Continuous flows driving Markov processes and multiplicative L^p-semigroups

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Abstract. We develop a method of driving a Markov processes through a continuous flow. In particular, at the level of the transition functions we investigate an approach of adding a first order operator to the generator of a Markov process, when the two generators commute. A relevant example is a measure-valued superprocess having a continuous flow as spatial motion and a branching mechanism which does not depend on the spatial variable. We prove that any flow is actually continuous in a convenient topology and we show that a Markovian multiplicative semigroup on an L^p space is generated by a continuous flow, completing the answer to the question whether it is enough to have a measurable structure, like a C_0-semigroup of Markovian contractions on an L^p-space with no fixed topology, in order to esnsure the existence of a right Markov process associated to the given semigroup. We extend from bounded to unbounded functions the weak generator (in the sense of Dynkin) and the corresponding martingale problem.

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1 Introduction

The solution of a first order differential equation in an Euclidean domain E, a typical example of continuous flow on E, may be regarded as a deterministic Markov process and its generator D acts on functions on E as a derivation, i.e., $D(u^2) = 2uDu$. It turns out this property remains valid for the generator of a right continuous flow on a general state space E, hence the approach herein considered provides a substitute for a gradient type operator in a general setting, possible infinite dimensional.

The purpose of this work is twofold. First, we study Markov processes which are driven by continuous flows, namely processes X^Φ admitting the structure

$$X_t^\Phi = \Phi_t(X_t), t \geqslant 0, \tag{1.1}$$

where Φ is a continuous flow and X is a Markov process on E. Second, we investigate multiplicative semigroups in an L^p-context and the associated continuous flows, completing the answer given in [5] to the question whether it is enough to have a measurable structure, like a C_0-semigroup of Markovian contractions on an L^p-space, with no fixed topology, in order to

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find a Markov process behind the given semigroup; see also [6], and [8]. We show that the additional property of being multiplicative on L^p (or equivalently, the L^p-generator to be a derivation) is enough for the existence of a continuous flow having the given L^p-semigroup as its transition function.

If L (resp. L^Φ) is the generator of X (resp. X^Φ) and (1.1) holds, then $L^\Phi = L + D$, so, we regard L^Φ as a modification of L with a drift type operator D. In this way, the weak generator (in the sense of E.B. Dynkin) of a Markov process but also of a right continuous flow are main tools in our approach. An example for which our method apply is obtained by taking L to be the fractional power (or more general, a Bochner subordination) of D. We present in particular a method of extending the domain of the weak generator from bounded to unbounded functions, enlarging the class of functions for which the associated martingale problem has a solution; for other related extensions of the weak generator see [31] and [32].

The motivation for the first aim is the application to the measure-valued superprocesses, cf. e.g. [35]. Recall that the state space of a superprocess $\widehat{X}$ is the set $M(E)$ of all positive finite measures on E and the evolution is given by a branching mechanism and a spatial motion which describe the movement of the particles between the branching moments. If the spatial motion is a right continuous flow and the branching mechanism does not depend on the spatial variable then the representation (1.1) holds on $M(E)$ by means of a second superprocess $\widehat{X^0}$ and of the flow on measures induced by Φ,

$$\widehat{X}_t = \Phi_t(\widehat{X^0_t}), t \geqslant 0.$$

Here, the superprocess $\widehat{X^0}$ is such that it has the same branching mechanism as $\widehat{X}$, however, it has no a spatial motion.

The structure and main results of the paper are as follows.

In Section 2 we present the basic facts on the right continuous flows and flows on a space with no fixed topology, called semi-dynamical systems. Theorem 2.4 shows that actually such a flow is continuous in a convenient topology, extending a result from [40]. As a consequence, the induced capacity is tight.

The results on the extended weak generator of a Markov process are exposed in Section 3, including the associated martingale problem. In Subsection 3.1 we study the extended weak generator of a semi-dynamical system. Finally, we show in Subsection 3.2, Proposition 3.9, that a continuous flow may be stopped at the first entry time in the complement of an open set, a procedure already used in [11] and [13]. Several technical proofs are included in the Appendix.

The theory of continuous flows driving Markov process is investigated in Section 4. The main result (Theorem 4.1) about the representation (1.1) and the drift modification of the weak generator of Markov process, is followed by the example on the Bochner subordination of a right continuous flow, stated in Corollary 4.2 from Subsection 4.1. The main application in this framework is given in Subsection 4.2.

Theorem 5.3 from Section 5 is the central result that relates multiplicative L^p-semigroups with continuous flows.

2 Semi-dynamical systems and right continuous flows

Transition functions, resolvent of kernels, and excessive functions. Let $(E, \mathcal{B}(E))$ be a Lusin measurable space, i.e., it is measurable isomorphic to a Borel subset of a metrizable compact space endowed with the Borel σ-algebra.

For a σ-algebra $\mathcal{G}$ we denote by $[\mathcal{G}]$ (resp. $p\mathcal{G}$) the vector space of all real-valued (resp. the set of all positive, numerical) $\mathcal{G}$-measurable functions on E. Also, for a set of real-valued functions $\mathcal{C}$ we denote by $\sigma(\mathcal{C})$ the σ-algebra generated by $\mathcal{C}$, by $[\mathcal{C}]$ the vector space spanned by $\mathcal{C}$, and by $p\mathcal{C}$ (resp. $b\mathcal{C}$) the set of all positive (resp. bounded) functions from $\mathcal{C}$.

We consider a sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}(E))$. A nonnegative, numerical, $\mathcal{B}(E)$-measurable function defined on E is called *$\mathcal{U}$-excessive* provided that

$$(2.1) \qquad \alpha U_\alpha u \leqslant u \quad \text{for all } \alpha > 0, \quad \text{ and } \quad \lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x), x \in E.$$

We denote by $\mathcal{E}(\mathcal{U})$ the set of all real-valued $\mathcal{U}$-excessive functions. If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha>0}$. A $\mathcal{U}_\beta$-excessive function is also called *β-excessive.* If w is a $\mathcal{U}_\beta$-supermedian function (*i.e.,* $\alpha U_{\beta+\alpha} w \leqslant w$ for all $\alpha > 0$), then its *$\mathcal{U}_\beta$-excessive regularisation* $\widehat{w}$ is given by $\widehat{w}(x) := \sup_\alpha \alpha U_{\beta+\alpha} w(x)$, $x\in E$.

Let $\mathbb{T} = (T_t)_{t\geqslant 0}$ be a sub-Markovian *transition function* on $(E, \mathcal{B}(E))$, that is

- T_t is a sub-Markovian kernel on E, $T_0 = Id$, $T_t \circ T_s = T_{t+s}$ for all $t, s > 0$;
- for every $f \in bp\mathcal{B}(E)$ the mapping $(x,t) \to T_t f(x)$ is $\mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+)-$measurable.

Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the resolvent of sub-Markovian kernels induced by $\mathbb{T} = (T_t)_{t\geqslant 0}$,

$$U_\alpha := \int_0^\infty e^{-\alpha t} T_t \,\mathrm{d}t, \text{ for all } \alpha > 0,$$

and let U be the *potential kernel* of $\mathbb{T}$ (and of $\mathcal{U}$), $U := \int_0^\infty T_t \,\mathrm{d}t$.
Recall that condition (2.1) is equivalent with

$$T_t u \leqslant u \text{ for all } t > 0 \text{ and } \lim_{t\searrow 0} T_t u(x) = u(x) \text{ for all } x \in E.$$

If $\beta > 0$ then clearly, $\mathcal{U}_\beta$ is the resolvent of kernels induced by the sub-Markovian transition function $\mathbb{T}_\beta = (e^{-\beta t} T_t)_{t\geqslant 0}$. Notice that the potential kernel of $\mathbb{T}_\beta$ is the bounded kernel U_β, in contrast with the potential kernel U of $\mathbb{T}$ which might be an unbounded kernel.

Assume now that E is a Lusin topological space (i.e., E is homeomorphic to a Borel subset of a metrizable compact space) and let $\mathcal{B}(E)$ its Borel σ-algebra. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x, \zeta)$ be a right Markov process on E having $(P_t)_{t\geqslant 0}$ as transition function, hence

$$P_t f(x) = \mathbb{E}^x(f(X_t), t < \zeta),\ t \geqslant 0,\ f \in p\mathcal{B}(E),$$

and let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the resolvent on $(E, \mathcal{B}(E))$ associated with $(P_t)_{t\geqslant 0}$. The *fine topology* is the coarsest topology on E making continuous all β-excessive functions for some (and equivalently for all) $\beta > 0$. Recall that in this context, a function f from $p\mathcal{B}(E)$ is finely continuous if and only if $t \to f(X_t)$ is a.s. right continuous on $[0, \zeta)$. Using this characterization and the fact that X is has right continuous paths, any continuous function on E is also finely continuous.

Semi-dynamical systems. Let $(E, \mathcal{B})$ be a Lusin measurable space and let $\Phi = (\Phi_t)_{t\geqslant 0}$ be a family of mappings $\Phi_t : E \to E, t \geqslant 0$. Then Φ is called *semi-dynamical system* on E provided that the following conditions are satisfied:

(sd1) $\Phi_{t+s}(x) = \Phi_t(\Phi_s(x))$ for all $s, t > 0$ and $x \in E$;

(sd2) $\Phi_0(x) = x$ for all $x \in E$;

(sd3) For each $t > 0$ the function $E \ni x \longmapsto \Phi_t(x)$ is $\mathcal{B}(E)/\mathcal{B}(E)$-measurable;

(sd4) There exists a countable set $\mathcal{C}_o \subset bp\mathcal{B}$ such that $\mathcal{C}_o$ separates the points of E and $\lim_{t\searrow 0} f(\Phi_t(x)) = f(x)$ for all $x \in E$ and $f \in \mathcal{C}_o$.

In the sequel, if $f \in [\mathcal{B}]$ and N is a kernel on $(E, \mathcal{B}(E))$, then by $Nf \in [\mathcal{B}(E)]$ we mean that $N|f| < \infty$, hence $N(f^+)$ and $N(f^-)$ are real-valued functions and $Nf = N(f^+) - N(f^-)$.

Remark 2.1. *Note that if $\Phi = (\Phi_t)_{t\geqslant 0}$ is a semi-dynamical system on E then the function $E\times[0,\infty) \ni (x,t) \longmapsto \Phi_t(x)$ is $\mathcal{B}(E)\otimes\mathcal{B}([0,\infty))/\mathcal{B}(E)$-measurable. This follows by a monotone class argument, observing first that from (sd4) it follows that for every $f \in \mathcal{C}_o$ the real-valued function $t \longmapsto f(\Phi_t(x))$ is right continuous on $[0,\infty)$.*

For each $t \geqslant 0$ define the Markovian kernel on E as

$$S_t f := f \circ \Phi_t \text{ for all } f \in p\mathcal{B}(E).$$

Then the family $\mathbb{S} = (S_t)_{t\geqslant 0}$ is a Markovian transition function on E, called the *transition function of* the semi-dynamical system $\Phi = (\Phi_t)_{t\geqslant 0}$.

Remark 2.2. *(i) The transition function $\mathbb{S} = (S_t)_{t\geqslant 0}$ of a semi-dynamical system $\Phi = (\Phi_t)_{t\geqslant 0}$ on E is* multiplicative, *that is, $S_t(fg) = (S_tf)(S_tg)$ for all $t \geqslant 0$ and $f, g \in bp\mathcal{B}(E)$.*

(ii) It is known that the converse of assertion (i) holds: Let $\mathbb{S} = (S_t)_{t\geqslant 0}$ be a Markovian transition function on E which is multiplicative and

(2.2) *there exists a countable set $\mathcal{C}_o \subset bp\mathcal{B}$ such that $\mathcal{C}_o$ separates the points of E,*

and $\lim_{t\searrow 0} S_t(x) = f(x)$ for all $x \in E$ and $f \in \mathcal{C}_o$. Then there exists a semi-dynamical system on E, having the transition function $\mathbb{S}$.
Indeed, for $x \in E$ and $t \geqslant 0$ let $S_{t,x}$ be the probability on E induced by the measure $f \longmapsto S_tf(x)$. If $A \in \mathcal{B}(E)$ then, S_t being multiplicative, we have $S_{t,x}(1_A) = (S_{t,x}(1_A))^2$, so, either $S_{t,x}(1_A) = 0$ or $S_{t,x}(1_A) = 1$. It follows that there exists $\Phi_t(x) \in E$ such that $S_{t,x} = \delta_{\Phi_t(x)}$. Since $S_tf \in bp\mathcal{B}(E)$ for all $f \in bp\mathcal{B}(E)$ it follows that (sd3) holds. The semigroup property of $(S_t)_{t\geqslant 0}$ implies that (sd1) is verified and from (2.2) it follows that (sd4) also holds. Finally, because $S_0 = Id$ we get (sd2).

(iii) Let $\mathcal{A}$ be a collection of bounded real-valued functions defined on E which is multiplicative (i.e., if $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$) and generates $\mathcal{B}(E)$. Let further $\mathbb{S} = (S_t)_{t\geqslant 0}$ be a sub-Markovian transition function on E such that $S_t(fg) = (S_tf)(S_tg)$ for all $f, g \in \mathcal{A}$. Then $\mathbb{S} = (S_t)_{t\geqslant 0}$ is multiplicative. Indeed, if we fix $x \in E$ and $g \in \mathcal{A}$ then, writing $g = g^+ - g^-$, the functionals $f \longmapsto S_t(fg)(x)$ and $f \longmapsto S_t(f)(x)S_t(g)(x)$ are differences of two positive finite measures which coincide on $\mathcal{A}$. By a monotone class argument we get $S_t(fg) = (S_tf)(S_tg)$ for all $f \in bp\mathcal{B}(E)$. Fixing now $f \in bp\mathcal{B}(E)$ and arguing as before, we conclude that the last equality holds for all $f, g \in bp\mathcal{B}(E)$.

(iv) *We have*

$$\text{(2.3)} \qquad \textit{if } \mathbb{S} = (S_t)_{t\geqslant 0} \textit{ is multiplicative and } v \textit{ is } \beta\textit{-excessive then } v^2 \textit{ is } 2\beta\textit{-excessive,}$$

where $\beta \geqslant 0$. *Indeed, since* $\mathbb{S} = (S_t)_{t\geqslant 0}$ *is multiplicative we have* $e^{-2\beta t} S_t(v^2) = (e^{-\beta t} S_t v)^2 \leqslant v^2$, *where the inequality holds because* v *is* β*-excessive. Then clearly* $\lim_{t\searrow 0} e^{-2\beta t} S_t(v^2) = \lim_{t\searrow 0}(S_t v)^2 = v^2$, *where the last equality follows from* $\lim_{t\searrow 0} S_t v = v$.

If E is a Lusin topological space and $\mathcal{B} = \mathcal{B}(E)$ is the Borel σ-algebra, then a family $\Phi = (\Phi_t)_{t\geqslant 0}$ of mappings on E is called *right continuous flow* (cf. [40], page 41) provided that $(sd1) - (sd3)$ hold and in addition:

$(sd4')$ For each $x \in E$ the function $t \longmapsto \Phi_t(x)$ is right continuous on $[0, \infty)$.

Clearly, any right continuous flow is a semi-dynamical system, because $(sd4')$ implies $(sd4)$, by taking $\mathcal{C}_o$ a countable subset of $bpC(E)$ which separates the points of E. If the function $t \longmapsto \Phi_t(x)$ is continuous on $[0, \infty)$ for all $x \in E$ then Φ is called *continuous flow.*

Remark 2.3. *One may regard a right continuous flow* $\Phi = (\Phi_t)_{t\geqslant 0}$ *as a deterministic right Markov process* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ *in the following way:* $\Omega := E$, $\mathcal{F} = \mathcal{F}_t := \mathcal{B}(E)$, $X_t(x) := \Phi_t(x)$ *for all* $x \in \Omega$ *and* $t \geqslant 0$, *and* $\mathbb{P}^x := \delta_x$.

Let $\mathcal{V} = (V_\alpha)_{\alpha>0}$ be the resolvent of kernels associated with $\mathbb{S}$, $V_\alpha f = \int_0^\infty e^{-\alpha t} f(\Phi_t)\,\mathrm{d}t$. We fix $\beta > 0$, a strictly positive function $f_o \in bp\mathcal{B}(E)$, and put $u_o := V_\beta f_o$. We define now the *capacity* induced by ϕ, by regarding ϕ as a (deterministic) right process. Let λ be a finite measure on E and consider the functional $M \longmapsto c_\lambda^\beta(M)$, $M \subset E$, defined as

$$c_\lambda^\beta(M) := \inf\left\{\int_E e^{-\beta D_G} u_o(\Phi_{D_G})\,\mathrm{d}\lambda : G \text{ open, } M \subset G\right\},$$

where D_G is the *first entry time* of G, $D_G(x) := \inf\{t \geqslant 0 : \Phi_t(x) \in G\}$, $x \in E$. For measurability properties of the first entry and hitting times in a set, for semi-dynamical systems with general state space see [19]. It turns out that c_λ^β is Choquet capacity on E; see e.g. [3] and also [15] and [4]. Recall that the capacity c_λ^β is called *tight* provided that there exists an increasing sequence $(K_n)_n$ of compact sets such that $\inf_n c_\lambda^\beta(K_n) = 0$.

We can state now the first main result, which shows that every semi-dynamical system becomes a continuous flow with respect to a convenient Lusin topology.

Theorem 2.4. *Let* $\Phi = (\Phi_t)_{t\geqslant 0}$ *be a semi-dynamical system on a Lusin measurable space* $(E, \mathcal{B})$. *Then there exists a Luzin topology* $\mathcal{T}$ *on* E *such that* $\mathcal{B} = \mathcal{B}(E)$ *is the Borel* σ*-algebra and* Φ *is a continuous flow with respect to this topology, such that the map* $x \longmapsto \Phi_t(x)$ *is continuous on* E *for all* $t \geqslant 0$. *For every finite measure* λ *on* E *and* $\beta > 0$ *the capacity* c_λ^β *is tight.*

Proof. Since by $(sd3)$ we have $\lim_{\alpha\to\infty} \alpha V_\alpha f = f$ pointwise on E for all $f \in \mathcal{C}_o$, it follows that $\mathcal{E}(\mathcal{V}_\beta)$ generates $\mathcal{B}(E)$, where $\beta > 0$. In addition, if $u, v \in \mathcal{E}(\mathcal{V}_\beta)$ then $u \wedge v := \inf(u, v)$ also belongs to $\mathcal{E}(\mathcal{V}_\beta)$, so, all the points of E are non-branch points with respect to $\mathcal{V}_\beta$.

The required Lusin topology $\mathcal{T}$ is going to be generated by a convex cone of bounded $\mathcal{V}_\beta$-excessive functions $\mathcal{R}$, called a *Ray cone.* Let us recall its usual construction, as, e.g., in [15],

the proof of Proposition 2.2: Let $\mathcal{R}_0 := V_\beta(\mathcal{C}_o) \cup \mathbb{Q}_+$. The Ray cone $\mathcal{R}$ is given by the closure in the sup norm of $\bigcup_{n\geqslant 0} \mathcal{R}_n$, where $\mathcal{R}_n$ is defined inductively as follows:
$\mathcal{R}_{n+1} := \mathbb{Q}_+ \cdot \mathcal{R}_n \cup (\sum_f \mathcal{R}_n) \cup (\bigwedge_f \mathcal{R}_n) \cup (\cup_{\alpha\in\mathbb{Q}_+^*} V_\alpha(\mathcal{R}_n)) \cup (\cup_{t\in\mathbb{Q}_+^*} S_t(\mathcal{R}_n)) \cup V_\beta((\mathcal{R}_n - \mathcal{R}_n)_+)$, where $\bigwedge_f \mathcal{R}_n$ is the set of all functions of the form $u_1 \wedge u_2 \wedge \cdots \wedge u_k$ with $u_i \in \mathcal{R}_n$, $i \leqslant k$, and $\sum_f \mathbb{Q}_+ \cdot \mathcal{R}_n$ is the set of all functions of the form $q_1u_1 + q_2u_2 + \cdots + q_ku_k$ with $q_i \in \mathbb{Q}_+$.

Note that $\mathcal{R}$ generates $\mathcal{B}(E)$ which is thus the Borel σ-algebra of $\mathcal{T}$. Since $t \longmapsto S_tV_\alpha f(x)$ is continuous and $S_t(u \wedge v) = S_tu \wedge S_tv$, it follows inductively that $t \longmapsto S_tu(x)$ is continuous on $[0, \infty)$ for all $x \in E$ and $u \in \bigcup_{n\geqslant 0} \mathcal{R}_n$, and therefore for all $u \in \mathcal{R}$. Hence $t \longmapsto u(\Phi_t(x))$ is continuous on $[0, \infty)$ for all $u \in \mathcal{R}$, that is, Φ is a $\mathcal{T}$-continuous flow.

We have $S_t(\mathcal{R}) \subset \mathcal{R}$ for all $t \in \mathbb{Q}_+$. So, clearly, S_tu is $\mathcal{T}$-continuous on E if $t \in \mathbb{Q}_+$ and therefore $x \longmapsto \Phi_t(x)$ is $\mathcal{T}$-continuous on E for all $t \in \mathbb{Q}_+$. Because for all $u \in \mathcal{R}$ the function $t \longmapsto S_tu(x)$ is decreasing, it follows that $S_tu = \sup_{\mathbb{Q}_+\ni t_n\searrow t} S_{t_n}u = \inf_{\mathbb{Q}_+\ni t_n\nearrow t} S_{t_n}u$ and thus the function S_tu is $\mathcal{T}$-continuous on E for all $t > 0$. We conclude that $x \longmapsto \Phi_t(x)$ is $\mathcal{T}$-continuous on E for all $t \geqslant 0$.

According to [37] and [4] (see also [38], [16], and [17]), the tightness property of the capacity c_λ^β is a direct consequence of the continuity of the trajectories of Φ in the topology $\mathcal{T}$. □

Remark 2.5. (*i*) *The Lusin topology from the above theorem is actually a* Ray topology *with respect to the resolvent* $(V_\alpha)_{\alpha>0}$ *of* $\mathbb{S}$*; for details see e.g. [4] and [6].*

(*ii*) *Theorem 2.4 extends a result about right continuous flows from [40], (47.8) at page 220.*

3 The extended weak generator

In this section we extend to unbounded real-valued functions the classical weak generator acting on bounded functions, considered by E.B. Dynkin (cf. [23] pag. 55); see also [26] and [35]. Notice that an extended generator was considered in [20] (and the references therein), however, only for bounded functions in the domain of the operator. Also, we shall complete the approach from [31].

Let $\mathbb{T} = (T_t)_{t\geqslant 0}$ be a sub-Markovian transition function with induced resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$, and set

(3.1) $\quad \mathcal{B}^0 = \mathcal{B}^0(\mathbb{T}) := \{f \in [\mathcal{B}] : T_t(|f|) < \infty \text{ for all } t > 0 \text{ and } f = \lim_{s\searrow 0} T_sf \text{ pointwise on } E\}$

Clearly, we have $[\mathcal{E}_\alpha] \subset \mathcal{B}^0 = \mathcal{B}^0(\mathbb{T}_\alpha)$ for every $\alpha \geqslant 0$. If $\mathbb{T} = (T_t)_{t\geqslant 0}$ is the transition function of a right Markov process with (Lusin topological) state space E, then every bounded finely continuous function belongs to $\mathcal{B}^0$, in particular, $bC(E) \subset \mathcal{B}^0$.

Define also

(3.2) $\quad \mathcal{B}_e := \{f \in [\mathcal{B}] : \exists\, h \in \mathcal{E} \text{ with } |f| \leqslant h \text{ and } f = \lim_{s\searrow 0} T_sf \text{ pointwise on } E\}$,

(3.3) $\quad \mathcal{B}_o = \mathcal{B}_o(\mathbb{T})$
$$:= \{f \in \mathcal{B}^0 : \forall \alpha > 0\ \exists\, t_o > 0, h_\alpha \in p\mathcal{B} \text{ such that } \sup_{0<s<t_o} T_s|f| \leqslant h_\alpha \text{ and } U_\alpha h_\alpha < \infty\}$$

(3.4) $\quad \mathcal{B}_{oo} = \mathcal{B}_{oo}(\mathbb{T})$
$$:= \{f \in \mathcal{B}_o : \forall t > 0\ \exists\, t_o > 0, h_t \in p\mathcal{B} \text{ such that } \sup_{0<s<t_o} T_s|f| \leqslant h_t \text{ and } T_th_t < \infty\}.$$

Several properties of the sets $\mathcal{B}^0$, $\mathcal{B}_e$, $\mathcal{B}_o$, and $\mathcal{B}_{oo}$ are collected in the following lemma, whose proof is included in Appendix (A.1).

Lemma 3.1. *The following assertions hold.*

(i) For each $\alpha > 0$ one has $U_\alpha(\mathcal{B}_o) \subset \mathcal{B}_{oo}$ and if $t > 0$ then $T_t(\mathcal{B}_{oo}) \subset \mathcal{B}_{oo}$. If $\beta > 0$ then $\mathcal{B}_o = \mathcal{B}_o(\mathbb{T}_\beta)$ and $\mathcal{B}_{oo} = \mathcal{B}_{oo}(\mathbb{T}_\beta)$.

(ii) If $\alpha, t > 0$ then $U_\alpha(\mathcal{B}_e) \subset \mathcal{B}_e \subset \mathcal{B}_{oo}$ and $T_t(\mathcal{B}_e) \subset \mathcal{B}_e$.

(iii) We have $b\mathcal{B}_o = b\mathcal{B}_{oo} = b\mathcal{B}_e = b\mathcal{B}^0$.

(iv) We have $[\mathcal{E}] \cup b[\mathcal{E}_\alpha] \subset \mathcal{B}_{oo}$, $\alpha > 0$. If $f \in [\mathcal{B}]$ is such that $U(|f|) < \infty$ then $Uf \in \mathcal{B}_{oo}$.

Corollary 3.2. *If $\mathbb{T} = (T_t)_{t \geqslant 0}$ is the transition function of a right Markov process with Lusin topological state space E, and $f \in C(E)$ is such that there exists $h \in \mathcal{E}$ with $|f| \leqslant h$, then $f \in \mathcal{B}_e$. In particular, $bC(E) \subset \mathcal{B}_e$.*

Further, let us consider

$$\mathcal{D}(L) := \Big\{ u \in \mathcal{B}_o : \forall \alpha > 0\ \exists t_o > 0, h_\alpha \in p\mathcal{B} \text{ with } \sup_{0<t<t_o} \left| \frac{T_t u - u}{t} \right| \le h_\alpha, U_\alpha h_\alpha < \infty, \tag{3.5}$$
$$\text{and } \lim_{t \searrow 0} \frac{T_t u - u}{t} \in \mathcal{B}_o \text{ pointwise on } E \Big\}$$

Clearly, $\mathcal{B}_o$, $\mathcal{B}_{oo}$, $\mathcal{B}_e$, and $\mathcal{D}(L)$ are vector spaces and define the linear operator

$$L : \mathcal{D}(L) \to \mathcal{B}_o, \quad Lu(x) := \lim_{t \searrow 0} \frac{T_t u(x) - u(x)}{t}, \ f \in \mathcal{D}(L),\ x \in E. \tag{3.6}$$

Define also

$$\mathcal{D}_o(L) := \{u \in \mathcal{D}(L) : Lu \in \mathcal{B}_{oo}\} \quad \text{and} \quad \mathcal{D}_e(L) := \{u \in \mathcal{D}(L) \cap \mathcal{B}_e : Lu \in \mathcal{B}_e\}. \tag{3.7}$$

The operator $(L, \mathcal{D}(L))$ is called the *extended weak generator* of $\mathbb{T} = (T_t)_{t \geqslant 0}$.

Remark 3.3. *(i) Recall the definition of the* weak generator $(L_w, \mathcal{D}(L_w))$ *considered in [23]: $\mathcal{D}(L_w)$ is the set of all bounded functions $f \in \mathcal{B}^0$ such that $\left(\frac{T_t f(x) - f(x)}{t}\right)_{t,x}$ is bounded for $x \in E$ and t in a neighbourhood of zero, there exists $\lim_{t \searrow 0} \frac{T_t f - f}{t}$ pointwise and the above limit is an element of $\mathcal{B}^0$. If $\alpha > 0$ then $\mathcal{D}(L_w) = U_\alpha(b\mathcal{B}^0)$, it is independent of $\alpha > 0$ and if $u = U_\alpha f$ with $f \in \mathcal{B}^0$, then $(\alpha - L_w)u = f$.*

(ii) In [31] an extended generator $(\overline{L}, \mathcal{D}(\overline{L}))$ of $\mathbb{T} = (T_t)_{t \geqslant 0}$ was considered by taking into account unbounded real-valued functions also, as follows: Let $u, g \in \mathcal{B}^0$, then u belongs to the domain $\mathcal{D}(\overline{L})$ of $\overline{L}$ and $g = \overline{L}u$ provided that

$$\forall\, t > 0, x \in E \textit{ we have } \int_0^t T_s(|g|)(x)\,\mathrm{d}s < \infty \textit{ and } T_t u(x) = u(x) + \int_0^t T_s g(x)\,\mathrm{d}s. \tag{3.8}$$

(iii) Assume that $\mathbb{T} = (T_t)_{t \geqslant 0}$ is the transition function of a right Markov process $X = (\Omega, \mathcal{F}_t, X_t, \mathbb{P}^x)$ with Lusin topological state space E. According to [31], Proposition 4.1 (see also [26], page 354, the proof of Theorem (4.1)), we have the following equivalent definition for the extended generator: If $u, g \in \mathcal{B}^0$ then $u \in \mathcal{D}(\overline{L})$ and $\overline{L}u = g$ if and only if for all $x \in E$ we have $\int_0^t T_s(|g|)(x)\,\mathrm{d}s < \infty$ for all $t > 0$ and $\left(u(X_t) - uX_0) - \int_0^t g(X_s)\,\mathrm{d}s\right)_{t \geqslant 0}$ is a $(\mathcal{F}_t)$-martingale under $\mathbb{P}^x$.

The next result collects properties of the extended weak generator. Several arguments used in the proof are similar to the case of the C_0-semigroups of contractions on a Banach space of functions; see, e.g., [25], Ch. 1, section 2. In particular, assertion (*viii*) below is a pointwise version of Theorem 1.3 from [23], Ch. I, section 3. For the reader convenience we present its proof in Appendix (A.2).

Proposition 3.4. *The following assertions hold for a sub-Markovian transition function* $\mathbb{T} = (T_t)_{t\geqslant 0}$*, its resolvent* $\mathcal{U} = (U_\alpha)_{\alpha>0}$*, and the extended weak generator* $(L, \mathcal{D}(L))$*.*

(i) If $\alpha > 0$ *then* $\mathcal{D}(L) = U_\alpha(\mathcal{B}_o)$ *and it is independent of* $\alpha > 0$*. If* $f \in \mathcal{B}^0(\mathbb{T})$*,* $\alpha \geqslant 0$ *and* $u = U_\alpha f$ *then* $(\alpha - L)u = f$*. If* $f \in b\mathcal{B}^0$*,* $t > 0$*, and* $u = \int_0^t T_s f\,\mathrm{d}s$ *then* $u \in \mathcal{D}(L)$ *and* $Lu = T_t f - f$*.*

(ii) The operator $(\overline{L}, \mathcal{D}(\overline{L}))$ *is well defined and we have* $\overline{L}u(x) = \lim_{t\searrow 0} \frac{T_t u(x)-u(x)}{t}$*,* $x \in E$*,* $u \in \mathcal{D}(\overline{L})$*.*

(iii) We have $\mathcal{D}(L_w) \subset \mathcal{D}_e(L) \subset \mathcal{D}(L) = \{u \in \mathcal{D}(\overline{L}) \cap \mathcal{B}_o : \overline{L}u \in \mathcal{B}_o\} \subset \mathcal{B}_{oo}$*,* $\overline{L}|_{\mathcal{D}(L)} = L$*, and* $L|_{\mathcal{D}(L_w)} = L_w$*.*

(iv) One has $\mathcal{D}_o(L) = U_\alpha(\mathcal{B}_{oo})$ *for each* $\alpha > 0$*. If* $t > 0$ *then* $T_t(\mathcal{D}_o(L)) \subset \mathcal{D}_o(L)$*,* $T_t(\mathcal{D}(\overline{L})) \subset \mathcal{D}(\overline{L})$*,* $\overline{L} \circ T_t = T_t \circ \overline{L}$ *on* $\mathcal{D}(\overline{L})$*, and* $L \circ T_t = T_t \circ L$ *on* $\mathcal{D}_o(L)$*.*

(v) If $\beta > 0$ *and* $(L^\beta, \mathcal{D}(L^\beta))$ *(resp.* $(\overline{L^\beta}, \mathcal{D}(\overline{L^\beta}))$*) denotes the extended weak generator (resp. the extended generator) of the transition function* $\mathbb{T}_\beta$*, then* $\mathcal{D}(L) \subset \mathcal{D}(L^\beta)$ *(resp.* $\mathcal{D}(\overline{L}) \subset \mathcal{D}(\overline{L^\beta})$*),* $L^\beta u = Lu - \beta u$ *for every* $u \in \mathcal{D}(L)$ *(resp.* $\overline{L^\beta}u = \overline{L}u - \beta u$ *for every* $u \in \mathcal{D}(\overline{L})$*), and* $\mathcal{D}_o(L) = \mathcal{D}_o(L^\beta)$*.*

(vi) We have $\mathcal{D}_e(L) = U_\alpha(\mathcal{B}_e) \subset \mathcal{D}_o(L)$ *for each* $\alpha > 0$ *and if* $t > 0$ *then* $T_t(\mathcal{D}_e(L)) \subset \mathcal{D}_e(L)$*.*

(vii) Let $\mathcal{D}^c_o(L) := \{u \in \mathcal{D}_o(L) : [0, \infty) \ni t \longmapsto LT_t u(x)$ *is continuous for each* $x \in E\}$*. If* $t, \alpha > 0$ *then* $T_t(\mathcal{D}^c_o(L)) \subset \mathcal{D}^c_o(L)$ *and* $U_\alpha(\mathcal{D}(L)) \subset \mathcal{D}^c_o(L)$*. If* $\beta > 0$ *then* $U_\beta U_\alpha(b[\mathcal{B}]) \subset \mathcal{D}^c_o(L)$*.*

(viii) If $u \in b\mathcal{D}^c_o(L)$ *then* $[0, \infty) \ni t \longmapsto T_t u(x)$ *is continuously differentiable for each* $x \in E$ *and* $(T_t u(x))' = LT_t u(x)$*. Moreover,* $u_t := T_t u$*,* $t \geqslant 0$*, is the unique solution of the equation*

$$\frac{\mathrm{d}u_t}{\mathrm{d}t} = Lu_t, t \geqslant 0, \tag{3.9}$$

such that $u_0 = u$*,* $u_t \in \mathcal{D}_o(L)$*,* $\|u_t\|_\infty$ *is bounded,* $Lu_t \in \mathcal{B}_{oo}$*, and* $[0, \infty) \ni t \longmapsto Lu_t(x)$ *is continuous for all* $x \in E$*.*

Corollary 3.5. *Assume that* $\mathbb{T} = (T_t)_{t\geqslant 0}$ *is the transition function of a right Markov process* $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ *with Lusin topological state space* E*. Then the following assertions hold.*

(i) If $f \in C(E)$ *is such that there exists* $h \in \mathcal{E}$ *with* $|f| \leqslant h$*, then* $U_\alpha f \in \mathcal{D}_e(L)$ *for each* $\alpha > 0$*. In particular,* $U_\alpha(bC(E)) \subset \mathcal{D}_e(L)$*. The above assertions are still true if we replace the continuity condition by the weaker one of fine continuity.*

(ii) If $\mathbb{T} = (T_t)_{t\geqslant 0}$ *is a* Feller semigroup, *i.e., each kernel* T_t*,* $t > 0$*, leaves invariant* $bC(E)$*, then* $U_\alpha(bC(E)) \subset \mathcal{D}^c_o(L)$*.*

(iii) The martingale problem associated with $(\overline{L}, \mathcal{D}(\overline{L}))$ has a solution. More precisely, for every $u \in \mathcal{D}(\overline{L})$ and $x \in E$, the process

$$\left(u(X_t) - u(X_0) - \int_0^t \overline{L}u(X_s)\,\mathrm{d}s\right)_{t\geqslant 0}$$

is a $(\mathcal{F}_t)$-martingale under $\mathbb{P}^x$.

The following additional property of $\mathbb{S} = (S_t)_{t\geqslant 0}$ will be considered further on:

(3.10) $\quad \exists\, \mathcal{C}_o \subset bp\mathcal{B}$ such that $1 \in \mathcal{C}_o, \mathcal{C}_o$ generates $\mathcal{B}$, and $\lim\limits_{t\searrow 0} S_t f(x) = f(x)$ for all $x \in E$.

Remark 3.6. *As a consequence of* (3.10) *we have for all $\alpha, \beta > 0$:*

(3.11) $\quad$ *If* (3.10) *holds then* $\sigma(V_\beta V_\alpha(b\mathcal{B}_{oo})) = \sigma(V_\alpha(b\mathcal{B}_{oo})) = \mathcal{B}$,

where $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is the resolvent of $\mathbb{S}$ and $\mathcal{B}_{oo} = \mathcal{B}_{oo}(\mathbb{S})$. Indeed, by (3.10) *if follows that for every $f \in \mathcal{C}_o$ we have pointwise $lim_{\alpha\to\infty}\alpha V_\alpha f = f$ and therefore $\mathcal{C}_o \subset bp\sigma(V_\alpha(bp\mathcal{B}))$, hence $\mathcal{B} = \sigma(\mathcal{C}_o) \subset \sigma(V_\alpha(bp\mathcal{B}))$, so, $\sigma(V_\alpha(b[\mathcal{B}])) = \mathcal{B}$. On the other hand by Lemma 3.1 (iv) we have $V_\alpha(b[\mathcal{B}]) \subset b\mathcal{B}_{oo} \subset b[\mathcal{B}]$ and therefore $\sigma(b\mathcal{B}_{oo}) = \mathcal{B}$. By Lemma 3.1 (i) $V_\alpha(b\mathcal{B}_{oo}) \subset \mathcal{B}_{oo}$ and therefore the vector space $V_\alpha(b\mathcal{B}_{oo})$ does not depend on $\alpha > 0$ and $\sigma(V_\alpha(b\mathcal{B}_{oo})) \subset \mathcal{B}_{oo}$. The converse inclusion also holds because for every $f \in b\mathcal{B}_o$ we have $\lim_{\alpha\to\infty} \alpha V_\alpha f = f$ pointwise on E and we conclude that the last equality from* (3.10) *is proven. Observe that the resolvent equation implies that the vector space $V_\beta V_\alpha(b\mathcal{B}_{oo})$ also does not depend on α and β. If $f \in b\mathcal{B}_o$ then $lim_{\beta\to\infty}\beta V_\beta V_\alpha f = V_\alpha f$ pointwise on E, hence $V_\alpha f \in b\sigma(V_\beta V_\alpha(b\mathcal{B}_{oo}))$ and therefore $\sigma(V_\alpha(b\mathcal{B}_{oo})) \subset \sigma(V_\beta V_\alpha(b\mathcal{B}_{oo}))$ and so, the first equality is also clear.*

Non-autonomous semi-dynamical systems. Let $(E, \mathcal{B})$ be a Lusin measurable space and let $\Phi = (\Phi_{s,t})_{t\geq s\geq 0}$ be a family of mappings $\Phi_{s,t} : E \to E, t \geq s \geq 0$. Inspired by e.g. [33], we say that Φ is a *non-autonomous semi-dynamical system* on E provided that the following conditions are satisfied:

(Nsd1) $\Phi_{s,t}(x) = \Phi_{r,t}(\Phi_{s,r}(x))$ for all $t \geq r \geq s \geq 0$ and $x \in E$;

(Nsd2) $\Phi_{s,s}(x) = x$ for all $s \geq 0$, $x \in E$;

(Nsd3) For each $t > 0$ the function $[0, \infty) \times E \ni (s, x) \longmapsto \Phi_{s,s+t}(x)$ is measurable;

(Nsd4) There exists a countable set $\mathcal{C}_o \subset bp\mathcal{B}$ such that $\mathcal{C}_o$ separates the points of E and $\lim_{t\searrow 0} f(\Phi_{s,s+t}(x)) = f(x)$ for all $s \geq 0, x \in E$ and $f \in \mathcal{C}_o$.

The paths of unique strong solutions to Ito SDEs on $\mathbb{R}^d$ which depend continuously on the initial data are typical examples of such non-autonomous semi-dynamical systems (see e.g. [27]).

Given a Φ as above, it is a straightforward to check that $\overline{\Phi} := \left(\overline{\Phi}_t\right)_{t\geq 0}$ defined by

$$\overline{\Phi}_t : [0, \infty) \times E \to [0, \infty) \times E, \quad \overline{\Phi}_t(s, x) := (s + t, \Phi_{s,s+t}(x)), \quad t, s \geq 0, x \in E,$$

is a semi-dynamical system on $[0, \infty) \times E$.

Thus, the results obtained in this work for (autonomous) semi-dynamical systems can be easily reinterpreted for non-autonomous semi-dynamical systems.

3.1 The extended weak generator of a semi-dynamical system

We have the following characterization of those Markovian transition functions that correspond to semi-dynamical systems:

Proposition 3.7. *Let $\mathbb{S} = (S_t)_{t\geqslant 0}$ be a Markovian transition function on $(E, \mathcal{B})$ and $(D, \mathcal{D}(D))$ be its extended weak generator. Then the following assertions are equivalent.*

(i) $\mathbb{S} = (S_t)_{t\geqslant 0}$ *is the transition function of a semi-dynamical system on E.*

(ii) *The transition function $\mathbb{S} = (S_t)_{t\geqslant 0}$ satisfies* (3.10) *and it is multiplicative, that is, for every $f, g \in bp\mathcal{B}$ and $t > 0$ we have $S_t(fg) = (S_t f)(S_t g)$.*

(iii) $\mathbb{S} = (S_t)_{t\geqslant 0}$ *satisfies* (3.10)*, $\mathcal{B}_e$ and $\mathcal{D}_e(D)$ are algebras, $\mathbb{S} = (S_t)_{t\geqslant 0}$ is multiplicative on $\mathcal{B}_e$, and if $u \in \mathcal{D}_e(D)$ then $Du^2 = 2uDu$.*

(iv) $\mathbb{S} = (S_t)_{t\geqslant 0}$ *satisfies* (3.10)*, $\mathcal{D}^c_b(D) := \{u \in b\mathcal{D}^c_o(D) : Du \in b\mathcal{B}_{oo}\}$ is an algebra, and if $u \in \mathcal{D}^c_b(D)$ then $Du^2 = 2uDu$.*

(v) $\mathbb{S} = (S_t)_{t\geqslant 0}$ *satisfies* (3.10) *and there exists an algebra $\mathcal{A} \subset \mathcal{D}^c_b(D)$ which generates $\mathcal{B}$, $S_t u \in \mathcal{A}$, $t > 0$, and $Du^2 = 2uDu$ for each $u \in \mathcal{A}$.*

Proof. The implication $(i) \to (ii)$ is clear; notice that $(sd4)$ implies that (3.10) holds.

$(ii) \to (iii)$. We show first that

$$\text{(3.12)} \qquad \text{if } \mathbb{S} = (S_t)_{t\geqslant 0} \text{ is multiplicative and } v \in \mathcal{E}_\beta \text{ then } v^2 \in \mathcal{E}_{2\beta},$$

where $\beta \geqslant 0$ and $\mathcal{E}_0 := \mathcal{E}$. Indeed, since $\mathbb{S} = (S_t)_{t\geqslant 0}$ is multiplicative we have $e^{-2\beta t} S_t(v^2) = (e^{-\beta t} S_t v)^2 \leqslant v^2$, where the inequlity holds because $v \in \mathcal{E}_\beta$. Then clearly $\lim_{t\searrow 0} e^{-2\beta t} S_t(v^2) = \lim_{t\searrow 0} (S_t v)^2 = v^2$, where the last equality follows from $\lim_{t\searrow 0} S_t v = v$.

As a consequence of (3.12) we have:

$$\text{(3.13)} \qquad \text{if } \mathbb{S} = (S_t)_{t\geqslant 0} \text{ is multiplicative then } \mathcal{B}_e \text{ is an algebra, i.e., if } f \in \mathcal{B}_e \text{ then } f^2 \in \mathcal{B}_e.$$

Indeed, if $f \in \mathcal{B}_e$ and $|f| \leqslant h \in \mathcal{E}$ then by (3.12) we get $f^2 \leqslant h^2 \in \mathcal{E}$ and because $S_s(f^2) = (S_s f)^2$ we also have $\lim_{s\searrow 0} S_s(f^2) = (\lim_{s\searrow 0} S_s f)^2 = f^2$. So, by Lemma 3.1 (ii) we conclude that f^2 also belongs to $\mathcal{B}_e$.

Let now $u \in \mathcal{D}_e(D)$, $|u| \leqslant h \in \mathcal{E}$. By (3.13) we get $u^2 \in \mathcal{B}_e$ and $S_t u^2 - u^2 = (S_t u - u(x))(S_t u + u)$, $t > 0$. We have $|S_t u + u| \leqslant 2h$ and $\sup_{0<t<t_o} |\frac{S_t u - u}{t}| \leqslant h_\alpha$, with $V_\alpha h_\alpha < \infty$ on E, where $\mathcal{V} = (V_\alpha)_{\alpha>0}$ is the resolvent of $\mathbb{S}$. Consequently, $\sup_{0<t<t_o} |\frac{S_t u^2 - u^2}{t}| \leqslant 2h_\alpha h$ and we have $V_\alpha(h_\alpha h) = \int_0^\infty e^{-\alpha s}(S_s h_\alpha) S_s h \leqslant h V_\alpha h_\alpha < \infty$ on E. Because $\lim_{t\searrow 0} S_t u = u$ pointwise on E, we conclude that for every $x \in E$ there exists the limit $\lim_{t\searrow 0} \frac{S_t u^2(x) - u^2(x)}{t} = \lim_{t\searrow 0} \frac{S_t u(x) - u(x)}{t} \lim_{t\searrow 0}(S_t u(x) + u(x)) = Du(x) 2u(x)$. So, $u^2 \in \mathcal{D}(D) \cap \mathcal{B}_e$ and $Du^2 = 2uDu$. Moreover, u and Du both belong to $\mathcal{B}_e$, therefore (3.13) implies that $Du^2 \in \mathcal{B}_e$, hence $u^2 \in \mathcal{D}_e(D)$.

$(iii) \to (iv)$ Notice first that $\mathcal{D}^c_b(D) \subset \mathcal{D}_e(D)$, because $1 \in \mathcal{E}$. If $u \in \mathcal{D}^c_b(D)$ then by the hypothesis (iii) we have $u^2 \in \mathcal{D}_e(D)$ and $Du^2 = 2uDu$. In addition, $u, Du \in b\mathcal{B}_{oo}$, hence Du^2 also belongs to $b\mathcal{B}_e$ which is an algebra included in $b\mathcal{B}_{oo}$. Consequently, $u^2 \in \mathcal{D}_o(D)$. Since

$DS_t(u^2) = 2(S_tu)(DS_tu)$ and the functions S_tu and DS_tu are continuous in t, we conclude that u^2 also belongs to $\mathcal{D}_b^c(D)$.

$(iv) \to (v)$ Assume that (iv) holds, then $\mathcal{D}_b^c(D)$ is multiplicative and we show that it generates $\mathcal{B}$. Indeed, by Proposition 3.4 (vii) we have $V_\beta V_\alpha(b\mathcal{B}_o) \subset \mathcal{D}_b^c(D)$. From (3.11) we get $\mathcal{B} = \sigma(V_\beta V_\alpha(b\mathcal{B}_o)) \subset \sigma(\mathcal{D}_b^c(D))$ and thus $\sigma(b\mathcal{D}_b^c(D)) = \mathcal{B}$.

$(v) \to (ii)$. Let now $u \in \mathcal{A}$ as in (v). If we put $v_t := (S_tu)^2$ then by hypothesis we have $v_t \in \mathcal{A}$, $t \geqslant 0$, $\sup_{0\leqslant t<\infty} \|v_t\|_\infty \leqslant \|u\|_\infty^2$, and $t \longmapsto Dv_t(x)$ is continuous for each $x \in E$. Using Proposition 3.4 $(viii)$ we obtain $\frac{\mathrm{d}v_t}{\mathrm{d}t} = 2S_tu \cdot DS_tu = Dv_t$, $t \geqslant 0$, with $v_0 = u^2$. By the uniqueness property of the equation (3.9) it follows that $(S_tu)^2 = S_tu^2$, hence $(S_tu)(S_tv) = S_t(uv)$ for all $u, v \in \mathcal{A}$. Applying Remark 2.2, (iii), we conclude that $\mathbb{S} = (S_t)_{t\geqslant 0}$ is multiplicative and therefor assertion (ii) holds.

$(ii) \to (i)$. The proof of this implication is straightforward, however, for the reader convenience we give some details here. Let $S_t(x, \cdot)$ be the probability on E induced by the Markovian kernel S_t and $x \in E$, $S_t(x, A) := S_t(1_A)(x)$ for all $A \in \mathcal{B}$. Taking $f = g = 1_A$ in the property of $\mathbb{S} = (S_t)_{t\geqslant 0}$ to be multiplicative we get $S_t(1_A) = (S_t(1_A))^2$ and therefore the number $S_t(x, A)$ should be either 0 or 1. Hence $S_t(x, \cdot)$ is a Dirac measure on E, concentrated at a point $\Phi_t(x) \in E$, $S_t(x, \cdot) = \delta_{\Phi_t(x)}$. We obtain $S_tf(x) = f(\Phi_t(x))$ for all $f \in p\mathcal{B}$, $x \in E$, and $t \geqslant 0$, and it is easy to check now that $\Phi = (\Phi_t)_{t\geqslant 0}$ verifies $(sd1) - (sd3)$, while $(sd4)$ follows from (2.2). So, $\Phi = (\Phi_t)_{t\geqslant 0}$ is a semi-dynamical system on E and $\mathbb{S} = (S_t)_{t\geqslant 0}$ is its transition function. □

The following result concerns the algebraic structure of the extended generator of a semi-dynamical system; its proof is deferred to Appendix (A.3).

Proposition 3.8. *Let $\mathbb{S} = (S_t)_{t\geqslant 0}$ be the transition function of a semi-dynamical system on $(E, \mathcal{B})$ and let $(\overline{D}, \mathcal{D}(\overline{D}))$ be its extended generator. If $f \in \mathcal{D}(\overline{D})$ and $\int_0^t S_s(|f\overline{D}f|)\,\mathrm{d}s < \infty$ for all $t > 0$ then $f^2 \in \mathcal{D}(\overline{D})$ and $\overline{D}f^2 = 2f\overline{D}f$. In particular, $b\mathcal{D}(\overline{D})$ is an algebra.*

Example: The classical case of an Euclidean gradient flow. Let $\mathbf{B} : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous vector field such that:

(B.i) For each $r > 0$ there exists a constant $c(r)$ such that for all $x, y \in \mathbb{R}^d$, $|x|, |y| \leq r$

$$\langle \mathbf{B}(x) - \mathbf{B}(y), x - y\rangle \leq c(r)|x - y|^2 \qquad (\textit{local weak monotonicity}).$$

(B.ii) There exists a constant c_0 such that for all $x \in \mathbb{R}^d$

$$\langle \mathbf{B}(x), x\rangle \leq c_0(1 + |x|^2) \qquad (\textit{weak coercivity}).$$

Then, by e.g. [Rockner-Wei Liu], Therem 3.1.1 (applied for $\sigma \equiv 0$), for each $x \in \mathbb{R}^d$ there exists a unique solution $(\Phi_t(x))_{t\geq 0} \in C([0, \infty); \mathbb{R}^d)$ to the equation

$$\begin{cases} \mathrm{d}\Phi_t(x) = \mathbf{B}(\Phi_t(x))\,\mathrm{d}t, \quad t \geqslant 0, \\ \Phi_0(x) = x. \end{cases} \tag{3.14}$$

$(\Phi_t)_{t\geq 0}$ is a semi-dynamical system as considered in Section 2, which can be regarded as a (deterministic) right process with transition function $(S_t)_{t\geq 0}$,

$$S_tf(x) = f(\Phi_t(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in b\mathcal{B}(\mathbb{R}^d).$$

Note that if $(D, \mathcal{D}(D))$ denotes the weak generator of the continuous flow $\Phi = (\Phi_t)_{t \geqslant 0}$, then it is clear that

$$Dv = \mathbf{B} \cdot \nabla v \text{ for all } v \in C_b^1(\mathbb{R}^m).$$

3.2 Stopped continuous flows

In this subsection (more precisely, in Proposition 3.9 below) we apply to continuous flows the classical technique of stopping a Markov process at its first entry time in a given set. This stopping technique has been used in [11], Remark 3.4, in studying stochastic fragmentation processes for particles with spatial position on a surface.

Let $\Phi = (\Phi_t)_{t \geqslant 0}$ be continuous flow on a Lusin topological space E and let $\mathcal{O}$ be an open susbset of E. Let T be *the first entry time* in $\mathcal{O}^c = E \setminus \mathcal{O}$,

$$T(x) = \inf\{t \geqslant 0 : \Phi_t(x) \in \mathcal{O}^c\}.$$

The following properties are immediate:

1. T is a *terminal time*, that is, the mapping $E \ni x \longmapsto T(x)$ is $\mathcal{B}(E)$-measurable and

$$t + T \circ \theta_t = T \text{ on } [t < T],$$

   or equivalently, $t + T(\Phi_t(x)) = T(x)$ if $t < T(x)$ for all $x \in E$.

2. If $x \in \overline{\mathcal{O}}$ then $\Phi_{T(x)}(x) \in \partial \mathcal{O}$.

3. If $x \in \overline{\mathcal{O}}^c$ then $T(x) = 0$, so, $\Phi_{T(x)}(x) = x$.

4. We have $\Phi_T(x)(x) \in \mathcal{O}^c$ for every $x \in E$.

For each $t \geqslant 0$ define the map $\Phi_t^o : E \to E$ as

$$\Phi_t^o(x) := \begin{cases} \Phi_t(x), & t < T(x) \\ \Phi_{T(x)}(x), & t \geqslant T(x) \end{cases}, x \in E.$$

The announced result of this subsection is the following collection of statements, whose proofs are presented in Appendix (A.4).

Proposition 3.9. *Then the following assertions hold.*

(i) The family $\Phi^o := (\Phi_t^o)_{t \geqslant 0}$ is a continuous flow on E and it is called the stopped flow *w.r.t. T. We have $\overline{\Phi}_t(x) = \Phi_t(x)$ if $t < T(x)$ and $\Phi_t^o(x) = x$ for every $x \in \mathcal{O}^c$ and $t \geqslant 0$.*

(ii) Let $(D, \mathcal{D}(D))$ (resp. $(D^o, \mathcal{D}(D^o))$) be the extended weak generator of the continuous flow Φ (resp. of the continuous flow Φ^o) on E. We have $D^o u = 0$ on $\mathcal{O}^c$ for all $u \in \mathcal{D}(D^o)$ and if in addition $u \in \mathcal{D}(D)$ then $D^o u = Du$ on $\mathcal{O}$.

(iii) The set $\overline{\mathcal{O}}$ is absorbing *for $\Phi^o := (\Phi_t^o)_{t \geqslant 0}$, that is, if $x \in \overline{\mathcal{O}}$ then $\Phi_t^o(x) \in \overline{\mathcal{O}}$ for all $t \geqslant 0$.*

(iv) Define the restriction *$\Phi^{\overline{\mathcal{O}}} = (\Phi_t^{\overline{\mathcal{O}}})_{t \geqslant 0}$ of Φ to $\overline{\mathcal{O}}$ as $\Phi^{\overline{\mathcal{O}}}(x) := \Phi_t^o(x)$ for all $x \in \overline{\mathcal{O}}$ and $t \geqslant 0$. Then $\Phi^{\overline{\mathcal{O}}}$ is a continuous flow on $\overline{\mathcal{O}}$.*

4 Continuous flow driving a Markov process

Let $(L, \mathcal{D}(L))$ and $(D, \mathcal{D}(D))$ be two extended weak generators on E. Define

$$\mathcal{D}(DL) := \{u \in \mathcal{D}(D) \cap \mathcal{D}(L) : Lu \in \mathcal{D}(D) \text{ and } DLu \in \mathcal{B}^0(\mathbb{T})\},$$

and $\mathcal{D}(LD)$ analogously.

We can present now the second main result of this paper.

Theorem 4.1. *Let $\mathbb{T} = (T_t)_{t \geqslant 0}$ be the transition function of a right (resp. Hunt) Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ with state space E and extended weak generator $(L, \mathcal{D}(L))$. Assume that there exists a multiplicative set $\mathcal{C}_1 \subset b\mathcal{C}(E)$ which generates $\mathcal{B}(E)$ such $T_t(\mathcal{C}_1) \subset C(E)$ for all $t > 0$. Let $\Phi = (\Phi_t)_{t \geqslant 0}$ be a right continuous flow on E such that the mapping $(x, t) \longmapsto \Phi_t(x)$ is continuous on $E \times [0, \infty)$, with transition function $\mathbb{S} = (S_t)_{t \geqslant 0}$ and extended weak generator $(D, \mathcal{D}(D))$. Suppose in addition that L* and *D* commute *in the sense that*

$$\mathcal{D}(DL) = \mathcal{D}(LD) =: \mathcal{D}_o \quad \text{and} \quad DL = LD \text{ on } \mathcal{D}_o.$$

Furthermore, set

$$X_t^{\Phi} := \Phi_t(X_t),\ t \geqslant 0.$$

Then the following assertions hold.

(i) *$X^{\Phi} := (\Omega, \mathcal{F}, \mathcal{F}_t, X_t^{\Phi}, \mathbb{P}^x)$ is a right (resp. Hunt) Markov process with state space E and the transition $\mathbb{T}^{\Phi} := (T_t^{\Phi})_{t \geqslant 0}$ defined as $T_t^{\Phi} := S_t T_t$ for all $t \geqslant 0$.*

(ii) *Let $\mathcal{D}_c := U_\alpha V_\beta(bC(E))$, $\alpha, \beta > 0$. Then $\mathcal{D}_o \subset \mathcal{D}_o(L) \cap \mathcal{D}_o(D) \cap \mathcal{D}_o(L^{\Phi})$, $\mathcal{D}_c \subset \mathcal{D}_o^c(L) \cap \mathcal{D}_o(D) \cap \mathcal{D}(L^{\Phi})$ and*

$$L^{\Phi} = L + D \text{ on } \mathcal{D}_c.$$

Proof. (i) We check first that X^{Φ} is a (simple) Markov process with $\mathbb{T}^{\Phi}$ as transition function. If $f \in bp\mathcal{B}$, μ is a probability on E, and $s, t \geqslant 0$ then by the Markov property of X we obtain $\mathbb{E}^{\mu}[f(X_{t+s}^{\Phi}|\mathcal{F}_t] = T_s(f(\Phi_{t+s}))(X_t) = T_s S_{t+s} f(X_t) = S_s T_s S_t f(X_t) = T_s^{\Phi} f(X_t^{\Phi})$. We have also $T_{t-s}^{\Phi} f(X_s^{\phi}) = S_s T_{t-s} S_{t-s} f(X_s) = T_{t-s} S_t f(X_s)$ if $s < t$. It follows that for all $t \geqslant 0$ $[s \longmapsto T_{t-s}^{\Phi} f(X_s^{\phi}) 1_{[0,t)}$ is not right continuous$] = [s \longmapsto T_{t-s}(S_t f)(X_s) 1_{[0,t)}$ is not right continuous$]$ and by Corollary (7.9) from [40] we conclude that X^{Φ} is a right process.

(ii) Observe that Corollary 3.2 implies that $bC(E) \subset \mathcal{B}_e(\mathbb{S}) \cap \mathcal{B}_e(\mathbb{T}) \cap \mathcal{B}_e(\mathbb{T}^{\phi})$. If $f \in bC(E)$, because T_t and V_β commute, by dominate convergence we get $\lim_{t \searrow 0} S_t U_\alpha f = U_\alpha(\lim_{t \searrow 0} S_t f) = f$. Therefore, by Lemma 3.1 (ii) we deduce that $U_\alpha f \in \mathcal{B}_{oo}(\mathbb{S})$ and consequently, if $u = U_\alpha V_\beta f$ then $u \in \mathcal{D}_o(D)$. Analogously, u belongs to $\mathcal{D}_o(L)$ too. In addition, $LT_t u = \alpha T_t u - V_\beta T_t f$, $DS_t u = \beta S_t u - U_\beta S_t f$ and so, the functions $LT_t u(x)$ and $DS_t u(x)$, $x \in E$, are continuous in t, hence $u \in \mathcal{D}_o^c(L) \cap \mathcal{D}_o^c(D)$.

Because $\lim_{t \searrow 0} \frac{T_t u - u}{t} = Lu$ if and only if $\lim_{t \searrow 0} \frac{e^{-\alpha t} T_t u - u}{t} = Lu - \alpha u$, we may suppose that the potential kernels U and V are bounded and that $u = UVf$, hence $U|f|$ and $V|f|$ are bounded functions. We have $\frac{T_t^{\Phi} u - u}{t} = S_t(\frac{T_t u - u}{t}) + \frac{S_t u - u}{t}$ and so, to show that $u \in \mathcal{D}(L^{\Phi})$ and $L^{\Phi} u = Lu + Du$, it is sufficient to prove that $\lim_{t \searrow 0} S_t(\frac{T_t u - u}{t}) = -Vf$ pointwise on E. We have $T_t u - u = -V(\int_0^t T_s f \, \mathrm{d}s)$, $S_t(\frac{T_t u - u}{t}) = -V(S_t \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s) = -Vf - V(S_t \frac{1}{t} \int_0^t T_s g \, \mathrm{d}s - f)$. Therefore, it remains to show that $\lim_{t \searrow 0} V(S_t \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s - f) = 0$ pointwise on E. We have $V(S_t \frac{1}{t} \int_0^t T_s f \, \mathrm{d}s - f) = \int_t^{\infty} S_{s'}(\frac{1}{t} \int_0^t (T_s f - f) \, \mathrm{d}s) \, \mathrm{d}s' - \int_0^t S_{s'} f \, \mathrm{d}s'$. Since $f \in bC(E)$, the second term from the right hand side of the last equality tends to zero when $t \searrow 0$. For the first term we have the estimation $\left|\int_t^\infty S_{s'}(\frac{1}{t}\int_0^t (T_s f - f)\,\mathrm{d}s)\,\mathrm{d}s'\right| \leqslant V(\frac{1}{t}\int_0^t |T_s f - f| ds)$ and because $\lim_{s\searrow 0} T_s f = f$ pointwise on E, the first term also vanishes when $t \searrow 0$. □

4.1 Right continuous flow driving its subordinate process

Let $\Phi = (\Phi_t)_{t\geqslant 0}$ be a right continuous flow on E, with transition function $\mathbb{S} = (S_t)_{t\geqslant 0}$. Let further $\mu = (\mu_t)_{t\geqslant 0}$ be a convolution semigroup on $\mathbb{R}_+$ and consider $\mathbb{S}^\mu = (S_t^\mu)_{t\geqslant 0}$, the subordinate of $(S_t)_{t\geqslant 0}$ in the sense of Bochner w.r.t. μ, defined as $S_t^\mu f := \int_0^\infty S_s f\, \mu_t(\mathrm{d}s)$, $t \geqslant 0$, $f \in p\mathcal{B}(E)$; for details see e.g. [39] and also [36]. In particular, the subordinate process $Y^\xi = (Y_t^\xi)_{t\geqslant 0}$ is defined as

$$Y_t^\xi(x,\omega) := \Phi_{\xi_t(\omega)}(x), \;\; t \geqslant 0, (x,\omega) \in E \times \Omega$$

and it turns out that $Y^\xi = (Y_t^\xi)_{t\geqslant 0}$ is a right Markov process with state space E, path space $E \times \Omega'$, and transition function $\mathbb{S}^\mu = (S_t^\mu)_{t\geqslant 0}$, where Ω' is the path space of the *subordinator* $(\xi_t)_{t\geqslant 0}$, the positive real-valued stationary stochastic process with path space Ω', with independent nonnegative increments induced by $\mu = (\mu_t)_{t\geqslant 0}$. So, Y^ξ is obtained by introducing jumps in the evolution of the given right continuous flow Φ, by means of the subordinator induced by $\mu = (\mu_t)_{t\geqslant 0}$.

We state now a consequence of Theorem 4.1 involving the right continuous flow Φ and the subordinate process Y^ξ.

Corollary 4.2. *Let $\mathbb{S} = (S_t)_{t\geqslant 0}$ be the transition function of a right continuous flow $\Phi = (\Phi_t)_{t\geqslant 0}$ on E. Let $(\xi_t)_{t\geqslant 0}$ be a positive real-valued stationary stochastic process with independent nonnegative increments induced by a convolution semigroup $\mu = (\mu_t)_{t\geq 0}$ on $\mathbb{R}_+$. Further, define*

$$Y_t^\Phi := \Phi_{t+\xi_t},\, t \geqslant 0.$$

Then the following assertions hold.

(i) $Y^\Phi := (E\times\Omega, Y_t^\Phi)$ is a right Markov process with state space E and the transition function $\mathbb{T}^\Phi := (T_t^\Phi)_{t\geqslant 0}$ defined as $T_t^\Phi := S_t S_t^\mu$ for all $t \geqslant 0$.

(ii) Let $(D, \mathcal{D}(D))$, $(D^\mu, \mathcal{D}(D^\mu))$, and $(L^\Phi, \mathcal{D}(L^\Phi))$ be the extended weak generators of $\mathbb{S}$, $\mathbb{S}^\mu$, and respectively $\mathbb{T}^\Phi$. Let further $\mathcal{D}_o := V_\alpha^\mu V_\beta(bC(E))$, $\alpha, \beta > 0$, where $\mathcal{V} = (V_\alpha)_{\alpha>0}$ (resp. $\mathcal{V}^\mu = (V_\alpha^\mu)_{\alpha>0}$) is the resolvent of $\mathbb{S}$ (resp. the resolvent of $\mathbb{S}^\mu$). Then $\mathcal{D}_o \subset \mathcal{D}_o^c(D^\mu) \cap \mathcal{D}_o(D) \cap \mathcal{D}(L^\Phi)$ and

$$L^\Phi = D^\mu + D \text{ on } \mathcal{D}_o.$$

Proof. We apply Theorem 4.1 for $X := Y^\Phi$. We clearly have $X_t^\Phi = \Phi_t(Y_t^\xi) = \Phi_t(\Phi_{\xi_t}) = \Phi_{t+\xi_t}$ and observe that the paths $t \longmapsto \Phi_{t+\xi_t(\omega)}(x)$ are right continuous, without assuming that the right continuous flow Φ is continuous. For all $t, t' > 0$ we have $S_{t'} S_t^\mu = S_t^\mu S_{t'} = \int_0^\infty S_{s+t'} \mu_t(\mathrm{d}s)$.

Assertion (ii) follows from Theorem 4.1 (ii). □

4.2 Continuous flow driving a superprocess

Let $\Psi : [0, \infty) \to \mathbb{R}$ be a branching mechanism,

$$\Psi(\lambda) = -b\lambda - c\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s) N(\mathrm{d}s),$$

where $b, c \in \mathbb{R}$, $c \geqslant 0$, and N is a measure on $(0, \infty)$ such that $N(u \wedge u^2) < \infty$. Consider the superpocess $\widehat{X^0}$ on the set $M(E)$ of all positive finite measures on E, having the branching mechanism Ψ and having no spatial motion. According to [18] the superprocess $\widehat{X^0}$ is called *pure branching*. For details on the measure-valued branching processes see [22], [24], [34] and also [26], [1], and [2].

Let further Φ be a continuous flow on E and consider the superprocess $\widehat{X}$ on $M(E)$, having the spatial motion Φ and the branching mechanism Ψ. By Φ we also denote the continuous flow on $M(E)$ (endowed with the weak topology) canonically induced by the given flow Φ on E.

It turns out that one can apply Theorem 4.1 on $M(E)$ for $\widehat{X^0}$ instead of X and the flow Φ on $M(E)$. We get the following representation of the superprocess $\widehat{X}$ by means of the pure branching superprocess $\widehat{X^0}$:

$$\widehat{X}_t = \Phi_t(\widehat{X^0_t}) \text{ for all } t \geqslant 0,$$

where the equality is in the distribution sense; see [18]. A similar result holds for non-local branching processes (in the sense of [12] and [14]) on the set of all finite configurations of the state space of the spatial motion; see also [13] for an associated nonlinear Dirichlet problem.

5 Multiplicative L^p-semigroups and continuous flows

Let $\Phi = (\Phi_t)_{t \geqslant 0}$ be a semi-dynamical system with state space $(E, \mathcal{B})$, $\mathbb{S} = (S_t)_{t \geqslant 0}$ its transition function, and $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ be the associated resolvent of kernels. Let further m be a positive σ-finite measure on E which *subinvariant* for $\mathbb{S}$, that is,

$$m \circ S_t \leqslant m \text{ for all } t > 0,$$

and fix $p \in [1, \infty)$. Then each kernel S_t, $t \geqslant 0$, induces a contraction on $L^p(E, m)$ which is *Markovian*, that is, if $f \in L^p(E, m)$, $0 \leqslant f \leqslant 1$ then $0 \leqslant S_t f \leqslant 1$ and there exists a sequence $(f_n)_n \subset L^p(E, m)$, $f_n \leqslant 1$ for all n, such that the sequence $(S_t f_n)_n$ is increasing m-a.e. to the constant function 1. It turns out that

(5.1) *the transition function $\mathbb{S}$ of a semi-dynamical system becomes a C_0-semigroup of Markovian contractions on $L^p(E, m)$ which in addition is* multiplicative on $L^p(E, m)$, *i.e.,*

$$S_t(fg) = (S_t f)(S_t g) \ \textit{ for all } \ f, g \in L^\infty(E, m) \cap L^p(E, m) \ \textit{ and } \ t \geqslant 0.$$

In this framework, Theorem 4.1 has a natural correspondent which goes as follows:

Proposition 5.1. *Let $\mathbb{T} = (T_t)_{t \geqslant 0}$ be the transition function of a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ with state space E and $\Phi = (\Phi_t)_{t \geqslant 0}$ a right continuous flow on E, with transition function $\mathbb{S} = (S_t)_{t \geqslant 0}$ as in Theorem 4.1. Let m be a positive σ-finite measure on E which is subinvariant for both $\mathbb{S}$ and $\mathbb{T}$ and let $p \in (1, \infty)$.*

Consider the generators $(L_p, \mathcal{D}(L_p))$, $(D_p, \mathcal{D}(D_p))$, *and* $(L_p^\Phi, \mathcal{D}(L_p^\Phi))$ *of* $\mathbb{T}$, $\mathbb{S}$, *and respectively* $\mathbb{T}^\Phi$ *as* C_0*-semigroups on* $L^p(E,m)$, *where* $\mathbb{T}^\Phi = (T_t^\Phi)_{t\geqslant 0}$ *is defined as*

$$T_t^\Phi := S_t T_t \text{ for all } t \geqslant 0.$$

Let further $\mathcal{D}_o := U_\alpha V_\beta(L^p(E,m))$, $\alpha, \beta > 0$, *where* $\mathcal{U}$ *and* $\mathcal{V}$ *are the resolvents of* $\mathbb{T}$ *and* $\mathbb{S}$ *on* $L^p(E,m)$. *Then the following assertions hold.*

(i) $\mathcal{D}_o$ *is a core of* L_p *and* D_p, $\mathcal{D}_o \subset \mathcal{D}(L_p) \cap \mathcal{D}(D_p) \cap \mathcal{D}(L_p^\Phi)$, *and*

$$L_p^\Phi = L_p + D_p \text{ on } \mathcal{D}_o.$$

(ii) Let $X_t^\phi := \Phi_t(X_t)$, $t \geqslant 0$, $g_0 \in L_+^{p'}(E,\mu)$ *(where* $\frac{1}{p} + \frac{1}{p'} = 1$*) be such that* $\int_E g_0 \,\mathrm{d}m = 1$, *and put* $\nu = g_0 \cdot m$. *Then* $(X_t^\Phi)_{t\geqslant 0}$ *solves the martingale problem for* $\big(L_p^\Phi, \mathcal{D}(L_p^\Phi)\big)$ *under* $\mathbb{P}^\nu = \int_E \mathbb{P}^x \nu(\mathrm{d}x)$, *that is, for every* $u \in \mathcal{D}(L_p^\Phi)$

$$\left(u(X_t^\Phi) - u(X_0^\Phi) - \int_0^t L_p^\Phi u(X_s^\Phi)\,\mathrm{d}s \right)_{t\geqslant 0}$$

is an $(\mathcal{F}_t)_{t\geqslant 0}$*-martingale under* $\mathbb{P}^\nu$.

Proof. Because $\mathcal{D}(L_p) = U_\alpha(L^p(E,m))$ and $\mathcal{D}(D_p) = V_\beta(L^p(E,m))$ we clearly have that D_o is dense in $L^p(E,m)$. Assertion (i) follows arguing as in the proof of Theorem 4.1 (ii).

Assertion (ii) is a consequence of Proposition 1.4 from [6]. □

Proposition 3.7 has an L^p-version as well, and its proof is given in Appendix (A.5).

Proposition 5.2. *Let* $(P_t)_{t\geqslant 0}$ *be a sub-Markovian strongly continuous semigroups of contractions on* $L^p(E,\mu)$. *Then the following assertions are equivalent.*

(i) The semigroup $(P_t)_{t\geqslant 0}$ *is multiplicative on* $L^p(E,m)$.

(ii) If $(L, D(L))$ *is the infinitesimal generator of* $(P_t)_{t\geqslant 0}$, *then*

$$u \in D(L) \cap L^\infty(E,\mu) \Rightarrow u^2 \in D(L) \quad \text{and} \quad Lu^2 = 2uLu.$$

Example. Let $E = [0,1) \cup (1,\infty)$, $\mu =$ Lebesgue measure on E and for $f \in L^p(E,\mu)$, let $P_t f := f(.+t)$. Then $(P_t)_{t\geqslant 0}$ is a sub-Markovian C_0- semigroup of contractions on $L^P(E,\mu)$ which is multiplicative. Let $E' = [0,\infty)$. Then clearly $(P_t)_{(t\geqslant 0)}$ coincides (on L^p) with the transition function of the semi-dynamical system on $E' \supset E$ given by uniform motion to the right.

The next theorem is the main result on multiplicative L^p-semigroups and continuous flows, and it represents a converse of statement (5.1).

Theorem 5.3. *Let* $p \in [1, +\infty)$ *and* $(\mathbf{S}_t)_{t\geqslant 0}$ *be a* C_0*-semigroup of Markovian contractions on* $L^p(E,\mu)$ *which is multiplicative, where* $(E,\mathcal{B})$ *is a Lusin measurable space and* μ *is a* σ*-finite measure on* $(E,\mathcal{B})$. *Then there exist a Lusin topological space* E' *with* $E \subset E'$, $E \in \mathcal{B}'$ *(the* σ*-algebra of all Borel subsets of* E'*),* $\mathcal{B} = \mathcal{B}'|_E$, *and a continuous flow with state space* E' *such that its transition function* $\mathbb{S} = (S_t)_{t\geqslant 0}$, *regarded on* $L^p(E',\overline{\mu})$, *coincides with* $(\mathbf{S}_t)_{t\geqslant 0}$, *where* $\overline{\mu}$ *is the measure on* $(E',\mathcal{B}')$ *extending* μ *by zero on* $E' \setminus E$.

Proof. Let $(\mathbf{V}_\alpha)_{\alpha>0}$ be the resolvent of sub-Markovian contractions on $L^p(E,\mu)$ associated with $(\mathbf{S}_t)_{t\geqslant 0}$. By Theorem 2.2 from [5] there exist a Lusin topological space E' with $E\subset E'$, $E\in\mathcal{B}'$ (the σ-algebra of all Borel subsets of E'), $\mathcal{B}=\mathcal{B}'|_E$, and a right Markov process X with state space E' such that its resolvent $(V_\alpha)_{\alpha>0}$, regarded on $L^p(E',\mu')$, coincides with $(\mathbf{V}_\alpha)_{\alpha>0}$, where μ' is the measure on $(E',\mathcal{B}')$ extending μ by zero on $E'\setminus E$.

Let $(P'_t)_{t\geqslant 0}$ be the transition function of X and $\mathcal{A}$ be a countable subset of $bp\mathcal{B}'\cap L^p(E',\mu')$ which is multiplicative and generates the σ-algebra $\mathcal{B}'$. Consider the set

$$F_o=\{x\in E' : P'_t(V_\beta f\cdot V_\beta g)=P'_t(V_\beta f)P'_t(V_\beta g) \text{ for all } t\in\mathbb{Q}_+ \text{ and } f,g\in\mathcal{A}\}$$

for some $\beta>0$. Clearly, P'_t coincides with $\mathbf{S}_t$ as an operator on $L^p(E',\mu')$ for each $t\geqslant 0$, hence it is multiplicative on $L^p(E',\mu')$ and therefore $\mu'(E'\setminus F_o)=0$. We have $F_o\in\mathcal{B}'$ and applying Lemma 2.8 from [7] we deduce that it is finely closed. By Lemma 2.1 and its proof from [5] there exists a finely closed set $F\in\mathcal{B}'$, $F\subset F_o$, such that $\mu'(E'\setminus F)=0$ and $V_\alpha(1_{E'\setminus F})=0$ on F. Since $V_\alpha(1_{E'\setminus F})>0$ on the finely open set $E'\setminus F$, if follows that F is an absorbing subset of E'. Therefore we may consider the restriction $(P_t)_{t\geqslant 0}$ of the transition function $(P'_t)_{t\geqslant 0}$ from E' to F, $P_tf:=P'_tf'|_F$, where $f'\in p\mathcal{B}'$ is such that $f'|_F=f$.

Because the functions $t\longmapsto P'_t(V_\beta f\cdot V_\beta g)$ and $t\longmapsto P'_t(V_\beta f)$ are right continuous on $[0,\infty)$ it follows that $P'_t(V_\beta f\cdot V_\beta g)=P'_t(V_\beta f)P'_t(V_\beta g)$ on F for all $t\geqslant 0$ and $f,g\in\mathcal{A}$. By a monotone class argument we get that $(P_t)_{t\geqslant 0}$ is a multiplicative transition function on F and condition (2.2) is satisfied. Consequently, Remark 2.2 implies that there exists a semi-dynamical system $\Phi^o=(\Phi^o_t)_{t\geqslant 0}$ on F having the transition function $(P_t)_{t\geqslant 0}$.

Let $\Phi=(\Phi_t)_{t\geqslant 0}$ on E' be the trivial extension of Φ^o from F to E', $\Phi_t(x):=\Phi^o_t(x)$ if $x\in F$ and $\Phi_t(x)=x$ if $x\in E'\setminus E$ for all $t\geqslant 0$. Since $(sd4)$ holds on F for Φ^o with the countable set $\mathcal{C}_o\subset bp\mathcal{B}$ then $(sd4)$ also holds for Φ on E' considering a countable set $\mathcal{C}'_o\subset bp\mathcal{B}'$ which separates the points of E' and $\mathcal{C}'_o|_E=\mathcal{C}_o$. So, $\Phi=(\Phi_t)_{t\geqslant 0}$ is a semi-dynamical system on E' and applying Theorem 2.4 we may replace the topology of E' with a a conveninent Ray one, such that Φ becomes a continuous flow on E' as claimed. □

Remark 5.4. *It is proven in [6] that under additional assumptions on the domain of the generator of a C_0-semigroup of sub-Markovian contractions on $L^p(E,m)$ the associated Markov process exists on E, so, it is not more necessary to consider a larger state space; for applications in significant examples see also [29], [30] and [21]. In this case, if the semigroup is multiplicative on $L^p(E,m)$, one can see that the associated continuous flow from Theorem 5.3 remains on E.*

Appendix

(A.1) Proof of Lemma 3.1. Observe first that if $f\in\mathcal{B}_o$ then $U_\alpha|f|\leqslant U_\alpha h_\alpha<\infty$ for all $\alpha>0$, so, $U_\alpha f\in[\mathcal{B}]$.

(i) Let $\alpha,\alpha'>0$ and $\alpha_o:=\inf(\alpha,\alpha')$. If $f\in\mathcal{B}_o$ then there exist $t_o>0$ and $h_{\alpha_o}\in p\mathcal{B}$ such that $\sup_{0<s<t_o}T_s|f|\leqslant h_{\alpha_o}$ with $U_{\alpha_o}h_{\alpha_o}<\infty$. Then one can see that $\sup_{0<s<t_o}T_s|U_\alpha f|\leqslant U_\alpha h_{\alpha_o}$. Since $\alpha_o\leqslant\alpha,\alpha'$ and $U_{\alpha_o}h_{\alpha_o}<\infty$, it follows that $U_\alpha h_{\alpha_o}$ and $U_{\alpha'}U_\alpha h_{\alpha_o}$ are also real-valued functions. We conclude that $U_\alpha f\in\mathcal{B}_o$. We have also $T_tU_\alpha h_{\alpha_o}\leqslant e^{\alpha t}U_\alpha h_{\alpha_o}<\infty$, $t>0$, hence $U_\alpha f\in\mathcal{B}_{oo}$.

Let now $f\in\mathcal{B}_{oo}$, $\alpha,t'>0$ and $t_o>0$ be such that $\sup_{0<s<t_o}T_s|f|\leqslant h:=\inf(h_t,h_{t+t'},h_\alpha)\in p\mathcal{B}$ with $T_th_t+T_{t+t'}h_{t+t'}+U_\alpha h_\alpha<\infty$. Then $T_t|T_sf|\leqslant T_th_t<\infty$ for every $s<t_o$. Since $\lim_{s\searrow 0}T_sf=f$, we deduce by dominated convergence that $T_t|f|<\infty$ and $\lim_{s\searrow 0}T_sT_tf=T_tf$.

We have also $\sup_{0<s<t_o} T_s|T_tf| \leqslant T_th < \infty$ with $U_\alpha T_th \leqslant e^{\alpha t}U_\alpha h < \infty$ and $T_{t'}T_th < \infty$. Therefore $T_tf \in \mathcal{B}_{oo}$.

Assertion (ii) follows because $U_\alpha h, T_th \in \mathcal{E}$, provided that $h \in \mathcal{E}$.

(iii) Let $f \in \mathcal{B}^0$ be bounded, so, we may assume that $|f| \leqslant 1$. Then $|f| \leqslant \widehat{1} := \lim_{t\searrow 0} T_t 1$ which is excessive and therefore f belongs to $\mathcal{B}_e$.

(iv) The first assertion follows from (ii) since $f = \lim_{t\searrow 0} T_tf$ if $f \in [\mathcal{E}] \cup b[\mathcal{E}_\alpha]$. If $U|f| < \infty$ then $Uf \in [\mathcal{E}]$ and therefore $Uf \in \mathcal{B}_{oo}$. □ □

(A.2) Proof of Proposition 3.4. (i) Since by assertion (i) of Lemma 3.1 we have $U_\alpha(\mathcal{B}_o) \subset \mathcal{B}_{oo}$, it is clear that the set $U_\alpha(\mathcal{B}_o)$ does not depend on $\alpha > 0$. Let $u = U_\alpha f$ with $f \in \mathcal{B}_o$. Then u also belongs to $\mathcal{B}_o$, hence in particular, $U_\alpha|u| < \infty$. Let further $t_o > 0$ and $h_\alpha \in p\mathcal{B}$ be such that $T_s|f| \leqslant h_\alpha$ for all $s < t_o$ and $U_\alpha h_\alpha < \infty$. We have $|T_tu - u| \leqslant (e^{\alpha t} - 1)|u| + h_\alpha e^{\alpha t}\int_0^t e^{-\alpha s}\,\mathrm{d}s$ if $t < t_o$. Therefore $\sup_{0<t<t_o} |\frac{T_tu-u}{t}| \leqslant h := \alpha|u| + h_\alpha$ and $U_\alpha h < \infty$. We also have $\frac{T_tu-u}{t} = \frac{e^{\alpha t}-1}{t}u + \frac{1-e^{\alpha t}}{\alpha t}f - \frac{e^{\alpha t}}{t}\int_0^t e^{-\alpha s}(T_sf - f)\,\mathrm{d}s$. Clearly, when $t \searrow 0$, the first term from the right hand side converges pointwise to αu, the second one to $-f$, while the third one converges to zero because $\lim_{s\searrow 0} T_sf = f$. We conclude that $u \in \mathcal{D}(L)$ and $Lu = \alpha u - f$. Conversely, if $u \in \mathcal{D}(L)$ then let $\alpha, t_o > 0$, and $h_\alpha \in p\mathcal{B}$ with $U_\alpha h_\alpha < \infty$ and $\sup_{0<t<t_o} |\frac{T_tu-u}{t}| \leqslant h_\alpha$. Let $v := Lu = \lim_{t\searrow 0} \frac{T_tu-u}{t} \in \mathcal{B}_o$. Because $U_\alpha h_\alpha < \infty$, by dominated convergence we get $\lim_{t\searrow 0} \frac{T_tU_\alpha u - U_\alpha u}{t} = U_\alpha v$. On the other hand, from the first part of the proof we have $U_\alpha u \in \mathcal{D}(L)$ and $\lim_{t\searrow 0} \frac{T_tU_\alpha u - U_\alpha u}{t} = L(U_\alpha u) = \alpha U_\alpha u - u$. We conclude that $u = U_\alpha(\alpha u - v) \in U_\alpha(\mathcal{B}_o)$.

To prove the last assertion of (i) we argue as in the proof of Proposition 1.5 (a) from [25]. We have $\frac{T_hu-u}{h} = \frac{1}{h}\int_0^t [T_{s+h}f - T_sf]\,\mathrm{d}s = \frac{1}{h}\int_t^{t+h} T_sf\,\mathrm{d}s - \frac{1}{h}\int_0^h Tsf\,\mathrm{d}s$. Because the function $s \longmapsto T_sf(x)$ is right continuous on $[0,\infty)$ for every $x \in E$, it follows that $\lim_{h\searrow 0} \frac{T_hu-u}{h} = T_tf - f$ pointwise on E. Since we also have $|\frac{T_hu-u}{h}| \leqslant 2\|f\|_\infty$ for all $h > 0$, we conclude that u belongs to $\mathcal{D}(L)$ and $Lu = T_tf - f$.

(ii) Let $x \in E$. Since $g \in \mathcal{B}^0$ we have $\lim_{t\searrow 0} T_tg(x) = g(x)$ and therefore $\lim_{t\searrow 0} \frac{T_tu(x)-u(x)}{t} = \lim_{t\searrow 0} \frac{1}{t}\int_0^t T_sg(x)\mathrm{d}s = g(x) = \overline{L}u(x)$.

(iii) We clearly have $\mathcal{D}(L_w) \subset \mathcal{D}_e(L)$ because $b\mathcal{B}^0 \subset \mathcal{B}_e$. Let $u = U_\alpha f \in \mathcal{D}(L)$. Then, by assertion (i) of Lemma 3.1 we get $u \in \mathcal{B}_{oo}$ and we have $\int_0^t T_s(\alpha u - f) = \alpha\int_0^t\int_0^\infty e^{-\alpha r}T_{r+s}f\,\mathrm{d}r\,\mathrm{d}s - \int_0^t T_sf\,\mathrm{d}s = \int_0^\infty (e^{\alpha r\wedge t} - 1)e^{-\alpha r}T_tf\,\mathrm{d}r - \int_0^t T_sf\,\mathrm{d}s = -u + e^t\int_t^\infty e^{-\alpha r}T_rf\,\mathrm{d}r = -u + T_tU_\alpha f = -u + T_tu$, where for the second equality we used Fubini's Theorem. We conclude that $u \in \mathcal{D}(\overline{L})$ and by assertion (ii) we clearly have $\overline{L}u = Lu$. Let now $u \in \mathcal{D}(\overline{L}) \cap \mathcal{B}_o$ such that $\overline{L}u \in \mathcal{B}_o$, let $\alpha > 0$ and $h_\alpha \in p\mathcal{B}$ with $U_\alpha h_\alpha < \infty$, be such that $T_s(|Lu|) \leqslant h_\alpha$ for all $s < t_o$ for some $t_o > 0$. Then $|\frac{T_tu-u}{t}| \leqslant \frac{1}{t}\int_0^t T_s(|Lu|)\mathrm{d}s \leqslant h_\alpha$ for all $t < t_o$. It follows that $u \in \mathcal{D}(L)$.

(iv) Let $u \in \mathcal{D}_o(L)$ and $\alpha > 0$. Then $u = U_\alpha(\alpha u - Lu)$ with $u, Lu \in \mathcal{B}_{oo}$, so, $u \in U_\alpha(\mathcal{B}_{oo})$. Conversely, if $u = U_\alpha f$ with $f \in \mathcal{B}_{oo}$, then by assertion (i) we have $u \in \mathcal{D}(L)$ and $Lu = \alpha u - f \in \mathcal{B}_{oo}$, hence $u \in \mathcal{D}_o(L)$.

Let $u = U_\alpha f \in \mathcal{D}_o(L)$, $f \in \mathcal{B}_{oo}$. According to Lemma 3.1 (i) we get $T_tf \in \mathcal{B}_{oo}$. Therefore $T_tu = U_\alpha T_tf$ also belongs to $\mathcal{D}_o(L)$ and we have $LT_tu = LU_\alpha T_tf = \alpha U_\alpha T_tf - T_tf = T_tLu$.

The proof of (v) is straightforward.

Assertion (vi) follows arguing as in the proof of (iv) and using Lemma 3.1 (ii).

(vii) The first inclusion follows from assertion (iv). Let now $u \in U_\alpha(\mathcal{D}(L)$, $u = U_\alpha U_\beta f$ with $f \in \mathcal{B}_o$. Then $LT_tu = \alpha T_tu - T_tU_\beta f$ and it is continuous in t, according with the following remark: *If* $g \in [\mathcal{B}]$ *is such that* $U_\alpha|g| < \infty$ *then the real-valued function* $t \longmapsto T_t U_\alpha g(x)$ *is continuous on* $[0, \infty)$ *for each* $x \in E$ *because* $T_t U_\alpha g = e^{\alpha t} \int_t^\infty e^{-\alpha s} T_s g \, \mathrm{d}s$.

To prove the last inclusion of assertion (*vii*), observe that by Lemma 3.1 (*iv*) we have $U_\alpha(b[\mathcal{B}]) \subset \mathcal{B}_{oo}$ and by assertion (*iv*) we obtain $U_\beta U_\alpha(b[\mathcal{B}]) \subset \mathcal{D}_o(L)$. The continuity property is obtained using again the above remark.

(*viii*) Let $u \in \mathcal{D}_o^c(L)$, $u = U_\alpha f$ with $f \in \mathcal{B}_{oo}$. Then by Lemma 3.1 (*i*) we have $T_t f \in \mathcal{B}_{oo}$ for each $t \geqslant 0$ and $LT_t u = \alpha T_t u - T_t f$. Because $t \longmapsto T_t u(x)$ is continuous, it follows that $T_t f(x)$ is also continuous in t on $[0, \infty)$ for each $x \in E$. We have $T_t u = e^{\alpha t}(u - \int_0^t e^{\alpha s} T_s f \, \mathrm{d}s)$ and from the above considerations the first statement of assertion (*viii*) follows. In particular, we proved that $u_t := T_t u$, $t \geqslant 0$, is a solution to the equation (3.9), satisfying the requested conditions: $T_0 = u$, $\|T_t u\|_\infty \leqslant \|u\|_\infty$, $T_t u \in \mathcal{D}_o(L)$ by the above assertion (*iv*), $LT_t \in \mathcal{B}_{oo}$, and $LT_t u(x)$ is continuous in t because we assumed that u belongs to $\mathcal{D}_o^c(L)$.

We show now the uniqueness property for the solution to the equation (3.9) and as announced, we use a classical argument, e.g., as in the proof of Theorem 1.3 from [23], Ch. I, section 3, page 28. Let u_t, $t \geqslant 0$, be a solution of (3.9) such that $u_0 = 0$, $u_t \in \mathcal{D}_o(L)$, $\|u_t\|_\infty$ is bounded, $Lu_t \in \mathcal{B}_{oo}$, and $Lu_t(x)$ is continuous in t for each $x \in E$. We have to show that $u_t = 0$ for each $t > 0$. Let $\alpha > 0$ and $v_t := e^{-\alpha t} u_t$. Then $\frac{\mathrm{d}v_t}{\mathrm{d}t} = (L - \alpha) v_t$ with $v_t \in \mathcal{D}_o(L)$. It follows that $U_\alpha(\frac{\mathrm{d}v_t}{\mathrm{d}t}) = -v_t$ for each $t > 0$ and therefore $\int_0^t v_s \, \mathrm{d}s = -U_\alpha(\int_0^t \frac{\mathrm{d}v_s}{\mathrm{d}s} \, \mathrm{d}s) = -U_\alpha v_t$. Consequently, $\int_0^t e^{-\alpha s} u_s(x) \, \mathrm{d}s = -e^{-\alpha t} U_\alpha u_t(x)$. Since $\|u_t\|_\infty$ is bounded, letting $t \to \infty$, it follows that the right hand side of the above equality tends to zero. We conclude that $\int_0^\infty e^{-\alpha s} u_s(x) \, \mathrm{d}s = 0$ for every $\alpha > 0$ and $x \in E$ and therefore $u_s(x) = 0$ for each $s > 0$ and $x \in E$. □

(A.3) Proof of Crefprop3.6. Let $g = \overline{D} f$ with $f \in \mathcal{D}(\overline{D})$ and $\int_0^t S_s(|f \overline{D} f|) \, \mathrm{d}s < \infty$ for all $t > 0$. We have to prove that $S_t f^2 = f^2 + 2 \int_0^t S_s f S_s g \, \mathrm{d}s$ for all $t > 0$, provided that $S_t f = f + \int_0^t S_s g \, \mathrm{d}s$. Indeed, we have $\int_0^t S_s f S_s g \, \mathrm{d}s = 2 \int_0^t [f + \int_0^s S_u g \, \mathrm{d}u] S_s g \, \mathrm{d}s = f \int_0^t S_s g \, \mathrm{d}s + \int_0^t \mathrm{d}s S_s g \int_0^s S_u g \, \mathrm{d}u = f \int_0^t S_s g \, \mathrm{d}s + \int_0^t \mathrm{d}u S_u g [\int_0^t S_s g \, \mathrm{d}s - \int_0^u S_s g \, \mathrm{d}s] = f \int_0^t S_s g \, \mathrm{d}s + \int_0^t \mathrm{d}u S_u g [\int_0^t S_s g \, \mathrm{d}s + f - S_u f] = 2f \int_0^t S_s g \, \mathrm{d}s - \int_0^t S_u f S_u g \, \mathrm{d}u = S_t f^2 + f^2 - 2f S_t f$. We conclude that $2 \int_0^t S_s f S_s g \, \mathrm{d}s = 2f(S_t f - f) + S_t f^2 + f^2 - 2f S_t f = S_t f^2 - f^2$. □

(A.4) Proof of Proposition 3.9. The proof of (*i*) is a straightforward verification.

(*ii*) Let $u \in \mathcal{D}(D^o)$ and $x \in \mathcal{O}^c$. Then by (*i*) we have $\Phi_t^o(x) = x$ and therefore $D^o u(x) = 0$. Let further $\mathbb{S} = (S_t)_{t \geqslant 0}$ (resp. $\mathbb{S}^o = (S_t^o)_{t \geqslant 0}$) be the transition function of Φ (resp. of Φ^o). If $u \in p\mathcal{B}(E)$ the $S_t^0 u(x) = S_t u(x)$ provided that $t < T(x)$ and $S_t^0 u(x) = u(\Phi_{T(x)}(x)$ if $t > T(x)$ and $T(x) < \infty$. If $u \in \mathcal{D}(D)$ and $x \in \mathcal{O}$ then there exists $\varepsilon > 0$ such that $\Phi_t(x) \in \mathcal{O}$ for all $t \leqslant \varepsilon$, hence $T(x) \geqslant \varepsilon$ and therefore $S_t^o u(x) = S_t u(x)$ for all $t \leqslant \varepsilon$. We conclude that $Du = D^o u$ on $\mathcal{O}$.

(*iii*) Let $x \in \overline{\mathcal{O}}$. If $x \in \partial\mathcal{O}$ then by (*i*) we have $\Phi_t^o(x) = x \in \overline{\mathcal{O}}$ for all $t \geqslant 0$. If $x \in \mathcal{O}$ then clearly $\Phi^o(x) = \Phi_t(x) \in \mathcal{O}$ for all $t < T(x)$. If $t \geqslant T(x)$ then $\Phi_t^o(x) = \Phi_{T(x)}(x) \in \partial\mathcal{O}$ by property (2) of T.

Assertion (*iv*) follows from (*iii*). □

(A.5) Proof of Proposition 5.2. Let $u \in D(L) \cap L^\infty(E, \mu)$. We have $\frac{P_t u^2 - u^2}{t} = \frac{P_t u - u}{t}(P_t u + u)$ and since $\frac{P_t u - u}{t}$ (resp. $P_t u$) is converging in $L^p(E, \mu)$ to Lu (resp. to u) as $t \to 0$, we deduce that $\frac{P_t u^2 - u^2}{t}$ is converging to $2uLu$, hence $u^2 \in D(L)$ and $Lu^2 = 2uLu$. Conversely, let $u \in D(L) \cap L^\infty(E, \mu)$ and put $u_t := (P_t u)^2 \in \mathcal{D}(L)$. Since $\frac{\mathrm{d}u_t}{\mathrm{d}t} = 2P_t u \cdot LP_t u = Lu_t$ and

$u_0 = u^2$, we get that $u_t = P_t u^2$, hence $(P_t u)^2 = P_t u^2$. It follows that $P_t(uv) = P_t u \cdot P_t v$ for all $u, v \in D(L) \cap L^\infty(E, \mu)$ and because $D(L) \cap L^\infty(E, \mu)$ is dense in $L^p(E, \mu)$ we conclude that the semigroup $(P_t)_{t \geqslant 0}$ is multiplicative on $L^p(E, \mu)$. □

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