequality and using the definition of F and A, we thus obtain

$$\mathbb{E}[\|\lambda_{k+1} - \lambda^\dagger\|^2 | \mathcal{F}_k] \leq \|\lambda_k - \lambda^\dagger\|^2 + \frac{C\gamma_k^2}{N}\|F(x_k) - y\|^2$$
$$+ \frac{(3+\eta)\kappa_0}{N}\gamma_k\|x_k - x^\dagger\|\|F(x_k) - y\|\|\lambda_k - \lambda^\dagger\|$$
$$- \frac{2\gamma_k}{N}\left\langle A^*(\lambda_k - \lambda^\dagger), x_k - x^\dagger\right\rangle.$$

By using the definition of ξ_k, the source condition (3), and the strong convexity of $\mathcal{R}$, we have

$$\left\langle A^*(\lambda_k - \lambda^\dagger), x_k - x^\dagger\right\rangle = \left\langle \xi_k - \xi^\dagger, x_k - x^\dagger\right\rangle \geq \Delta_k + \sigma\|x_k - x^\dagger\|^2.$$

Therefore

$$\mathbb{E}[\|\lambda_{k+1} - \lambda^\dagger\|^2|\mathcal{F}_k] \leq \|\lambda_k - \lambda^\dagger\|^2 + \frac{C\gamma_k^2}{N}\|F(x_k) - y\|^2 - \frac{2\sigma\gamma_k}{N}\|x_k - x^\dagger\|^2 - \frac{2\gamma_k}{N}\Delta_k$$
$$+ \frac{(3+\eta)\kappa_0}{N}\gamma_k\|x_k - x^\dagger\|\|F(x_k) - y\|\|\lambda_k - \lambda^\dagger\|$$
$$\leq \|\lambda_k - \lambda^\dagger\|^2 + \frac{C\gamma_k^2}{N}\|F(x_k) - y\|^2 - \frac{2\gamma_k}{N}\Delta_k$$
$$+ \frac{(3+\eta)^2\kappa_0^2}{8\sigma N}\gamma_k\|F(x_k) - y\|^2\|\lambda_k - \lambda^\dagger\|^2.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will use Lemma 4.1 and Lemma 4.2 to derive the almost sure convergence rate of Algorithm 1. To this end, we need the following Robbins-Siegmund theorem ([2, 25]).

Theorem 4.3 (Robbins-Siegmund theorem)**.** *In a probability space consider a filtration $\{\mathcal{F}_k\}$ and four non-negative sequences of $\{\mathcal{F}_k\}$-adapted processes $\{V_k\}$, $\{U_k\}$, $\{Z_k\}$ and $\{\alpha_k\}$ such that $\sum_k Z_k < \infty$ and $\sum_k \alpha_k < \infty$ almost surely. If for any integers $k \geq 0$, there holds*

$$\mathbb{E}[V_{k+1} \mid \mathcal{F}_k] + U_k \leq (1 + \alpha_k)V_k + Z_k,$$

then $\{V_k\}$ converges and $\sum_{k=0}^{\infty} U_k$ is finite almost surely.

Now we are ready to show the main convergence rate result on Algorithm 1 under the source condition (3).

Theorem 4.4. *Let Assumption 1 and Assumption 4 hold. Let $\{\gamma_k\}$ be a sequence of positive numbers such that*

$$\bar{\gamma} := \sup_{k\geq0} \gamma_k < \frac{4\sigma(1-\eta)}{L^2} \quad \text{and} \quad \sum_{k=0}^{\infty} \gamma_k = \infty. \qquad (5)$$

Consider Algorithm 1 and assume the source condition (3) holds. Then

$$\Delta_k = O\left(s_k^{-1}\right) \quad \text{and} \quad \|x_k - x^\dagger\| = O\left(s_k^{-1/2}\right) \quad \text{almost surely,}$$

where $s_k := \sum_{l=0}^{k} \gamma_l$.

Proof. According to Lemma 4.1, taking the conditional expectation on $\mathcal{F}_k$ gives

$$\mathbb{E}[\Delta_{k+1}|\mathcal{F}_k] \le \Delta_k - \frac{c_6\gamma_k}{N}\|F(x_k) - y\|^2.$$

By taking the full expectation, we further obtain

$$\frac{c_6\gamma_k}{N}\mathbb{E}[\|F(x_k) - y\|^2] \le \mathbb{E}[\Delta_k] - \mathbb{E}[\Delta_{k+1}].$$

Consequently

$$\frac{c_6}{N}\mathbb{E}\left[\sum_{k=0}^{\infty}\gamma_k\|F(x_k) - y\|^2\right] \le \sum_{k=0}^{\infty}\left(\mathbb{E}[\Delta_k] - \mathbb{E}[\Delta_{k+1}]\right) \le \Delta_0 < \infty.$$

This implies that $\sum_{k=0}^{\infty}\gamma_k\|F(x_k) - y\|^2 < \infty$ almost surely. Therefore, we may apply Robbins-Siegmund theorem to the inequality in Lemma 4.2 to conclude that $\sum_{k=0}^{\infty}\gamma_k\Delta_k < \infty$ almost surely. By using the monotonicity of $\{\Delta_k\}$ given in Lemma 4.1 we obtain

$$s_k\Delta_k = \left(\sum_{l=0}^{k}\gamma_l\right)\Delta_k \le \sum_{l=0}^{k}\gamma_l\Delta_l \le \sum_{l=0}^{\infty}\gamma_l\Delta_l < \infty \quad \text{almost surely.}$$

Therefore

$$\Delta_k = O\left(s_k^{-1}\right) \quad \text{almost surely.}$$

By the strong convexity of $\mathcal{R}$, we then obtain $\|x_k - x^\dagger\| = O(s_k^{-1/2})$ almost surely. The proof is complete. $\qquad\square$

Remark 4.5. When X is a Hilbert space and $\mathcal{R}(x) = \frac{1}{2}\|x\|^2$, Algorithm 1 reduces to the stochastic gradient method studied in [15]. Under the diminishing step-size $\gamma_k = \gamma_0(k+1)^{-\alpha}$ with $\alpha \in (0,1)$, [15, Theorem 4.8] shows that, under the source condition (3), there holds the convergence rate

$$\mathbb{E}[\|x_k - x^\dagger\|^2] = O\left((k+1)^{-\min\{1-\alpha, \alpha-\varepsilon\}}\right)$$

for some small $\varepsilon \in (0, \alpha/2)$. Besides Assumption 4 and smallness of $\|\lambda^\dagger\|$ and γ_0, the proof in [15] also relies on the technical condition

$$\mathbb{E}\left[\|(I - R_{z_t})G(x_k)\|^2\right]^{1/2} \le C\mathbb{E}\left[\|x_k - x^\dagger\|^2\right]^{\theta/2}\mathbb{E}\left[\|G(x_k)\|^2\right]^{1/2},$$
$$\mathbb{E}\left[\|(I - R_{z_t}^*)G(x_k)\|^2\right]^{1/2} \le C\mathbb{E}\left[\|x_k - x^\dagger\|^2\right]^{\theta/2}\mathbb{E}\left[\|G(x_k)\|^2\right]^{1/2}$$

for some $\theta \in (0,1]$; one may refer to [15, Assumption 2.4] for the meaning of R_{z_t} and $G(x)$. Unfortunately there is no any justification available for this technical condition. In contrast, our result requires neither such a technical condition nor the smallness of $\|\lambda^\dagger\|$. Our convergence rate is valid for any step-size $\{\gamma_k\}$ satisfying (5). For the step-size scheme $\gamma_k = \gamma_0(k+1)^{-\alpha}$ with $\alpha \in (0,1)$, our Theorem 4.4 gives the nicer almost sure convergence rate

$$\|x_k - x^\dagger\|^2 = O\left((k+1)^{-(1-\alpha)}\right).$$

Our result even allows to use constant step-size schemes to give much better almost sure convergence rate

$$\|x_k - x^\dagger\|^2 = O\left(1/(k+1)\right).$$

Remark 4.6. For the stochastic mirror descent method for nonlinear ill-posed system with noisy data, it is not yet clear how to derive convergence rates. We hope to address this issue in a future work.

5. Numerical results

In this section we present some computational results to validate our theoretical convergence rate results on the method (3) when it is terminated by the discrepancy principle (6) with $\tau > (1 + \eta)/(1 - \eta)$. The step-sizes γ_k^δ are chosen by various rules which are described below:

- *Rule 1*: $\gamma_k^\delta = \gamma/L^2$ with $\gamma = 1.98(1 - \eta - (1 + \eta)/\tau)$;
- *Rule 2*: $\gamma_k^\delta = \max\left\{\frac{\gamma\|r_k^\delta\|^2}{\|F'(x_k^\delta)^*r_k^\delta\|^2}, \bar{\gamma}\right\}$ with $\gamma = 1.98(1 - \eta - (1 + \eta)/\tau)$ and $\bar{\gamma} = 600$;
- *Rule 3*: $\gamma_k^\delta = \max\left\{\frac{\gamma_0((1-\eta)\|r_k^\delta\|-(1+\eta)\delta)\|r_k^\delta\|}{\|F'(x_k^\delta)^*r_k^\delta\|^2}, \bar{\gamma}\right\}$ with $\gamma_0 = 1.98$ and $\bar{\gamma} = 600$.

Here $r_k := F(x_k^\delta) - y^\delta$. In the following computations, the regularization functionals $\mathcal{R}$ that we use always satisfy Assumption 1 with $\sigma = 1/2$. Thus, the step-size selections for γ_k^δ, as outlined above, are legitimate in accordance with Corollary 3.13 and Theorem 3.14.

Example 5.1. Consider the linear integral equation of the first kind

$$(Ax)(s) := \int_0^1 \phi(t, s)x(t)dt = y(s), \quad s \in [0, 1],$$

where $\phi(t, s) := 1 + t + s$, and assume the sought solution $x^\dagger$ is a probability density function on $[0, 1]$, i.e. $x^\dagger \geq 0$ a.e. on $[0, 1]$ and $\int_0^1 x^\dagger = 1$. Clearly A is a compact linear operator from $L^1[0, 1]$ to $L^2[0, 1]$ and

$$\|Ax\|_{L^2[0,1]}^2 \leq \int_0^1 \max_{0 \leq t \leq 1} |\phi(t, s)|^2 dt \|x\|_{L^1[0,1]}^2 = \frac{19}{3}\|x\|_{L^1[0,1]}^2$$

which implies $\|A\| \leq L := \sqrt{19/3}$. To determine such a solution we use the regularization functional $\mathcal{R}$ given in (d) of Example 3.2 with $\Omega = [0, 1]$. The corresponding method (3) becomes ([17])

$$\xi_{k+1}^\delta = \xi_k^\delta - \gamma_k^\delta A^*(Ax_k^\delta - y^\delta), \qquad x_{k+1}^\delta = e^{\xi_{k+1}^\delta}\left/\int_0^1 e^{\xi_{k+1}^\delta}\right., \tag{6}$$

where y^δ is a noisy data satisfying $\|y^\delta - Ax^\dagger\|_{L^2[0,1]} \leq \delta$ with a noise level $\delta > 0$. Clearly, Assumption 2 and Assumption 3 hold with $\eta = 0$ and $\kappa_0 = 0$. It is easy to check that if $x^\dagger$ satisfies the condition

$$1 + \log x^\dagger \in \text{Ran}(A^*), \tag{7}$$

then it satisfies the source condition (7) with $\xi_0 = 0$. Consequently, for the integer k_δ output by the discrepancy principle, if $x^\dagger$ satisfies (7) then we can expect the convergence rate $\|x_{k_\delta}^\delta - x^\dagger\| = O(\sqrt{\delta})$ according to our theoretical results.

To numerically check if the convergence rate is achieved, we assume the sought solution $x^\dagger$ is given by

$$x^\dagger(t) := e^{1.5a-1+at}, \quad t \in [0.1],$$

where $a = 0.4949075935 \cdots$ is such that $e^{1.5a-1}(e^a - 1)/a = 1$ which guarantees $\int_0^1 x^\dagger = 1$ so that $x^\dagger$ is a probability density function on $[0, 1]$. It is easy to see that $1 + \log x^\dagger = A^*\lambda^\dagger$ with $\lambda^\dagger \equiv a$ on $[0, 1]$ and thus the source condition (7) holds. In our numerical

TABLE 1. Numerical results for Example 5.1, where `iter` denotes the integer k_δ output by the discrepancy principle (6) with $\tau = 1.01$, `err` denotes the approximation error, i.e. `err` $= \|x_{k_\delta}^\delta - x^\dagger\|_{L^1}$, and `ratio` $=$ `err`$/\sqrt{\delta}$.

δ	Rule 1			Rule 2			Rule 3		
	iter	err	ratio	iter	err	ratio	iter	err	ratio
5e-2	743	5.0723e-2	0.2268	160	5.0659e-2	0.2266	66	5.0275e-2	0.2248
5e-3	2630	5.0405e-3	0.0713	1649	5.0215e-3	0.0710	189	4.9796e-3	0.0704
5e-4	4509	4.6703e-4	0.0209	3495	4.6610e-4	0.0208	280	4.6441e-4	0.0208
5e-5	6387	8.2451e-5	0.0117	5348	8.2431e-5	0.0117	368	8.2427e-5	0.0117

computation, we use noisy data y^δ with various noise level $\delta > 0$ to reconstruct $x^\dagger$. To carry out the computation, all integrals over $[0, 1]$ are approximated by the trapezoidal rule by partitioning $[0, 1]$ into 5000 subintervals of equal length; such a fine grid is used for the purpose of reducing the effect of discretization so that we can observe the convergence rate in terms of δ more accurately. In Table 1 we present the computational results obtained using the method (9) with the initial guess $\xi_0 \equiv 0$, leading to $x_0 \equiv 1$. The method is terminated according to the discrepancy principle (6) with $\tau = 1.01$, and the step-size γ_k^δ is selected using Rule 1, Rule 2 and Rule 3. The computational outcomes corroborate our theoretical findings discussed in Corollary 3.13 and Theorem 3.14. Moreover, Table 1 indicates that Rule 3 generally requires fewer iterations than Rule 1 and Rule 2.

Example 5.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Consider determining the coefficient c in the boundary value problem

$$-\triangle u + cu = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{8}$$

from an $L^2(\Omega)$-measurement of u, where $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Assuming the sought solution $c^\dagger$ is in $L^2(\Omega)$, this nonlinear inverse problem reduces to solving $F(c) = u$, where $F : L^2(\Omega) \to L^2(\Omega)$ is defined by $F(c) := u(c)$ with $u(c) \in H^1(\Omega) \subset L^2(\Omega)$ being the unique weak solution of (8). It is known that F is well-defined on the set

$$D(F) := \{c \in L^2(\Omega) : \|c - \hat{c}\|_{L^2} \le \varepsilon_0 \text{ for some } \hat{c} \ge 0 \text{ a.e.}\}$$

for some positive constant $\varepsilon_0 > 0$. Furthermore, F is Fréchet differentiable, the Fréchet derivative of F and the adjoint are given by

$$F'(c)h = -A(c)^{-1}(hu(c)) \quad \text{and} \quad F'(c)^*w = -u(c)A(c)^{-1}w$$

for $c \in D(F)$ and $h, w \in L^2(\Omega)$, where $A(c) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by $A(c)u = -\triangle u + cu$. Additionally, F satisfies Assumption 2 (iii) for a number $\eta > 0$ which can be very small if $\rho > 0$ is sufficiently small (see [9]). If $u(c^\dagger) \ge \kappa$ for some constant $\kappa > 0$, then Assumption 3 holds in a neighborhood of $c^\dagger$; see [13].

We are interested in the situation that the sought solution $c^\dagger \in L^2(\Omega)$ is nonnegative. Thus, to reconstruct $c^\dagger$, we use the functional $\mathcal{R}$ given in (a) of Example 3.2 with $C := \{c \in L^2(\Omega) : c \ge 0 \text{ a.e.}\}$. Correspondingly the method (3) becomes

$$\xi_{k+1}^\delta = \xi_k^\delta - \gamma_k^\delta F'(c_k^\delta)^*(F(c_k^\delta) - u^\delta), \qquad c_{k+1}^\delta = \max\{\xi_{k+1}^\delta, 0\}, \tag{9}$$

where $u^\delta \in L^2(\Omega)$ is a noisy data satisfying $\|u^\delta - u(c^\dagger)\|_{L^2(\Omega)} \le \delta$ with a noise level $\delta > 0$. The source condition (7) can be equivalently stated as

$$c^\dagger = P_C(\xi_0 + F'(c^\dagger)^*\lambda^\dagger) = \max\{\xi_0 + F'(c_0)^*\lambda^\dagger, 0\} \tag{10}$$

for some $\lambda^\dagger \in L^2(\Omega)$, where P_C denotes the orthogonal projection from $L^2(\Omega)$ onto C.

TABLE 2. Numerical results for Example 5.2, where `iter` denotes the integer k_δ output by the discrepancy principle (6) with $\tau = 1.1$, `err` denotes the approximation error, i.e. $\texttt{err} = \|c_{k_\delta}^\delta - c^\dagger\|_{L^2}$, and $\texttt{ratio} = \texttt{err}/\sqrt{\delta}$.

δ	Rule 2			Rule 3		
	iter	err	ratio	iter	err	ratio
1e-2	19	1.7471e-1	1.7471	2	1.7357e-1	1.7357
1e-3	97	4.0901e-2	1.2934	26	4.1922e-2	1.3257
1e-4	227	8.3865e-3	0.8386	84	8.3326e-3	0.8333
1e-5	548	2.5209e-3	0.7972	365	2.5909e-3	0.8193
1e-6	5423	9.3739e-4	0.9374	5741	9.4716e-3	0.9472

To test the convergence rate result, we take $\Omega = [0,1]^2$ and assume the sought solution $c^\dagger$ is given by

$$c^\dagger(x, y) = \left(\max\{1 - 9(x^2 + y^2), 0\}\right)^2$$

and the right hand side f is given by $f = -4 + (1 + x^2 + y^2)c^\dagger(x, y)$. Then $u(c^\dagger) = 1 + x^2 + y^2$. Take the initial guess $\xi_0 = 0$. Then the source condition (10) holds with

$$\lambda^\dagger = A(c^\dagger)\left(-\frac{c^\dagger}{u(c^\dagger)}\right) = \triangle\left(\frac{c^\dagger}{u(c^\dagger)}\right) - \frac{(c^\dagger)^2}{u(c^\dagger)} \in L^2(\Omega).$$

Thus, according to our theory, if the step-size γ_k^δ is chosen according to those rules stated in Corollary 3.13 and Theorem 3.14, and if k_δ denotes the integer output by the discrepancy principle, we can expect $\|c_{k_\delta}^\delta - c^\dagger\|_{L^2(\Omega)} = O(\sqrt{\delta})$ as $\delta \to 0$. To validate this result, we use noisy data u^δ with various noise level $\delta > 0$ to reconstruct $c^\dagger$ by the method (9). To perform the computation, we set $\eta = 0.04$ and divide Ω into 128×128 small squares of equal size. All partial differential equations are solved approximately using a multigrid method ([12]) with finite difference discretization. In Table 2 we report the computational results by the method (9), terminated by the discrepancy principle (6) with $\tau = 1.1$, with the step-size γ_k^δ chosen by Rule 2 and Rule 3; we do not report the computation result for γ_k^δ chosen by Rule 1 because the upper bound L is hard to obtain. The computational results confirm our theoretical result presented in Corollary 3.13 and Theorem 3.14. Furthermore, Table 2 indicates that Rule 3 usually requires less number of iterations than Rule 2 particularly when the noise level is moderate or not too small.

Acknowledgement

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