WEBS GENERATED BY PRODUCTS OF CONVEX AND HOMOGENEOUS FOLIATIONS ON $\mathbb{P}^2$

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Abstract. This paper investigates flat webs on the projective plane. We present two methods for constructing such webs: the first involves taking the product of two convex reduced foliations and invariant lines, while the second consists of taking the product of finitely many convex homogeneous foliations and invariant lines. In both cases, we demonstrate that the dual web is flat.

Introduction

Locally, in affine coordinates on the projective plane, a foliation is defined by a polynomial 1-form $\omega = a(x,y)dx+b(x,y)dy$ with isolated zeros. Similarly, a k-web is defined by a k-symmetric polynomial 1-form with non identically zero discriminant and isolated zeros as follows

$$\Omega = \sum_{i+j=k} a_{ij}(x,y)dx^i dy^j.$$

In the context of this article, we are interested in the curvature of webs. Given a web $\mathcal{W}$ on a complex surface, its curvature is a meromorphic 2-form with poles contained in the discriminant $\Delta(\mathcal{W})$, denoted by $K(\mathcal{W})$. This 2-form satisfies $\varphi^* K(\mathcal{W}) = K(\varphi^*\mathcal{W})$ for any holomorphism φ. When the curvature is zero we say that $\mathcal{W}$ is flat. A key observation: since there are no holomorphic 2-forms on $\mathbb{P}^2$, the web is flat if and only if the curvature is holomorphic at the generic points of the irreducible components of $\Delta(\mathcal{W})$. A important result of web geometry, due to Blaschke-Dubourdien, characterizes the local equivalence of a (germ of) 3-web $\mathcal{W}$ on $\mathbb{C}^2$ with the trivial 3-web defined by $dx \cdot dy \cdot (dx - dy)$ through the vanishing of curvature of $\mathcal{W}$. The usually definition of curvature for a k-web $\mathcal{W}$ with $k > 3$ is the sum of the curvatures of all 3-subwebs of $\mathcal{W}$, and we say that the k-web is flat when this sum is zero. To our knowledge there is not a characterization of the flatness of a web, nevertheless is a necessary condition for the maximality of the rank of the web, according to a result of Mihaileanu.

For a global context, given a k-web $\mathcal{W}$ of degree d on the projective plane we can consider the Legendre transform of $\mathcal{W}$, it will be denoted by $\operatorname{Leg}\mathcal{W}$. This is a d-web of degree k on the dual projective plane $\check{\mathbb{P}}^2$ which we explain now. Take a generic line l on $\mathbb{P}^2$ and consider the tangency locus $\operatorname{Tang}(\mathcal{W}, l) = \{p_1, \cdots, p_d\} \subset \mathbb{P}^2$ between $\mathcal{W}$ and l. We can think the dual $\check{p}_i$ as lines on $\check{\mathbb{P}}^2$ passing through the point l of $\check{\mathbb{P}}^2$. Then the set of tangent lines of $\operatorname{Leg}\mathcal{W}$ at l is just $T_l \operatorname{Leg}\mathcal{W} = \bigcup_{i=1}^{d} \check{p}_i$. More precisely, let (p,q) be the affine chart of $\check{\mathbb{P}}^2$ correspond to the line $\{y = px + q\} \subset \mathbb{P}^2$. If $\mathcal{W}$ is defined by an implicit affine equation $F(x, y; p) = 0$ with $p = dy/dx$ then $\operatorname{Leg}\mathcal{W}$ is given by the implicit differential equation

$$\check{F}(p, q; x) := F(x, px + q; p), \qquad \text{with} \qquad x = -\frac{dq}{dp}.$$

When the web $\mathcal{W}$ decomposes as the product of two webs $\mathcal{W}_1 \boxtimes \mathcal{W}_2$ then we have that the following decomposition also occurs: $\operatorname{Leg}\mathcal{W} = \operatorname{Leg}\mathcal{W}_1 \boxtimes \operatorname{Leg}\mathcal{W}_2$. For more details about the definition of Legendre transform, see for instance [5, Part I Chapter II Section 2.5].

In particular, we are especially interested in convex foliations. For us a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is convex if its inflection divisor is completely invariant by $\mathcal{F}$. Even more, if this divisor is also reduced we will say that $\mathcal{F}$ is reduced convex foliation.

Our first result is about the dual of a web generated by product of some invariant lines and two reduced convex foliations under some hypotheses.

Theorem A. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be reduced convex foliations on $\mathbb{P}^2_{\mathbb{C}}$ of degrees d_1 and d_2, such that, $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ is formed by invariant lines by $\mathcal{F}_1$ and $\mathcal{F}_2$. If $l_1, \cdots, l_n$ are invariant lines of $\mathcal{F}_1$ and $\mathcal{F}_2$ then the transform $\operatorname{Leg}\mathcal{W}$ of the web $\mathcal{W} = l_1 \boxtimes \cdots \boxtimes l_n \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2$ is flat.*

We present some reduced convex foliations that satisfies the theorem conditions and provide a family of examples of flat webs of this type.

In the second part of this paper, we will work with homogeneous foliations. A foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2$ is called homogeneous if there exists a system of affine coordinates (x, y) where $\mathcal{H}$ is defined by a homogeneous vector field $A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ with $A, B \in \mathbb{C}[x, y]_d$ and $gcd(A, B) = 1$. In the same direction as the first result, we want to determine the flatness of the Legendre transform of a web generated by the product of invariant lines and many convex homogeneous foliations. The result can be phrased as follows.

Theorem B. *Let $\mathcal{H}_1, \cdots, \mathcal{H}_n$ be convex homogeneous foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees d_i with $i = 1, \cdots, n$, such that, $\operatorname{Tang}(\mathcal{H}_i, \mathcal{H}_j) = L_\infty \cup C^{inv}_{ij}$ where $\forall i \neq j$ in $\{1, \cdots, n\}$, C^{inv}_{ij} is a product of invariant lines by $\mathcal{H}_i$, $\mathcal{H}_j$. If $l_1, \cdots, l_k$ are invariant lines by all the foliations then the dual web $\operatorname{Leg}\mathcal{W}$ of $\mathcal{W} = l_1 \boxtimes \cdots \boxtimes l_k \boxtimes \mathcal{H}_1 \boxtimes \cdots \boxtimes \mathcal{H}_n$ is flat.*

These results are expansion of those presented by Bedrouni in [1]. In Theorem A, we added a convex reduced foliation, while the Theorem B can be seen as a generalization of [1, Theorem 2].

1. Preliminaries

Let $\mathcal{F}$ be a satured foliation on $\mathbb{P}^2$, we can define the Gauss map of $\mathcal{F}$ as the rational map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ such that $\mathcal{G}_{\mathcal{F}}(p) = T_p\mathcal{F}$, which is well defined outside of $\operatorname{Sing}\mathcal{F}$, called of the singular set of $\mathcal{F}$. If the homogeneous form $\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$ defines the foliation $\mathcal{F}$, then the Gauss map can be described as

$$\mathcal{G}_{\mathcal{F}}(p) = [a(p) : b(p) : c(p)].$$

The inflection divisor of $\mathcal{F}$, denoted by $I(\mathcal{F})$, is defined by the vanishing of the determinant

$$\det \begin{pmatrix} x & y & z \\ X(x) & X(y) & X(z) \\ X^2(x) & X^2(y) & X^2(z) \end{pmatrix}$$

with X being the vector field defining $\mathcal{F}$. This divisor has some relevant properties for us:

(i) $I(\mathcal{F})$ does not depend on a particular choice of system of homogeneous coordinates on $\mathbb{P}^2$;
(ii) If C is an irreducible algebraic invariant curve of $\mathcal{F}$ then $C \subset I(\mathcal{F})$, if and only if, C is an invariant line;
(iii) The degree of $I(\mathcal{F})$ is exactly $3d$, where d is the degree of $\mathcal{F}$.

For a discussion in a more general context and additional details about inflection divisor see [7]. The following result will be useful for us.

Lemma 1.1 ([4], Lemma 2.2)**.** *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Then*

$$\Delta(\operatorname{Leg}\mathcal{F}) = \mathcal{G}_{\mathcal{F}}(I(\mathcal{F})) \cup \check{\Sigma}(\mathcal{F})$$

where $\check{\Sigma}(\mathcal{F})$ consists in the dual lines of the special singularities

$$\Sigma(\mathcal{F}) = \{s \in \operatorname{Sing}(\mathcal{F}) : \nu(\mathcal{F}, s) \geq 2 \text{ or } s \text{ is a radial singularity of } \mathcal{F}\}$$

of $\mathcal{F}$ and $\nu(\mathcal{F}, s)$ stands for the algebraic multiplicity of $\mathcal{F}$ at s.

We also denote by $\Sigma_{\mathcal{F}}^{\text{rad}}$ the set of radial singularities and by $\check{\Sigma}_{\mathcal{F}}^{\text{rad}}$ the dual lines. It is convenient to define a similar nomenclature: we will denote by $\Sigma_{\mathcal{F}}^{l}$ where l is a line, the set of singularities of $\mathcal{F}$ in l i.e. $\operatorname{Sing}\mathcal{F} \cap l$ and by $\check{\Sigma}_{\mathcal{F}}^{l}$ the respective dual lines. In order to study product of foliations, we shall need the following proposition.

Proposition 1.2. *Let $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ be a (d_1+d_2)-web in $\mathbb{P}^2_{\mathbb{C}}$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are foliations of respective degrees d_1 and d_2. If $\operatorname{Tang}(\mathcal{F}_1,\mathcal{F}_2) = C^{inv} \cup C^{tr}$, where C^{inv} is the invariant part for both $\mathcal{F}_1$ and $\mathcal{F}_2$, and C^{tr} is the transversal part, then*

$$\Delta(\operatorname{Leg}\mathcal{W}) = \Delta(\operatorname{Leg}\mathcal{F}_1) \cup \Delta(\operatorname{Leg}\mathcal{F}_2) \cup \mathcal{G}_{\mathcal{F}_i}(C^{inv}) \cup \mathcal{G}_{\mathcal{F}_i}(C^{tr})$$

for each $i = 1, 2$.

Proof.: We know that $\Delta(\operatorname{Leg}\mathcal{W}) = \Delta(\operatorname{Leg}\mathcal{F}_1) \cup \Delta(\operatorname{Leg}\mathcal{F}_2) \cup \operatorname{Tang}(\operatorname{Leg}\mathcal{F}_1, \operatorname{Leg}\mathcal{F}_2)$. Clearly we have $\mathcal{G}_{\mathcal{F}_i}(C^{\text{inv}}) \cup \mathcal{G}_{\mathcal{F}_i}(C^{\text{tr}}) \subset \operatorname{Tang}(\operatorname{Leg}\mathcal{F}_1, \operatorname{Leg}\mathcal{F}_2)$. Now, take an irreducible curve $C \subseteq \operatorname{Tang}(\operatorname{Leg}\mathcal{F}_1, \operatorname{Leg}\mathcal{F}_2) \setminus (\Delta(\operatorname{Leg}\mathcal{F}_1) \cup \Delta(\operatorname{Leg}\mathcal{F}_2))$ and $\check{l} \in C$ a generic point. Then there is a point $p \in l$ such that $p \in \operatorname{Tang}(\mathcal{F}_i, l)$ for $i = 1, 2$ and we would have that $p \in \operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ and $\mathcal{G}_{\mathcal{F}_i}(p) = \check{l}$. Thus, $C \subseteq \mathcal{G}_{\mathcal{F}_i}(C^{\text{inv}}) \cup \mathcal{G}_{F_i}(C^{\text{tr}})$. ■

For the our results we will need the following Lemma

Lemma 1.3. *Let $\mathcal{F}_1, \cdots, \mathcal{F}_k$ be foliations on $\mathbb{P}^2_{\mathbb{C}}$. If s is radial singularity of order ν_i of $\mathcal{F}_i$ for $i = 1, \cdots, k$, then*

$$\operatorname{Leg}(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \mathcal{W}_1 \boxtimes \mathcal{W}_2$$

where

i. $\mathcal{W}_1$ is a ν-web leaving $\check{s}$ invariant, with $\nu = \nu_1 + \cdots + \nu_k$;
ii. $\check{s}$ is a component totally invariant of $\Delta(\mathcal{W}_1)$ of minimal order $\nu - 1$;
iii. $\mathcal{W}_2$ is transversal to $\check{s}$.

Proof.: Following [6, Proposition 3.3] we can write, around $s = (0,0)$, the foliations by

$$\check{F}_1(p,q;x) = q + a_1^1(p,q)qx + \cdots + a_{\nu_1-1}^1(p,q)qx^{\nu_1-1} + a_{\nu_1}^1(p,q)x^{\nu_1} + \cdots + a_{d_1}^1(p,q)x^{d_1}$$

$$\vdots$$

$$\check{F}_k(p,q;x) = q + a_1^k(p,q)qx + \cdots + a_{\nu_k-1}^k(p,q)qx^{\nu_k-1} + a_{\nu_k}^k(p,q)x^{\nu_k} + \cdots + a_{d_2}^k(p,q)x^{d_k}$$

with $a_{\nu_i}^i(p,0)$ not identically zero $\forall i = 1, \cdots, k$. Therefore,

$$\check{F}(p,q;x) = qG(p,q;x) + a_\nu(p,q)x^\nu + \cdots + a_d(p,q)x^d$$

where $d = d_1 + \cdots + d_k$ and the degree of $G(p,q;x)$ in x less than or equal to $\nu - 1$ and $a_\nu(p,0)$ not zero.
Then, we have that $\check{s}$ appears in the discriminant $\Delta(\mathcal{W})$ with minimal order $\nu - 1$. Thus, we can to write $\operatorname{Leg}(\mathcal{W}) = \mathcal{W}_1 \boxtimes \mathcal{W}_2$ with $\mathcal{W}_1$ a (ν)-web leaving $\check{s}$ completely invariant by $\Delta(\mathcal{W}_1)$ and $\mathcal{W}_2$ transversal to $\check{s}$. ■

Product with not invariant line. Given a foliation $\mathcal{F}$, it is expected that for a generic non-invariant line l the web $\operatorname{Leg}(l \boxtimes \mathcal{F})$ is not flat. To simplify the presentation, we will show this in the following case.

Proposition 1.4. *Let $\mathcal{F}$ be a foliation of degree $d = 2$ and l a generic line, then the dual web of $\mathcal{W} = l \boxtimes \mathcal{F}$, $\operatorname{Leg}\mathcal{W}$ is not flat.*

The proof will be done by contradiction. Since being flat is a closed condition, we shall assume that $\operatorname{Leg}\mathcal{W}$ is flat for all $l \in \check{\mathbb{P}}^2$.

We can assume that:

i. $l \cap \operatorname{Sing} \mathcal{F} = \emptyset$, that is $\check{\Sigma}^{l}_{\mathcal{F}} = \emptyset$

ii. $\mathcal{G}_{\mathcal{F}}(l) \not\subseteq \check{\Sigma}^{rad}_{\mathcal{F}}$, in fact, for p a tangency point between l and $\mathcal{F}$ we have that $\mathcal{G}_{\mathcal{F}}(p) = \check{l} \notin \check{\Sigma}^{rad}_{\mathcal{F}}$.

Set $\mathcal{G}_{\mathcal{F}}(l) = C$ and take a generic point $t \in C$ such that the line $\check{t}$ is not invariant by $\mathcal{F}$. Let $\operatorname{Tang}(\mathcal{F}, \check{t}) = p_1 + p_2$, with $p_1 \in \check{t}$, since $I(\mathcal{F})$ is formed by invariant lines we have that p_1 and p_2 are not inflection points. On the order hand, condition (ii) guarantees that these points are not radial singularities either. Therefore, in a neighborhood of t we can write

$$\operatorname{Leg}(\mathcal{W}) = (\operatorname{Leg} l \boxtimes \mathcal{F}_1) \boxtimes \mathcal{F}_2$$

where $\mathcal{F}_1$ is tangent to $\operatorname{Leg} l$ along C and $\mathcal{F}_2$ is transversal to $\mathcal{W}_2 := \operatorname{Leg} l \boxtimes \mathcal{F}_1$

By [6, Theorem 1], since $K(\operatorname{Leg} \mathcal{W})$ is holomorphic in a generic point of C we have that C is invariant by $\mathcal{W}_2$ or by $\beta_{\mathcal{W}_2}(\mathcal{F}_2) = \mathcal{F}_2$. In the first case C is $\operatorname{Leg} l$-invariant and l would be invariant by $\mathcal{F}$, so this is not the case. Thus C is invariant by $\mathcal{F}_2$ and so $\check{C}$ is an invariant curve for $\mathcal{F}$. Clearly C depends of the line l and then so will $\check{C}$. Since $\mathcal{F}$ can have at most a pencil of invariant curves we would have an infinite number of lines l with the same image $\mathcal{G}_{\mathcal{F}}(l)$, which is impossible.

2. Convex Reduced Foliations

In this section we study webs that are dual to the product of convex reduced foliations and (maybe) some lines. We say that $\mathcal{F}$ is **convex** if $I(\mathcal{F})$ is formed by invariant lines. If moreover $I(\mathcal{F})$ is a reduced divisor we will say that $\mathcal{F}$ is **convex reduced**.

Proposition 2.1. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be convex reduced foliations on $\mathbb{P}^2_{\mathbb{C}}$ such that,* $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ *is a union of invariant lines by $\mathcal{F}_1$ and $\mathcal{F}_2$. Then the web $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ satisfies*

$$\Delta(\operatorname{Leg} \mathcal{W}) = \check{\Sigma}^{rad}_{\mathcal{F}_1} \cup \check{\Sigma}^{rad}_{\mathcal{F}_2}.$$

Proof.: As $\mathcal{F}_1$, $\mathcal{F}_2$ are convex and reduced, we have that all the singularities are non-degenerate ([3, Lemma 6.8]), thus $\Delta(\operatorname{Leg} \mathcal{F}_i) = \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_i}$ for each $i = 1, 2$. furthermore, being $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2) = l_1 \cup \cdots \cup l_k$, with each l_j invariant, and due to the fact that the Gauss map of invariant lines being points, we obtain the desired equality as a consequence of Proposition 1.2. ∎

Now we establish the flatness of the product of two convex reduced foliations under certain hypothesis.

Proposition 2.2. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be convex reduced foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees $d_1, d_2 \geq 3$, such that,* $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ *in a union of invariant lines and denote $\mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$. Then, the web* $\operatorname{Leg} \mathcal{W} = \operatorname{Leg} \mathcal{F}_1 \boxtimes \operatorname{Leg} \mathcal{F}_2$ *is flat.*

Proof.: By Proposition 2.1, we have that $\Delta(\operatorname{Leg} \mathcal{W}) = \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_1} \cup \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_2}$. Take $\check{s} \in \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_1} \setminus \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_2}$ and a generic line l (particularly, not invariant) passing through s. By [6, Proposition 3.3] we can write locally around l: $\operatorname{Leg} \mathcal{F}_1 = \mathcal{W}_\nu \boxtimes \mathcal{W}_{d_1-\nu}$ with l a totally invariant component of $\Delta(\mathcal{W}_\nu)$ of minimal multiplicity $\nu - 1$ and $\mathcal{W}_{d_1-\nu}$ transverse to $\check{s}$. As $s \notin \Sigma^{\mathrm{rad}}_{\mathcal{F}_2}$ we have that $\operatorname{Tang}(\mathcal{F}_2, l, s) \leq 1$.

- If $s \notin \operatorname{Sing} \mathcal{F}_2$, we can assume $l \cap \operatorname{Sing} \mathcal{F}_2 = \emptyset$, then $\operatorname{Leg} \mathcal{F}_2$ is completely transversal to $\check{s}$. On the other hand, if one direction of $\mathcal{W}_{d_1-\nu}$ and $\operatorname{Leg} \mathcal{F}_2$ coincide, this direction would correspond to a point of $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2) = l_1 \cup \cdots \cup l_k$ therefore, we would have $l = l_j$ for some $j = 1, \cdots, k$, which does not occur since l is generic. Thus, $\operatorname{Leg} \mathcal{F}_2$ is transverse to $\mathcal{W}_{d-\nu}$ and $\operatorname{Leg} \mathcal{W} = \mathcal{W}_\nu \boxtimes (\mathcal{W}_{d_1-\nu} \boxtimes \operatorname{Leg} \mathcal{F}_2)$ with $\mathcal{W}_{d_1-\nu} \boxtimes \operatorname{Leg} \mathcal{F}_2$ transverse to $\check{s}$. Therefore, by [6, Proposition 2.6], the curvature of $\operatorname{Leg} \mathcal{W}$ is holomorphic along $\check{s}$.
- If $s \in \operatorname{Sing} \mathcal{F}_2$ we have $\operatorname{Tang}(\mathcal{F}_2, l, s) = 1$ and we decompose $\operatorname{Leg} \mathcal{F}_2 = \mathcal{W}_1 \boxtimes \mathcal{W}_{d_2-1}$. By the previous argument, $\mathcal{W}_{d_2-1}$ is transverse to $\mathcal{W}_{d_1-\nu}$. Thus, by [2, Proposition 3.9], the curvature of $\operatorname{Leg} \mathcal{W} = (\mathcal{W}_1 \boxtimes \mathcal{W}_\nu) \boxtimes (\mathcal{W}_{d_1-\nu} \boxtimes \mathcal{W}_{d_2-1})$ is holomorphic along $\check{s}$.

It remains to analyze the case $\check{s} \in \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_1} \cap \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_2}$, but by Lemma 1.3 and [6, Proposition 2.6], we obtain that $\mathrm{Leg}\,\mathcal{W}$ is holomorphic along $\check{s}$. ■

Example 2.3. *Let $\mathcal{F}_d$ be the Fermat foliation of degree d, it is defined by the vector field*

$$V_d = (x^d - x)\frac{\partial}{\partial x} + (y^d - y)\frac{\partial}{\partial y}.$$

By [6, Section 5] we have that $\mathcal{F}_d$ is convex reduced and that the $3d$ invariant lines are: $L_\infty, x = 0, y = 0, x = \xi, y = \xi, y = \xi x$, with $\xi^{d-1} = 1$.

Take now $\mathcal{F}_l$, $\mathcal{F}_d$ Fermat foliations of degrees l and d with $d = 2l-1$. Note that, $\deg(\mathrm{Tang}(\mathcal{F}_l, \mathcal{F}_d))$ $= d + l + 1 = 3l$*. On the other hand, with $U_{l-1}(\mathbb{C})$ being the unity group composed by $(l-1)$ roots of unity, we have that $U_{l-1}(\mathbb{C}) \subseteq U_{2(l-1)}(\mathbb{C}) = U_{d-1}(\mathbb{C})$ thus, $\mathcal{I}_{\mathcal{F}_l} \subseteq \mathcal{I}_{\mathcal{F}_d}$. Therefore,* $\mathrm{Tang}(\mathcal{F}_l, \mathcal{F}_d) = \mathcal{I}_{\mathcal{F}_l}$ *contains only invariant lines of both foliations. We conclude by Proposition 2.2 that* $\mathrm{Leg}(\mathcal{F}_d \boxtimes \mathcal{F}_l)$ *is flat.*

The following proposition shows that this is the only way to obtain such examples with Fermat foliations.

Proposition 2.4. *Let $\mathcal{F}_l$, $\mathcal{F}_d$ be Fermat foliations, with $l < d$. Then,* $\mathrm{Tang}(\mathcal{F}_l, \mathcal{F}_d)$ *is reduced and formed by invariant lines of $\mathcal{F}_l$ and $\mathcal{F}_d$ if and only if $d = 2l - 1$.*

Proof.: If $\mathrm{Tang}(\mathcal{F}_l, \mathcal{F}_d) = l_1 \cup \cdots \cup l_s$ where l_j are distinct invariant lines of $\mathcal{F}_l$ and $\mathcal{F}_d$, we know that $\mathcal{F}_l$ and $\mathcal{F}_d$ have $3l$ and $3d$ invariant lines, respectively, being them

$$l_\infty, \quad x = 0,, \quad y = 0, \quad x = \xi_l, \quad y = \xi_l, \quad y = \xi_l x, \qquad \text{with } \xi_l^{l-1} = 1$$

$$l_\infty, \quad x = 0,, \quad y = 0, \quad x = \xi_d, \quad y = \xi_d, \quad y = \xi_d x, \qquad \text{with } \xi_d^{d-1} = 1$$

Let $k = gcd(l-1, d-1)$, then $\mathcal{F}_l$ and $\mathcal{F}_d$ have $3k + 3$ common invariant lines and we can write $l - 1 = k \cdot l_1$ and $d - 1 = k \cdot d_1$ with $gcd(l_1, d_1) = 1$. As $\deg(\mathrm{Tang}(\mathcal{F}_l, \mathcal{F}_d)) = d + l + 1$ we have that $d + l = 3k + 2$, thus $l_1 + d_1 = 3$. Therefore, $l = k + 1$ and $d = 2k + 1$ i.e $d = 2l - 1$.
From Example 2.3, we have the other implication. ■

Now we consider product of foliations and one invariant line.

Proposition 2.5. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be convex reduced foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees d_1 and d_2, such that,* $\mathrm{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ *is a union of invariant lines. For any invariant line l consider the web $\mathcal{W} = l \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2$. Then* $\mathrm{Leg}\,\mathcal{W} = \mathrm{Leg}\,l \boxtimes \mathrm{Leg}\,\mathcal{F}_1 \boxtimes \mathrm{Leg}\,\mathcal{F}_2$ *is flat.*

Proof.: It follows from Proposition 2.1 and [2, Section 2] that $\Delta(\mathrm{Leg}\,\mathcal{W}) = \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_1} \cup \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_2} \cup \check{\Sigma}^{l}_{\mathcal{F}_1} \cup \check{\Sigma}^{l}_{\mathcal{F}_2}$, where $\Sigma^l_{\mathcal{F}} = \mathrm{Sing}\,\mathcal{F} \cap l$. Consider the following cases

i. $s \in \Sigma^{\mathrm{rad}}_{\mathcal{F}_1} \setminus (\Sigma^{\mathrm{rad}}_{\mathcal{F}_2} \cup l)$ then we repeat the argument of Proposition 2.2 and obtain that $K(\mathrm{Leg}\,\mathcal{W})$ is holomorphic along $\check{s}$. The case in which $\check{s} \in \check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_2} \setminus (\check{\Sigma}^{\mathrm{rad}}_{\mathcal{F}_1} \cup l)$ is analogous.

ii. If $s \in (\Sigma^{\mathrm{rad}}_{\mathcal{F}_1} \cap \Sigma^{\mathrm{rad}}_{\mathcal{F}_2}) \setminus l$ with radiality orders ν_1 and ν_2 respectively. Since $s \notin l$ we obtain that the direction of $\mathrm{Leg}\,l$ on the general point of $\check{s}$ does not coincide with any other of $\mathrm{Leg}\,\mathcal{F}_1$ and $\mathrm{Leg}\,\mathcal{F}_2$. By lemma 1.3 we can write

$$\mathrm{Leg}\,\mathcal{W} = (\mathcal{W}_{\nu_1+\nu_2}) \boxtimes (\mathrm{Leg}\,l \boxtimes \mathcal{W}_{d_1+d_2-\nu_1-\nu_2})$$

and apply [2, Proposition 2.6] to conclude that $\mathrm{Leg}\,\mathcal{W}$ is flat.

iii. If $s \in l \setminus \Sigma^{\mathrm{rad}}_{\mathcal{F}_2}$ with ν_1 being the radiality order of s in $\mathcal{F}_1$, then we write $\mathrm{Leg}\,\mathcal{F}_1 = \mathcal{W}_{\nu_1} \boxtimes \mathcal{W}_{d_1-\nu_1}$. Note that the directions of $\mathrm{Leg}\,\mathcal{F}_2$ and $\mathcal{W}_{d_1-\nu_1}$ do not coincide at generic $\check{t} \in \check{s}$, and $\mathrm{Tang}(\mathcal{F}_2, t, s) \leq 1$ (because $s \notin \Sigma^{\mathrm{rad}}_{\mathcal{F}_2}$). Then,

- If $s \notin \operatorname{Sing}\mathcal{F}_2$, we have that,

$$\operatorname{Leg}\mathcal{W} = (\operatorname{Leg} l \boxtimes \mathcal{W}_{\nu_1}) \boxtimes (\operatorname{Leg}\mathcal{F}_2 \boxtimes \mathcal{W}_{d_1-\nu_1})$$

  where $\operatorname{Leg}\mathcal{F}_2 \boxtimes \mathcal{W}_{d_1-\nu_1}$ is smooth and transversal to $\check{s}$. Now, we apply [2, Proposition 3.9] if $\nu_1 \geq 2$ and [6, Theorem 1] if $\nu_1 = 1$ in order to obtain the flatness.
- If $s \in \operatorname{Sing}\mathcal{F}_2$, we have $\operatorname{Tang}(\mathcal{F}_2, t, s) = 1$ and we can write

$$\operatorname{Leg}\mathcal{W} = (\operatorname{Leg} l \boxtimes \mathcal{W}_1 \boxtimes \mathcal{W}_{\nu_1}) \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1})$$

  with $\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1}$ smooth and transversal to $\check{s}$ and $\mathcal{W}_1$ a foliation tangent to $\check{s}$. If $\nu_1 \geq 2$ the flatness of $\operatorname{Leg}\mathcal{W}$ will follow from [2, Remark 3.10]. On other hand, if $\nu_1 = 1$ we conclude as before from Lemma 1.3 and [2, Proposition 3.9].

iv. If $s \in \Sigma^{\mathrm{rad}}_{\mathcal{F}_1} \cap \Sigma^{\mathrm{rad}}_{\mathcal{F}_2} \cap l$ with ν_1 and ν_2 being the orders of s, respectively, in $\mathcal{F}_1$ and $\mathcal{F}_2$, then by Lemma 1.3 we can write

$$\operatorname{Leg}\mathcal{W} = (\operatorname{Leg} l \boxtimes \mathcal{W}_{\nu_1+\nu_2}) \boxtimes (\mathcal{W}_{d_1+d_2-\nu_1-\nu_2})$$

with $\mathcal{W}_{d_1+d_2-\nu_1-\nu_2}$ smooth and transversal to $\check{s}$. We conclude again by [2, Prop 3.9].

■

Corollary 2.6. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be convex reduced foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees d_1 and d_2, such that,* $\operatorname{Tang}(\mathcal{F}_1, \mathcal{F}_2)$ *is a union of invariant lines. For invariant lines l_1, l_2 consider the web $\mathcal{W} = l_1 \boxtimes l_2 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2$. Then* $\operatorname{Leg}\mathcal{W} = \operatorname{Leg} l_1 \boxtimes \operatorname{Leg} l_2 \boxtimes \operatorname{Leg}\mathcal{F}_1 \boxtimes \operatorname{Leg}\mathcal{F}_2$ *is flat.*

Proof.: We write in a generic point, $\mathcal{W}_1 = \operatorname{Leg}\mathcal{F}_1 = \mathcal{G}_1 \boxtimes \cdots \boxtimes \mathcal{G}_n$ and $\mathcal{W}_2 = \operatorname{Leg}\mathcal{F}_2 = \mathcal{H}_1 \boxtimes \cdots \boxtimes \mathcal{H}_m$ where $\mathcal{G}_i$ and $\mathcal{H}_j$ are foliations for each $i = 1, \cdots, n$ and $j = 1 \cdots, m$. Then,

$$\begin{aligned} K(\operatorname{Leg}\mathcal{W}) &= \sum_{i=1}^{n} K(\operatorname{Leg} l_1 \boxtimes \operatorname{Leg} l_2 \boxtimes \mathcal{G}_i) + \sum_{j=1}^{m} K(\operatorname{Leg} l_1 \boxtimes \operatorname{Leg} l_2 \boxtimes \mathcal{H}_j) \\ &+ \sum_{i<j} K(\operatorname{Leg} l_1 \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) + \sum_{i<j} K(\operatorname{Leg} l_2 \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) + \sum_{i<j} K(\operatorname{Leg} l_1 \boxtimes \mathcal{H}_i \boxtimes \mathcal{H}_j) \\ &+ \sum_{i<j} K(\operatorname{Leg} l_2 \boxtimes \mathcal{H}_i \boxtimes \mathcal{H}_j) + \sum_{i=1,j=1}^{n,m} K(\operatorname{Leg} l_1 \boxtimes \mathcal{G}_i \boxtimes \mathcal{H}_j) \\ &+ \sum_{i=1,j=1}^{n,m} K(\operatorname{Leg} l_2 \boxtimes \mathcal{G}_i \boxtimes \mathcal{H}_j) + K(\mathcal{W}_1 \boxtimes \mathcal{W}_2) \end{aligned}$$

On the other hand we have

$$K(\operatorname{Leg} l \boxtimes \mathcal{W}_1) = \sum_{i<j} K(\operatorname{Leg} l \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) + K(\mathcal{W}_1)$$

$$\begin{aligned} K(\operatorname{Leg} l_1 \boxtimes \operatorname{Leg} l_2 \boxtimes \mathcal{W}_1) &= \sum_{i=1}^{n} K(\operatorname{Leg} l_1 \boxtimes \operatorname{Leg} l_2 \boxtimes \mathcal{G}_i) + \sum_{i<j} K(\operatorname{Leg} l_1 \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) \\ &+ \sum_{i<j} K(\operatorname{Leg} l_2 \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) + K(\mathcal{W}_1) \end{aligned}$$

$$\begin{aligned} K(\operatorname{Leg} l \boxtimes \mathcal{W}_1 \boxtimes \mathcal{W}_2) &= \sum_{i<j} K(\operatorname{Leg} l \boxtimes \mathcal{G}_i \boxtimes \mathcal{G}_j) + \sum_{i<j} K(\operatorname{Leg} l \boxtimes \mathcal{H}_i \boxtimes \mathcal{H}_j) \\ &+ \sum_{i=1,j=1}^{n,m} K(\operatorname{Leg} l \boxtimes \mathcal{G}_i \boxtimes \mathcal{H}_j) + K(\mathcal{W}_1 \boxtimes \mathcal{W}_2) \end{aligned}$$

Putting all these together with the fact that $K(\mathcal{W}_i) = K(\mathcal{W}_1 \boxtimes \mathcal{W}_2) = K(\mathrm{Leg}(l_i \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2)) = 0$, and also $K(\mathrm{Leg}(l_1 \boxtimes l_2 \boxtimes \mathcal{F}_i)) = 0$ by [1, Theorem 1], we obtain $K(\mathrm{Leg}\,\mathcal{W}) = 0$. ■

Proof. of Theorem A:
By [1, Lemma 2.1], we have that

$$K(\mathcal{W}' \boxtimes \mathcal{W}'') = K(\mathcal{W}') - (n-2)\sum_{i=1}^{n} K(\mathrm{Leg}\, l_i \boxtimes \mathcal{W}'') + \sum_{1\le i<j\le n} K(\mathrm{Leg}\, l_i \boxtimes \mathrm{Leg}\, l_j \boxtimes \mathcal{W}'') + \binom{n-1}{2} K(\mathcal{W}'')$$

where $\mathcal{W}' = \mathrm{Leg}\, l_i \boxtimes \cdots \boxtimes \mathrm{Leg}\, l_n$ and $\mathcal{W}'' = \mathrm{Leg}\,\mathcal{F}_1 \boxtimes \mathrm{Leg}\,\mathcal{F}_2$.
As $K(\mathcal{W}') = 0$ and by Propositions 2.2, 2.5, and Corollary 2.6, we have that $K(\mathrm{Leg}\,\mathcal{W}) = 0$. ■

Remark 2.7. *Although we can look a statement with k convex reduced foliations, the Proposition 2.4 shows that it is not possible to increase the number of foliations by repeating the argument of Example 2.3 with Fermat Foliations. On the other hand, to the best of our knowledge, starting from degree 8 there is no other know convex reduced foliations besides Fermat foliation, see for example [6, Problem 9.1].*

3. Homogeneous Foliations

This section is devoted to prove Theorem B. We begin by a extension of [4, Lemma 3.1] for product of non-saturated homogeneous foliations.

Lemma 3.1. *Let $\mathcal{W} = l_1 \boxtimes \cdots \boxtimes l_k \boxtimes \mathcal{H}_1 \boxtimes \cdots \boxtimes \mathcal{H}_n$ be the product of homogeneous foliations and lines passing through the origin. If the curvature of $\mathrm{Leg}\,\mathcal{W}$ is holomorphic on $\check{\mathbb{P}}^2 \setminus \check{O}$, then $\mathrm{Leg}\,\mathcal{W}$ is flat.*

Proof.: Fix coordinates (a, b) in $\check{\mathbb{P}}^2$ associated to the line $\{ax + by = 1\}$ in $\mathbb{P}^2$. Set $\omega_i = A_i dx + B_i dy$ the homogeneous form defining $\mathcal{H}_i$ and $l_j = \{a_j x + b_j y = 0\}$.
Thus the web $\mathcal{W}$ is defined by the form $\Omega = F_k \cdot \omega_1 \cdots \omega_n$, with $F_k(x, y) = (a_1 x + b_1 y) \cdot \ldots \cdot (a_k x + b_k y)$. Note that $\mathcal{W}$ is invariant by the homothecies $h_\lambda(x, y) = (\lambda x, \lambda y)$, in fact

$$h_\lambda^*(\Omega) = (\lambda^k F_k) \cdot (\lambda^{d_1+1}\omega_1) \cdots (\lambda^{d_n+1}\omega_n) = \lambda^{d+n+k}\Omega$$

where $d = d_1 + \cdots + d_n$. So we have that $\mathrm{Leg}\,\mathcal{W}$ is invariant by the dual maps $\check{h}_\lambda(a, b) = (\frac{a}{\lambda}, \frac{b}{\lambda})$ and so

$$\check{h}_\lambda^*(K(\mathrm{Leg}\,\mathcal{W})) = K(\mathrm{Leg}\,\mathcal{W}).$$

How in affine coordinates (a, b) the $K(\mathrm{Leg}\,\mathcal{W})$ is a rational 2-form with poles at maximum in $\check{O} = L_\infty$, we can write the curvature as $K(\mathrm{Leg}\,\mathcal{W}) = p(a, b)da \wedge db$ where p is a polynomial. Therefore $\lambda^2 p(a, b) = p(\frac{a}{\lambda}, \frac{b}{\lambda})$ and the lemma follows. ■

Proposition 3.2. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be convex homogeneous foliations on $\mathbb{P}^2$ such that, $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2) = L_\infty \cup l_1 \cup \cdots \cup l_k$ where l_j is an invariant line $\forall j = 1, \cdots, k$. Consider the web $\mathcal{W} = \mathcal{H}_1 \boxtimes \mathcal{H}_2$. Then $\mathrm{Leg}\,\mathcal{W} = \mathrm{Leg}\,\mathcal{H}_1 \boxtimes \mathrm{Leg}\,\mathcal{H}_2$ is flat.*

Proof.: It follows from Proposition 1.2 and [3, lemma 3.2] that

$$\Delta(\mathrm{Leg}\,\mathcal{W}) = \check{\Sigma}^{rad}_{\mathcal{H}_1} \cup \check{\Sigma}^{rad}_{\mathcal{H}_2} \cup \check{O}$$

If $s \in (\Sigma^{rad}_{\mathcal{H}_1} \cup \Sigma^{rad}_{\mathcal{H}_2})$, we do a similar analysis to that done in the Theorem 2.2 and obtain that $K(\mathrm{Leg}\,\mathcal{W})$ is holomorphic over $\check{s}$. Thus $K(\mathrm{Leg}\,\mathcal{W})$ is holomorphic on $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \check{O}$ and by Lemma 3.1 we have that $\mathrm{Leg}\,\mathcal{W}$ is flat. ■

We can prove a similar result with three foliations.

Proposition 3.3. *Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ be convex homogeneous foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees d_1, d_2 and d_3, such that,* $\operatorname{Tang}(\mathcal{H}_i,\mathcal{H}_j) = L_\infty \cup C_{ij}^{inv}$ *where* $\forall i,j = 1,2,3$, C_{ij}^{inv} *is a product of invariant lines. Consider the web* $\mathcal{W} = \mathcal{H}_1 \boxtimes \mathcal{H}_2 \boxtimes \mathcal{H}_3$. *Then* $\operatorname{Leg}\mathcal{W} = \operatorname{Leg}\mathcal{H}_1 \boxtimes \operatorname{Leg}\mathcal{H}_2 \boxtimes \operatorname{Leg}\mathcal{H}_3$ *is flat.*

Proof.: As before the discriminant $\Delta(\operatorname{Leg}\mathcal{W})$ is

$$\Delta(\operatorname{Leg}\mathcal{W}) = \check{\Sigma}^{rad}_{\mathcal{H}_1} \cup \check{\Sigma}^{rad}_{\mathcal{H}_2} \cup \check{\Sigma}^{rad}_{\mathcal{H}_3} \cup \check{O}$$

Take $s \in \Sigma^{rad}_{\mathcal{H}_1} \setminus (\Sigma^{rad}_{\mathcal{H}_2} \cup \Sigma^{rad}_{\mathcal{H}_3})$, we have that s is a radial singularity of order ν_1 of $\mathcal{H}_1$ and given a generic line l (particularly, not invariant) on $\check{s}$, we can write $\operatorname{Leg}\mathcal{H}_1 = \mathcal{W}_{\nu_1} \boxtimes \mathcal{W}_{d_1-\nu_1}$ with $\mathcal{W}_{\nu_1}$ leaving $\check{s}$ invariant and with $\Delta(\mathcal{W}_{\nu_1})$ having minimal multiplicity along $\check{s}$ and the $(d_1-\nu_1)$-web $\mathcal{W}_{d_1-\nu_1}$ is transverse to $\check{s}$. As $s \notin \Sigma^{rad}_{\mathcal{H}_2} \cup \Sigma^{rad}_{\mathcal{H}_3}$ we have that $\operatorname{Tang}(\mathcal{H}_2, l, s)$ and $\operatorname{Tang}(\mathcal{H}_3, l, s)$ are less or equal than 1.

- If $s \notin (\operatorname{Sing}\mathcal{H}_2 \cup \operatorname{Sing}\mathcal{H}_3)$, we can assume $l \cap (\operatorname{Sing}\mathcal{H}_2 \cup \operatorname{Sing}\mathcal{H}_3) = \emptyset$, then $\operatorname{Leg}\mathcal{H}_2$ and $\operatorname{Leg}\mathcal{H}_3$ are completely transversal to $\check{s}$. On the other hand, for example, if one direction of $\mathcal{W}_{d_1-\nu_1}$ and $\operatorname{Leg}\mathcal{H}_2$ coincide, this direction would correspond to a point of $\operatorname{Tang}(\mathcal{H}_1,\mathcal{H}_2) = L_\infty \cup l_1 \cup \cdots \cup l_k$ therefore, we would have $l = l_j$ for some j, which does not occur.Thus, $\operatorname{Leg}\mathcal{H}_2$ and $\operatorname{Leg}\mathcal{H}_3$ are transverse to $\mathcal{W}_{d-\nu_1}$ and in the same way they are mutually transversal, thus in the neighborhood of l we can write $\operatorname{Leg}\mathcal{W} = \mathcal{W}_{\nu_1} \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \operatorname{Leg}\mathcal{H}_2 \boxtimes \operatorname{Leg}\mathcal{H}_3)$ with $\mathcal{W}_{d_1-\nu_1} \boxtimes \operatorname{Leg}\mathcal{H}_2 \boxtimes \operatorname{Leg}\mathcal{H}_3$ smooth and transverse to $\check{s}$. Therefore, by [6, Proposition 2.6], the curvature of $\operatorname{Leg}\mathcal{W}$ is holomorphic along $\check{s}$.
- If $s \in (\operatorname{Sing}\mathcal{H}_2 \setminus \operatorname{Sing}\mathcal{H}_3)$ we have $\operatorname{Tang}(\mathcal{H}_2, l, s) = 1$ and we decompose $\operatorname{Leg}\mathcal{H}_2 = \mathcal{F}_2 \boxtimes \mathcal{W}_{d_2-1}$ where $\mathcal{F}_2$ is a foliation tangent to $\check{s}$ and $\mathcal{W}_{d_2-1}$ is a web transverse to $\check{s}$. By the previous argument, $\mathcal{W}_{d_2-1}$ and $\operatorname{Leg}\mathcal{H}_3$ are transverse to $\mathcal{W}_{d_1-\nu_1}$ and to each other as well. Thus, by [2, Proposition 3.9], the curvature of $\operatorname{Leg}\mathcal{W} = (\mathcal{F}_2 \boxtimes \mathcal{W}_{\nu_1}) \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1} \boxtimes \operatorname{Leg}\mathcal{H}_3)$ is holomorphic along $\check{s}$.
- If $s \in (\operatorname{Sing}\mathcal{H}_2 \cap \operatorname{Sing}\mathcal{H}_3)$ we have $\operatorname{Tang}(\mathcal{H}_2, l, s) = \operatorname{Tang}(\mathcal{H}_3, l, s) = 1$ and we decompose $\operatorname{Leg}\mathcal{H}_2 = \mathcal{F}_2 \boxtimes \mathcal{W}_{d_2-1}$ and $\operatorname{Leg}\mathcal{H}_3 = \mathcal{F}_3 \boxtimes \mathcal{W}_{d_3-1}$ as before. By the previous argument, $\mathcal{W}_{d_2-1}$ and $\mathcal{W}_{d_3-1}$ are transverse to $\mathcal{W}_{d_1-\nu_1}$. Thus, by [2, Remark 3.10], the curvature of $\operatorname{Leg}\mathcal{W} = (\mathcal{F}_2 \boxtimes \mathcal{F}_3 \boxtimes \mathcal{W}_{\nu_1}) \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1} \boxtimes \mathcal{W}_{d_3-1})$ is holomorphic along $\check{s}$.

Now, take $s \in (\Sigma^{rad}_{\mathcal{H}_1} \cap \Sigma^{rad}_{\mathcal{H}_2}) \setminus \Sigma^{rad}_{\mathcal{H}_3}$, we have that s is a radial singularity of orders ν_1 and ν_2 of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and we have local decompositions as before: $\operatorname{Leg}\mathcal{H}_i = \mathcal{W}_{\nu_i} \boxtimes \mathcal{W}_{d_i-\nu_i}$. We need to analyze two situations:

- If $s \notin \operatorname{Sing}\mathcal{H}_3$ by Lemma 1.3 and [6, Proposition 2.6], we obtain that the curvature of $\operatorname{Leg}\mathcal{W} = (\mathcal{W}_{\nu_1} \boxtimes \mathcal{W}_{\nu_2}) \boxtimes (\operatorname{Leg}\mathcal{H}_3 \boxtimes \mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-\nu_2})$ is holomorphic along $\check{s}$
- if $s \in \operatorname{Sing}\mathcal{H}_3$, we can decompose $\operatorname{Leg}\mathcal{H}_3 = \mathcal{F}_3 \boxtimes \mathcal{W}_{d_3-1}$ with $\mathcal{F}_3$ a foliation tangent to $\check{s}$. Again by Lemma 1.3 and [2, Remark 3.10] the curvature of $\operatorname{Leg}\mathcal{W} = (\mathcal{W}_{\nu_1} \boxtimes \mathcal{W}_{\nu_2} \boxtimes \mathcal{F}_3) \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-\nu_2} \boxtimes \mathcal{W}_{d_3-1}$ is holomorphic along $\check{s}$.

In the case where $\check{s}$ is radial singularity of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ of respective orders ν_1, ν_2, ν_3 we obtain that $\operatorname{Leg}\mathcal{W}$ is holomorphic along $\check{s}$ by lemma 1.3 and [6, Proposition 2.6].

Then, $K(\operatorname{Leg}\mathcal{W})$ is holomorphic on $\check{\mathbb{P}}^2 \setminus \check{O}$ and by Lemma 3.1 we obtain the flatness of $\operatorname{Leg}\mathcal{W}$. ■

Now we can also consider the product with invariant lines.

Proposition 3.4. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be convex homogeneous foliations on $\mathbb{P}^2_{\mathbb{C}}$ of respective degrees d_1 and d_2, such that,* $\operatorname{Tang}(\mathcal{H}_1,\mathcal{H}_2) = L_\infty \cup l_1 \cup \cdots \cup l_k$ *where l_j is a line invariant* $\forall j = 1, \cdots, k$, *and let l be an invariant line. Consider the web* $\mathcal{W} = l \boxtimes \mathcal{H}_1 \boxtimes \mathcal{H}_2$. *Then* $\operatorname{Leg}\mathcal{W} = \operatorname{Leg} l \boxtimes \operatorname{Leg}\mathcal{H}_1 \boxtimes \operatorname{Leg}\mathcal{H}_2$ *is flat.*

Proof.: Firstly, we consider $l = L_\infty$. By [3], we can locally decompose $\operatorname{Leg}(L_\infty \boxtimes \mathcal{H}_1 \boxtimes \mathcal{H}_2)$ as

$$\operatorname{Leg}(L_\infty \boxtimes \mathcal{H}_1 \boxtimes \mathcal{H}_2) = \operatorname{Leg}(L_\infty) \boxtimes \mathcal{W}_{d_1+d_2}$$

where $\mathcal{W}_{d_1+d_2} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_{d_1+d_2}$ and, for each i, $\mathcal{F}_i$ is given by $\check{\omega}_i := \lambda_i(p)dq - qdp$ with $\lambda_i(p) = p - p_i(p)$ and $\{p_i(p)\} = \underline{\mathcal{G}}_{\mathcal{H}}^{-1}(p)$. The radial foliation $\mathrm{Leg}(L_\infty)$ is defined by the vector field $X := q\frac{\partial}{\partial q}$ and this is a transverse symmetry of the web $\mathcal{W}_{d_1+d_2}$. By [2, Theorem 3.4] and Proposition 3.2, we have that $K(\mathrm{Leg}(L_\infty \boxtimes \mathcal{H}_1 \boxtimes \mathcal{H}_2)) = K(\mathrm{Leg}(\mathcal{H}_1 \boxtimes \mathcal{H}_2)) = 0$.

Now, assume l is an invariant line passing through the origin, we can decompose the discriminant

$$\Delta(\mathrm{Leg}\,\mathcal{W}) = \Delta(\mathrm{Leg}\,\mathcal{H}_1)\cup\Delta(\mathrm{Leg}\,\mathcal{H}_2)\cup\mathrm{Tang}(\mathrm{Leg}\,l, \mathrm{Leg}\,\mathcal{H}_1)\cup\mathrm{Tang}(\mathrm{Leg}\,l, \mathrm{Leg}\,\mathcal{H}_2)\cup\mathrm{Tang}(\mathrm{Leg}\,\mathcal{H}_1, \mathrm{Leg}\,\mathcal{H}_2)$$

Since l is invariant by $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathrm{Tang}(\mathrm{Leg}\,l, \mathrm{Leg}\,\mathcal{H}_1) = \mathrm{Tang}(\mathrm{Leg}\,l, \mathrm{Leg}\,\mathcal{H}_2) = \check{O} \cup \check{a}$ with $a = l \cap L_\infty$. On the other hand, since $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2)$ is a union of invariant lines, we have that

$$\Delta(\mathrm{Leg}\,\mathcal{W}) = \check{\Sigma}^{rad}_{\mathcal{H}_1} \cup \check{\Sigma}^{rad}_{\mathcal{H}_2} \cup \check{O} \cup \check{a}.$$

Note that, for $s \in (\Sigma^{rad}_{\mathcal{H}_1} \cup \Sigma^{rad}_{\mathcal{H}_2}) \setminus l$ a similar analysis to that done in Proposition 2.5 leads us the holomorphy of $K(\mathrm{Leg}\,\mathcal{W})$ along $\check{s}$. Take now a generic line t (particularly, t is not invariant by $\mathcal{H}_1$ and $\mathcal{H}_2$) passing through a, we have three situations:

- If $a \notin (\Sigma^{rad}_{\mathcal{H}_1} \cup \Sigma^{rad}_{\mathcal{H}_2})$, since $\mathcal{H}_1$ and $\mathcal{H}_2$ are convex homogeneous, $\mathrm{Tang}(\mathcal{H}_1, t, a) = \mathrm{Tang}(\mathcal{H}_2, t, a) = 1$ thus we can write locally around $\check{t}$: $\mathrm{Leg}\,\mathcal{H}_1 = \mathcal{F}_1 \boxtimes \mathcal{W}_{d_1-1}$ and $\mathrm{Leg}\,\mathcal{H}_2 = \mathcal{F}_2 \boxtimes \mathcal{W}_{d_2-1}$ with $\check{a}$ invariant by $\mathcal{F}_1$ and $\mathcal{F}_2$ and $\mathcal{W}_{d_1-1} \boxtimes \mathcal{W}_{d_2-1}$ smooth and transverse to $\check{a}$. Therefore
$$\mathrm{Leg}\,\mathcal{W} = (\mathrm{Leg}\,l \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2) \boxtimes (\mathcal{W}_{d_1-1} \boxtimes \mathcal{W}_{d_2-1}).$$
By Lemma 1.3 and [2, Proposition 3.9], we conclude that $K(\mathrm{Leg}\,\mathcal{W})$ is holomorphic along $\check{a}$.
- If $a \in \Sigma^{rad}_{\mathcal{H}_1} \setminus \Sigma^{rad}_{\mathcal{H}_2}$, being ν_1 the radiality order of a as singularity of $\mathcal{H}_1$, we can write locally: $\mathrm{Leg}\,\mathcal{H}_1 = \mathcal{W}_{\nu_1} \boxtimes \mathcal{W}_{d_1-\nu_1}$ and $\mathrm{Leg}\,\mathcal{H}_2 = \mathcal{F} \boxtimes \mathcal{W}_{d_2-1}$ as before. Thus
$$\mathrm{Leg}\,\mathcal{W} = (\mathrm{Leg}\,l \boxtimes \mathcal{W}_{\nu_1} \boxtimes \mathcal{F}) \boxtimes (\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1})$$
with $\mathcal{F} \boxtimes \mathcal{W}_{d_1-\nu_1}$ tangent to $\check{a}$ and $\mathcal{W}_{d_1-\nu_1} \boxtimes \mathcal{W}_{d_2-1}$ transverse to $\check{a}$. By [2, Remark 3.10] we get the holomorphy of $K(\mathrm{Leg}\,\mathcal{W})$ along $\check{a}$.
- The case that $a \in (\check{\Sigma}^{rad}_{\mathcal{H}_1} \cap \check{\Sigma}^{rad}_{\mathcal{H}_2})$ is similar to what we did in item i.v. of Proposition 2.5.

Therefore, $K(\mathrm{Leg}\,\mathcal{W})$ is holomorphic on $\check{\mathbb{P}}^2 \setminus \check{O}$ and, $\mathrm{Leg}\,\mathcal{W}$ is flat by Lemma 3.1. ■

To finish the proof of Theorem B we need some general results about curvature.

Proposition 3.5. *Let $\mathcal{W} = \mathcal{W}_1 \boxtimes \cdots \boxtimes \mathcal{W}_n$ be a product of webs. If for each $i, j, l = 1, \cdots, n$ we have:*

i. $K(\mathcal{W}_i) = 0$*;*
ii. $K(\mathcal{W}_i \boxtimes \mathcal{W}_j) = 0$*, $i \neq j$;*
iii. $K(\mathcal{W}_i \boxtimes \mathcal{W}_j \boxtimes \mathcal{W}_l) = 0$ *with i, j, l pairwise distinct.*

Then $K(\mathcal{W}) = 0$.

Proof.: Firstly, we can decompose each web as $\mathcal{W}_i = \mathcal{F}^i_1 \boxtimes \cdots \boxtimes \mathcal{F}^i_{n_i}$ in a neighborhood of a generic point outside the discriminant. By the hypothesis we have:

$$\begin{aligned} 0 = K(\mathcal{W}_i \boxtimes \mathcal{W}_j) &= K(\mathcal{W}_i) + K(\mathcal{W}_j) + \sum_{\substack{1\le k<r\le n_i\\1\le s\le n_j}} K(\mathcal{F}^i_k \boxtimes \mathcal{F}^i_r \boxtimes \mathcal{F}^j_s) + \sum_{\substack{1\le k<r\le n_j\\1\le s\le n_i}} K(\mathcal{F}^j_k \boxtimes \mathcal{F}^j_r \boxtimes \mathcal{F}^i_s) \\ &= \sum_{\substack{1\le k<r\le n_i\\1\le s\le n_j}} K(\mathcal{F}^i_k \boxtimes \mathcal{F}^i_r \boxtimes \mathcal{F}^j_s) + \sum_{\substack{1\le k<r\le n_j\\1\le s\le n_i}} K(\mathcal{F}^j_k \boxtimes \mathcal{F}^j_r \boxtimes \mathcal{F}^i_s)\end{aligned}$$

and

$$
\begin{aligned}
0=K(\mathcal{W}_i\boxtimes\mathcal{W}_j\boxtimes\mathcal{W}_k) &= \sum_{\substack{1\le k<r\le n_i\\ 1\le s\le n_j}} K(\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r\boxtimes\mathcal{F}^j_s)+\sum_{\substack{1\le k<r\le n_j\\ 1\le s\le n_i}} K(\mathcal{F}^j_k\boxtimes\mathcal{F}^j_r\boxtimes\mathcal{F}^i_s)\\
&\quad+\sum_{\substack{1\le k<r\le n_i\\ 1\le s\le n_l}} K(\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r\boxtimes\mathcal{F}^l_s)+\sum_{\substack{1\le k<r\le n_l\\ 1\le s\le n_i}} K(\mathcal{F}^l_k\boxtimes\mathcal{F}^l_r\boxtimes\mathcal{F}^i_s)\\
&\quad+\sum_{\substack{1\le k<r\le n_l\\ 1\le s\le n_j}} K(\mathcal{F}^l_k\boxtimes\mathcal{F}^l_r\boxtimes\mathcal{F}^j_s)+\sum_{\substack{1\le k<r\le n_j\\ 1\le s\le n_l}} K(\mathcal{F}^j_k\boxtimes\mathcal{F}^j_r\boxtimes\mathcal{F}^l_s)\\
&\quad+\sum_{i<j<l}\sum_{k,r,s=1}^{n_i,n_j,n_k} K(\mathcal{F}^i_k\boxtimes\mathcal{F}^j_r\boxtimes\mathcal{F}^l_s)+K(\mathcal{W}_i)+K(\mathcal{W}_j)+K(\mathcal{W}_k)\\
&= \sum_{i<j<l}\sum_{k,r,s=1}^{n_i,n_j,n_k} K(\mathcal{F}^i_k\boxtimes\mathcal{F}^j_r\boxtimes\mathcal{F}^l_s)
\end{aligned}
$$

Putting all these together and expanding the curvature of $\mathcal{W}$ we obtain $K(\mathcal{W})=0$. ■

Corollary 3.6. *Let $\mathcal{W}=\mathcal{F}\boxtimes\mathcal{W}_1\boxtimes\cdots\boxtimes\mathcal{W}_n$ be a product of a foliation and n webs. If for each $i,j,l=1,\cdots,n$ the Proposition 3.5 hypotheses are satisfies and we have:*

i. $K(\mathcal{F}\boxtimes\mathcal{W}_i)=0$;
ii. $K(\mathcal{F}\boxtimes\mathcal{W}_i\boxtimes\mathcal{W}_j)=0$, $i\neq j$.

Then $K(\mathcal{W})=0$.

Proof.: Writing as before $\mathcal{W}_i=\mathcal{F}^i_1\boxtimes\cdots\boxtimes\mathcal{F}^i_{n_i}$, we have that

$$
0=K(\mathcal{F}\boxtimes\mathcal{W}_i) = K(\mathcal{W}_i)+\sum_{1\le k<r\le n_i} K(\mathcal{F}\boxtimes\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r)=\sum_{1\le k<r\le n_i} K(\mathcal{F}\boxtimes\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r)
$$

and

$$
\begin{aligned}
0=K(\mathcal{F}\boxtimes\mathcal{W}_i\boxtimes\mathcal{W}_j) &= K(\mathcal{W}_i)+K(\mathcal{W}_j)+\sum_{\substack{1\le k<r\le n_i\\ 1\le s\le n_j}} K(\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r\boxtimes\mathcal{F}^j_s)\\
&\quad+\sum_{\substack{1\le k<r\le n_j\\ 1\le s\le n_i}} K(\mathcal{F}^j_k\boxtimes\mathcal{F}^j_r\boxtimes\mathcal{F}^i_s)+\sum_{1\le k<r\le n_i} K(\mathcal{F}\boxtimes\mathcal{F}^i_k\boxtimes\mathcal{F}^i_r)\\
&\quad+\sum_{1\le k<r\le n_i} K(\mathcal{F}\boxtimes\mathcal{F}^j_k\boxtimes\mathcal{F}^j_r)+\sum_{\substack{1\le k<\le n_i\\ 1\le r\le n_j}} K(\mathcal{F}\boxtimes\mathcal{F}^i_k\boxtimes\mathcal{F}^j_r)\\
&= \sum_{\substack{1\le k\le n_i\\ 1\le r\le n_j}} K(\mathcal{F}\boxtimes\mathcal{F}^i_k\boxtimes\mathcal{F}^j_r)
\end{aligned}
$$

Now a simple computation shows that $K(\mathcal{W})=0$. ■

Corollary 3.7. *Let $\mathcal{W}=\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\mathcal{W}_1\boxtimes\cdots\boxtimes\mathcal{W}_n$ be a product of two foliations and n webs. With the same hypotheses as the previous lemma for $\mathcal{F}_1$ and $\mathcal{F}_2$, and additionally: if $K(\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\mathcal{W}_i)=0$ for each $i=1,\cdots,n$. Then $K(\mathcal{W})=0$.*

Proof.: With the same notation as before we have

$$\begin{aligned} 0 = K(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{W}_i) &= K(\mathcal{W}_i) + \sum_{1\le k<r\le n_i} K(\mathcal{F}_1 \boxtimes \mathcal{F}_k^i \boxtimes \mathcal{F}_r^i) + \sum_{1\le k<r\le n_i} K(\mathcal{F}_2 \boxtimes \mathcal{F}_k^i \boxtimes \mathcal{F}_r^i) \\ &\quad + \sum_{k=1}^{n_i} K(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_k^i) \\ &= \sum_{k=1}^{n_i} K(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_k^i) \end{aligned}$$

Combining this with the information obtained in the previous lemmas, we conclude that $K(\mathcal{W}) = 0$.

■

Proof. of Theorem B: Set $\mathcal{W}' = \mathrm{Leg}\, l_i \boxtimes \cdots \boxtimes \mathrm{Leg}\, l_k$ and $\mathcal{W}'' = \mathrm{Leg}\,\mathcal{H}_1 \boxtimes \cdots \boxtimes \mathrm{Leg}\,\mathcal{H}_n$. By [1, lemma 2.1], we have that

$$K(\mathcal{W}'\boxtimes\mathcal{W}'') = K(\mathcal{W}') - (k-2)\sum_{i=1}^{k} K(\mathrm{Leg}\, l_i \boxtimes \mathcal{W}'') + \sum_{1\le i<j\le k} K(\mathrm{Leg}\, l_i \boxtimes \mathrm{Leg}\, l_j \boxtimes \mathcal{W}'') + \binom{k-1}{2} K(\mathcal{W}'').$$

By [6, Theorem 4.2], Propositions 3.2, 3.3, and 3.4 we are in condition to apply Corollary 3.6 and conclude that $K(\mathrm{Leg}\, l_i \boxtimes \mathcal{W}'') = 0$. Similarly, we use in addition [1, Theorem 2] to be able to use Corollary 3.7 and obtain $K(\mathrm{Leg}\, l_i \boxtimes \mathrm{Leg}\, l_j \boxtimes \mathcal{W}'') = 0$. Finally, by Proposition 3.5 we have $K(\mathcal{W}'') = 0$ and we finish our proof. ■

Example 3.8. *Let ω_d be the form $\omega_d = y^d dx - x^d dx$. It is shown in [3, Proposition 4.1] that this form defines a convex homogeneous foliation of degree d with invariant lines: L_∞, $x = 0$, $y = 0$, $y = \xi x$, with $\xi^{d-1} = 1$.*

Take now $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ the foliations defined by ω_d, ω_{d+1} and ω_{d+2} respectively, with $d = 2k+1 \ge 3$. We have that the tangencies between each pair of foliations are: $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_2) = x^d y^d (y - x)$, $\mathrm{Tang}(\mathcal{H}_1, \mathcal{H}_3) = x^d y^d (y^2 - x^2)$, $\mathrm{Tang}(\mathcal{H}_2, \mathcal{H}_3) = x^{d+1} y^{d+1} (y - x)$. *Therefore, we are in the hypothesis of Theorem B and for any curve C formed by the reduced product of any combination of the lines L_∞, $x = 0$, $y = 0$ and $y = x$ we conclude that* $\mathrm{Leg}(C \boxtimes \mathcal{H}_1 \boxtimes \mathcal{H}_2 \boxtimes \mathcal{H}_3)$ *is a flat web.*

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