

A NOTE ON A RECENT ATTEMPT TO SOLVE THE SECOND PART OF HILBERT'S 16TH PROBLEM

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ABSTRACT. For a given natural number n , the second part of Hilbert's 16th Problem asks whether there exists a finite upper bound for the maximum number of limit cycles that planar polynomial vector fields of degree n can have. This maximum number of limit cycle, denoted by $H(n)$, is called the n th Hilbert number. It is well-established that $H(n)$ grows asymptotically as fast as $n^2 \log n$. A direct consequence of this growth estimation is that $H(n)$ cannot be bounded from above by any quadratic polynomial function of n . Recently, the authors of the paper [Exploring limit cycles of differential equations through information geometry unveils the solution to Hilbert's 16th problem. *Entropy*, 26(9), 2024] affirmed to have solved the second part of Hilbert's 16th Problem by claiming that $H(n) = 2(n-1)(4(n-1)-2)$. Since this expression is quadratic in n , it contradicts the established asymptotic behavior and, therefore, cannot hold. In this note, we further explore this issue by discussing some counterexamples.

1. INTRODUCTION

For a given natural number n , the second part of Hilbert's 16th Problem asks whether there is a finite upper bound for the number of limit cycles that planar polynomial vector fields of degree n can possess. More precisely, let

$$H(n) := \sup\{\pi(P, Q) : \deg(P), \deg(Q) \leq n\},$$

where $\pi(P, Q)$ denotes the number of limit cycles of the polynomial differential system

$$(1) \quad \begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y). \end{cases}$$

Recall that a limit cycle of (1) is a (non-stationary) periodic orbit that is isolated from other periodic orbits (see [8, Definition 9]). Thus, the second part of Hilbert's 16th Problem consists of proving that $H(n) < \infty$ for all $n \in \mathbb{N}$ (see [8, Chapter 2]). The value $H(n)$ is called the n th Hilbert number.

The most significant advancement in understanding the asymptotic behavior of the function $H(n)$ was made by Christopher and Lloyd in [2], who introduced a method showing that $H(n)$ grows as fast as $n^2 \log n$. This classical result has been revisited and improved by several works, including [1, 4, 5]. In particular, Han and Li in [4] refined Christopher and Lloyd's result, demonstrating that $H(n)$ grows at least as fast as $(n+2)^2 \log(n+2)/(2 \log 2)$ by establishing that

$$\liminf_{n \rightarrow \infty} \frac{H(n)}{(n+2)^2 \log(n+2)} \geq \frac{1}{2 \log 2}.$$

This remains the best-known lower estimation for the asymptotic growth of $H(n)$.

A direct conclusion from this asymptotic growth estimation is that $H(n)$ cannot be bounded from above by any quadratic polynomial function in n , as the expression $(n+2)^2 \log(n+2)/(2 \log 2)$ surpasses any degree two polynomial in n for sufficiently large values of n .

Recently, the authors of the paper [3] affirmed to have solved the second part of Hilbert's 16th Problem by claiming that

$$(2) \quad H(n) = 2(n-1)(4(n-1)-2),$$

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for $n \geq 2$ (see [3, Theorem 4]). They make use of the following scalar curvature associated to a Fisher information metric:

$$(3) \quad R = \frac{1}{\sqrt{G}} \left[\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{G}} \frac{\partial G_{22}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{G}} \frac{\partial G_{11}}{\partial y} \right) \right],$$

where

$$G_{11} = 2 \left[\left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial Q}{\partial x} \right)^2 \right], \quad G_{22} = 2 \left[\left(\frac{\partial P}{\partial y} \right)^2 + \left(\frac{\partial Q}{\partial y} \right)^2 \right], \quad \text{and } G = G_{11} G_{22}.$$

Their approach relies on [3, Definition 1], which aims to provide an alternative definition for limit cycles, referred to as being “in the framework of GBT”. It begins by establishing that

(A) *a limit cycle is the periodic state of (1) in which R is positive in the neighborhood of the equilibrium points of (1) and $|R|$ is singular.*

By $|R|$ singular, they mean the existence of zeros of the denominator of $|R|$ that makes $|R|$ to diverge to infinity. Thus, it is also asserted that

(B) *if R is positive in the neighborhood of the equilibrium points of (1) and the magnitude of R diverges to infinity at symmetrical singularities with respect to the origin, then (1) possesses only one limit cycle. Nonetheless, if R is positive in the neighborhood of the equilibrium points of (1) and the magnitude of R diverges to infinity at different singularities, then (1) has more than one limit cycle such that the total number of distinctive divergences of $|R|$ to infinity provides the maximum number of limit cycles of (1).*

Subsequent to Definition 1, it is stated that such a definition “agrees with the definition of limit cycles in the framework of classical bifurcation theory”, that is (non-stationary) periodic orbits isolated from other periodic orbits. In this way, the approach employed in [3] to obtain (2) consisted in counting the number of divergences of $|R|$ to infinity, as highlighted in the proof of [3, Theorem 4].

As previously mentioned, the function $H(n)$ cannot be bounded from above by any quadratic polynomial in n . Therefore, the relationship (2), which is quadratic in n , cannot hold. To explore this issue further, we present counterexamples in the following sections. Section 2 discusses a well-known example from the literature that contradicts (2), along with references to other known examples that serves as counterexamples to (2). In Section 3, we provide examples of polynomial systems that exhibit limit cycles but do not satisfy (A), and vice versa. This demonstrates that (A) is neither necessary nor sufficient for the existence of limit cycles of (1) and, therefore, is not equivalent to the standard definition of limit cycles. As a result, the definition of limit cycles proposed in [3] is not applicable to the study of the second part of Hilbert’s 16th problem, meaning that the number of singularities of $|R|$ does not determine the maximum number of limit cycles in (1), as suggested by assertion (B).

2. KNOWN COUNTEREXAMPLES IN THE LITERATURE

The objective of this section is not to construct new counterexamples to the main conclusion (2) of [3], but rather to highlight known examples from the literature that serve as counterexamples for it.

In [5, Section 3], Li et al. revisited the class of polynomial differential systems originally studied by Christopher and Lloyd [2], addressing a minor issue in the original analysis. This correction did not affect the leading term $n^2 \log n$ of the lower estimation for the asymptotic growth of $H(n)$. Their approach, as well as Christopher and Lloyd’s approach, consists of constructing a sequence of recursively defined polynomial differential systems (PH_k) of degree $2^k - 1$, each possessing at least S_k limit cycles, where

$$S_k = 4^{k-1} \left(k - \frac{13}{6} \right) + 2^k - \frac{1}{3}.$$

This sequence implies that

$$(4) \quad H(2^k - 1) \geq S_k = 4^{k-1} \left(k - \frac{13}{6} \right) + 2^k - \frac{1}{3}.$$

However, the conclusion (2) from [3] provides that

$$H(2^k - 1) = 4(2^k - 2)(2^{k+1} - 5),$$

which contradicts (4) for $k \geq 35$. This means that system PH_k , for $k \geq 35$, has more limit cycles than predicted by the main result of [3]. The other sequences of polynomial systems discussed in [5, Sections 4 and 5] also provide counterexamples to (2).

The works [4] and, more recently, [1] also provide similar lower estimations for the asymptotic growth of $H(n)$. Both works present sequences of polynomial differential systems with specified degrees and numbers of limit cycles, differing in the mechanisms used to generate these limit cycles. Counterexamples to (2) can be derived from these sequences in a way analogous to the approach outlined above.

3. POSSIBLE ISSUE FOR THE PROPOSED METHOD

We begin by presenting three examples of polynomial differential systems where the existence of limit cycles is guaranteed, but assertion (A) does not hold. Specifically, in these examples, either R is negative in a neighborhood of the unique equilibrium point, or R is positive in a neighborhood of the unique equilibrium point, but $|R|$ is not singular. These examples demonstrate that limit cycles satisfying (A) do not encompass all possible limit cycles in polynomial systems. As a result, the maximum number of limit cycles satisfying (A) for a polynomial system of degree n does not provide an upper bound for $H(n)$. This likely explains why the main result (2) of [3] does not agree with the established lower estimations for the asymptotic growth of $H(n)$, as discussed in the previous section.

Example 1. We start by considering the following cubic vector field

$$(5) \quad \begin{cases} \dot{x} = -y + x(x^2 + y^2 - 1), \\ \dot{y} = x + y(x^2 + y^2 - 1), \end{cases}$$

which has a single equilibrium point, located at the origin $(0,0)$. This vector field also has a unique limit cycle surrounding the origin. To see that, it is enough to write system (5) in polar coordinates $(x,y) = (r \cos(\theta), r \sin(\theta))$ as follows:

$$\begin{cases} \dot{r} = r(r^2 - 1), \\ \dot{\theta} = 1. \end{cases}$$

This implies that system (5) has a unique limit cycle which is unstable and whose orbit corresponds to the unit circle with center at the origin. Now, computing the function R we get

$$R(x,y) = \frac{R_1(x,y)}{R_2(x,y)},$$

where

$$\begin{aligned} R_1(x,y) = & 72x^{10} - 216x^8y^2 - 204x^8 - 320x^7y - 3056x^6y^4 + 464x^6y^2 + 368x^6 + 192x^5y^3 + 192x^5y \\ & - 3056x^4y^6 + 2360x^4y^4 - 304x^4y^2 - 240x^4 - 192x^3y^5 - 216x^2y^8 + 464x^2y^6 - 304x^2y^4 - 96x^2y^2 \\ & + 96x^2 + 320xy^7 - 192xy^5 + 72y^{10} - 204y^8 + 368y^6 - 240y^4 + 96y^2 - 16 \quad \text{and} \end{aligned}$$

$$R_2(x,y) = \left((3x^2 + y^2 - 1)^2 + (2xy + 1)^2 \right)^2 \left((x^2 + 3y^2 - 1)^2 + (2xy - 1)^2 \right)^2.$$

Observe that R_2 does not vanish at the origin, implying that R is continuous in its neighborhood. Additionally, since $R(0,0) = -1 < 0$, continuity ensures that $R(x,y)$ remains negative in a neighborhood of the origin, which corresponds to the unique equilibrium point of (5). Therefore, system (5) provides an example of a limit cycle that does not satisfy assertion (A).

Example 2. Using the approach from Example 1, we can easily construct polynomial systems with any number of limit cycles and a unique equilibrium point, where R is negative in its neighborhood. For instance, the following polynomial system has a single equilibrium point at the origin and two nested limit cycles surrounding it:

$$(6) \quad \begin{cases} \dot{x} = -y + x(x^2 + y^2 - 1)(x^2 + y^2 - 4), \\ \dot{y} = x + y(x^2 + y^2 - 1)(x^2 + y^2 - 4). \end{cases}$$

Indeed, by applying a polar change of variables, one can deduce that (6) has exactly two limit cycles: an asymptotically stable one, whose orbit corresponds to the unit circle centered at the origin; and an unstable one whose orbit corresponds to a circle of radius two, also centered at the origin. The expression for R is cumbersome and thus omitted here, but

following the same reasoning of Example 1, we conclude that R is continuous in a neighborhood of the origin, with $R(0,0) = -80/289 < 0$, implying that R remains negative near the origin. Therefore, system (6) provides examples of limit cycles that do not satisfy assertion (A).

Example 3. Now, consider the system (5) under the following linear change of variables: $(x, y) = (u, u + v/2)$. This yields the transformed system:

$$(7) \quad \begin{cases} \dot{u} = -2u - \frac{v}{2} + 2u^3 + u^2v + \frac{uv^2}{4}, \\ \dot{v} = 4u + 2u^2v + uv^2 + \frac{v^3}{4}. \end{cases}$$

Of course, system (7) has a unique equilibrium point at the origin $(0,0)$ and a unique limit cycle surrounding it. Computing the function R for system (7), we obtain

$$R(u, v) = \frac{R_1(u, v)}{R_2(u, v)},$$

where

$$\begin{aligned} R_1(u, v) = & 32 \left(-663552u^{10} - 8638464u^9v - 25353216u^8v^2 - 7421952u^8 - 37943808u^7v^3 - 18733056u^7v \right. \\ & - 36060032u^6v^4 - 22151168u^6v^2 + 5670912u^6 - 23658048u^5v^5 - 18140416u^5v^3 + 10874880u^5v \\ & - 10971920u^4v^6 - 11152128u^4v^4 + 7196416u^4v^2 - 2199552u^4 - 3555048u^3v^7 - 4852576u^3v^5 \\ & + 2186496u^3v^3 - 4174848u^3v - 772632u^2v^8 - 1359232u^2v^6 + 296160u^2v^4 - 2595840u^2v^2 \\ & + 219136u^2 - 103056uv^9 - 222052uv^7 + 49248uv^5 - 828032uv^3 + 472064uv - 6399v^{10} - 18528v^8 \\ & \left. + 18596v^6 - 126272v^4 + 134912v^2 + 61440 \right) \quad \text{and} \end{aligned}$$

$$R_2(u, v) = \left((24u^2 + 8uv + v^2 - 8)^2 + 16(4uv + v^2 + 4)^2 \right)^2 \left((8u^2 + 8uv + 3v^2)^2 + 4(2u^2 + uv - 1)^2 \right)^2.$$

Again, R_2 does not vanish at the origin, so R is continuous in its neighborhood. Moreover, since $R(0,0) = 6/5 > 0$, continuity ensures that $R(u, v)$ is positive in a neighborhood of the origin, corresponding to the unique equilibrium point of (7). Additionally, since R_2 is a product of sums of squares, it follows that $R_2(u, v) = 0$ if and only if (u, v) satisfies one of the following systems of algebraic equations:

$$S_1 : \begin{cases} 24u^2 + 8uv + v^2 - 8 = 0 \\ 4uv + v^2 + 4 = 0 \end{cases} \quad \text{or} \quad S_2 : \begin{cases} 8u^2 + 8uv + 3v^2 = 0 \\ 2u^2 + uv - 1 = 0. \end{cases}$$

We begin by analyzing S_1 . First, note that if (u, v) is a solution of S_1 , then $v \neq 0$. Solving the second equation of S_1 for u and substituting into the first equation yields the algebraic equation $17v^4 + 152v^2 + 384 = 0$, which has no real solutions. Next, for system S_2 , if (u, v) is a solution, then $u \neq 0$. Solving the second equation of S_2 for v and substituting into the first equation leads to the algebraic equation $3 - 4u^2 + 4u^4 = 0$, which also has no real solutions. This shows that the denominator R_2 of R does not vanish, and hence $|R|$ has no singularities. Therefore, system (7) provides another example of a limit cycle that does not satisfy assertion (A).

From the above examples, we observed that assertion (A) is not necessary for the existence of limit cycles, as there are polynomial systems with limit cycles where (A) does not hold. Nevertheless, we can still ask whether (A) is a sufficient condition for the existence of limit cycles. The following example provides a negative answer to this question.

Example 4. Consider the following quadratic polynomial system:

$$(8) \quad \begin{cases} \dot{x} = -y + x^2, \\ \dot{y} = x + xy. \end{cases}$$

This system and its properties have been extensively studied in the literature, as it appears as a normal form for a class of isochronous quadratic systems, commonly referred to as S_2 (see [6, 7]). This system has a unique equilibrium point at the origin, which is a center, meaning that there exists a neighborhood U around the origin where all orbits in

$U \setminus \{(0, 0)\}$ are periodic. Clearly, no periodic orbit in U is a limit cycle, as none are isolated from other periodic orbits. In fact, this system does not have any limit cycles. By computing the function R for system (8), we obtain

$$R(x, y) = \frac{1}{(x^2 + 1)^2 (4x^2 + (y + 1)^2)}.$$

Observe that R is continuous in a neighborhood of the origin, as its denominator does not vanish at $(0, 0)$. Since $R(0, 0) = 1$, continuity ensures that R remains positive in a neighborhood around the origin, which is the unique equilibrium point of (8). Furthermore, $|R|$ is singular at $(x, y) = (0, -1)$. Thus, system (8) provides an example where assertion (A) holds for every periodic orbit within U , despite the absence of limit cycles.

4. CONCLUSION

In this note, we have demonstrated that the recent attempt to solve the second part of Hilbert's 16th problem, as presented in [3], contains significant issues. We began by exploring counterexamples which demonstrate that the quadratic expression proposed for $H(n)$ contradicts the well-established asymptotic behavior of this function, which states that $H(n)$ grows as fast as $(n + 2)^2 \log(n + 2) / (2 \log 2)$. Moreover, we discussed how the alternative definition of limit cycles (A), used in [3], is not applicable to the study of the second part of Hilbert's 16th problem, as it is neither necessary nor sufficient for the existence of limit cycles in (1), according to the standard definition, which refers to (non-stationary) periodic orbits isolated from other periodic orbits.

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