FUNDAMENTAL GROUPOID SCHEMES AND SEMI-FINITE BUNDLES

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ABSTRACT. In this article, we study the various fundamental groupoid schemes and semi-finite bundles on complex tori and Riemann surfaces. We have computed fundamental groupoid of anisotropic conic, Klein bottle and abelian varieties. We also study the relation among various fundamental groupoid schemes by considering their representations.

1. INTRODUCTION

Let X be a proper, connected and reduced scheme over a perfect field k. In the case when X has a k-rational point x, Madhav Nori [14] defined a fundamental group scheme of (X, x) corresponding to the neutral Tannakian category of essentially finite bundles. If X does not have a rational point, then we can't get a natural fibre functor, but one can get a functor that takes value in the category of quasi-coherent sheaves over some k-scheme S. By Tannaka duality, it corresponds to an affine k-groupoid scheme acting transitively on S. In this article, we have studied groupoid versions of certain Tannakian categories. In particular, we have computed fundamental groupoid of anisotropic conic, Klein bottle and abelian varieties. We also describe the relation among various fundamental groupoid scheme by considering their representations.

Let X be a complex manifold with the structure sheaf $\mathcal{O}_X$. Recall that Riemann-Hilbert correspondence assigns $\mathbb{C}\pi_1(X)$-module to locally constant sheaf of $\mathbb{C}$-modules. By taking tensor product with $\mathcal{O}_X$, we get a locally free sheaf of $\mathcal{O}_X$-module. In this way, we have a functor

$$\mathbb{C}\pi_1(X)\text{–mod} \to \mathcal{O}_X\text{–mod}.$$

Given a surjective group homomorphism $\alpha : \pi_1(X) \to \Sigma$, we can restrict this functor to the category $\mathbb{C}\Sigma$–mod by taking composition with the natural functor $\tilde{\alpha}^* : \mathbb{C}\Sigma\text{–mod} \to \mathbb{C}\pi_1(X)\text{–mod}$, the resulting functor

$$S : \mathbb{C}\Sigma\text{–mod} \to \mathcal{O}_X\text{–mod}$$

is known as the *Schottky functor*.

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In [8], it is proven that the Schottky functor induces an equivalence between the category of unipotent modules and the category of unipotent bundles on both complex tori and compact Riemann surfaces. For complex tori, we generalize this result to the category of semi-finite bundles, which strictly contains the category of unipotent bundles. We also establish an analogous result of [8, Theorem 1.2] for products of compact Riemann surfaces. Let G be a connected affine algebraic group. Recall that a principal G-bundle $j : P \to X$ is called *finite* if every vector bundle associated to a G-representation on X is finite. We give an analogous Schottky result for principal bundles (see Proposition 6.12).

2. Preliminaries

A groupoid is a category in which every morphism is an isomorphism. For example, a topological fundamental groupoid; where objects are the points of topological space and morphisms are homotopy classes of paths between two points. One can define a groupoid scheme by considering the functor of points taking values in groupoid. Let us give a precise definition of a groupoid scheme. Let k be a field and S be any k-scheme.

Definition 2.1. *An affine k-groupoid scheme G acting transitively on a k-scheme S is a k-scheme G with*

- *a faithfully flat affine morphism $(t, s) : G \longrightarrow S \times_k S$ of $S \times_k S$-schemes,*
- *product morphism $m : G \times_{{}^{s}S^{t}} G \longrightarrow G$ of $S \times_k S$-schemes,*
- *unit element morphism $e : S \longrightarrow G$ over $S \times_k S$, where S is over $S \times_k S$ by a diagonal morphism $\Delta : S \longrightarrow S \times_k S$, and*
- *inverse element morphism $i : G \longrightarrow G$ of $S \times_k S$-scheme such that $s \circ i = t$ and $t \circ i = s$*

which satisfying following axioms:

(1) *Associativity: $m \circ (m \times id) = m \circ (id \times m)$.*
(2) *Identity: $m \circ (e \times id) = m \circ (id \times e)$.*
(3) *Inverse: $m \circ (i \times id) = e \circ s$.*

Let us understand the above definition by taking the functor of points view. Let $(y, x) : T \to S \times S$ be any morphisms. Consider the category $\mathcal{G}(T)$ whose objects are $S(T)$ and morphisms are $G_{y,x}(T)$ for any two object $x, y \in S(T)$; where $G_{y,x}$ defined by the following diagram:

$$\begin{array}{ccc} G_{y,x} & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle (t,s)} \\ T & \xrightarrow{(y,x)} & S \times_k S \end{array}$$

The category $\mathcal{G}(T)$ is a groupoid. We will use the above notations throughout this section.

Example 2.2. Consider $G = S \times_k S$ with the map $(pr_1, pr_2) : S \times_k S \to S \times_k S$. Then, this forms a groupoid scheme acting on a k-scheme S; where pr_1 and pr_2 denote the first and second projection map, respectively.

Example 2.3. If G is an affine group scheme over a field k with the structure map $f : G \to \operatorname{Spec} k$. Then, $(t, s) = (f, f)$ gives a natural structure of k-groupoid scheme acting transitively on $\operatorname{Spec} k$. In this sense, groupoid scheme is a generalization of a group scheme.

Observe that, if $t = s$ in Definition 2.1, then the morphism $(t, s) : G \to S \times_k S$ factor through diagonal morphism $\Delta : S \to S \times_k S$. In this case, G becomes group scheme over S. Given a groupoid scheme G over S, we can define the corresponding diagonal group scheme G^{Δ} over S and, for any morphism of k-scheme $f : T \to S$ we have a groupoid scheme G_T over T by the following square diagrams.

$$\begin{array}{ccc} G^{\Delta} & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle (t,s)} \\ S & \xrightarrow{\Delta} & S \times_k S \end{array} \qquad\qquad \begin{array}{ccc} G_T & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle (t,s)} \\ T \times_k T & \xrightarrow{(f,f)} & S \times_k S \end{array}$$

2.1. **Representations of groupoid.** A representation of G is a pair of quasi-coherent sheaf V of $\mathcal{O}_S$-module together with an action ρ of G on V, which is compatible with multiplication and base change. That means, for any two objects $(y, x) : T \to S \times_k S$, we have a map $\rho_{y,x} : G_{y,x}(T) \to \operatorname{Iso}_T(x^*V \to y^*V)$ which is compatible with multiplication and base change, and if $x = y$ then $\rho_{x,x}(\mathrm{Id}_x) = \mathrm{Id}_{x^*V}$. The category of quasi-coherent representations of G on S is denoted by $\mathbf{Rep}(S : G)$.

Let $f : T' \to T$ be a morphism of k-schemes. We get two object $(y \circ f, x \circ f) : T' \to S \times_k S$ by taking composition with $(y, x) : T \to S \times_k S$. There is natural map $f^{\#} : G_{y,x}(T) \to G_{y\circ f, x\circ f}(T')$ given by $g : T \to G_{y,x}$ goes to $(pr_1 \circ g \circ f, \mathrm{id}_{T'}) : T' \to G_{y\circ f, x\circ f}$; where $pr_1 : G_{y,x} \to G$ is the first projection map. The following diagram gives compatibility with base change $f : T' \to T$.

$$\begin{array}{ccc} G_{y,x}(T) & \xrightarrow{\rho_{y,x}} & \operatorname{Iso}_T(x^*V \to y^*V) \\ {\scriptstyle f^{\#}}\downarrow & & \downarrow{\scriptstyle f^*} \\ G_{y\circ f, x\circ f}(T') & \xrightarrow[\rho_{y\circ f, x\circ f}]{} & \operatorname{Iso}_{T'}(f^*x^*V \to f^*y^*V) \end{array}$$

From the above, we can conclude that any representation comes from the representation $\rho_{t,s} : G_{t,s}(G) \to \operatorname{Iso}_G(s^*V \to t^*V)$ by the base change. Any given $g : T \to G_{y,x}$, take $f = pr_1 \circ g : T \to G$ in the above diagram, then one can determined $\rho_{y,x}(g)$ by the base change map f. Here, the map $f^{\#} : G_{t,s}(G) \to G_{t\circ f, s\circ f}(T) = G_{y,x}(T)$ and $f^{\#}(\mathrm{id}_G, \mathrm{id}_G) = g$. That means, $\rho_{t,s}(G)(\mathrm{id}_G, \mathrm{id}_G) : s^*V \simeq t^*V$ determines the representation of G on V.

Remark 2.4. The category $\mathbf{Rep}(S : G)$ satisfies the base-change property [5, Remark 1.8]. i.e. for any $T \to S$, we have that $\mathbf{Rep}(S : G)$ is equivalent to $\mathbf{Rep}(T : G_T)$; where G_T denote the base change of G. In particular, if $G = S \times_k S$ with $(t, s) = (pr_1, pr_2)$ are first and second projection map, then for any affine open set U of S, we have $\mathbf{Rep}(S : S \times_k S) \simeq \mathbf{Rep}(U : U \times_k U)$ by the base change $U \hookrightarrow S$.

3. Tannakian category

In this section, we recall some basic notion of Tannakian category from [5]. Let k be a field and $\mathcal{T}$ be a k-linear abelian rigid tensor category with $\mathrm{End}(\mathbb{1}) = k$.

Definition 3.1. *A fiber functor of $\mathcal{T}$ on a k-scheme S is an exact k-linear tensor functor $\omega : \mathcal{T} \longrightarrow \mathbf{QCoh}(S)$ provided with a functorial isomorphisms $\omega(X) \otimes \omega(Y) \longrightarrow \omega(X \otimes Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{T})$, which is compactible with associativity, commutativity and unity.*

Note that a fiber functor commutes with dual. The axioms on $\mathcal{T}$ force that ω takes value in locally free sheaves of finite rank (see [5, 1.9]).

Definition 3.2. *A category $\mathcal{T}$ is called Tannakian over k if $\mathcal{T}$ admits a fiber functor $\omega : \mathcal{T} \longrightarrow \mathbf{QCoh}(S)$ for a non-empty k-scheme S. If $S = \operatorname{Spec} k$, then we call $\mathcal{T}$ to be a neutral Tannakian category.*

Example 3.3. Let G be an affine group scheme over a field k. Then, the category of linear representations of G denote by $\mathbf{Rep}_k(G)$ with the forgetful functor on the category $\mathbf{Vec}_k$ of finite dimensional k-vector spaces forms a neutral Tannakian category over k. More generally, for any affine k-groupoid scheme G acting transitively on S, the category $\mathbf{Rep}(S : G)$ with the forgetful functor on $\mathbf{QCoh}(S)$ is a Tannakian category.

Let $\omega : \mathcal{T} \to \mathbf{QCoh}(S)$ be a fiber functor. Let $pr_1, pr_2 : S \times_k S \to S$ denote the first and second projection morphisms respectively. Then, they induce a functor $pr_1^*, pr_2^* : \mathbf{QCoh}(S) \to \mathbf{QCoh}(S \times_k S)$ by the pull-back. Define

$$\mathbf{Aut}_k^{\otimes}(\omega) := \mathbf{Isom}_{S \times_k S}^{\otimes}(pr_2^*\omega, pr_1^*\omega).$$

This is a representable functor over $S \times_k S$, which yields a groupoid scheme acting on S (see the following theorem by Deligne).

Theorem 3.4. [5, Theorem 1.12] *Let $\mathcal{T}$ be a k-linear abelian rigid tensor category on a field k and let $\omega : \mathcal{T} \to \mathbf{QCoh}(S)$ be a fiber functor of $\mathcal{T}$ on a k-scheme S. Then,*

1. *the groupoid $\mathbf{Aut}_k^{\otimes}(\omega)$ is affine and faithfully flat on $S \times_k S$.*
2. *ω induces an equivalence of $\mathcal{T}$ with a category $\mathbf{Rep}(S : \mathbf{Aut}_k^{\otimes}(\omega))$.*
3. *let G be an affine k-groupoid acting transitively on S and faithfully flat on $S \times_k S$. Let $\tilde{\omega} : \mathbf{Rep}(S : G) \to \mathbf{QCoh}(S)$ defined by forgetting an action of G. Then, we have $G \simeq \mathbf{Aut}_k^{\otimes}(\tilde{\omega})$.*

The above theorem provides an equivalence between an affine groupoid acting transitively on S with faithfully flat on $S \times_k S$ and Tannakian categories whose fiber functor takes value in $\mathbf{QCoh}(S)$. Now, let us recall a theorem from [5] which tells about an existence of fiber functor, when k has a zero characteristic.

Definition 3.5. *Let X be an object in $\mathcal{T}$. The dimension of X is defined by $\dim(X) := ev \circ \delta$, where $\delta : \mathbb{1} \longrightarrow X^{\vee} \otimes X$ and $ev : X^{\vee} \otimes X \longrightarrow \mathbb{1}$.*

Theorem 3.6. [5, Theorem 7.1] *Let $\mathcal{T}$ be a k-linear abelian rigid tensor category on a field k of characteristic zero. Then, the following conditions are equivalent:*

(1) *$\mathcal{T}$ is Tannakian.*
(2) *For all $X \in \mathrm{Ob}(\mathcal{T})$, we have $\dim(X) \in \mathbb{Z}_{\geq 0}$.*
(3) *For all $X \in \mathrm{Ob}(\mathcal{T})$, there exist $n \geq 0$ such that $\wedge^n X = 0$.*

As an application of the above theorem, we can say that any k-linear abelian rigid tensor full subcategory of the category of vector bundles forms Tannakian category. For a fiber functor, as mentioned in the proof of above theorem, there exist a ring element A in the induced category of $\mathcal{T}$ such that $V \otimes A \simeq A^{\dim(V)}$ for all $V \in \mathrm{Ob}(\mathcal{T})$. So, one can define a functor $V \mapsto \Gamma(V \otimes A)$ which takes value in the category of $\Gamma(A)$-modules, which is equivalent to $\mathbf{QCoh}(\mathrm{Spec}\,\Gamma(A))$.

Remark 3.7. If k has a positive characteristic, then the above result hold if the category $\mathcal{T}$ satisfies an extra condition of finiteness (see [7]).

4. Fundamental Groupoid Schemes

In this section, we will define fundamental groupoid versions of fundamental group schemes by using the Tannakian duality. Let us recall some definitions and fix some notations. Throughout this section, let X be a smooth scheme of finite type defined over a field k of characteristic zero with $\mathrm{H}^0(X, \mathcal{O}_X) = k$.

Definition 4.1. *A vector bundle V on X is called finite bundle if there are two different polynomials $f \neq g \in \mathbb{N}[t]$ such that $f(V) \simeq g(V)$.*

Definition 4.2. *A vector bundle V on X is called unipotent if there is a filtration*

$$V = V_0 \supset V_1 \supset V_2 \supset ... \supset V_{n-1} \supset V_n = 0$$

such that each successive quotient $V_i/V_{i-1} \simeq \mathcal{O}_X$ for all i.

Definition 4.3. *A vector bundle V on X is called semi-finite if there is a filtration*

$$V = V_0 \supset V_1 \supset V_2 \supset ... \supset V_{n-1} \supset V_n = 0$$

such that each successive quotient V_i/V_{i-1} is indecomposable finite bundle for all i.

Definition 4.4. *A vector bundle V on X is called numerically flat if V and V^* are numerically effective bundles. That means, tautological line bundles $\mathcal{O}_{\mathbb{P}(V)}(1)$ and $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ are numerically effective (nef).*

A vector bundle V on X is called *Nori-semistable* if for any smooth projective curve C and a non-constant morphism $f : C \to X$, the pull-back f^*V is semi-stable bundle of degree zero.

Consider the categories $\mathcal{C}^{\mathrm{N}}(X)$, $\mathcal{C}^{\mathrm{uni}}(X)$, $\mathcal{C}^{\mathrm{EN}}(X)$, and $\mathcal{C}^{\mathrm{nf}}(X)$, which are full subcategories of the category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves on X with objects consisting of finite, unipotent, semi-finite, and numerically flat bundles, respectively. These categories satisfy the following relations:

$$\mathcal{C}^{\mathrm{N}}(X), \mathcal{C}^{\mathrm{uni}}(X) \subset \mathcal{C}^{\mathrm{EN}}(X) \subset \mathcal{C}^{\mathrm{nf}}(X).$$

Each of these categories is a k-linear, abelian, rigid tensor category. If X has a k-rational point x, then the category $\mathcal{C}^{\star}(X)$, with the fiber functor x^*, forms a neutral Tannakian category, where $\star = \mathrm{N}, \mathrm{uni}, \mathrm{EN}, \mathrm{nf}$. By Tannakian duality, these categories correspond to an affine group scheme $\pi^{\star}(X, x)$ for $\star = \mathrm{N}, \mathrm{uni}, \mathrm{EN}$, and $\pi^{\mathrm{S}}(X, x)$ for $\star = \mathrm{nf}$. The affine group schemes $\pi^{\mathrm{N}}(X, x)$, $\pi^{\mathrm{uni}}(X, x)$, $\pi^{\mathrm{EN}}(X, x)$, and $\pi^{\mathrm{S}}(X, x)$ are called the *Nori fundamental group scheme*, *unipotent fundamental group scheme*, *extended Nori fundamental group scheme*, and *S-fundamental group scheme*, respectively (see [14, 17, 11, 12] for more details).

Let S be a k-scheme, and let $\omega : \mathcal{C}^{\star} \to \mathbf{QCoh}(S)$ be a fiber functor. By Tannakian duality, this corresponds to a k-groupoid scheme $\Pi^{\star}(X, \omega)$ acting on S. The groupoid schemes $\Pi^{\mathrm{N}}(X, \omega)$, $\Pi^{\mathrm{uni}}(X, \omega)$, $\Pi^{\mathrm{EN}}(X, \omega)$, and $\Pi^{\mathrm{S}}(X, \omega)$ are called the *Nori fundamental groupoid scheme*, *unipotent fundamental groupoid scheme*, *extended Nori fundamental groupoid scheme*, and *S-fundamental groupoid scheme*, respectively.

4.1. **Fundamental groupoid of an anisotropic conic.** Let X be a Riemann surface and $\sigma : X \to X$ be an anti-holomorphic involution. The pair (X, σ) determines the real curve. If $X = \mathbb{P}^1_{\mathbb{C}}$, then there are only two possibilities for σ upto equivalents (see [10, Chapter-II, Exercise-4.7(e)]).

1. Let $\sigma_1 : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ given by $[z_1 : z_2] \mapsto [\overline{z_1} : \overline{z_2}]$ be an anti-holomorphic involution. The pair $(\mathbb{P}^1_{\mathbb{C}}, \sigma_1)$ gives a real projective line $\mathbb{P}^1_{\mathbb{R}}$.
2. Let $\sigma_2 : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ given by $[z_1 : z_2] \mapsto [-\overline{z_2} : \overline{z_1}]$ be an anti-holomorphic involution. The pair $(\mathbb{P}^1_{\mathbb{C}}, \sigma_2)$ gives an anisotropic conic given by $C : \{x_0^2 + x_1^2 + x_2^2 = 0\}$ contained in $\mathbb{P}^2_{\mathbb{R}}$. Note that C does not have any $\mathbb{R}$-point.

We want to find the various fundamental groupoid schemes for the above two real curves. For this, we need a classification of vector bundles which is given in [3]. By [3, Theorem 5.3], $\mathcal{C}^{\mathrm{nf}}(C)$ contains only trivial bundles on C and so $\mathcal{C}^{\mathrm{EN}}(C)$, $\mathcal{C}^{\mathrm{N}}(C)$ and $\mathcal{C}^{\mathrm{uni}}(C)$. Consider a fiber functor

$$\tau : \mathcal{C}^{\mathrm{nf}}(C) \longrightarrow \mathbf{QCoh}(C)$$

defined by an inclusion. Let $\overline{x} : \operatorname{Spec} \mathbb{C} \to C$ be a geometric point, then we have $\tau' := \overline{x}^* \circ \tau : \mathcal{C}^{\mathrm{nf}}(C) \to \mathbf{Vec}_{\mathbb{C}}$ a fiber functor taking value in $\mathbf{Vec}_{\mathbb{C}}$. By Tannaka duality, we get an S-fundamental $\mathbb{R}$-groupoid scheme $\Pi^{\mathrm{S}}(C, \tau') \simeq \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ which act transitively on $\operatorname{Spec} \mathbb{C}$ for the corresponding Tannakian category $(\mathcal{C}^{\mathrm{nf}}(C), \tau')$. By Remark 2.4, we have

$$\Pi^{\mathrm{S}}(C, \tau) \simeq C \times_{\mathbb{R}} C$$

and so $\Pi^{\mathrm{EN}}(C, \tau) \simeq \Pi^{\mathrm{N}}(C, \tau) \simeq \Pi^{\mathrm{uni}}(C, \tau) \simeq C \times_{\mathbb{R}} C$.

Remark 4.5. Applying a similar computation for $\mathbb{P}^1_{\mathbb{R}}$, with a rational point $x : \operatorname{Spec} \mathbb{R} \to \mathbb{P}^1_{\mathbb{R}}$ and a fiber functor x^*, yields $\Pi^{\mathrm{S}}(\mathbb{P}^1_{\mathbb{R}}, x^*) \simeq \operatorname{Spec} \mathbb{R} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{R} \simeq \operatorname{Spec} \mathbb{R}$. Thus, we obtain the trivial group scheme over $\mathbb{R}$. More generally, if X is k-scheme with a k-rational point x such that the category $\mathcal{C}^{\mathrm{nf}}(X)$ contains only trivial bundles, then $\Pi^{\mathrm{S}}(X, x^*) \simeq \operatorname{Spec} k$.

Remark 4.6. Suppose X is a k-scheme such that the category $\mathcal{C}^{\mathrm{nf}}(X)$ contains only trivial bundles. If X does not have any k-point, then $\Pi^{\mathrm{S}}(X, \overline{x}^*) \simeq \operatorname{Spec} \overline{k} \times_k \operatorname{Spec} \overline{k}$, where $\overline{x} : \mathcal{C}^{\mathrm{nf}}(X) \to \operatorname{Spec} \overline{k}$ is a geometric point. For example, X is nondegenerate conic [4] or Brauer-Severi varieties [15, 16].

4.2. **Fundamental groupoid of a Klein bottle.** A *Klein bottle* X is a geometrically connected smooth projective curve of genus one defined over $\mathbb{R}$ and it does not have any real point. In other words, it is a smooth elliptic curve $X_{\mathbb{C}}$ over $\mathbb{C}$ with an anti-holomorphic involution which does not have any fixed point.

Let L be a torsion (or finite) line bundle on X. Therefore, there exist a smallest positive integer $n \in \mathbb{N}$ such that $L^{\otimes n} \simeq \mathcal{O}_X$ and for any $1 \leq m \leq n-1$, we have $L^{\otimes m} \not\simeq \mathcal{O}_X$. Let $\mathcal{C}_L$ be the smallest $\mathbb{R}$-linear abelian rigid tensor full subcategory of $\mathbf{Coh}(X)$ containing a line bundle L. By following the construction [14, Page no. 83], we have

$$\mathrm{Ob}(\mathcal{C}_L) = \left\{ \bigoplus_{i=1}^{i=k} V_i : V_i \in \{\mathcal{O}_X, L, L^{\otimes 2}, ..., L^{\otimes n-1}\} \text{ and } k \in \mathbb{N} \right\}.$$

Consider the following fiber functor corresponding to the identity $e : \operatorname{Spec} \mathbb{C} \to X_{\mathbb{C}}$ element of $X_{\mathbb{C}}$:

$$\omega : \mathcal{C}_L \longrightarrow \mathbf{Coh}(X_{\mathbb{C}}) \longrightarrow \mathbf{Vec}_{\mathbb{C}}$$

The pair $(\mathcal{C}_L, \omega)$ forms Tannakian category over $\mathbb{R}$. By Theorem 3.4, we have a groupoid scheme $(t, s) : \Pi \to \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ acting transitively on $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. Let Π_t denote the groupoid scheme Π with the base scheme $\operatorname{Spec} \mathbb{C}$ by considering $pr_1 \circ (t, s) : \Pi \to \operatorname{Spec} \mathbb{C}$. We claim that $\Pi_t(\operatorname{Spec} \mathbb{C}) = \frac{\mathbb{Z}}{n\mathbb{Z}} \times \mathrm{Gal}(\mathbb{C}|\mathbb{R})$. Consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{C} & \xrightarrow{\quad a \quad} & \Pi \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle (t,s)} \\ \operatorname{Spec} \mathbb{C} & \xleftarrow{\;pr_1\;} & \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \end{array} \tag{4.1}$$

Note that $\Pi_t(\operatorname{Spec} \mathbb{C}) = \{a : \text{the diagram (4.1) commutes}\}$. We have only two morphisms, $(\mathrm{id}, \mathrm{id}) : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ and $(\mathrm{id}, \overline{\mathrm{id}}) : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \\ & {\scriptstyle \mathrm{id}} \searrow & \big\downarrow{\scriptstyle pr_1} \\ & & \operatorname{Spec} \mathbb{C} \end{array} \tag{4.2}$$

commutes; where $\overline{\mathrm{id}}$ denote the morphism correspond to complex conjugate. Thus, we can partition the set $\Pi_t(\operatorname{Spec} \mathbb{C})$ as the following disjoint union:

$$\Pi_t(\operatorname{Spec} \mathbb{C}) = \{a : (t, s) \circ a = (\mathrm{id}, \mathrm{id})\} \cup \{a : (t, s) \circ a = (\mathrm{id}, \overline{\mathrm{id}})\}.$$

By definition, Π is functor

$$\Pi : (\mathbf{Sch}|_{\operatorname{Spec}\mathbb{C}\times_{\mathbb{R}}\operatorname{Spec}\mathbb{C}}) \to \mathbf{Sets}$$

given by $(h : T \to \operatorname{Spec}\mathbb{C} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C}) \mapsto \mathbf{Isom}^{\otimes}_{T}(h^* pr_2^* \omega, h^* pr_1^* \omega)$; where $(\mathbf{Sch}|_{\operatorname{Spec}\mathbb{C}\times_{\mathbb{R}}\operatorname{Spec}\mathbb{C}})$ denote the category of schemes over $\operatorname{Spec}\mathbb{C} \times_{\mathbb{R}} \operatorname{Spec}\mathbb{C}$ and **Sets** denote the category of sets.

Therefore, we have

$$\{a : (t, s) \circ a = (\mathrm{id}, \mathrm{id})\} = \Pi(\mathrm{id}, \mathrm{id}) = \frac{\mathbb{Z}}{n\mathbb{Z}}$$

and

$$\{a : (t, s) \circ a = (\mathrm{id}, \overline{\mathrm{id}})\} = \Pi(\mathrm{id}, \overline{\mathrm{id}}) = \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

Hence, $\Pi_t(\operatorname{Spec}\mathbb{C}) = \frac{\mathbb{Z}}{n\mathbb{Z}} \times \operatorname{Gal}(\mathbb{C}|\mathbb{R})$. Similarly, let $\mathcal{C}_{tor}$ be the smallest Tannakian category containing all torsion line bundles on X having a fiber functor same as $\omega : \mathcal{C}_{tor} \to \mathbf{Vec}_{\mathbb{C}}$, then we have $\Pi_t(\operatorname{Spec}\mathbb{C}) = \frac{\mathbb{Q}}{\mathbb{Z}} \times \operatorname{Gal}(\mathbb{C}|\mathbb{R})$ of the corresponding groupoid.

4.3. **Fundamental groupoid of an abelian variety.** Throughout this section, let A be an abelian variety over a field k of characteristic zero. Recall that, $\mathcal{C}^{\mathrm{uni}}(A), \mathcal{C}^{\mathrm{N}}(A), \mathcal{C}^{\mathrm{EN}}(A)$ are k-linear abelian rigid tensor category. To apply Tannaka duality, consider the fiber functor define by inclusion in $\mathbf{QCoh}(A)$.

$$i_{\mathrm{uni}}, i_{\mathrm{N}}, i_{\mathrm{EN}} : \mathcal{C}^{\mathrm{uni}}(A), \mathcal{C}^{\mathrm{N}}(A), \mathcal{C}^{\mathrm{EN}}(A) \longrightarrow \mathbf{QCoh}(A)$$

The corresponding groupoid k-scheme denoted by $\Pi^{\mathrm{uni}}(A), \Pi^{\mathrm{N}}(A), \Pi^{\mathrm{EN}}(A)$ which acts transitively on A. By definition, we have

$$\Pi^{\mathrm{uni}}(A) = \mathbf{Aut}^{\otimes}_{k}(i_{\mathrm{uni}}) = \mathbf{Isom}^{\otimes}_{A\times_k A}(pr_2^* i_{\mathrm{uni}}, pr_1^* i_{\mathrm{uni}})$$

$$\Pi^{\mathrm{N}}(A) = \mathbf{Aut}^{\otimes}_{k}(i_{\mathrm{N}}) = \mathbf{Isom}^{\otimes}_{A\times_k A}(pr_2^* i_{\mathrm{N}}, pr_1^* i_{\mathrm{N}})$$

$$\Pi^{\mathrm{EN}}(A) = \mathbf{Aut}^{\otimes}_{k}(i_{\mathrm{EN}}) = \mathbf{Isom}^{\otimes}_{A\times_k A}(pr_2^* i_{\mathrm{EN}}, pr_1^* i_{\mathrm{EN}})$$

where $pr_1, pr_2 : A \times_k A \longrightarrow A$ denote the first and second projection respectively. The following theorem establishes a relation among them.

Theorem 4.7. *Let A be an abelian variety defined over an algebraically closed field k of characteristic zero. Then, we have*

$$\Pi^{\mathrm{EN}}(A) \simeq \Pi^{\mathrm{uni}}(A) \times_k \Pi^{\mathrm{N}}(A).$$

Proof. Let ω denote the any one of $i_{\mathrm{uni}}, i_{\mathrm{N}}, i_{\mathrm{EN}}$ and **Groupoid** denote the category of groupoid. Then $\mathbf{Aut}^{\otimes}_{k}(\omega)$ is a functor

$$\mathbf{Aut}^{\otimes}_{k}(\omega) : (\mathbf{Sch}|_{A\times_k A}) \longrightarrow \mathbf{Groupoid}$$

defined by $\phi : Y \to A \times_k A$ maps to $\mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* \omega, \phi^* pr_1^* \omega)$. We want to prove that, a groupoid $\mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{EN}}, \phi^* pr_1^* i_{\mathrm{EN}})$ is equivalent to a groupoid $\mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{uni}}, \phi^* pr_1^* i_{\mathrm{uni}})$

$\times\, \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{N}}, \phi^* pr_1^* i_{\mathrm{N}})$. Note that product of two groupoid is again a groupoid. Let us define a functor

$$\mathbb{F} : \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{EN}}, \phi^* pr_1^* i_{\mathrm{EN}}) \longrightarrow \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{uni}}, \phi^* pr_1^* i_{\mathrm{uni}}) \times \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{N}}, \phi^* pr_1^* i_{\mathrm{N}})$$

defined by $\eta : \phi^* pr_2^* i_{\mathrm{EN}} \to \phi^* pr_1^* i_{\mathrm{EN}}$ map to $(\eta|_{\mathcal{C}^{\mathrm{uni}}(A)}, \eta|_{\mathcal{C}^{\mathrm{N}}(A)})$. On the other way, let us define a functor

$$\mathbb{G} : \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{uni}}, \phi^* pr_1^* i_{\mathrm{uni}}) \times \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{N}}, \phi^* pr_1^* i_{\mathrm{N}}) \longrightarrow \mathbf{Isom}^{\otimes}_{Y}(\phi^* pr_2^* i_{\mathrm{EN}}, \phi^* pr_1^* i_{\mathrm{EN}})$$

define by $(\lambda_1 : \phi^* pr_2^* i_{\mathrm{uni}} \to \phi^* pr_1^* i_{\mathrm{uni}}, \lambda_2 : \phi^* pr_2^* i_{\mathrm{N}} \to \phi^* pr_1^* i_{\mathrm{N}})$ map to $\lambda_1 \otimes \lambda_2 : \phi^* pr_2^*(i_{\mathrm{uni}} \otimes i_{\mathrm{N}}) \to \phi^* pr_1^*(i_{\mathrm{uni}} \otimes i_{\mathrm{N}})$.

We have $(\lambda_1 \otimes \lambda_2)|_{\mathcal{C}^{\mathrm{uni}}(A)} \simeq \lambda_1$ and $(\lambda_1 \otimes \lambda_2)|_{\mathcal{C}^{\mathrm{N}}(A)} \simeq \lambda_2$. Hence, $\mathbb{F} \circ \mathbb{G} = \mathbf{Id}$, and $\mathbb{G} \circ \mathbb{F} = \mathbf{Id}$ follows from the fact that any semi-finite bundle can be written as the tensor product of unipotent and finite bundle in a unique way up to isomorphism [1, Remark 3.7]. □

5. Representations

The notion of closed immersion and faithfully flat morphism between affine group schemes are completely characterize by their representations (see [6, Proposition 2.21]). In this section, we will extend this to an affine groupoid case. Let k be a field and $\overline{k}$ denote its algebraic closure. Let $S = \operatorname{Spec} \overline{k}$ be an affine scheme over a field k. Let $(G_1, s_1, t_1, m_1, e_1, i_1)$ and $(G_2, s_2, t_2, m_2, e_2, i_2)$ are two affine k-groupoid schemes acting transitively on a k-scheme S. Therefore, we have following morphisms for $i = 1, 2$:

- a faithfully flat affine morphism $(t_i, s_i) : G_i \to S \times_k S$
- a morphism of $S \times_k S$-scheme $m_i : G_i \times_{{}^sS^t} G_i \to G_i$, $e_i : S \to G_i$ and $inv_i : G_i \to G_i$.

Let $G_1 = \operatorname{Spec} L_1$ and $G_2 = \operatorname{Spec} L_2$; where L_1 and L_2 are faithfully flat $\overline{k} \otimes_k \overline{k}$-algebra as (t_i, s_i) are faithfully flat affine morphisms. A morphism between two groupoid schemes is defined as an obvious way.

Definition 5.1. *A morphism $f : (G_1, s_1, t_1, m_1, e_1, i_1) \to (G_2, s_2, t_2, m_2, e_2, i_2)$ between two k-groupoid schemes acting on a k-scheme S is a morphism of k-scheme $f : G_1 \to G_2$ satisfying $s_2 \circ f = s_1, t_2 \circ f = t_1, f \circ m_1 = m_2 \circ (f, f), e_2 = f \circ e_1$ and $f \circ inv_1 = inv_2 \circ f$.*

Let $f : G_1 \to G_2$ be a morphism between k-groupoid schemes acting on a k-scheme S. Then, it induced a morphism on diagonal group schemes $f^{\Delta} = f \times \mathrm{id}_S : G_1^{\Delta} \to G_2^{\Delta}$ by the base change $\Delta : S \to S \times_k S$. Note that, f is an isomorphism if and only if f^{Δ} is an isomorphism (see [5, 3.5.2]).

Definition 5.2. *A morphism $f : G_1 \to G_2$ of groupoid is called faithfully flat (resp. closed immersion) if it is faithfully flat (resp. closed immersion) as a k-scheme morphism.*

Lemma 5.3. *A morphism $f : G_1 \to G_2$ is faithfully flat if and only if $f^{\Delta} : G_1^{\Delta} \to G_2^{\Delta}$ is faithfully flat.*

Proof. Suppose $f : G_1 \to G_2$ is faithfully flat map, then $f^{\#} : L_2 \to L_1$ is a faithfully flat algebra map. Hence, L_1 is faithfully flat L_2-module. Consider a base change under the map $L_2 \to L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}$, then $L_1 \otimes_{L_2} (L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}) \simeq L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}$ is faithfully flat $L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}$-module. Hence, $f^{\Delta} : G_1^{\Delta} \to G_2^{\Delta}$ is a faithfully flat. Conversely, assume that, the map $f^{\#} \otimes \mathrm{id} : L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \to L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}$ is a faithfully flat. We have the following commutative diagram:

$$\begin{array}{ccc} L_2 & \xrightarrow{f^{\#}} & L_1 \\ \downarrow & & \downarrow \\ L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} & \xrightarrow{f^{\#}\otimes \mathrm{id}} & L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \end{array} \tag{5.1}$$

Consider the map $g : L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \to L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \otimes_k \overline{k} \simeq L_1$ by the inclusion, which is faithfully flat. Note that this map makes the above diagram commutative. Hence, $f^{\#} : L_2 \to L_1$ is faithfully flat as it is compositions of faithfully flat maps. $\square$

Lemma 5.4. *A morphism $f : G_1 \to G_2$ is closed immersion if and only if $f^{\Delta} : G_1^{\Delta} \to G_2^{\Delta}$ is closed immersion.*

Proof. We have $\Delta : S \to S \times_k S$ is closed immersion as S is an affine scheme. The projection map $G_i^{\Delta} \to G_i$ is the base change of $\Delta : S \to S \times_k S$ by $(t_i, s_i) : G_i \to S \times_k S$, so $G_i^{\Delta} \to G_i$ is closed immersion. If f is closed immersion, then f^{Δ} is closed immersion by the diagram (5.1). Let f^{Δ} be a closed immersion. It is morphism of affine schemes so $f^{\#} \otimes \mathrm{id} : L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \to L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k}$ is surjective. This implies that the map $f^{\#} \otimes \mathrm{id} \otimes \mathrm{id} : L_2 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \otimes_k \overline{k} \to L_1 \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \otimes_k \overline{k}$ is surjective and compactible with $f^{\#}$ by considering standard isomophism $L_i \otimes_{\overline{k}\otimes_k\overline{k}} \overline{k} \otimes_k \overline{k} \to L_i$. Hence, $f^{\#}$ is a surjective. $\square$

Let $\omega_f : \mathbf{Rep}(S : G_2) \to \mathbf{Rep}(S : G_1)$ denote the induced tensor functor on the representation categories defined by $(V, \rho) \mapsto (V, f \circ \rho)$.

Theorem 5.5. *Let $f : G_1 \to G_2$ be a morphism between two affine k-groupoid scheme G_1 and G_2 acting transitively on a k-scheme $S = \operatorname{Spec} \overline{k}$. Let $\omega_f : \mathbf{Rep}(S : G_2) \to \mathbf{Rep}(S : G_1)$ be the induced functor. Then,*

1. *f is faithfully flat if and only if ω_f is fully faithful and every subobject of $\omega_f(X)$, for $X \in \mathrm{Ob}(\mathbf{Rep}(S : G_2))$, is isomorphic to image of a subobject of X.*
2. *f is closed immersion if and only if every object of $\mathbf{Rep}(S : G_1)$ is isomorphic to a subquotient of an object $\omega_f(X)$, for $X \in \mathrm{Ob}(\mathbf{Rep}(S : G_2))$.*

Proof. (1) Let $f : G_1 \to G_2$ be a faithfully flat morphism. Then, we have an equivalence between $\mathbf{Rep}(S : G_2)$ and the subcategory of $\mathbf{Rep}(S : G_1)$ such that the representation of G_1 factor through the map f. Hence, ω_f has the stated property. Conversely, suppose that ω_f has the stated properties. Note that any representation (V, ρ) of G_2 is determined by the pair $(V, s_2^*V \simeq t_2^*V)$; where V is quasi-coherent $\mathcal{O}_S$-module. The image of (V, ρ)

under the functor ω_f is determind by $(V, f^*s_2^*V \simeq f^*t_2^*V)$ as $t_2 \circ f = t_1$ and $s_2 \circ f = s_1$. We have the following commutative diagram:

$$\begin{array}{ccc} G_1^{\Delta} & \xrightarrow{f^{\Delta}} & G_2^{\Delta} \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{f} & G_2 \end{array} \tag{5.2}$$

The above diagram gives the same condition on the functor $\omega_{f^{\Delta}}$ which implies that f^{Δ} is faithfully flat [6, Proposition 2.21]. Hence, so f by Lemma 5.3.

(2)Let $\mathcal{D}$ be the full subcategory of $\mathbf{Rep}(S : G_1)$ whose objects are subquotients of $\omega_f(X)$ for $X \in \mathrm{Ob}(\mathbf{Rep}(S : G_2))$. Clearly, $(\mathcal{D}, \omega_1|_{\mathcal{D}})$ is Tannakian category. The sequence $\mathbf{Rep}(S : G_2) \to \mathcal{D} \to \mathbf{Rep}(S : G_1)$ induced a sequence on a groupoid schemes $G_1 \to G_{\mathcal{D}} \to G_2$; where $G_{\mathcal{D}}$ denote the corresponding groupoid scheme acting transitively on S. By taking a diagonal group scheme, we have the same properties for the representation categories of diagonal group schemes. Then, the assertion follows by [6, Proposition 2.21] and Lemma 5.4. □

Let X be a k-scheme of finite type with $\mathrm{H}^0(X, \mathcal{O}_X) = k$. Let $X_{\overline{k}}$ denote the base change and let $x : \operatorname{Spec} \overline{k} \to X_{\overline{k}}$ be a rational point of $X_{\overline{k}}$. Consider the fiber functor by taking pull back via the composition map $\operatorname{Spec} \overline{k} \to X_{\overline{k}} \to X$ for all the categories

$$\omega : \mathcal{C}^{\mathrm{nf}}(X), \mathcal{C}^{\mathrm{EN}}(X), \mathcal{C}^{\mathrm{N}}(X), \mathcal{C}^{\mathrm{uni}}(X) \to \mathbf{QCoh}(X_{\overline{k}}) \to \mathbf{Vec}_{\overline{k}}.$$

Corollary 5.6. *We have the following diagram of groupoid schemes with all the maps are faithfully flat:*

$$\begin{array}{ccccc} \Pi^{\mathrm{S}}(X,\omega) & \longrightarrow & \Pi^{\mathrm{EN}}(X,\omega) & \longrightarrow & \Pi^{\mathrm{N}}(X,\omega) \\ & & \downarrow & & \\ & & \Pi^{\mathrm{uni}}(X,\omega) & & \end{array}$$

6. Semi-finite bundles on complex tori and Riemann surfaces

In this section, we have extended some results of [8] on the category of semi-finite bundles.

Schottky functor. Let X be a connected topological space which is locally simply connected. Let $\mathbb{C}_X$ denote the constant sheaf corresponding to ring $\mathbb{C}$. Let $\mathcal{F}$ be the sheaf of rings with the morphism $i : \mathbb{C}_X \to \mathcal{F}$ of sheaves. Let $\pi_1(X)$ denote the topological fundamental group of X with respect to some base point. Let Σ be a group with the surjective group homomorphism $\alpha : \pi_1(X) \to \Sigma$. This homomorphism induces a morphism of group $\mathbb{C}$-algebra $\tilde{\alpha} : \mathbb{C}\pi_1(X) \to \mathbb{C}\Sigma$ which further gives a pull-back functor from the category of $\mathbb{C}\Sigma$-modules to the category of $\mathbb{C}\pi_1(X)$-modules:

$$\tilde{\alpha}^* : \mathbb{C}\Sigma\text{–mod} \longrightarrow \mathbb{C}\pi_1(X)\text{–mod}.$$

The essential image of this functor is called *Schottky modules.*

Definition 6.1. *A $\mathbb{C}\pi_1(X)$-module M is called Schottky module if there is a $\mathbb{C}\Sigma$-module N such that the induced $\mathbb{C}\pi_1(X)$-module structure on N given by the map $\tilde{\alpha} : \mathbb{C}\pi_1(X) \to \mathbb{C}\Sigma$ agrees with M.*

There is natural functor from the category of representations of fundamental group to the category of vector bundles. Specifically, it comes from Riemann-Hilbert correspondence (see Theorem 6.3).

Definition 6.2. *A sheaf of ring $\mathcal{F}$ on X is called locally constant sheaf if there is a open cover $\{U_i\}_{i\in I}$ of X such that the restriction of $\mathcal{F}$ to each open set U_i is constant sheaf on U_i.*

Theorem 6.3. [18, Theorem 2.5.15] *There is an equivalence between the category* $\mathrm{LC}(\mathbb{C})$ *of locally constant sheaves of $\mathbb{C}$-modules and the category of $\mathbb{C}\pi_1(X)$-modules.*

Consider the Riemann-Hilbert functor $RH : \mathbb{C}\pi_1(X)\text{–mod} \longrightarrow \mathrm{LC}(\mathbb{C})$ and the functor $- \otimes_{\mathbb{C}_X} \mathcal{F} : \mathrm{LC}(\mathbb{C}) \longrightarrow \mathcal{F} - \mathrm{Mod}$. Let us take the following compositions of functors:

$$S := (- \otimes_{\mathbb{C}_X} \mathcal{F}) \circ RH \circ \tilde{\alpha}^* : \mathbb{C}\Sigma\text{–mod} \longrightarrow \mathcal{F}\text{–mod}.$$

The above functor is known to be *Schottky functor*. Note that the functor S takes value in locally free sheaves of $\mathcal{F}$-modules. If X is a complex manifold and $\mathcal{F} = \mathcal{O}_X$ be the sheaf of holomorphic functions on X, then given a $\mathbb{C}\Sigma$-modules M, we have a holomorphic vector bundle $S(M)$ on X.

Proposition 6.4. [8, Proposition 2.1] *Let X be a complex manifold. Then, the functor $S : \mathbb{C}\Sigma\text{–mod} \longrightarrow \mathcal{O}_X\text{–mod}$ is faithful, exact, additive and commutes with direct sums and tensor products.*

Definition 6.5. *A $\mathbb{C}\Sigma$-module M is called finite if there exist a two different polynomials $f \neq g \in \mathbb{N}[t]$ such that $f(M) \simeq g(M)$; where product is defined as tensor product and sum is given by direct sum.*

Let $K(\mathbb{C}\Sigma\text{–mod})$ denote the Grothendieck group associated to monoid $\mathbb{C}\Sigma\text{–mod}/\sim_{\mathrm{iso}}$. We have an analogues result of finite bundles [14, Chapter-I, Lemma 3.1] for the finite modules.

Lemma 6.6. *Let M be a $\mathbb{C}\Sigma$-module. Then, the following are equivalent:*

(1) *M is finite.*
(2) *M is integral over $\mathbb{Z}$ in $K(\mathbb{C}\Sigma\text{–mod})$.*
(3) *$M \otimes 1$ is integral over $\mathbb{Q}$ in $K(\mathbb{C}\Sigma\text{–mod}) \otimes_{\mathbb{Z}} \mathbb{Q}$.*
(4) *$\{$indecomposable component of $M^{\otimes n} : n \in \mathbb{N}\}$ is finite set.*

Proof. Similar to [14, Chapter-I, Lemma 3.1]. □

Note that unipotent and semi-finite modules define in obvious way. In the next two section, we will study the behavior of this functor S when we restrict it on the category of finite and semi-finite modules for the base space complex tori and Riemann surfaces.

6.1. **Semi-finite Schottky bundles on complex tori.** In this section, we will prove the analogue result of [8, Theorem 1.1] for a bigger category of semi-finite bundles. Let $X = V/\Lambda$ be a complex torus with the dimension of V is g. There is a basis $\{e_1, ..., e_g\}$ of V and a basis $\{\lambda_1, ..., \lambda_{2g}\}$ of Λ such that $\lambda_i = \sum_{j=1}^{g} \pi_{ij} \cdot e_j$ and the period matrix Π of X is $\Pi = (\pi_{ij})^t_{2g\times g} = (Z, I)$; where $I_{g\times g}$ is identity and $Z_{g\times g}$ is symmetric complex matrix (see [9, Section 8]).

Let Σ be a free abelian group with g generators $B_1, ..., B_g$. Define $\alpha : \pi_1(X) \simeq \Lambda \to \Sigma$ by $\lambda_i \mapsto B_i$ and $\lambda_{i+g} \mapsto 0$ for all $i = 1, ..., g$. Then, we have the following Schottky functor:

$$S : \mathbb{C}\Sigma\text{–mod} \longrightarrow \mathcal{O}_X\text{–mod}$$

Let $\mathbf{F}_{\mathbb{C}\Sigma}, \mathbf{U}_{\mathbb{C}\Sigma}$ and $\mathbf{SF}_{\mathbb{C}\Sigma}$ denote the full subcategory of $\mathbb{C}\Sigma$–mod with the objects are finite, unipotent and semi-finite $\mathbb{C}\Sigma$-modules, respectively. In [8, Theorem 1.1], they proved that the functor S gives an equivalence between $\mathbf{U}_{\mathbb{C}\Sigma}$ and $\mathcal{C}^{\text{uni}}(X)$. In Theorem 6.9, we proved an equivalence between $\mathbf{SF}_{\mathbb{C}\Sigma}$ and $\mathcal{C}^{\text{EN}}(X)$.

Remark 6.7. By [1, Theorem 3.3], we conclude that any finite bundle on X is a direct sum of torsion line bundles and any semi-finite bundle on X is direct sum $\bigoplus_i L_i \otimes U_i$; where L_i's are torsion line bundles and U_i's are unipotent bundles. This implies that, every semi-finite bundle is homogeneous by [13, Theorem 4.17].

Lemma 6.8. *There is an equivalence between the following categories:*

$$S : \mathbf{F}_{\mathbb{C}\Sigma} \simeq \mathcal{C}^{\text{N}}(X).$$

Proof. Let V be a finite bundle on X, then V is direct sum of torsion line bundle by Remark 6.7. A torsion line bundle has degree zero, so it is Schottky by [8, Lemma 7.1]. Hence, the functor S is essentially surjective as S is compatible with direct sum. The functor S is faithful by Proposition 6.4. Thus, it is enough to prove that $\phi : \text{Hom}_{\mathbb{C}\Sigma}(\mathbb{C}, M) \to \text{Hom}_{\mathcal{O}_X}(S(\mathbb{C}), S(M)) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L) \simeq \Gamma(X, L)$ is a surjective map as Hom commutes with direct sum and by adjoint property; where M is torsion module and L is torsion line bundle. Suppose $L \neq \mathcal{O}_X$. If L has a global section which vanishes at some point, then $L^{\otimes n}$ also has a global section which vanishes at some point. This leads to a contradiction because $L^{\otimes n} \simeq \mathcal{O}_X$ for some n. Hence, $\Gamma(X, L) = 0$ for $L \neq \mathcal{O}_X$. If $L = \mathcal{O}_X$, then $\text{Hom}_{\mathbb{C}\Sigma}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C} \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$. Hence, ϕ is a surjective map. □

Theorem 6.9. *Let X be a complex torus of dimension g, and let Σ be a free abelian group of rank g. Then, the Schottky functor induced an equivalence between the category $\mathbf{SF}_{\mathbb{C}\Sigma}$ of semi-finite $\mathbb{C}\Sigma$-modules and the category $\mathcal{C}^{\text{EN}}(X)$ of semi-finite bundles on X:*

$$S : \mathbf{SF}_{\mathbb{C}\Sigma} \simeq \mathcal{C}^{\text{EN}}(X).$$

Proof. Let V be a semi-finite bundle on X. By Remark 6.7, an indecomposable component of V is $L \otimes U$; where L is torsion line bundle and U is unipotent bundle. The category $\mathcal{C}^{\text{EN}}(X)$ is rigid, so Hom distributes over tensor product. Hence, the functor S is full and essentially surjective by Lemma 6.8 and [8, Theorem 1.1]. Therefore, S is an equivalence as it is faithful by Proposition 6.4. □

6.2. **Semi-finite Schottky bundles on Riemann surfaces.** Let X be a compact Riemann surface of genus g. Let $\pi_1(X)$ denote the fundamental group of it. There is a basis $\{a_1, ..., a_g, b_1, ..., b_g\}$ of $\pi_1(X)$ satisfying the relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$. Let $\Sigma = F_g$ be the free group of g generators $B_1, ..., B_g$. Let $\alpha : \pi_1(X) \to \Sigma$ be a homomorphism defined by $a_i \mapsto 1$ and $b_i \mapsto B_i$ for all $1 \leq i \leq g$. This map defines the *Schottky functor* for X.

$$S : \mathbb{C}F_g\text{–mod} \longrightarrow \mathcal{O}_X\text{–mod}$$

Let $\tilde{X} \to X$ be the universal cover of X, then it is a principal $\pi_1(X)$-bundle on X. Let G be a connected affine algebraic group over $\mathbb{C}$. For any $\rho : \pi_1(X) \to G$, we can define an associated principal G-bundle by $P_\rho := (\tilde{X} \times G)/\sim$; where $(\tilde{x}, g) \sim (\tilde{x} \cdot h, \rho(h)^{-1} g)$ for $\tilde{x} \in \tilde{X}, g \in G$ and $h \in \pi_1(X)$.

Definition 6.10. *A principal G-bundle P is called Schottky bundle if $P \simeq P_\rho$ for some $\rho \in \mathrm{Hom}(\pi_1(X), G)$ and $\rho(\mathrm{Ker}\,\alpha) \subset Z(G)$.*

In [2], Biswas defined finite principal G-bundle and proved some equivalent criteria of it. Here, we adopted one of his equivalent criteria as a definition.

Definition 6.11. *A principal G-bundle is called finite if all the vector bundle associated to representations of G is finite.*

Proposition 6.12. *Let P be a finite principal G-bundle with $Z(G)$ is trivial. Then, P is G-Schottky bundle iff all bundles associated to a G-representation are Schottky.*

Proof. Suppose P is G-Schottky, then there is a homomorphism $\rho : \pi_1(X) \to G$ such that $P \simeq P_\rho$ and $\rho(\gamma) = 1$ for all $\gamma \in \mathrm{Ker}(\alpha)$. Let W be any finite dimensional complex G-module. By the condition $\rho(\mathrm{Ker}(\alpha)) = 1$, we have a homomorphism $\Sigma \to G$ which gives W the structure of $\mathbb{C}\Sigma$-module. Hence, the corresponding vector bundle is Schottky.

Conversely, assume that all bundles associated to a G-representation are Schottky. Since P admits a flat connection [2, Theorem 1.1], there is a homomorphism $\theta : \pi_1(X) \to G$ such that $P \simeq P_\theta$. Consider the adjoint representation $Ad : G \to GL(\mathfrak{g})$. The vector bundle corresponding to Ad is the bundle corresponding to the representation $Ad \circ \theta : \pi_1(X) \to GL(\mathfrak{g})$, which is Schottky by assumption. Hence, $Ad \circ \theta(\mathrm{Ker}(\alpha)) \in Z(G) = 1$, which implies $\theta(\mathrm{Ker}(\alpha)) = 1$. □

Definition 6.13. *Let $x_0 \in X$ be a point. A flat vector bundle V on X is called abelian if there exist a flat bundle W on $X \times X$ such that $W|_{X \times \{x_0\}} \simeq W|_{\{x_0\} \times X} \simeq V$.*

Proposition 6.14. *All flat abelian bundles come from representations of $\pi_1(X)_{ab}$. Conversely, every bundle corresponding to the representation of $\pi_1(X)_{ab}$ is abelian.*

Proof. Let V be a flat abelian bundle on X with the corresponding representation $\rho_V \in \mathbb{C}[\pi_1(X)]$–mod. There is a flat vector bundle V' on $X \times X$ with the corresponding representation $\rho_{V'} \in \mathbb{C}[\pi_1(X) \times \pi_1(X)]$–mod such that $V'|_{\{x_0\} \times X} \simeq V$ and $V'|_{X \times \{x_0\}} \simeq V$ for a point $x_0 \in X$. Hence, $\rho_{V'}(e,a) = \rho_{V'}(a,e) = \rho_V(a)$ for all $a \in \pi_1(X)$ and that implies $\rho_{V'}(a,b) = \rho_{V'}(a,e)\cdot\rho_{V'}(e,b) = \rho_{V'}(e,a)\cdot\rho_{V'}(b,e) = \rho_{V'}(b,a)$ for all $a,b \in \pi_1(X)$.

$$\begin{aligned}
\rho_V(aba^{-1}b^{-1}) &= \rho_{V'}(aba^{-1}b^{-1}, e)\\
&= \rho_{V'}(a,e)\cdot\rho_{V'}(b,e)\cdot\rho_{V'}(a^{-1},e)\cdot\rho_{V'}(b^{-1},e)\\
&= \rho_{V'}(a,e)\cdot\rho_{V'}(e,b)\cdot\rho_{V'}(a^{-1},e)\cdot\rho_{V'}(b^{-1},e)\\
&= \rho_{V'}(a,b)\cdot\rho_{V'}(a^{-1},e)\cdot\rho_{V'}(b^{-1},e)\\
&= \rho_{V'}(e,b)\cdot\rho_{V'}(a,e)\cdot\rho_{V'}(a^{-1},e)\cdot\rho_{V'}(b^{-1},e)\\
&= \rho_{V'}(b,e)\cdot\rho_{V'}(b^{-1},e)\\
&= \rho_{V'}(e,e)
\end{aligned}$$

Conversely, consider the multiplication map $m : \pi_1(X)_{ab} \times \pi_1(X)_{ab} \to \pi_1(X)_{ab}$ and let $\theta \in \mathbb{C}\pi_1(X)_{ab} - \text{Mod}$ be a representation. We can treat $\tilde{\theta} := \theta$ as a representation of $\pi_1(X)\times\pi_1(X)$ via the quotient map $\pi_1(X)\times\pi_1(X) \to \pi_1(X)_{ab}\times\pi_1(X)_{ab}$ composition with the map m. Note that $\tilde{\theta}|\{e\}\times\pi_1(X) \simeq \tilde{\theta}|\pi_1(X)\times\{e\} \simeq \theta$. This completes the proof. □

Let X and Y be two compact Riemann surfaces of genus g and h, respectively. Then, the fundamental group $\pi_1(X)$ is generated by $\{a_i, b_i : 1 \le i \le g\}$ satisfying the relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$, and the fundamental group $\pi_1(Y)$ is generated by elements $\{c_j, d_j : 1 \le j \le h\}$ satisfying the relation $\prod_{j=1}^{h} c_j d_j c_j^{-1} d_j^{-1} = 1$. We can identify $\pi_1(X\times Y)$ with $\pi_1(X)\times\pi_1(Y)$. We want to define a Schottky functor for the product $X\times Y$. Let F_g and F_h denote the free group generated by $\{A_1,\ldots,A_g\}$ and $\{C_1,\ldots,C_h\}$, respectively. Consider a surjective homomorphism

$$\alpha : \pi_1(X)\times\pi_1(Y) \to F_g \times F_h$$

of groups defined by $a_i \mapsto A_i\,, b_i \mapsto 1\,, c_j \mapsto C_i$ and $d_j \mapsto 1$ for all $1 \le i \le g$ and $1 \le j \le h$.

The homomorphism α gives ring homomorphism $\tilde{\alpha} : \mathbb{C}[\pi_1(X)\times\pi_1(Y)] \to \mathbb{C}[F_g\times F_h]$. Further, it induces a pull-back functor $\tilde{\alpha}^*$ for the corresponding modules category.

$$\tilde{\alpha}^* : \mathbb{C}[F_g\times F_h]\text{–mod} \longrightarrow \mathbb{C}[\pi_1(X)\times\pi_1(Y)]\text{–mod}$$

By Riemann-Hilbert correspondence, we have an equivalence

$$RH : \mathbb{C}[\pi_1(X)\times\pi_1(Y)]\text{–mod} \longrightarrow \text{LC}(\mathbb{C})$$

between the category of $\mathbb{C}[\pi_1(X)\times\pi_1(Y)]$-modules to the category $\text{LC}(\mathbb{C})$ of locally constant sheaves of $\mathbb{C}$-modules on $X\times Y$. Let $\mathbb{C}_{X\times Y}$ denote the constant sheaf of rings on $X\times Y$ defined by $\mathbb{C}$. Then, we have a sheaf morphism $i : \mathbb{C}_{X\times Y} \to \mathcal{O}_{X\times Y}$. This morphism i induces a functor

$$i^* : \text{LC}(\mathbb{C}) \longrightarrow \mathcal{O}_{X\times Y}\text{–mod}$$

defined by $\mathcal{F} \mapsto \mathcal{F}\otimes_{\mathbb{C}_{X\times Y}} \mathcal{O}_{X\times Y}$. Note that this functor takes value in locally free sheaves of $\mathcal{O}_{X\times Y}$-modules. The Schottky functor S is defined as compositions $i^*\circ RH\circ\tilde{\alpha}^*$.

$$S := i^*\circ RH\circ\tilde{\alpha}^* : \mathbb{C}[F_g\times F_h]\text{–mod} \longrightarrow \mathcal{O}_{X\times Y}\text{–mod}$$

Let $\mathbf{U}_{\mathbb{C}\Theta} \subset \mathbb{C}[F_g \times F_h]$–mod denote the full subcategory of unipotent modules; where $\Theta := F_g \times F_h$.

Remark 6.15. The product formula for the unipotent fundamental group scheme is true [14, Chapter-IV, Lemma 8]. i.e. we have $\pi^{\mathrm{uni}}(X \times Y, x \times y) \simeq \pi^{\mathrm{uni}}(X, x) \times \pi^{\mathrm{uni}}(Y, y)$. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ denote the first and second projection map respectively. By product formula and tensor product of Tannakian categories [5, 5.18.1], we conclude that any unipotent vector bundle on $X \times Y$ is isomorphic to $p^*U_1 \otimes q^*U_2$; where U_1 and U_2 are unipotent vector bundles on X and Y, respectively.

Proposition 6.16. *We have the following equivalence of categories:*

$$S : \mathbf{U}_{\mathbb{C}\Theta} \to \mathcal{C}^{\mathrm{uni}}(X \times Y)$$

Proof. First, we will prove essential surjection of this functor S. Let U be any unipotent vector bundle on $X \times Y$. By Remark 6.15, there is $U_1 \in \mathcal{C}^{\mathrm{uni}}(X)$ and $U_2 \in \mathcal{C}^{\mathrm{uni}}(Y)$ such that $U \simeq p^*U_1 \otimes q^*U_2$. By [8, Theorem 1.2], there exist an unipotent $\mathbb{C}F_g$-module M_1 and $\mathbb{C}F_h$-module M_2 such that $S_1(M_1) = p^*U_1$ and $S_2(M_2) = q^*U_2$; where S_1 and S_2 denote the Schottky functor for X and Y, respectively. Let $M := M_1 \otimes_{\mathbb{C}} M_2$ be a $\mathbb{C}[F_g \times F_h] \simeq \mathbb{C}F_g \otimes_{\mathbb{C}} \mathbb{C}F_h$-module. We only need to prove $S(M) = U$. Note that $M_1 \otimes_{\mathbb{C}} M_2$ is isomorphic to $M_1 \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}\Theta} \mathbb{C} \otimes_{\mathbb{C}} M_2$ and as the functor S is compactible with tensor product, we have $S(M) = S(M_1) \otimes S(M_2) = U$. Hence, S is essential surjective. The fullness of the functor S follows by applying the similar argument mentioned in the proof of [8, Theorem 1.1]. □

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