HARMONIC FORMS ON ALE RICCI-FLAT 4-MANIFOLDS

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Abstract. In this paper, we compute the expansion of harmonic functions and 1-forms on ALE Ricci-flat 4-manifolds.

1. Introduction

This paper studies ALE Ricci-flat 4-manifolds. In [BKN89], Bando, Kasue, and Nakajima studied the expansion of the metric near infinity and provided a conjecture that all simply-connected ALE Ricci-flat 4-manifolds must be hyper-Kähler, which has been classified by Kronheimer, see [Kro89a] and [Kro89b]. There has been only a little progress on this conjecture; see [LV16] and [Li23]. More recently, in [BH23], Biquard and Hein improved the expansion of the metric near infinity and proved that the renormalized volume $\mathcal{V} = \lim_{R\to\infty} Vol(B_R, g_X) - Vol(B_R, g_{\mathbb{R}^4/\Gamma})$ is non-positive. They used a function with a constant Laplacian to prove their results.

In [BH23], Biquard and Hein also sketched the computation of expansions of harmonic 1-forms asymptotic to $x^i dx^j - x^j dx^i$, and used this to study Killing fields. The main goal of this paper is to generalize their result to harmonic 1-forms asymptotic to $x^i dx^j$.

Our first theorem deals with functions

Theorem 1.1. *We fix $\epsilon \in (0,1)$. For any $a_{ij} \in \mathbb{R}$, $i,j = 1,\cdots,4$, such that $a_{ij} = a_{ji}$, such that $\sum_{i,j=1}^4 a_{ij}x^ix^j$ is invariant under Γ, there exists a unique smooth function u_a on X such that*

$$\Pi^*\Phi^*u_a - \sum_{i,j=1}^4 a_{ij}x^ix^j \in W^{k,2}_{-2+\epsilon}(\mathbb{R}^4\backslash B_{2R}(0)) \tag{1}$$

for all $k \geq 0$, and $\Delta_X u_a = (d\delta_X + \delta_X d)u_a = -2\sum_{i,j=1}^4 a_{ij}\delta_{ij}$ on X. Moreover, the expansion of u_a is given by the following:

$$\Pi^*\Phi^*u_a - \sum_{i,j=1}^4 a_{ij}\big(x^ix^j - \tilde{\eta}_{ij} + \frac{|\Gamma|\mathcal{V}\delta_{ij}}{2\pi^2r^2}\big) \in W^{k,2}_{-3+\epsilon}(\mathbb{R}^4\backslash B_{2R}(0)), \tag{2}$$

where $r = \sqrt{\sum_{i=1}^4 (x^i)^2}$, $\tilde{\eta}_{ij}$ is defined in (78) and (89).

We have a similar result for 1-forms.

Theorem 1.2. *We fix $\epsilon \in (0,1)$. For any $a_{ij} \in \mathbb{R}$, $i,j = 1,\cdots,4$, such that $\sum_{i,j=1}^4 a_{ij}x^idx^j$ is invariant under Γ, there exists a unique smooth 1-form ω_a on X such that*

$$\Pi^*\Phi^*\omega_a - \sum_{i,j=1}^4 a_{ij}x^idx^j \in W^{k,2}_{-3+\epsilon}(\mathbb{R}^4\backslash B_{2R}(0)) \tag{3}$$

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for all $k \geq 0$, and $\Delta_X \omega_a = (d\delta_X + \delta_X d)\omega_a = 0$ on X. Moreover, the expansion of ω_a is given by the following:

$$\Pi^*\Phi^*\omega_a - \sum_{i,j=1}^{4} a_{ij}\big(x^i dx^j - \tilde{\mu}_{ij} + \sum_{k,l=1}^{4} \frac{Con[i,j,l,k]x^k}{r^4} dx^l\big) \in W^{k,2}_{-4+\epsilon}(\mathbb{R}^4 \backslash B_{2R}(0)), \tag{4}$$

where $\tilde{\mu}_{ij}$ is defined in (154), and $Con[i,j,k,l]$ satisfies the equations in (223), (224) and appendix A.

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2. The settings of ALE-manifolds

In this part, we prepare the basic elements for our calculation.

Let (X, g) be an ALE Ricci-flat 4-manifold. This means that there exists a finite subgroup Γ of $SO(4)$ acting freely on $\mathbb{S}^3$, together with a quotient map $\Pi : \mathbb{R}^4 \backslash B_R(0) \to (\mathbb{R}^4 \backslash B_R(0))/\Gamma$, and a diffeomorphism $\Phi : (\mathbb{R}^4 \backslash B_R(0))/\Gamma \to X \backslash U$ for some bounded open subset $U \subset X$ such that for all $k \in \mathbb{N}_0$,

$$|\nabla^k_{g_0}(\Pi^*\Phi^* g - g_0)|_{g_0} = O(r^{-4-k}) \text{ as } r \to \infty, \tag{5}$$

where g_0 denotes the Euclidean metric on $\mathbb{R}^4$.

From Theorem B in [BH23], for any $k_0 \in \mathbb{N}$, there exists a decomposition

$$\Pi^*\Phi^* g - g_0 = h + h', \tag{6}$$

where the leading term

$$h : \mathbb{R}^4 \backslash B_R(0) \to \mathrm{Sym}^2 \mathbb{R}^4 \tag{7}$$

is a Γ-equivariant harmonic function that decays at infinity, and

$$h = h^+ + h^-; \quad \sum_{k=0}^{k_0} r^k |\nabla^k_{g_0} h'|_{g_0} \leq C(k_0) r^{-5} \tag{8}$$

for any given $k_0 \in \mathbb{N}$.

In the formula above, h^+ is a symmetric 2-tensor on $\mathbb{R}^4 \backslash B_R(0)$ of the form

$$-\frac{3}{2} r^6 h^+ = \zeta_{11}(2\alpha_1^2 - \alpha_2^2 - \alpha_3^2) + \zeta_{22}(2\alpha_2^2 - \alpha_3^2 - \alpha_1^2) + \zeta_{33}(2\alpha_3^2 - \alpha_1^2 - \alpha_2^2) \tag{9}$$

$$+ \zeta_{12}(\alpha_1 \odot \alpha_2) + \zeta_{13}(\alpha_1 \odot \alpha_3) + \zeta_{23}(\alpha_2 \odot \alpha_3), \tag{10}$$

Where $f \odot g = f \otimes g + g \otimes f$ denotes the symmetic product, $\alpha_j = I_j(rdr)$[1], and (I_1, I_2, I_3) is the standard triple of complex structures on $\mathbb{R}^4$ given by the following:

$$I_1(x^1, x^2, x^3, x^4) = (-x^2, x^1, -x^4, x^3), \tag{11}$$

$$I_2(x^1, x^2, x^3, x^4) = (-x^3, x^4, x^1, -x^2), \tag{12}$$

$$I_3 = I_1 I_2. \tag{13}$$

[1]Actually, one should understand this as $\sum_{i=1}^{4} 2x^i I_j^*(dx^i)$, and one can omit the 2 for simplicity, and remark that since $I_3 = I_1 I_2$, we have $I_3^* = I_2^* I_1^*$.

and where (ζ_{ij}) is any symmetric 3×3 matrix, and $h^- = \mathcal{R}^* h^+$ for some h^+ of the form above and some $\mathcal{R} \in \mathrm{O}(4)\backslash\mathrm{SO}(4)$.
Now we choose a particular $\mathcal{R} = \mathrm{diag}(1,-1,-1,-1)$, and use ξ to denote the other ζ appearing in $h^- = \mathcal{R}^* h^+$. Thus we have the following explicit expression of h^+ and h^- as a matrix w.r.t the basis $\{dx^i \otimes dx^j\}$ in the region $\{r \geq R\}$:

$$(-\frac{3}{2}r^6h^+)_{11} = (x^2)^2(2\zeta_{11}-\zeta_{22}-\zeta_{33}) + 6x^2(\zeta_{12}x^3 - \zeta_{13}x^4) - \tag{14}$$

$$(x^3)^2(\zeta_{11}-2\zeta_{22}+\zeta_{33}) - 6\zeta_{23}x^3x^4 - (x^4)^2(\zeta_{11}+\zeta_{22}-2\zeta_{33}), \tag{15}$$

$$(-\frac{3}{2}r^6h^+)_{12} = x^1(x^2(-2\zeta_{11}+\zeta_{22}+\zeta_{33}) - 3\zeta_{12}x^3 + 3\zeta_{13}x^4) \tag{16}$$

$$- 3\left(\zeta_{13}x^2x^3 + \zeta_{12}x^2x^4 + \zeta_{23}(x^3)^2 + \zeta_{22}x^3x^4 - \zeta_{33}x^3x^4 - \zeta_{23}(x^4)^2\right), \tag{17}$$

$$(-\frac{3}{2}r^6h^+)_{13} = x^1(-3\zeta_{12}x^2 + x^3(\zeta_{11}-2\zeta_{22}+\zeta_{33}) + 3\zeta_{23}x^4) \tag{18}$$

$$+ 3\left(\zeta_{13}(x^2)^2 + \zeta_{23}x^2x^3 + x^2x^4(\zeta_{11}-\zeta_{33}) + x^4(\zeta_{12}x^3 - \zeta_{13}x^4)\right), \tag{19}$$

$$(-\frac{3}{2}r^6h^+)_{14} = 3x^2(\zeta_{13}x^1 + x^3(\zeta_{22}-\zeta_{11}) - \zeta_{23}x^4) + 3\zeta_{23}x^1x^3 + x^1x^4(\zeta_{11}+\zeta_{22}-2\zeta_{33}) \tag{20}$$

$$+ 3\zeta_{12}(x^2)^2 + 3x^3(\zeta_{13}x^4 - \zeta_{12}x^3), \tag{21}$$

$$(-\frac{3}{2}r^6h^+)_{22} = (x^1)^2(2\zeta_{11}-\zeta_{22}-\zeta_{33}) + 6x^1(\zeta_{13}x^3 + \zeta_{12}x^4) - (x^3)^2(\zeta_{11}+\zeta_{22}-2\zeta_{33}) \tag{22}$$

$$+ 6\zeta_{23}x^3x^-(x^4)^2(\zeta_{11}-2\zeta_{22}+\zeta_{33}), \tag{23}$$

$$(-\frac{3}{2}r^6h^+)_{23} = 3\zeta_{12}(x^1)^2 - 3x^2(\zeta_{13}x^1 + \zeta_{23}x^4) + 3x^1(\zeta_{23}x^3 + x^4(\zeta_{22}-\zeta_{11})) \tag{24}$$

$$+ x^2x^3(\zeta_{11}+\zeta_{22}-2\zeta_{33}) - 3x^4(\zeta_{13}x^3 + \zeta_{12}x^4), \tag{25}$$

$$(-\frac{3}{2}r^6h^+)_{24} = -3\zeta_{13}(x^1)^2 - 3x^1(\zeta_{12}x^2 + x^3(\zeta_{33}-\zeta_{11}) + \zeta_{23}x^4) - 3\zeta_{23}x^2x^3 \tag{26}$$

$$+ x^2x^4(\zeta_{11}-2\zeta_{22}+\zeta_{33}) + 3x^3(\zeta_{13}x^3 + \zeta_{12}x^4), \tag{27}$$

$$(-\frac{3}{2}r^6h^+)_{33} = -\left((x^1)^2(\zeta_{11}-2\zeta_{22}+\zeta_{33})\right) - 6x^1(\zeta_{23}x^2 + \zeta_{12}x^4) \tag{28}$$

$$- (x^2)^2(\zeta_{11}+\zeta_{22}-2\zeta_{33}) + 6\zeta_{13}x^2x^4 + (x^4)^2(2\zeta_{11}-\zeta_{22}-\zeta_{33}), \tag{29}$$

$$(-\frac{3}{2}r^6h^+)_{34} = 3\left(x^1x^2(\zeta_{33}-\zeta_{22}) + x^1(\zeta_{13}x^4 - \zeta_{23}x^1) + \zeta_{23}(x^2)^2 + \zeta_{12}x^2x^4\right) \tag{30}$$

$$+ x^3(3\zeta_{12}x^1 - 3\zeta_{13}x^2 + x^4(-2\zeta_{11}+\zeta_{22}+\zeta_{33})), \tag{31}$$

$$(-\frac{3}{2}r^6h^+)_{44} = -\left((x^1)^2(\zeta_{11}+\zeta_{22}-2\zeta_{33})\right) + x^1(6\zeta_{23}x^2 - 6\zeta_{13}x^3) \tag{32}$$

$$- (x^2)^2(\zeta_{11}-2\zeta_{22}+\zeta_{33}) - 6\zeta_{12}x^2x^3 + (x^3)^2(2\zeta_{11}-\zeta_{22}-\zeta_{33}), \tag{33}$$

$$(-\frac{3}{2}r^6h^-)_{11} = (x^2)^2(2\xi_{11}-\xi_{22}-\xi_{33}) + 6x^2(\xi_{12}x^3 - \xi_{13}x^4) - (x^3)^2(\xi_{11}-2\xi_{22}+\xi_{33}) \tag{34}$$

$$- 6\xi_{23}x^3x^4 - (x^4)^2(\xi_{11}+\xi_{22}-2\xi_{33}), \tag{35}$$

$$(-\frac{3}{2}r^6h^-)_{12} = x^1(x^2(-2\xi_{11}+\xi_{22}+\xi_{33}) - 3\xi_{12}x^3 + 3\xi_{13}x^4) \tag{36}$$

$$+ 3\left(\xi_{13}x^2x^3 + \xi_{12}x^2x^4 + \xi_{23}(x^3)^2 + \xi_{22}x^3x^4 - \xi_{33}x^3x^4 - \xi_{23}(x^4)^2\right), \tag{37}$$

$$(-\frac{3}{2}r^6h^-)_{13} = x^1(-3\xi_{12}x^2 + x^3(\xi_{11} - 2\xi_{22} + \xi_{33}) + 3\xi_{23}x^4) \tag{38}$$

$$- 3\left(\xi_{13}(x^2)^2 + \xi_{23}x^2x^3 + x^2x^4(\xi_{11} - \xi_{33}) + x^4(\xi_{12}x^3 - \xi_{13}x^4)\right), \tag{39}$$

$$(-\frac{3}{2}r^6h^-)_{14} = 3x^2(\xi_{13}x^1 + x^3(\xi_{11} - \xi_{22}) + \xi_{23}x^4) + 3\xi_{23}x^1x^3 + x^1x^4(\xi_{11} + \xi_{22} - 2\xi_{33}) \tag{40}$$

$$- 3\xi_{12}(x^2)^2 + 3x^3(\xi_{12}x^3 - \xi_{13}x^4), \tag{41}$$

$$(-\frac{3}{2}r^6h^-)_{22} = (x^1)^2(2\xi_{11} - \xi_{22} - \xi_{33}) - 6x^1(\xi_{13}x^3 + \xi_{12}x^4) - (x^3)^2(\xi_{11} + \xi_{22} - 2\xi_{33}) \tag{42}$$

$$+ 6\xi_{23}x^3x^4 - (x^4)^2(\xi_{11} - 2\xi_{22} + \xi_{33}), \tag{43}$$

$$(-\frac{3}{2}r^6h^-)_{23} = 3\xi_{12}(x^1)^2 + 3\xi_{13}x^1x^2 - 3\xi_{23}x^1x^3 + 3x^1x^4(\xi_{11} - \xi_{22}) \tag{44}$$

$$+ x^2x^3(\xi_{11} + \xi_{22} - 2\xi_{33}) - 3\xi_{23}x^2x^4 - 3x^4(\xi_{13}x^3 + \xi_{12}x^4), \tag{45}$$

$$(-\frac{3}{2}r^6h^-)_{24} = -3\xi_{13}(x^1)^2 + 3x^1(\xi_{12}x^2 + x^3(\xi_{33} - \xi_{11}) + \xi_{23}x^4) - 3\xi_{23}x^2x^3 \tag{46}$$

$$+ x^2x^4(\xi_{11} - 2\xi_{22} + \xi_{33}) + 3x^3(\xi_{13}x^3 + \xi_{12}x^4), \tag{47}$$

$$(-\frac{3}{2}r^6h^-)_{33} = -\left((x^1)^2(\xi_{11} - 2\xi_{22} + \xi_{33})\right) + 6x^1(\xi_{23}x^2 + \xi_{12}x^4) \tag{48}$$

$$- (x^2)^2(\xi_{11} + \xi_{22} - 2\xi_{33}) + 6\xi_{13}x^2x^4 + (x^4)^2(2\xi_{11} - \xi_{22} - \xi_{33}), \tag{49}$$

$$(-\frac{3}{2}r^6h^-)_{34} = -3x^3(\xi_{12}x^1 + \xi_{13}x^2) + 3x^2(x^1(\xi_{22} - \xi_{33}) + \xi_{12}x^4) \tag{50}$$

$$- 3x^1(\xi_{23}x^1 + \xi_{13}x^4) + 3\xi_{23}(x^2)^2 + x^3x^4(-2\xi_{11} + \xi_{22} + \xi_{33}), \tag{51}$$

$$(-\frac{3}{2}r^6h^-)_{44} = -\left((x^1)^2(\xi_{11} + \xi_{22} - 2\xi_{33})\right) + 6x^1(\xi_{13}x^3 - \xi_{23}x^2) \tag{52}$$

$$- (x^2)^2(\xi_{11} - 2\xi_{22} + \xi_{33}) - 6\xi_{12}x^2x^3 + (x^3)^2(2\xi_{11} - \xi_{22} - \xi_{33}). \tag{53}$$

With the basic metric structure, we use the standard cut-off trick to extend r as a smooth function on X such that $r \geq 1$ on X. Then one can define the weighted Sobolev norm

$$||\omega||^2_{L^2_\nu(X)} := \int_X |\omega|^2 r^{-4-2\nu} dVol_X, \qquad ||\omega||^2_{W^{k,2}_\nu(X)} := \sum_{m=0}^{k} ||\nabla^m \omega||^2_{L^2_{\nu-m(X)}}, \tag{54}$$

where ω is a tensor field on X.

We need the following results in our proof, which are well-known to experts in this field. For example, see [Mel93] for the proof. Note that in a slightly different setting, the first author and his collaborators have provided a self-contained proof of a similar result, see Proposition 4.5 in [CVZ23].

Proposition 2.1. *Let (X, g) be an ALE manifold of order 4. Then the following properties hold.*

(1) *For any $\nu \in \mathbb{R} \setminus \mathbb{Z}$ and $k \in \mathbb{N}$, there exist constants $R(X,\nu) > 0$ and $C(X,\nu,k) > 0$ such that for any p-form $\omega \in W^{k+2,2}_\nu(X)$,*

$$\|\omega\|_{W^{k+2,2}_\nu(X)} \leq C \cdot (\|\Delta_X \omega\|_{W^{k,2}_{\nu-2}(X)} + \|\omega\|_{L^2(\{r\leq 3R\}\subset X)}). \tag{55}$$

(2) *For any $\nu \in \mathbb{R} \setminus \mathbb{Z}$ and $k \in \mathbb{N}$, the operator*

$$\Delta_X : W^{k+2,2}_\nu(X) \to W^{k,2}_{\nu-2}(X) \tag{56}$$

is a Fredholm operator. Then for any p-form $\omega \in W^{k,2}_{\nu-2}(X)$,

$$\Delta_X \tau = \omega \tag{57}$$

has a solution $\tau \in W^{k+2,2}_{\nu}(X)$ *if and only if for all* $\psi \in \mathcal{H}^p_{-2-\nu}(X)$,

$$\int_X (\omega, \psi)_X dVol_X = 0, \tag{58}$$

where $\mathcal{H}^p_{-2-\nu}(X)$ *is the space of all harmonic p-forms on X in* $L^2_{-2-\nu}(X)$. *Note that* $\mathcal{H}^p_{-2-\nu}(X) \subset W^{k,2}_{-2-\nu}(X)$ *for all* $k > 0$ *by elliptic estimates.*

(3) *For any* $\nu \in \mathbb{R} \setminus \mathbb{Z}$, $k \in \mathbb{N}$, *and p-form* $\omega \in W^{k,2}_{\nu}(X)$, *there exists some* $\tau \in W^{k+2,2}_{\nu+2}(X)$ *such that* $\Delta_{\mathbb{R}^4} \tau = \omega$ *when* $r \geq 2R$.

(4) *Let* $\nu, \nu' \in \mathbb{R} \setminus \mathbb{Z}$ *and* $\nu - \nu' \in (0,1)$. *Consider any form* $\omega \in W^{k,2}_{\nu}(\mathbb{R}^4 \setminus B(R))$ *such that* $\Delta_{\mathbb{R}^4}\omega = 0$ *when* $r \geq 2R$. *If* $\mathbb{Z} \cap [\nu', \nu] = \emptyset$, *then* $\omega \in W^{k,2}_{\nu'}(\mathbb{R}^4 \setminus B(R))$. *If there is some* $q \in \mathbb{Z} \cap (\nu', \nu)$, *then* ω *can be written as the sum of a* $\mathbb{R}^4$*-harmonic form* $\sum_{i_1 < \ldots < i_p} r^q u(\theta) dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ *and an element in* $W^{k,2}_{\nu'}(\mathbb{R}^4 \setminus B(R))$, *where* θ *denotes the coordinate on* $\mathbb{S}^3$.

3. Harmonic functions

In this section, we prove Theorem 1.1; for simplicity, we assume that $\Gamma = \{1\}$, other cases are all similar.

3.1. **Case 1.** We first consider the functions $(x^i)^2$. Recall the formula (here we will denote the $(i,j)-$term of $\Pi^*\Phi^* g$ as g_{ij}, and the determinant of (g_{ij}) as G).

$$\Delta_X = -\frac{1}{\sqrt{G}} \partial_i (\sqrt{G} g^{ij} \partial_j), \quad \Gamma^j_{ji} = \frac{1}{\sqrt{G}} \partial_i (\sqrt{G}). \tag{59}$$

We get

$$\Delta_X((x^i)^2) = -\left(\left(\Gamma^k_{kj} g^{jl} + \partial_j (g^{jl}) \right) \partial_l + g^{jk} \partial_j \partial_k \right) (x^i)^2 \tag{60}$$

$$= -\left(\Gamma^k_{kj} g^{ji} + \partial_j (g^{ji}) \right) 2x^i - 2g^{ii}. \tag{61}$$

In the previous expansion (6), $h = O(r^{-4})$, $h' = O(r^{-5})$, so we could do Taylor expansion to get the leading term, i.e.

$$g^{ij} = \delta_{ij} - h_{ij} - h'_{ij} + O(r^{-8}). \tag{62}$$

Thus

$$\Gamma^k_{kj} = \frac{1}{2} g^{kl} (g_{kl,j} + g_{jl,k} - g_{kj,l}) \tag{63}$$

$$= \frac{1}{2} (\delta_{kl} - h_{kl} - h'_{kl}) \left((h_{kl,j} + h'_{kl,j}) + (h_{jl,k} + h'_{jl,k}) - (h_{jk,l} + h'_{jk,l}) \right) + O(r^{-9}) \tag{64}$$

$$= \frac{1}{2} \delta_{kl} \left(h_{kl,j} + h_{jl,k} - h_{jk,l} \right) + O(r^{-6}) \tag{65}$$

$$= \frac{1}{2} h_{kk,j} + O(r^{-6}), \tag{66}$$

and

$$\partial_j (g^{ji}) = -h_{ji,j} + O(r^{-6}). \tag{67}$$

We get

$$\Delta_X((x^i)^2) = -\sum_{k=1}^{4}\left(\frac{1}{2}h_{kk,i} - h_{ki,k}\right)2x^i - 2(1-h_{ii}) + O(r^{-5}). \tag{68}$$

Note that $\sum_{k=1}^4 h_{kk,i} = \sum_{k=1}^4 h_{ki,k} = 0$. One can get this by direct calculation, or by noticing that h is trace-free and divergence-free.
Thus

$$\Delta_X((x^i)^2) = -2 + 2h_{ii} + O(r^{-5}). \tag{69}$$

By Proposition 2.1, for any $\nu \notin \mathbb{Z}$, the Laplacian

$$\Delta_X : W^{2,2}_{\nu}(X) \to L^2_{\nu-2}(X) \tag{70}$$

is a Fredholm operator. Let χ be a cutoff function on X such that

$$\chi = \begin{cases} 0, & \text{if } r < R, \\ 1, & \text{if } r \geq 2R. \end{cases} \tag{71}$$

We claim that there exists a function u_{ii} on X, such that $\chi(x^i)^2 - u_{ii} \in W^{2,2}_{\nu}(X)$ for ν to be determined, and

$$\Delta_X(\chi(x^i)^2 - u_{ii}) = 2 + \Delta_X(\chi(x^i)^2) = 2h_{ii} + O(r^{-5}). \tag{72}$$

That is, $2 + \Delta_X(\chi(x^i)^2) \in \mathrm{image}(\Delta_X) \subset L^2_{\nu-2}(X)$.
To see this, we first note that there must be

$$||2 + \Delta_X(\chi(x^i)^2)||_{L^2_{\nu-2}(X)} < \infty, \tag{73}$$

which holds if and only if $\nu < 0$.
By Proposition 2.1, it suffices to make $\mathcal{H}^p_{-2-\nu}(X) = 0$. By standard elliptic estimate, one gets

$$\mathcal{H}^p_{-2-\nu}(X) = \ker(\Delta_X : W^{k,2}_{-\nu-2}(X) \to W^{k-2,2}_{-\nu-4}(X)).\ \forall k \geq 0. \tag{74}$$

In this section, $p = 0$. By the maximum principle, in order to make $\ker(\Delta_X)$ trivial, it suffices to make the L^2-norm of ω decay, that is, $\nu > -2$. Thus, in the following, we will take $\nu = -2 + \epsilon$, where $0 < \epsilon \ll 1$. In the next section, $p = 1$, we take $\nu = -3 + \epsilon$. Then for $H^p_{-2-\nu} = H^p_{1-\epsilon}$, by Proposition 2.1, the leading order must be dx^i. As long as Γ is non-trivial, such a thing cannot be $\Gamma-$invariant. So the leading term vanishes, and it decays. Now we apply the Bochner formula

$$-\frac{1}{2}\Delta_X|\omega|^2 = -\langle \Delta\omega, \omega\rangle + |\nabla\omega|^2 + Ric(\sharp\omega, \sharp\omega) \tag{75}$$

$$= |\nabla\omega|^2 \geq 0, \tag{76}$$

where we used the fact that ω is harmonic and X is Ricci-flat.
With the expansion of the metric, one can easily replace Δ_X with $\Delta_{\mathbb{R}^4}$, that is,

$$\Delta_{\mathbb{R}^4}(\chi(x^i)^2 - u_{ii}) = 2h_{ii} + O(r^{-5}). \tag{77}$$

One can verify that

$$\tilde{\eta}_{ii} = h_{ii} \cdot \frac{r^2}{4} \tag{78}$$

satisfies $\Delta_{\mathbb{R}^4}\tilde{\eta}_{ii} = 2h_{ii}$. Therefore

$$\Delta_{\mathbb{R}^4}(\chi(x^i)^2 - u_{ii} - \tilde{\eta}_{ii}) = O(r^{-5}) \in W^{2,2}_{-5+\epsilon}, r \geq 2R. \tag{79}$$

By Proposition 2.1, one can find $\tilde{\tilde{\eta}}_{ii} \in W^{2,2}_{-3+\epsilon}$ such that $\Delta_{\mathbb{R}^4}(\tilde{\tilde{\eta}}_{ii}) = \Delta_{\mathbb{R}^4}(\chi(x^i)^2 - u_{ii} - \tilde{\eta}_{ii}),\ r \geq 2R$. That is,

$$\Delta_{\mathbb{R}^4}((x^i)^2 - u_{ii} - \tilde{\eta}_{ii} - \tilde{\tilde{\eta}}_{ii}) = 0,\ r \geq 2R. \tag{80}$$

By Proposition 2.1, one can get

$$(x^i)^2 - u_{ii} - \tilde{\eta}_{ii} - \tilde{\tilde{\eta}}_{ii} = (r^{-2} \text{ ordered homogeneous harmonic function}) + O(r^{-3}) \tag{81}$$

$$= -\frac{C_{ii}}{r^2} + O(r^{-3}),\ r \geq 2R. \tag{82}$$

Therefore one finally gets

$$u_{ii} = (x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2} + O(r^{-3+\epsilon}),\ r \geq 2R, \tag{83}$$

where we have omitted $\tilde{\tilde{\eta}}_{ii}$.

3.2. **Case 2.** As for the harmonic functions of type $x^i x^j$, the calculation is similar.

$$\Delta_X(\chi x^i x^j) = -\left(\Gamma^k_{kl} g^{li} + \partial_l(g^{li})\right) x^j - \left(\Gamma^k_{kl} g^{lj} + \partial_l(g^{lj})\right) x^i - 2g^{ij} + O(r^{-5}) \tag{84}$$

$$= \sum_{k=1}^{4}(\frac{1}{2}h_{kk,i} + h_{ki,k})x^j + \sum_{k=1}^{4}(\frac{1}{2}h_{kk,j} + h_{kj,k})x^i + 2h_{ij} + O(r^{-5}) \tag{85}$$

$$= 2h_{ij} + O(r^{-5}),\ r \geq 2R. \tag{86}$$

By the arguments above, one can find a function $v_{ij} \in W^{2,2}_{-2+\epsilon}(X)$, such that $\Delta_X(\chi x^i x^j) = \Delta_X(v_{ij})$. Moreover, one may get the harmonic function

$$u_{ij} = \chi x^i x^j - v_{ij} \tag{87}$$

$$= x^i x^j - \tilde{\eta}_{ij} + \frac{C_{ij}}{r^2} + O(r^{-3+\epsilon}),\ r \geq 2R, \tag{88}$$

where

$$\tilde{\eta}_{ij} = h_{ij} \cdot \frac{r^2}{4} \tag{89}$$

satisfies $\Delta_{\mathbb{R}^4}\tilde{\eta}_{ij} = 2h_{ij}$.

3.3. **Determine the constant.** Integrating $\Delta_X F$ on the region Ω_ρ such that $\Pi^*\Phi^*(\Omega_\rho - U) = [-\rho, \rho]^4 - B_R(0)$, one gets

$$\int_{\Omega_\rho} \Delta_X F dVol_X \tag{90}$$

$$= \int_{\Omega_\rho} *_X(\delta_X dF) \tag{91}$$

$$= \int_{\Omega_\rho} *_X(- *_X d *_X dF) \tag{92}$$

$$= -\int_{\Omega_\rho} d *_X dF \tag{93}$$

$$= -\int_{\partial\Omega_\rho} *_X dF. \tag{94}$$

With the convention that

$$\omega \wedge *_X \eta = \langle \omega, \eta \rangle dVol_X, \quad \langle dx^i, dx^j \rangle = g^{ij}, \tag{95}$$

we get

$$*_X dF = \sum_{j,k=1}^{4} \frac{\partial F}{\partial x^k} g^{kj} \sqrt{G} (-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^4 \tag{96}$$

where $G = \det(g_{ij}) = 1 + \sum_i h_{ii} + O(r^{-5}) = 1 + O(r^{-5})$ because h is trace-free. Combine this with $g^{ij} = \delta_{ij} - h_{ij} + O(r^{-5})$, one get

$$\int_{\partial\Omega_\rho} *_X dF = \sum_{j=1}^{4} \int_{\partial\Omega_\rho} \left(\frac{\partial F}{\partial x^j} - \sum_{k=1}^{4} h_{kj} \frac{\partial F}{\partial x^k} \right) (-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^4 \tag{97}$$

$$+ O(\rho^{-1+\epsilon}) \tag{98}$$

3.4. **Case 1.** We integrate on both sides of the equation

$$\Delta_X(u_{ii}) = \Delta_X(\chi(x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}) + O(r^{-5+\epsilon}). \tag{99}$$

On the LHS, we get

$$\int_{\Omega_\rho} \Delta_X(u_{ii}) dVol_X = -2Vol_X(\Omega_\rho). \tag{100}$$

Denote $\omega_j = (-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^4$. On the RHS, we get

$$\int_{\Omega_\rho} \left(\Delta_X(\chi(x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}) + O(r^{-5+\epsilon}) \right) dVol_X \tag{101}$$

$$= -\int_{\partial\Omega_\rho} *_X d((x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}) + O(\rho^{-1+\epsilon}) \tag{102}$$

$$= -\sum_{j=1}^{4} \int_{\partial\Omega_\rho} \left(\frac{\partial}{\partial x^j}((x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}) - \sum_{k=1}^{4} h_{kj} \frac{\partial}{\partial x^j}((x^i)^2) \right) \omega_j + O(\rho^{-1+\epsilon}) \tag{103}$$

$$= 4\pi^2 C_{ii} - 2Vol_{g_0}([-\rho, \rho]^4) + \sum_{j=1}^{4} \int_{\partial\Omega_\rho} \left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii}) + \sum_{k=1}^{4} h_{kj} \frac{\partial}{\partial x^j}((x^i)^2) \right) \omega_j \tag{104}$$

$$+ O(\rho^{-1+\epsilon}), \tag{105}$$

where we have used

$$\sum_{j=1}^{4} \int_{\partial\Omega_\rho} \frac{\partial}{\partial x^j}(\frac{1}{r^2}) \omega_j = -4\pi^2, \sum_{j=1}^{4} \int_{\partial\Omega_\rho} \frac{\partial}{\partial x^j}((x^i)^2) \omega_j = -2Vol_{g_0}([-\rho, \rho]^4). \tag{106}$$

Notice that the leading terms of $\frac{\partial}{\partial x^j}((x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}) - \sum_{k=1}^{4} h_{kj} \frac{\partial}{\partial x^j}((x^i)^2)$ are all -3-ordered, thus by letting $\rho \to \infty$, the errors vanish, and the integral left stays invariant, and we may

replace Ω_ρ with $\Omega = \Omega_1$.
Now we claim that

$$\sum_{j=1}^{4}\int_{\partial\Omega_\rho}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)\omega_j=0.$$

Indeed,

$$\sum_{j=1}^{4}\int_{\partial\Omega_\rho}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)\omega_j \tag{107}$$

$$=\sum_{j=1}^{4}\sum_{l=1}^{4}\int_{\Sigma_l^+ +\Sigma_l^-}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)\omega_j \tag{108}$$

$$=\sum_{j=1}^{4}\int_{\Sigma_j^+ +\Sigma_j^-}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)(-1)^{j-1}dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^4 \tag{109}$$

$$=\sum_{j=1}^{4}\int_{\Sigma_j^+}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)dx^1\cdots\widehat{dx^j}\cdots dx^4 \tag{110}$$

$$-\sum_{j=1}^{4}\int_{\Sigma_j^-}\left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2)\right)dx^1\cdots\widehat{dx^j}\cdots dx^4, \tag{111}$$

$$\tag{112}$$

where $\Sigma_l^\pm=\{x^l=\pm1\}\times[-1,1]^3$, e.g. $\Sigma_1^+=\{x^1=1\}\times[-1,1]^3$.
For convenience, we denote

$$Q_{i,j}:=\frac{\partial}{\partial x^j}(\tilde{\eta}_{ii})+\sum_{k=1}^{4}h_{kj}\frac{\partial}{\partial x^j}((x^i)^2). \tag{113}$$

Without loss of generality, we set $i=1$, and we have

$$r^6Q_{1,1}=\frac{2}{3}\zeta_{11}x^1\left(4x^1x^2-2x^1(x^3+x^4)-2(x^2)^2+6x^2x^3-6x^2x^4+(x^3)^2+(x^4)^2\right) \tag{114}$$

$$+\frac{2}{3}\zeta_{22}x^1\left(-2x^1x^2-2x^1(x^4-2x^3)+(x^2)^2-6x^2x^3-2(x^3)^2+6x^3x^4+(x^4)^2\right) \tag{115}$$

$$+\frac{2}{3}\zeta_{33}x^1\left(-2x^1x^2-2x^1(x^3-2x^4)+(x^2)^2+6x^2x^4+(x^3)^2-6x^3x^4-2(x^4)^2\right) \tag{116}$$

$$+\frac{2}{3}\xi_{11}x^1\left(4x^1x^2-2x^1(x^3+x^4)-2(x^2)^2-6x^2x^3+6x^2x^4+(x^3)^2+(x^4)^2\right) \tag{117}$$

$$+\frac{2}{3}\xi_{22}x^1\left(-2x^1x^2-2x^1(x^4-2x^3)+(x^2)^2+6x^2x^3-2(x^3)^2-6x^3x^4+(x^4)^2\right) \tag{118}$$

$$+\frac{2}{3}\xi_{33}x^1\left(-2x^1x^2-2x^1(x^3-2x^4)+(x^2)^2-6x^2x^4+(x^3)^2+6x^3x^4-2(x^4)^2\right) \tag{119}$$

$$+\frac{2}{3}\zeta_{12}x^1\left(6x^1x^2+6x^1x^3-6(x^2)^2-6x^2x^3+6x^2x^4+6(x^3)^2-6x^3x^4\right) \tag{120}$$

$$+\frac{2}{3}\xi_{12}x^1\left(6x^1x^2+6x^1x^3+6(x^2)^2-6x^2x^3-6x^2x^4-6(x^3)^2+6x^3x^4\right) \tag{121}$$

$$+\frac{2}{3}\zeta_{13}x^1\left(-6x^1x^2-6x^1x^4-6(x^2)^2+6x^2x^3+6x^2x^4-6x^3x^4+6(x^4)^2\right) \tag{122}$$
$$+\frac{2}{3}\xi_{13}x^1\left(-6x^1x^2-6x^1x^4+6(x^2)^2-6x^2x^3+6x^2x^4+6x^3x^4-6(x^4)^2\right) \tag{123}$$
$$+\frac{2}{3}\zeta_{23}x^1\left(-2x^1(3x^3+3x^4)-6x^2x^3+6x^2x^4+6(x^3)^2+6x^3x^4-6(x^4)^2\right) \tag{124}$$
$$+\frac{2}{3}\xi_{23}x^1\left(-2x^1(3x^3+3x^4)+6x^2x^3-6x^2x^4-6(x^3)^2+6x^3x^4+6(x^4)^2\right), \tag{125}$$
$$r^6Q_{1,2}=\frac{1}{3}\zeta_{11}\left(-2(x^1)^2x^2+2(x^2)^3-x^2\left(4(x^3)^2+4(x^4)^2\right)\right) \tag{126}$$
$$+\frac{1}{3}\zeta_{22}\left((x^1)^2x^2-(x^2)^3-x^2\left((x^4)^2-5(x^3)^2\right)\right) \tag{127}$$
$$+\frac{1}{3}\zeta_{33}\left((x^1)^2x^2-(x^2)^3-x^2\left((x^3)^2-5(x^4)^2\right)\right) \tag{128}$$
$$+\frac{1}{3}\xi_{11}\left(-2(x^1)^2x^2+2(x^2)^3-x^2\left(4(x^3)^2+4(x^4)^2\right)\right) \tag{129}$$
$$+\frac{1}{3}\xi_{22}\left((x^1)^2x^2-(x^2)^3-x^2\left((x^4)^2-5(x^3)^2\right)\right) \tag{130}$$
$$+\frac{1}{3}\xi_{33}\left((x^1)^2x^2-(x^2)^3-x^2\left((x^3)^2-5(x^4)^2\right)\right) \tag{131}$$
$$+\frac{1}{3}\zeta_{12}\left(-3(x^1)^2x^3+9(x^2)^2x^3-3x^3\left((x^3)^2+(x^4)^2\right)\right) \tag{132}$$
$$+\frac{1}{3}\zeta_{13}\left(3(x^1)^2x^4-9(x^2)^2x^4+3x^4\left((x^3)^2+(x^4)^2\right)\right) \tag{133}$$
$$+\frac{1}{3}\xi_{12}\left(-3(x^1)^2x^3+9(x^2)^2x^3-3x^3\left((x^3)^2+(x^4)^2\right)\right) \tag{134}$$
$$+\frac{1}{3}\xi_{13}\left(3(x^1)^2x^4-9(x^2)^2x^4+3x^4\left((x^3)^2+(x^4)^2\right)\right) \tag{135}$$
$$-4\zeta_{23}x^2x^3x^4 \tag{136}$$
$$-4\xi_{23}x^2x^3x^4. \tag{137}$$

$r^6Q_{1,3}$ and $r^6Q_{1,4}$ are similar to $r^6Q_{1,2}$, and thus omitted. Then one can fine the integrals equal zero by noticing their oddity.
Thus we finally get

$$4\pi^2C_{ii}=\lim_{\rho\to\infty}2(Vol_{g_0}([-\rho,\rho]^4)-Vol_X(\Omega_\rho))=:-2\mathcal{V}, \tag{138}$$

where $\mathcal{V}$ is the renormalized volume, which is finite according to Biquard and Hein's works [BH23].

3.5. **Case 2.** Integrate both sides of the equation

$$\Delta_X(\chi x^ix^j-v_{ij})=0. \tag{139}$$

That is, integrate both sides of the equation

$$\Delta_X(\chi x^ix^j)=\tilde{\eta}_{ij}-\frac{C_{ij}}{r^2}+O(r^{-3+\epsilon}). \tag{140}$$

Notice that the integral of the 1-order term $\sum_{j=1}^4 \frac{\partial}{\partial x^k}(x^i x^j)(-1)^{j-1} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^4$ is obviously zero, and all the other leading terms are of -3-order, thus by a similar procedure, we get

$$2\pi^2 C_{ij} = \sum_{j=1}^4 \int_{\Sigma_j^+} \left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ij}) + \sum_{k=1}^4 h_{kj} \frac{\partial}{\partial x^k}(x^i x^j) \right) dx^1 \cdots \widehat{dx^j} \cdots dx^4 \tag{141}$$

$$- \sum_{j=1}^4 \int_{\Sigma_j^-} \left(\frac{\partial}{\partial x^j}(\tilde{\eta}_{ij}) + \sum_{k=1}^4 h_{kj} \frac{\partial}{\partial x^k}(x^i x^j) \right) dx^1 \cdots \widehat{dx^j} \cdots dx^4. \tag{142}$$

Thus one can determine C_{ij} by calculating the integral on the RHS. By calculations similar to case 1, they are zero.

4. Harmonic 1-forms

In this part, we consider the (invariant) harmonic 1-forms on ALE manifolds. Namely, we only need to consider $x^{i_1} dx^{i_2}$. For simplicity, we still assume that $\Gamma = \{1\}$.

4.1. **Find the expansions.** The process is similar to that described in section 2.1, but it requires more calculation. Before proceeding, let's determine our convention for the Hodge stars, i.e.

$$*_X(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_{\substack{j_{k+1} < \cdots < j_n, \\ j_r \neq j_s \ for \ r \neq s}} \sqrt{G} g^{i_1 j_1} \cdots g^{i_k j_k} \varepsilon_{j_1 \cdots j_n} dx^{j_{k+1}} \wedge \cdots \wedge dx^{j_n}, \tag{143}$$

where $\varepsilon_{j_1 \cdots j_n}$ is the Levi-Civita symbol with $\varepsilon_{1 \cdots n} = 1$ and $\varepsilon_{j_1 \cdots j_n} \neq 0$ if and only if every j_r are different with each other, and we will omit this restriction when writing our followinng summation for simpliciy. We begin with Proposition 2.1 to get $\tilde{\omega}_{i_1 i_2} \in W^{2,2}_{-3+\epsilon}(X)$ that satisfies

$$\Delta_X \tilde{\omega}_{i_1 i_2} = \Delta_X(\chi x^{i_1} dx^{i_2}) = (d\delta_X + \delta_X d)(\chi x^{i_1} dx^{i_2}). \tag{144}$$

Firstly we notice that, for a 1-form $\tilde{\omega} = \tilde{\omega}_i dx^i$, $\delta_X \tilde{\omega} = -\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j}(\tilde{\omega}_i g^{ij} \sqrt{G})$. Thus

$$\delta_X(\chi x^{i_1} dx^{i_2}) = -\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j}(\chi x^{i_1} g^{i_2 j} \sqrt{G}) \tag{145}$$

$$= -\frac{\partial}{\partial x^j}(\chi x^{i_1} g^{i_2 j}) + O(r^{-5}) \tag{146}$$

$$= -\delta_{i_1 i_2} + h_{i_1 i_2} + \sum_{j=1}^4 x^{i_1} h_{i_2 j, j} + O(r^{-5}), \ r \geq 2R, \tag{147}$$

and therefore

$$d\delta_X(\chi x^{i_1} dx^{i_2}) = \sum_{k=1}^4 \frac{\partial}{\partial x^k}(h_{i_1 i_2} + \sum_{j=1}^4 x^{i_1} h_{i_2 j, j}) dx^k + O(r^{-6}) \ , r \geq 2R. \tag{148}$$

For the second term, we notice that for a 2-form $\tilde{\omega} = \sum_{i,j} \tilde{\omega}_{ij} dx^i \wedge dx^j$, we have $\delta_X \tilde{\omega} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left((\tilde{\omega}_{kl} - \tilde{\omega}_{lk}) g^{ik} g^{jl} \sqrt{G} \right) g_{im} dx^m$. Thus

$$\delta_X d(x^{i_1} dx^{i_2}) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left((\delta_{i_1 k} \delta_{i_2 l} - \delta_{i_1 l} \delta_{i_2 k}) g^{ik} g^{jl} \sqrt{G} \right) g_{im} dx^m \tag{149}$$

$$= - \sum_{i,j,k,l=1}^{4} (\delta_{i_1 k} \delta_{i_2 l} - \delta_{i_1 l} \delta_{i_2 k}) \frac{\partial}{\partial x^j} \left(\delta_{ik} h_{jl} + h_{ik} \delta_{jl} \right) dx^i + O(r^{-6}), \ r \geq 2R. \tag{150}$$

As a result, we get

$$\Delta_X (\chi x^{i_1} dx^{i_2}) = \sum_{k=1}^{4} \frac{\partial}{\partial x^k} (h_{i_1 i_2} + \sum_{j=1}^{4} x^{i_1} h_{i_2 j, j}) dx^k \tag{151}$$

$$+ \sum_{i,j,k,l=1}^{4} (\delta_{i_1 l} \delta_{i_2 k} - \delta_{i_1 k} \delta_{i_2 l}) \frac{\partial}{\partial x^j} \left(\delta_{ik} h_{jl} + h_{ik} \delta_{jl} \right) dx^i + O(r^{-6}) \tag{152}$$

$$:= L_{i_1 i_2} + O(r^{-6}), \ r \geq 2R. \tag{153}$$

One can verify that

$$\tilde{\mu}_{i_1 i_2} = L_{i_1 i_2} \cdot \frac{r^2}{12} \tag{154}$$

satisfies $\Delta_{\mathbb{R}^4} \tilde{\mu}_{i_1 i_2} = L_{i_1 i_2}$.
By Proposition 2.1, we get the expansion

$$\omega_{i_1 i_2} := \chi x^{i_1} dx^{i_2} - \tilde{\omega}_{i_1 i_2} = x^{i_1} dx^{i_2} - \tilde{\mu}_{i_1 i_2} \tag{155}$$

$$- \sum_{k,l=1}^{4} \frac{Con[i_1, i_2, l, k] x^k}{r^4} dx^l + O(r^{-4+\epsilon}), \ r \geq 2R. \tag{156}$$

4.2. **Determine the relation of the constants.** In this section, we will derive the restriction equations using divergence arguments and integration.
We first consider divergence arguments. For the harmonic 1-form $\omega_{i_1 i_2} = \chi x^{i_1} dx^{i_2} - \tilde{\omega}_{i_1 i_2}$, $\delta_X \omega_{i_1 i_2}$ is a decaying harmonic function plus $-\delta_{i_1 i_2}$. By applying the maximum principle, $\delta_X \omega_{i_1 i_2} = -\delta_{i_1 i_2}$. Note that because $\tilde{\omega}_{i_1 i_2}$ is of order -3, one can calculate the divergence of $\tilde{\omega}_{i_1 i_2}$ using the Euclidean divergence. That is,

$$\delta_X \omega_{i_1 i_2} + \delta_{i_1 i_2} = \delta_X (\chi x^{i_1} dx^{i_2} - \tilde{\omega}_{i_1 i_2}) + \delta_{i_1 i_2} \tag{157}$$

$$= \sum_{l=1}^{4} \frac{\partial}{\partial x^l} (\tilde{\mu}_{i_1 i_2; l} + \sum_{k=1}^{4} \frac{Con[i_1, i_2, l, k] x^k}{r^4}) + h_{i_1 i_2} \tag{158}$$

$$+ \sum_{j=1}^{4} x^{i_1} h_{i_2 j, j} + O(r^{-5+\epsilon}) \tag{159}$$

$$= 0. \tag{160}$$

Take ω_{12} for example, and one will get

$$0 = -3Con[1,2,1,1] + Con[1,2,2,2] + Con[1,2,3,3] + Con[1,2,4,4]; \tag{161}$$

$$0 = 4\zeta_{11} - 2(\zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 18Con[1,2,1,2] + 18Con[1,2,2,1]); \tag{162}$$

$$0 = \zeta_{12} + \xi_{12} - 6(Con[1,2,1,3] + Con[1,2,3,1]); \tag{163}$$
$$0 = \zeta_{13} + \xi_{13} + 6(Con[1,2,1,4] + Con[1,2,4,1]); \tag{164}$$
$$0 = -2\zeta_{11} + \zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 9Con[1,2,1,2] + 9Con[1,2,2,1]; \tag{165}$$
$$0 = Con[1,2,1,1] - 3Con[1,2,2,2] + Con[1,2,3,3] + Con[1,2,4,4]; \tag{166}$$
$$0 = \zeta_{13} - \xi_{13} - 6(Con[1,2,2,3] + Con[1,2,3,2]); \tag{167}$$
$$0 = \zeta_{12} - \xi_{12} - 6(Con[1,2,2,4] + Con[1,2,4,2]); \tag{168}$$
$$0 = \zeta_{12} + \xi_{12} - 6(Con[1,2,1,3] + Con[1,2,3,1]); \tag{169}$$
$$0 = \zeta_{13} - \xi_{13} - 6(Con[1,2,2,3] + Con[1,2,3,2]); \tag{170}$$
$$0 = 2\zeta_{23} - 2\xi_{23} + 3(Con[1,2,1,1] + Con[1,2,2,2] - 3Con[1,2,3,3] + Con[1,2,4,4]); \tag{171}$$
$$0 = \zeta_{22} - \zeta_{33} - \xi_{22} + \xi_{33} - 6Con[1,2,3,4] - 6Con[1,2,4,3]; \tag{172}$$
$$0 = \zeta_{13} + \xi_{13} + 6(Con[1,2,1,4] + Con[1,2,4,1]); \tag{173}$$
$$0 = \zeta_{12} - \xi_{12} - 6(Con[1,2,2,4] + Con[1,2,4,2]); \tag{174}$$
$$0 = \zeta_{22} - \zeta_{33} - \xi_{22} + \xi_{33} - 6Con[1,2,3,4] - 6Con[1,2,4,3]; \tag{175}$$
$$0 = -2\zeta_{23} + 2\xi_{23} + 3(Con[1,2,1,1] + Con[1,2,2,2] + Con[1,2,3,3] \tag{176}$$
$$- 3Con[1,2,4,4]). \tag{177}$$

Next , we consider the integral. For two harmonic 1-forms $\omega_{i_1i_2}$ and $\omega_{i_3i_4}$, combined with the divergence arguments, we have

$$0 = \int_\Omega \langle \Delta_X \omega_{i_1i_2}, \omega_{i_3i_4}\rangle dVol_X - \int_\Omega \langle \Delta_X \omega_{i_3i_4}, \omega_{i_1i_2}\rangle dVol_X \tag{178}$$
$$= (\int_{\partial\Omega} *_X d\omega_{i_1i_2} \wedge \omega_{i_3i_4} + \int_{\partial\Omega} *_X d *_X \omega_{i_1i_2} \wedge *_X \omega_{i_3i_4}) \tag{179}$$
$$- (\int_{\partial\Omega} *_X d\omega_{i_3i_4} \wedge \omega_{i_1i_2} + \int_{\partial\Omega} *_X d *_X \omega_{i_3i_4} \wedge *_X \omega_{i_1i_2}) \tag{180}$$
$$= \int_{\partial\Omega} *_X d\omega_{i_1i_2} \wedge \omega_{i_3i_4} - \int_{\partial\Omega} *_X d\omega_{i_3i_4} \wedge \omega_{i_1i_2} \tag{181}$$
$$= \int_{\partial\Omega} \alpha_{i_1i_2i_3i_4} - \int_{\partial\Omega} \alpha_{i_3i_4i_1i_2}. \tag{182}$$

Where we have used the leading term arguments, and the $\alpha_{i_1i_2i_3i_4}$ denotes the leading term of $*_X d\omega_{i_1i_2} \wedge \omega_{i_3i_4}$. Besides, one can calculate $\alpha_{i_1i_2i_3i_4}$ explicitly, say,

$$\alpha_{i_1i_2i_3i_4} = - \sum_{\substack{j_3<j_4,\\ j_r\neq j_s \ for \ r\neq s}} \sum_{r,s,t=1}^{4} \Bigg(\frac{\partial}{\partial x^s}(\tilde{\mu}_{i_1i_2;r} + \sum_{k=1}^{4} \frac{Con[i_1,i_2,r,k]x^k}{r^4}) x^{i_3}\delta_{i_4t}\delta_{sj_1}\delta_{rj_2} \tag{183}$$
$$+ \delta_{i_1s}\delta_{i_2r}(\tilde{\mu}_{i_3i_4;t} + \sum_{k=1}^{4} \frac{Con[i_3,i_4,t,k]x^k}{r^4})\delta_{sj_1}\delta_{rj_2} \tag{184}$$
$$+ \delta_{i_1s}\delta_{i_2r}x^{i_3}\delta_{i_4t}h_{sj_1}\delta_{rj_2} + \delta_{i_1s}\delta_{i_2r}x^{i_3}\delta_{i_4t}\delta_{sj_1}h_{rj_2}\Big) \varepsilon_{j_1j_2j_3j_4} dx^{j_3} \wedge dx^{j_4} \wedge dx^t. \tag{185}$$

For example, take $(i_1, i_2, i_3, i_4) = (1, 2, 3, 4)$, then we will get the expression of α_{1234}, which is a 6-page-long equation.

In this enormous equation, the terms with nonzero integrals are similar to the following types:

$$\frac{x^2(x^1)^4}{r^8}dx^1\wedge dx^3\wedge dx^4,\quad \frac{x^2(x^1)^2(x^3)^2}{r^8}dx^1\wedge dx^3\wedge dx^4, \tag{186}$$

$$\frac{(x^2)^3(x^1)^2}{r^8}dx^1\wedge dx^3\wedge dx^4,\quad \frac{(x^2)^5}{r^8}dx^1\wedge dx^3\wedge dx^4, \tag{187}$$

$$\text{integrated on } \Sigma_2^{\pm}. \tag{188}$$

In addition, the exterior normal vector of Σ_2^+ and Σ_2^- has opposite orientations. Thus, after letting $x^2=\pm 1$ and integrating respectively, one will get the same result. For example,

$$\int_{\Sigma_2^+}\frac{x^2(x^1)^4}{r^8}dx^1\wedge dx^3\wedge dx^4=\int_{\Sigma_2^-}\frac{x^2(x^1)^4}{r^8}dx^1\wedge dx^3\wedge dx^4=\int_{[-1,1]^3}\frac{(x^1)^4}{r^8}dx^1dx^3dx^4. \tag{189}$$

Therefore, let's denote our integrals as (here $r^2=1+(x^1)^2+(x^3)^2+(x^4)^2$)

$$\sigma_1=\int_{[-1,1]^3}\frac{(x^1)^4}{r^8}dx^1dx^3dx^4, \tag{190}$$

$$\sigma_2=\int_{[-1,1]^3}\frac{(x^1)^2(x^3)^2}{r^8}dx^1dx^3dx^4, \tag{191}$$

$$\sigma_3=\int_{[-1,1]^3}\frac{(x^1)^2}{r^8}dx^1dx^3dx^4, \tag{192}$$

$$\sigma_4=\int_{[-1,1]^3}\frac{1}{r^8}dx^1dx^3dx^4, \tag{193}$$

and we will get the following equation:

$$0=8(\sigma_1+2\sigma_2+\sigma_3)Con[1,2,3,4]-8(2\sigma_1+4\sigma_2+5\sigma_3+\sigma_4)Con[1,2,4,3] \tag{194}$$

$$-8(\sigma_1+2\sigma_2+\sigma_3)Con[3,4,1,2]+8(2\sigma_1+4\sigma_2+5\sigma_3+\sigma_4)Con[3,4,2,1]. \tag{195}$$

We point out that

$$8(3\sigma_1+6\sigma_2+6\sigma_3+\sigma_4)=2\pi^2. \tag{196}$$

To see this,

$$(3\sigma_1+6\sigma_2+6\sigma_3+\sigma_4) \tag{197}$$

$$=\int_{[-1,1]^3}\left.\frac{3(x^1)^4x^2+6(x^1)^2(x^3)^2x^2+6(x^1)^2(x^2)^3+(x^2)^5}{r^8}\right|_{x^2=1}dx^1dx^3dx^4 \tag{198}$$

$$=\int_{[-1,1]^3}\frac{1}{r^8}\Bigg(x^2(\sum_{i\neq 2}(x^i)^4+\sum_{i<j,i\neq 2,j\neq 2}2(x^i)^2(x^j)^2 \tag{199}$$

$$+\sum_{i\neq 2}2(x^i)^2(x^2)^4+(x^2)^4)\Bigg)\Bigg|_{x^2=1}dx^1dx^3dx^4 \tag{200}$$

$$=\int_{[-1,1]^3}\left.\frac{x^2}{r^4}\right|_{x^2=1}dx^1dx^3dx^4. \tag{201}$$

Thus

$$8(3\sigma_1 + 6\sigma_2 + 6\sigma_3 + \sigma_4) \tag{202}$$
$$= 8\int_{[-1,1]^3} \frac{x^2}{r^4}\Big|_{x^2=1} dx^1dx^3dx^4 \tag{203}$$
$$= \sum_{j=1}^{4}\int_{\Sigma_j^+} \frac{x^j}{r^4}dx^1\cdots\widehat{dx^j}\cdots dx^4 - \sum_{j=1}^{4}\int_{\Sigma_j^-} \frac{x^j}{r^4}dx^1\cdots\widehat{dx^j}\cdots dx^4 \tag{204}$$
$$= -\frac{1}{2}\int_{\partial[-1,1]^4} *_{\mathbb{R}^4} d(\frac{1}{r^2}) \tag{205}$$
$$= \frac{1}{2}\int_{[-1,1]^4} \Delta_{\mathbb{R}^4}(\frac{1}{r^2}) \tag{206}$$
$$= \frac{1}{2}\int_{B_1(0)} \Delta_{\mathbb{R}^4}(\frac{1}{r^2}) \tag{207}$$
$$= \frac{1}{2}\int_{\mathbb{S}^3} -\frac{\partial}{\partial r}(\frac{1}{r^2})dS \tag{208}$$
$$= \int_{\mathbb{S}^3} dS = |\mathbb{S}^3| = 2\pi^2. \tag{209}$$

Applying the previous fact that

$$Con[1,2,3,4] + Con[1,2,4,3] = Con[3,4,1,2] + Con[3,4,2,1], \tag{210}$$

one gets

$$0 = 2\pi^2(Con[1,2,3,4] - Con[3,4,1,2]). \tag{211}$$

Gathering the two parts, we will get a massive system of equations about 256 variables $Con[i,j,k,l]$.

5. The massive system of equations about 256 variables

The variables can be categorized into 5 types by symmetry:

$$\text{I.}Con[i,i,i,i]; \tag{212}$$
$$\text{II.}Con[i,i,i,j], Con[i,i,j,i], Con[i,j,i,i], Con[j,i,i,i]; \tag{213}$$
$$\text{III.}Con[i,i,j,j], Con[i,j,i,j], Con[i,j,j,i]; \tag{214}$$
$$\text{IV.}Con[i,i,j,k], Con[i,j,i,k], Con[i,j,k,i], Con[j,k,i,i], Con[j,i,k,i], Con[j,i,i,k]; \tag{215}$$
$$\text{V.}Con[i,j,k,l]. \tag{216}$$

The system consists of three parts: divergence arguments, integrals, and function differentials. We will make full use of them to simplify our results.

5.1. **Type I.** Recall our expansion:

$$u_{ii} = (x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2} + O(r^{-3+\epsilon}), \tag{217}$$
$$C_{ii} = C = \frac{\mathcal{V}}{2\pi^2}, \quad (i = 1,2,3,4) \tag{218}$$

$$\omega_{i_1 i_2} = \chi x^{i_1} dx^{i_2} - \tilde{\omega}_{i_1 i_2} = x^{i_1} dx^{i_2} - \tilde{\mu}_{i_1 i_2} - \sum_{k,l=1}^{4} \frac{Con[i_1, i_2, l, k] x^k}{r^4} dx^l + O(r^{-4+\epsilon}). \tag{219}$$

We differentiate the first equation and get

$$du_{ii} = d\left((x^i)^2 - \tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}\right) + O(r^{-4+\epsilon}),\ r \geq 2R. \tag{220}$$

On the other hand, du_{ii} is a harmonic 1-form asymptotic to $2x^i dx^i$. Thus we get

$$du_{ii} = 2\omega_{ii}, \tag{221}$$

that is,

$$2\left(\tilde{\mu}_{ii} + \sum_{k,l=1}^{4} \frac{Con[i,i,l,k] x^k}{r^4} dx^l\right) + d\left(-\tilde{\eta}_{ii} + \frac{C_{ii}}{r^2}\right) = 0. \tag{222}$$

Routine calculations give

$$Con[i,i,i,i] = -C = -\frac{\mathcal{V}}{2\pi^2}. \tag{223}$$

5.2. **Type II.** Not only do we get $Con[i,i,i,i]$, but we also get

$$Con[i,i,i,j] = Con[i,i,j,i] = 0,\ Con[i,j,i,i] + Con[j,i,i,i] = 0. \tag{224}$$

5.3. **Type III.** Without loss of generality, we set $i = 1, j = 2$. We immediately obtain

$$18C + 2\zeta_{11} - \zeta_{22} - \zeta_{33} + 2\xi_{11} - \xi_{22} - \xi_{33} + 18Con[1,1,2,2] = 0. \tag{225}$$

However, $Con[1,2,1,2], Con[1,2,2,1], Con[2,1,2,1]$ and $Con[2,1,1,2]$ are interconnected. In fact, they satisfy the following equations:

$$-2\zeta_{11} + \zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 18Con[1,2,2,1] + 18Con[2,1,2,1] = 0; \tag{226}$$

$$-2\zeta_{11} + \zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 18Con[1,2,1,2] + 18Con[2,1,1,2] = 0; \tag{227}$$

$$-2\zeta_{11} + \zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 18Con[1,2,2,1] + 18Con[1,2,1,2] = 0; \tag{228}$$

$$-2\zeta_{11} + \zeta_{22} + \zeta_{33} - 2\xi_{11} + \xi_{22} + \xi_{33} + 18Con[2,1,2,1] + 18Con[2,1,1,2] = 0, \tag{229}$$

where the third and fourth equation come from divergence arguments. The rank of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{230}$$

is 3, therefore one can use $Con[1,2,1,2]$, ζ_{ii}, and ξ_{ii} to express the other 3 variables. Indeed,

$$Con[1,2,1,2] = Con[2,1,2,1]; \tag{231}$$

$$Con[1,2,2,1] = Con[2,1,1,2] \tag{232}$$

$$= -Con[1,2,1,2] + \frac{2\zeta_{11} - \zeta_{22} - \zeta_{33} + 2\xi_{11} - \xi_{22} - \xi_{33}}{18}, \tag{233}$$

and one can't get any more information from inner product arguments.

5.4. **Type IV-i.** We point out that

$$Con[i,j,i,k], Con[i,j,k,i], Con[j,i,k,i] and Con[j,i,i,k] \tag{234}$$

are interconnected.
Without loss of generality, let $i=1, j=2, k=3$. Consider the 8 variables, say,

$$Con[1,2,1,3], Con[2,1,1,3], Con[1,3,1,2], Con[3,1,1,2], \tag{235}$$
$$Con[1,2,3,1], Con[2,1,3,1], Con[1,3,2,1], Con[3,1,2,1]. \tag{236}$$

They satisfy

$$\zeta_{12}+\xi_{12}-6(Con[1,2,1,3]+Con[2,1,1,3])=0; \tag{237}$$
$$\zeta_{12}+\xi_{12}-6(Con[1,3,1,2]+Con[3,1,1,2])=0; \tag{238}$$
$$\zeta_{12}+\xi_{12}-6(Con[1,2,3,1]+Con[2,1,3,1])=0; \tag{239}$$
$$\zeta_{12}+\xi_{12}-6(Con[1,3,2,1]+Con[3,1,2,1])=0; \tag{240}$$
$$\zeta_{12}+\xi_{12}-6(Con[1,2,1,3]+Con[1,2,3,1])=0; \tag{241}$$
$$\zeta_{12}+\xi_{12}-6(Con[1,3,1,2]+Con[1,3,2,1])=0; \tag{242}$$
$$\zeta_{12}+\xi_{12}-6(Con[2,1,1,3]+Con[2,1,3,1])=0; \tag{243}$$
$$\zeta_{12}+\xi_{12}-6(Con[3,1,1,2]+Con[3,1,2,1])=0. \tag{244}$$

The rank of the matrix

$$\begin{pmatrix} 1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&1&0\\ 1&0&1&0&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&1&0 \end{pmatrix} \tag{245}$$

is 6. We solve the linear equations, and get

$$Con[2,1,3,1]=Con[1,2,1,3]; \tag{246}$$
$$Con[3,1,2,1]=Con[1,3,1,2]; \tag{247}$$
$$Con[1,2,3,1]=Con[2,1,1,3]=\frac{\zeta_{12}+\xi_{12}}{6}-Con[1,3,1,2]; \tag{248}$$
$$Con[1,3,2,1]=Con[3,1,1,2]=\frac{\zeta_{12}+\xi_{12}}{6}-Con[1,2,1,3]. \tag{249}$$

Thus, only $Con[1,2,1,3]$ and $Con[1,3,1,2]$ remain, and it's time to make use of the equations from the inner product:

$$0=8(\sigma_1+2\sigma_2+\sigma_3)Con[1,2,1,3]-8(2\sigma_1+4\sigma_2+5\sigma_3+\sigma_4)Con[1,2,3,1] \tag{250}$$
$$-8(\sigma_1+2\sigma_2+\sigma_3)Con[1,3,1,2]+8(2\sigma_1+4\sigma_2+5\sigma_3+\sigma_4)Con[1,3,2,1], \tag{251}$$

which implies $Con[1,2,1,3]=Con[1,3,1,2]$.

5.5. **Type IV-ii.** Similarly, $Con[i,i,j,k]$ and $Con[j,k,i,i]$ are interconnected. Without loss of generality, let $i=1, j=2, k=3$. Consider the 16 variables, say,

$$Con[1,1,2,3], Con[1,1,3,2], Con[2,3,1,1], Con[3,2,1,1], \tag{252}$$
$$Con[2,2,2,3], Con[2,2,3,2], Con[2,3,2,2], Con[3,2,2,2], \tag{253}$$
$$Con[3,3,2,3], Con[3,3,3,2], Con[2,3,3,3], Con[3,2,3,3], \tag{254}$$
$$Con[4,4,2,3], Con[4,4,3,2], Con[2,3,4,4], Con[3,2,4,4]. \tag{255}$$

We already known that

$$Con[2,2,2,3] = Con[2,2,3,2] = Con[3,3,2,3] = Con[3,3,3,2] = 0. \tag{256}$$

Besides, we have

$$\zeta_{12} + \xi_{12} + 6Con[1,1,2,3] = 0; \tag{257}$$
$$\zeta_{12} + \xi_{12} + 6Con[1,1,3,2] = 0; \tag{258}$$
$$\zeta_{12} + \xi_{12} - 6Con[4,4,2,3] = 0; \tag{259}$$
$$\zeta_{12} + \xi_{12} - 6Con[4,4,3,2] = 0. \tag{260}$$

That is,

$$Con[1,1,2,3] = Con[1,1,3,2] = -\frac{\zeta_{12} + \xi_{12}}{6}; \tag{261}$$
$$Con[4,4,2,3] = Con[4,4,3,2] = \frac{\zeta_{12} + \xi_{12}}{6}. \tag{262}$$

Now we deal with $Con[2,3,i,i]$ and $Con[3,2,i,i]$ (i=1,2,3,4).
We have the following equations:

$$0 = \zeta_{12} + \xi_{12} + 3(Con[2,3,1,1] + Con[3,2,1,1]); \tag{263}$$
$$0 = Con[2,3,2,2] + Con[3,2,2,2]; \tag{264}$$
$$0 = Con[2,3,3,3] + Con[3,2,3,3]; \tag{265}$$
$$0 = \zeta_{12} + \xi_{12} - 3(Con[2,3,4,4] + Con[3,2,4,4]); \tag{266}$$
$$0 = -2\zeta_{12} - 2\xi_{12} + 3(-3Con[2,3,1,1] + Con[2,3,2,2] + Con[2,3,3,3] \tag{267}$$
$$+ Con[2,3,4,4]); \tag{268}$$
$$0 = Con[2,3,1,1] - 3Con[2,3,2,2] + Con[2,3,3,3] + Con[2,3,4,4]; \tag{269}$$
$$0 = Con[2,3,1,1] + Con[2,3,2,2] - 3Con[2,3,3,3] + Con[2,3,4,4]; \tag{270}$$
$$0 = 2\zeta_{12} + 2\xi_{12} + 3(Con[2,3,1,1] + Con[2,3,2,2] + Con[2,3,3,3] - 3Con[2,3,4,4]); \tag{271}$$
$$0 = -2\zeta_{12} - 2\xi_{12} + 3(-3Con[3,2,1,1] + Con[3,2,2,2] + Con[3,2,3,3] \tag{272}$$
$$+ Con[3,2,4,4]); \tag{273}$$
$$0 = Con[3,2,1,1] - 3Con[3,2,2,2] + Con[3,2,3,3] + Con[3,2,4,4]; \tag{274}$$
$$0 = Con[3,2,1,1] + Con[3,2,2,2] - 3Con[3,2,3,3] + Con[3,2,4,4]; \tag{275}$$
$$0 = 2\zeta_{12} + 2\xi_{12} + 3(Con[3,2,1,1] + Con[3,2,2,2] + Con[3,2,3,3] - 3Con[3,2,4,4]). \tag{276}$$

The rank of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -3 & 0 \\ 0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -3 \end{pmatrix} \tag{277}$$

is 7. Thus we may get

$$Con[2,3,1,1] = -Con[3,2,4,4]; \tag{278}$$

$$Con[2,3,4,4] = -Con[3,2,1,1] = \frac{\zeta_{12}+\xi_{12}}{3} + Con[2,3,1,1]; \tag{279}$$

$$Con[2,3,2,2] = Con[2,3,3,3] = -Con[3,2,2,2] = -Con[3,2,3,3] \tag{280}$$

$$= \frac{\zeta_{12}+\xi_{12}}{6} + Con[2,3,1,1], \tag{281}$$

and one can't get any more information from inner product arguments.

5.6. **Type V.** Without loss of generality, we consider the 8 variables, say,

$$Con[1,2,3,4], Con[1,2,4,3], Con[2,1,3,4], Con[2,1,4,3], \tag{282}$$

$$Con[3,4,1,2], Con[4,3,1,2], Con[3,4,2,1], Con[4,3,2,1]. \tag{283}$$

They satisfy

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[1,2,3,4]+6Con[2,1,3,4] = 0; \tag{284}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[1,2,4,3]+6Con[2,1,4,3] = 0; \tag{285}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[3,4,1,2]+6Con[4,3,1,2] = 0; \tag{286}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[3,4,2,1]+6Con[4,3,2,1] = 0; \tag{287}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[1,2,3,4]+6Con[1,2,4,3] = 0; \tag{288}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[2,1,3,4]+6Con[2,1,4,3] = 0; \tag{289}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[3,4,1,2]+6Con[3,4,2,1] = 0; \tag{290}$$

$$-\zeta_{22}+\zeta_{33}+\xi_{22}-\xi_{33}+6Con[4,3,1,2]+6Con[4,3,2,1] = 0. \tag{291}$$

Similarly, we solve the equations, and get

$$Con[2,1,4,3] = Con[1,2,3,4]; \tag{292}$$

$$Con[4,3,2,1] = Con[3,4,1,2]; \tag{293}$$

$$Con[1,2,4,3] = Con[2,1,3,4] = \frac{\zeta_{22}-\zeta_{33}-\xi_{22}+\xi_{33}}{6} - Con[1,2,3,4]; \tag{294}$$

$$Con[3,4,2,1] = Con[4,3,1,2] = \frac{\zeta_{22} - \zeta_{33} - \xi_{22} + \xi_{33}}{6} - Con[3,4,1,2]. \tag{295}$$

By inner product arguments, we get

$$0 = 8(\sigma_1 + 2\sigma_2 + \sigma_3)Con[1,2,3,4] - 8(2\sigma_1 + 4\sigma_2 + 5\sigma_3 + \sigma_4)Con[1,2,4,3] \tag{296}$$

$$- 8(\sigma_1 + 2\sigma_2 + \sigma_3)Con[3,4,1,2] + 8(2\sigma_1 + 4\sigma_2 + 5\sigma_3 + \sigma_4)Con[3,4,2,1], \tag{297}$$

which implies $Con[1,2,3,4] = Con[3,4,1,2]$.

Appendix A. Lists of constants

By symmetry, we can compute all the constraints on $Con[i,j,k,l]$.

A.1. **The full list of constants of type III.**

$$Con[1,2,1,2] = Con[2,1,2,1]; \tag{298}$$

$$Con[1,2,2,1] = Con[2,1,1,2] \tag{299}$$

$$= -Con[1,2,1,2] + \frac{2\zeta_{11} - \zeta_{22} - \zeta_{33} + 2\xi_{11} - \xi_{22} - \xi_{33}}{18}, \tag{300}$$

$$Con[1,3,1,3] = Con[3,1,3,1]; \tag{301}$$

$$Con[1,3,3,1] = Con[3,1,1,3] \tag{302}$$

$$= -Con[1,3,1,3] + \frac{2\zeta_{22} - \zeta_{11} - \zeta_{33} + 2\xi_{22} - \xi_{11} - \xi_{33}}{18}, \tag{303}$$

$$Con[1,4,1,4] = Con[4,1,4,1]; \tag{304}$$

$$Con[1,4,4,1] = Con[4,1,1,4] \tag{305}$$

$$= -Con[1,4,1,4] + \frac{2\zeta_{33} - \zeta_{11} - \zeta_{22} + 2\xi_{33} - \xi_{11} - \xi_{22}}{18}, \tag{306}$$

$$Con[2,3,2,3] = Con[3,2,3,2]; \tag{307}$$

$$Con[2,3,3,2] = Con[3,2,2,3] \tag{308}$$

$$= -Con[2,3,2,3] + \frac{2\zeta_{33} - \zeta_{11} - \zeta_{22} + 2\xi_{33} - \xi_{11} - \xi_{22}}{18}, \tag{309}$$

$$Con[2,4,2,4] = Con[4,2,4,2]; \tag{310}$$

$$Con[2,4,4,2] = Con[4,2,2,4] \tag{311}$$

$$= -Con[2,4,2,4] + \frac{2\zeta_{22} - \zeta_{11} - \zeta_{33} + 2\xi_{22} - \xi_{11} - \xi_{33}}{18}, \tag{312}$$

$$Con[3,4,3,4] = Con[4,3,4,3]; \tag{313}$$

$$Con[3,4,4,3] = Con[4,3,3,4] \tag{314}$$

$$= -Con[3,4,3,4] + \frac{2\zeta_{11} - \zeta_{22} - \zeta_{33} + 2\xi_{11} - \xi_{22} - \xi_{33}}{18}. \tag{315}$$

A.2. **The full list of constants of type IV-i.**

$$Con[2,1,3,1] = Con[1,2,1,3] = Con[3,1,2,1] = Con[1,3,1,2]; \tag{316}$$

$$Con[1,2,3,1] = Con[2,1,1,3] = Con[1,3,2,1] = Con[3,1,1,2] \tag{317}$$

$$= \frac{\zeta_{12} + \xi_{12}}{6} - Con[1,2,1,3]. \tag{318}$$

$$Con[2,1,4,1] = Con[1,2,1,4] = Con[4,1,2,1] = Con[1,4,1,2]; \tag{319}$$
$$Con[1,2,4,1] = Con[2,1,1,4] = Con[1,4,2,1] = Con[4,1,1,2] \tag{320}$$
$$= \frac{\zeta_{13} + \xi_{13}}{6} - Con[1,2,1,4]. \tag{321}$$
$$Con[3,1,4,1] = Con[1,3,1,4] = Con[4,1,3,1] = Con[1,4,1,3]; \tag{322}$$
$$Con[1,3,4,1] = Con[3,1,1,4] = Con[1,4,3,1] = Con[4,1,1,3] \tag{323}$$
$$= \frac{\zeta_{23} + \xi_{23}}{6} - Con[1,3,1,4]. \tag{324}$$
$$Con[3,2,1,2] = Con[2,3,2,1] = Con[1,2,3,2] = Con[2,1,2,3]; \tag{325}$$
$$Con[2,3,1,2] = Con[3,2,2,1] = Con[2,1,3,2] = Con[1,2,2,3] \tag{326}$$
$$= \frac{\zeta_{13} - \xi_{13}}{6} - Con[2,3,2,1]. \tag{327}$$
$$Con[4,2,1,2] = Con[2,4,2,1] = Con[1,2,4,2] = Con[2,1,2,4]; \tag{328}$$
$$Con[2,4,1,2] = Con[4,2,2,1] = Con[2,1,4,2] = Con[1,2,2,4] \tag{329}$$
$$= \frac{\zeta_{12} - \xi_{12}}{6} - Con[2,4,2,1]. \tag{330}$$
$$Con[4,2,3,2] = Con[2,4,2,3] = Con[3,2,4,2] = Con[2,3,2,4]; \tag{331}$$
$$Con[2,4,3,2] = Con[4,2,2,3] = Con[2,3,4,2] = Con[3,2,2,4] \tag{332}$$
$$= \frac{\zeta_{23} + \xi_{23}}{6} - Con[2,4,2,3]. \tag{333}$$
$$Con[1,3,2,3] = Con[3,1,3,2] = Con[2,3,1,3] = Con[3,2,3,1]; \tag{334}$$
$$Con[3,1,2,3] = Con[1,3,3,2] = Con[3,2,1,3] = Con[2,3,3,1] \tag{335}$$
$$= \frac{-\zeta_{23} + \xi_{23}}{6} - Con[3,1,3,2]. \tag{336}$$
$$Con[1,3,4,3] = Con[3,1,3,4] = Con[4,3,1,3] = Con[3,4,3,1]; \tag{337}$$
$$Con[3,1,4,3] = Con[1,3,3,4] = Con[3,4,1,3] = Con[4,3,3,1] \tag{338}$$
$$= \frac{-\zeta_{12} + \xi_{12}}{6} - Con[3,1,3,4]. \tag{339}$$
$$Con[2,3,4,3] = Con[3,2,3,4] = Con[4,3,2,3] = Con[3,4,3,2]; \tag{340}$$
$$Con[3,2,4,3] = Con[2,3,3,4] = Con[3,4,2,3] = Con[4,3,3,2] \tag{341}$$
$$= \frac{\zeta_{13} + \xi_{13}}{6} - Con[3,2,3,4]. \tag{342}$$
$$Con[1,4,2,4] = Con[4,1,4,2] = Con[2,4,1,4] = Con[4,2,4,1]; \tag{343}$$
$$Con[4,1,2,4] = Con[1,4,4,2] = Con[4,2,1,4] = Con[2,4,4,1] \tag{344}$$
$$= \frac{\zeta_{23} - \xi_{23}}{6} - Con[4,1,4,2]. \tag{345}$$
$$Con[1,4,3,4] = Con[4,1,4,3] = Con[3,4,1,4] = Con[4,3,4,1]; \tag{346}$$
$$Con[4,1,3,4] = Con[1,4,4,3] = Con[4,3,1,4] = Con[3,4,4,1] \tag{347}$$
$$= \frac{-\zeta_{13} + \xi_{13}}{6} - Con[4,1,4,3]. \tag{348}$$

$$Con[2,4,3,4] = Con[4,2,4,3] = Con[3,4,2,4] = Con[4,3,4,2]; \tag{349}$$

$$Con[4,2,3,4] = Con[2,4,4,3] = Con[4,3,2,4] = Con[3,4,4,2] \tag{350}$$

$$= \frac{\zeta_{12} + \xi_{12}}{6} - Con[4,2,4,3]. \tag{351}$$

A.3. **The full list of constants of type IV-ii.**

$$Con[1,2,3,3] = -Con[2,1,4,4]; \tag{352}$$

$$Con[1,2,4,4] = -Con[2,1,3,3] = \frac{-\zeta_{23} + \xi_{23}}{3} + Con[1,2,3,3]; \tag{353}$$

$$Con[1,2,2,2] = Con[1,2,1,1] = -Con[2,1,2,2] = -Con[2,1,1,1] \tag{354}$$

$$= \frac{-\zeta_{23} + \xi_{23}}{6} + Con[1,2,3,3]. \tag{355}$$

$$Con[1,3,2,2] = -Con[3,1,4,4]; \tag{356}$$

$$Con[1,3,4,4] = -Con[3,1,2,2] = \frac{\zeta_{13} - \xi_{13}}{3} + Con[1,3,2,2]; \tag{357}$$

$$Con[1,3,1,1] = Con[1,3,3,3] = -Con[3,1,1,1] = -Con[3,1,3,3] \tag{358}$$

$$= \frac{\zeta_{13} - \xi_{13}}{6} + Con[1,3,2,2]. \tag{359}$$

$$Con[1,4,2,2] = -Con[4,1,3,3]; \tag{360}$$

$$Con[1,4,3,3] = -Con[4,1,2,2] = \frac{\zeta_{12} - \xi_{12}}{3} + Con[1,4,2,2]; \tag{361}$$

$$Con[1,4,1,1] = Con[1,4,4,4] = -Con[4,1,1,1] = -Con[4,1,4,4] \tag{362}$$

$$= \frac{\zeta_{12} - \xi_{12}}{6} + Con[1,4,2,2]. \tag{363}$$

$$Con[2,3,1,1] = -Con[3,2,4,4]; \tag{364}$$

$$Con[2,3,4,4] = -Con[3,2,1,1] = \frac{\zeta_{12} + \xi_{12}}{3} + Con[2,3,1,1]; \tag{365}$$

$$Con[2,3,2,2] = Con[2,3,3,3] = -Con[3,2,2,2] = -Con[3,2,3,3] \tag{366}$$

$$= \frac{\zeta_{12} + \xi_{12}}{6} + Con[2,3,1,1]. \tag{367}$$

$$Con[2,4,3,3] = -Con[4,2,1,1]; \tag{368}$$

$$Con[2,4,1,1] = -Con[4,2,3,3] = \frac{\zeta_{13} + \xi_{13}}{3} + Con[2,4,3,3]; \tag{369}$$

$$Con[2,4,2,2] = Con[2,4,4,4] = -Con[4,2,2,2] = -Con[4,2,4,4] \tag{370}$$

$$= \frac{\zeta_{13} + \xi_{13}}{6} + Con[2,4,3,3]. \tag{371}$$

$$Con[3,4,2,2] = -Con[4,3,1,1], \tag{372}$$

$$Con[3,4,1,1] = -Con[4,3,2,2] = \frac{\zeta_{23} + \xi_{23}}{3} + Con[3,4,2,2], \tag{373}$$

$$Con[3,4,3,3] = Con[3,4,4,4] = -Con[4,3,3,3] = -Con[4,3,4,4] \tag{374}$$

$$= \frac{\zeta_{23} + \xi_{23}}{6} + Con[3,4,2,2]. \tag{375}$$

A.4. **The full list of constants of type V.**

$$Con[2,1,4,3] = Con[1,2,3,4] = Con[4,3,2,1] = Con[3,4,1,2], \tag{376}$$

$$Con[1,2,4,3] = Con[2,1,3,4] = Con[3,4,2,1] = Con[4,3,1,2] \tag{377}$$

$$= \frac{\zeta_{22} - \zeta_{33} - \xi_{22} + \xi_{33}}{6} - Con[1,2,3,4]; \tag{378}$$

$$Con[3,1,4,2] = Con[1,3,2,4] = Con[4,2,3,1] = Con[2,4,1,3], \tag{379}$$

$$Con[1,3,4,2] = Con[3,1,2,4] = Con[2,4,3,1] = Con[4,2,1,3] \tag{380}$$

$$= \frac{\zeta_{33} - \zeta_{11} - \xi_{33} + \xi_{11}}{6} - Con[1,3,2,4]; \tag{381}$$

$$Con[4,1,3,2] = Con[1,4,2,3] = Con[3,2,4,1] = Con[2,3,1,4], \tag{382}$$

$$Con[1,4,3,2] = Con[4,1,2,3] = Con[2,3,4,1] = Con[3,2,1,4] \tag{383}$$

$$= \frac{\zeta_{11} - \zeta_{22} - \xi_{11} + \xi_{22}}{6} - Con[1,4,2,3]. \tag{384}$$

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