Petz–Rényi relative entropy in QFT from modular theory

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Abstract: We consider the generalization of the Araki–Uhlmann formula for relative entropy to Petz–Rényi relative entropy. We compute this entropy for a free scalar field in the Minkowski wedge between the vacuum and a coherent state, as well as for a thermal state. In contrast to the relative entropy which in these cases only depends on the sympletic form and thus reduces to the classical entropy of a wave packet, the Petz–Rényi relative entropy also depends on the symmetric part of the two-point function and is thus genuinely quantum.

Contents

1 Introduction

Using Tomita–Takesaki modular theory [1, 2], Araki and Uhlmann [3–5] have shown that the relative entropy between two cyclic and separating states Ψ and Φ can be computed as the expectation value of the relative modular Hamiltonian $\ln \Delta_{\Psi|\Phi}$ according to

$$\mathcal{S}(\Psi \| \Phi) = -\langle \Psi | \ln \Delta_{\Psi|\Phi} | \Psi \rangle \,. \tag{1.1}$$

The relative modular Hamiltonian depends on the two states as well as the von Neumann algebra $\mathfrak{A}$ describing the part of the system that one is interested in. In applications, $\mathfrak{A}$ is usually the algebra of fields restricted to a certain spacetime region, in which case the above formula computes the relative entanglement entropy between this region and its complement. Using (1.1), relative entropy has been computed in a number of examples [6–9]. Relative entropy and more generally the modular Hamiltonian have also been useful in deriving various constraints on quantum field theory in diverse settings. The literature on this topic is vast, and we refer the reader to the recent works [10–18] for various aspects and references to earlier work.

Formula (1.1) is a direct generalization of the quantum-mechanical one, where the relative entropy between two density matrices ρ and σ is given by

$$\mathcal{S}(\rho \| \sigma) = \operatorname{tr}(\rho \ln \rho - \rho \ln \sigma) \,. \tag{1.2}$$

Namely, on the tensor product Hilbert space that describes the bipartite quantum system (of the part of interest and its complement) the relative modular Hamiltonian has the very simple form [19]

$$\ln \Delta_{\Psi|\Phi} = \ln\left(\rho_\Phi \otimes \rho_\Psi^{-1} \right) , \tag{1.3}$$

where ρ_Φ and ρ_Ψ are the reduced density matrices obtained from the pure states Φ and Ψ defined on the tensor product space by tracing over either the part of interest or its complement. It is then easy to see that expression (1.1) reduces to (1.2), and in fact this was the original motivation for (1.1).

Operationally, relative entropy measures the difference between the two states in the sense that the probability to wrongly ascribe the state Φ to the system in the state Ψ after N measurements decays asymptotically for large N like $\mathrm{e}^{-N\mathcal{S}(\Psi\|\Phi)}$ [20]. However, there are other measures of the difference between states, such as the Petz–Rényi relative entropy [21–23]

$$\mathcal{S}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\ln\operatorname{tr}\Big(\rho^\alpha\sigma^{1-\alpha}\Big)\,, \tag{1.4}$$

defined for $\alpha \in (0,1)$.[1] In the limit $\alpha \to 1$, one recovers the relative entropy (1.2). Since ρ and σ do not commute, other generalizations are possible which in a limit reduce to the relative entropy, such as sandwiched Rényi relative entropy [24, 25]

$$\tilde{\mathcal{S}}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\ln\operatorname{tr}\Big(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\Big)^\alpha \tag{1.5}$$

defined for $\alpha \in (0,1)\cup(1,\infty)$. Using the formula (1.3), it is easy to see that the Petz–Rényi relative entropy (1.4) can be written as [22, 23]

$$\mathcal{S}_\alpha(\Psi\|\Phi) = \frac{1}{\alpha-1}\ln\langle\Psi|\Delta_{\Psi|\Phi}^{1-\alpha}|\Psi\rangle\,, \tag{1.6}$$

while for the sandwiched Rényi relative entropy (1.5) such a generalization is much harder [26–29]. Let us note that this equation is to be understood from spectral calculus, which means that we actually define

$$\mathcal{S}_\alpha(\Psi\|\Phi) = \frac{1}{\alpha-1}\ln\int_0^\infty \lambda^{1-\alpha}\,\mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\,, \tag{1.7}$$

where E_λ is the spectral resolution of the (positive) operator $\Delta_{\Psi|\Phi}$. In the remainder of this work, we thus concentrate on the Petz–Rényi relative entropy in the form (1.7). We show that it is well-defined in general for $\alpha \in [0,1)$ and that the limit $\alpha \to 1$ exists, that it can be computed using analytic continuation of the modular flow, and that it can be obtained for free fields using the standard subspace approach. Lastly, we consider the concrete example of a free scalar field in the Minkowski wedge.

2 Petz–Rényi relative entropy

2.1 General results

We recall that the relative Tomita operator $S_{\Psi|\Phi}$ is defined as the closure of the map

$$S_{\Psi|\Phi}a|\Psi\rangle = a^\dagger|\Phi\rangle \quad \text{for all} \quad a \in \mathfrak{A} \tag{2.1}$$

for a von Neumann algebra $\mathfrak{A}$ acting on a Hilbert space $\mathcal{H}$ and two states $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$, which we both assume for simplicity to be cyclic and separating for $\mathfrak{A}$ as well as normalized.

[1] If $\rho \leq C\sigma$ for some constant $C > 0$, the definition can be further extended to $\alpha \in (1,\infty)$.

The polar decomposition $S_{\Psi|\Phi} = J_{\Psi|\Phi}\Delta_{\Psi|\Phi}^{\frac{1}{2}}$ then defines the relative modular conjugation $J_{\Psi|\Phi}$ and the relative modular operator $\Delta_{\Psi|\Phi}$. It also follows that $|\Psi\rangle \in \mathcal{D}\left(\Delta_{\Psi|\Phi}^{\frac{1}{2}}\right)$, which we will use in the following without explicitly mentioning it.

Since $\Delta_{\Psi|\Phi}$ is positive and $\lambda^r < 1 + \lambda$ for $r \in [0, 1]$ and $\lambda \geq 0$, we obtain

$$\begin{aligned} 0 &\leq \int_0^\infty \lambda^r \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle < \int_0^\infty (1+\lambda) \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \\ &= \||\Psi\rangle\|^2 + \left\|\Delta_{\Psi|\Phi}^{\frac{1}{2}}|\Psi\rangle\right\|^2 = 1 + \left\|J_{\Psi|\Phi}^{-1}|\Phi\rangle\right\|^2 = 1 + \||\Phi\rangle\|^2 = 2\,. \end{aligned} \tag{2.2}$$

We see that the spectral integral is uniformly bounded (independently of r), and it follows that it is continuous as a function of $r \in [0, 1]$, keeping in mind that the states $|\Psi\rangle$ and $|\Phi\rangle$ are fixed. We would also like to take derivatives with respect to r, which we may exchange with the integration by dominated convergence provided we can find a dominating integrable function. For this, we note that for $k \in \mathbb{N}$ we have

$$\begin{aligned} \left|\lambda^r \ln^k \lambda\right| &\leq \Theta(1-\lambda)\lambda^r \ln^k \lambda^{-1} + \Theta(\lambda - 1)\lambda^r \ln^k \lambda \\ &\leq \Theta(1-\lambda) \ln^k \lambda^{-1} + \Theta(\lambda-1)\frac{\lambda^{r+k\epsilon}}{\epsilon^k \, \mathrm{e}^k} \end{aligned} \tag{2.3}$$

for all $\epsilon > 0$. This is an integrable function for all $r \in [0, r_1]$ with $0 < r_1 < 1$, where we may take $\epsilon = (1 - r_1)/k$. Since r_1 is arbitrary, this shows that arbitrary derivatives with respect to r can be taken inside the integration in the range $r \in [0, 1)$:

$$\partial_r^k \int_0^\infty \lambda^r \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = \int_0^\infty \lambda^r \ln^k \lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\,. \tag{2.4}$$

However, continuity of the derivatives as $r \to 1$, which corresponds to $\alpha \to 0$, is not guaranteed.

It follows that the Petz–Rényi relative entropy $\mathcal{S}_\alpha(\Psi\|\Phi)$ (1.7) is well-defined for $\alpha \in [0, 1)$. In the limit $\alpha \to 1^-$, l'Hôpital's rule shows that

$$\begin{aligned} \lim_{\alpha\to1^-} \mathcal{S}_\alpha(\Psi\|\Phi) &= \lim_{\alpha\to1^-} \partial_\alpha \ln \int_0^\infty \lambda^{1-\alpha} \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \\ &= -\lim_{\alpha\to1^-} \left[\left(\int_0^\infty \lambda^{1-\alpha} \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\right)^{-1} \int_0^\infty \lambda^{1-\alpha} \ln\lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\right] \\ &= -\int_0^\infty \ln\lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = -\langle\Psi|\ln\Delta_{\Psi|\Phi}|\Psi\rangle = \mathcal{S}(\Psi\|\Phi)\,, \end{aligned} \tag{2.5}$$

where we also used that $\int_0^\infty \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = \||\Psi\rangle\|^2 = 1$. That is, in the limit $\alpha \to 1$ the Petz–Rényi relative entropy reduces to the Araki–Uhlmann relative entropy (1.1), which is known to be positive. Note that for $\alpha = 0$ we have

$$\mathcal{S}_0(\Psi\|\Phi) = -\ln \int_0^\infty \lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = -\ln \||\Phi\rangle\|^2 = 0\,. \tag{2.6}$$

To show that Petz–Rényi relative entropy $\mathcal{S}_\alpha(\Psi\|\Phi)$ is non-negative for $\alpha \in [0, 1]$, we can use that it is monotonously increasing with α [27, App. B]. This follows because $\frac{F(r)}{r}$ is monotonically decreasing for $r \in (0,1)$ if $F(0) \geq 0$ and F is a concave function, since then

$$F(a) = F\left(\frac{a}{b} b + \left(1 - \frac{a}{b}\right) 0\right) \geq \frac{a}{b} F(b) + \left(1 - \frac{a}{b}\right) F(0) \geq \frac{a}{b} F(b) \quad \text{for} \quad 0 \leq a \leq b\,. \tag{2.7}$$

So it remains to show that $(1-\alpha)\mathcal{S}_\alpha(\Psi\|\Phi)$ is concave as a function of $r = 1-\alpha$, i.e., that

$$F(r) = -\ln \int_0^\infty \lambda^r \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \tag{2.8}$$

is convex, and that $F(0) \geq 0$. The second property follows easily from

$$F(0) = -\ln \int_0^\infty \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = -\ln \| |\Psi\rangle \|^2 = 0\,. \tag{2.9}$$

For convexity, we use that $F(r)$ is a differentiable function for $r \in (0,1)$ by the above results, we can compute its second derivative which reads

$$\begin{aligned}
&\left[\int_0^\infty \lambda^r \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\right]^2 F''(r) \\
&= \left[\left(\int_0^\infty \lambda^r \ln\lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\right)^2 - \int_0^\infty \lambda^r \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \int_0^\infty \lambda^r \ln^2\lambda \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\right] \\
&= -\frac{1}{2}\int_0^\infty \int_0^\infty \lambda^r \mu^r (\ln\lambda - \ln\mu)^2 \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \, \mathrm{d}\langle\Psi|E_\mu|\Psi\rangle \leq 0\,,
\end{aligned} \tag{2.10}$$

and it follows that $F(r)$ is concave as required.

Lastly, we would like to show that one can compute the Petz–Rényi entropy using an analytic continuation of the modular flow. Consider thus the function

$$f(t) = \langle\Psi|\Delta_{\Psi|\Phi}^{\mathrm{i}t}|\Psi\rangle = \int_0^\infty \lambda^{\mathrm{i}t} \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\,, \tag{2.11}$$

defined initially for $t \in \mathbb{R}$. We want to prove that it admits an analytic extension into the complex strip $\mathscr{I} = \{z = t - \mathrm{i}r \in \mathbb{C} \colon t \in \mathbb{R}, r \in (0,1)\}$ with a continuous extension to the boundaries $r = 0$ and $r = 1$. The proof of the existence of such an extension substantially relies on the the analyticity of the exponential function, and we follow [30, Lemma 9.2.12]. First of all, we notice that the function $f(t)$ can be extended to a bounded function of z on the strip $\mathscr{I}$, thanks to the bound $\left|\lambda^{\mathrm{i}z}\right| = \lambda^r$ and the estimates (2.2) for $r \in [0,1]$. To show that this extension is also analytic, we define a family of complex-valued functions $\{f_n\}$ for $n \in \mathbb{N}$ by

$$f_n(z) = \int_{[n^{-1},n]} \lambda^{\mathrm{i}z} \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle\,. \tag{2.12}$$

For every $n \in \mathbb{N}$ it is easy to see that the functions $f_n(z)$ are analytic on $\mathbb{C}$ thanks to the uniform bound $\left|\partial_z^k f_n(z)\right| \leq n^{|\Im\mathfrak{m}\, z|} \ln^k n$. In addition, for $z \in \mathscr{I}$ it holds that

$$|f(z) - f_n(z)| \leq \int_{[0,n^{-1})\cup(n,\infty)} (1+\lambda) \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle \leq \int_0^\infty (1+\lambda) \, \mathrm{d}\langle\Psi|E_\lambda|\Psi\rangle = 2\,, \tag{2.13}$$

using that $\left|\lambda^{\mathrm{i}z}\right| \leq \lambda^r < 1 + \lambda$. The difference between $f(z)$ and $f_n(z)$ is thus uniformly bounded, and we can apply the dominated convergence theorem to conclude that the sequence $\{f_n\}$ converges uniformly to $f(z)$ for $z \in \mathscr{I}$. It then follows that the extension $f(z)$ is analytic in the strip $\mathscr{I}$ and continuous on the boundaries.

Summarizing the above results, we obtain

Proposition 1. *The Petz–Rényi relative entropy $\mathcal{S}_\alpha(\Psi\|\Phi)$ defined by (1.7) is well-defined for $\alpha \in [0,1]$, and a differentiable function of α for all $\alpha \in [0,1)$. It vanishes for $\alpha = 0$ and increases monotously with α, and is thus non-negative for all $\alpha \in [0,1]$ (but may become infinite). In the limit $\alpha \to 1^-$, it reduces to the Araki–Uhlmann relative entropy (1.1), which is thus an upper bound for the Petz–Rényi relative entropy. It can be computed by analytically continuing the modular flow (2.11).*

In general, relative modular operators and Hamiltonians are difficult to compute explicitly. One exception is if both $|\Psi\rangle$ and $|\Phi\rangle$ are excitations of a common state $|\Omega\rangle$, which is also cyclic and separating for $\mathfrak{A}$. That is, assuming that $|\Psi\rangle = UU'|\Omega\rangle$ and $|\Phi\rangle = VV'|\Omega\rangle$ with invertible $U, V \in \mathfrak{A}$ and invertible $U', V' \in \mathfrak{A}'$, equation (2.1) for the relative Tomita operator can be written as

$$U^\dagger (V')^{-1} S_{\Psi|\Phi} U' (V^{-1})^\dagger V^\dagger a U |\Omega\rangle = (V^\dagger a U)^\dagger |\Omega\rangle = S_\Omega V^\dagger a U |\Omega\rangle \,, \tag{2.14}$$

where we used that U' and V' are in the commutant $\mathfrak{A}'$ and thus commute with a, U, V and their adjoints which are in the algebra $\mathfrak{A}$. Since U and V are invertible, this is equivalent to

$$\left(U^\dagger (V')^{-1} S_{\Psi|\Phi} U' (V^{-1})^\dagger - S_\Omega \right) b |\Omega\rangle = 0 \quad \text{for all} \quad b \in \mathfrak{A} \,, \tag{2.15}$$

and since $|\Omega\rangle$ is separating, we obtain

$$S_{\Psi|\Phi} = (U^{-1})^\dagger V' S_\Omega (U')^{-1} V^\dagger \,. \tag{2.16}$$

The relative modular Hamiltonian and relative modular conjugation can then be obtained from the polar decomposition of $S_{\Psi|\Phi}$. In particular if U and V' are unitary, it follows that

$$\Delta_{\Psi|\Phi} = S^\dagger_{\Psi|\Phi} S_{\Psi|\Phi} = [(U')^{-1} V^\dagger]^\dagger \Delta_\Omega (U')^{-1} V^\dagger \,, \tag{2.17}$$

and if also U' and V are unitary, spectral calculus shows that

$$\Delta^r_{\Psi|\Phi} = V U' \Delta^r_\Omega (U')^\dagger V^\dagger \,. \tag{2.18}$$

For unitary excitations of a common state $|\Omega\rangle$, the Petz–Rényi relative entropy (1.7) therefore reduces to

$$\mathcal{S}_\alpha(UU'\Omega\|VV'\Omega) = \frac{1}{\alpha - 1} \ln \int_0^\infty \lambda^{1-\alpha} \, \mathrm{d}\left\langle V^\dagger U \Omega \middle| E_\lambda \middle| V^\dagger U \Omega \right\rangle , \tag{2.19}$$

where now E_λ is the spectral resolution of the (positive) operator Δ_Ω. In particular, it does not depend on U' and V', the excitations in the commutant. This shows that the Petz–Rényi relative entropy, similarly to the Araki–Uhlmann relative entropy, only depends on the states defined on the algebra $\mathfrak{A}$ and not on their specific vector representatives in the Hilbert space $\mathcal{H}$. Indeed, the two vectors $|\Omega\rangle$ and $U'|\Omega\rangle$ define the same state ω on $\mathfrak{A}$ via $\omega(a) = \langle\Omega|a|\Omega\rangle = \langle\Omega|(U')^\dagger a U'|\Omega\rangle$ for all $a \in \mathfrak{A}$.

2.2 Free bosonic QFTs

Let us now specialize to a free bosonic quantum field theory, and the corresponding second-quantized symmetric Fock space $\mathcal{F}$ constructed over the one-particle Hilbert space $\mathcal{H}$. On $\mathcal{F}$ there exists a representation of the abstract Weyl algebra, which is the completion of the free algebra generated by the identity $\mathbb{1}$ and the Weyl operators $W(f)$, quotiened by the relations

$$[W(f)]^* = W(-f) \,, \quad W(f)W(g) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(f,g)} W(f+g) \,. \tag{2.20}$$

Here, f are elements of a symplectic space (S, σ) and σ is the corresponding symplectic form; in the case that we are interested in, f are vectors in some real-linear subspace $\mathcal{L}$ of the one-particle Hilbert space $\mathcal{H}$ and σ is twice the imaginary part of the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ on $\mathcal{H}$ and non-degenerate. If the Hilbert space is not given, it is possible to construct it (and the symmetric Fock space) starting from the symplectic space together with a quasi-free state ω on the Weyl algebra $\mathcal{A}$ [31, App. A]. This state is defined by the action

$$\omega(W(f)) = \mathrm{e}^{-\frac{1}{2}\mu(f,f)} \,, \tag{2.21}$$

where μ is a real symmetric bilinear form on $S \times S$ that satisfies $\mu(f,f)\mu(h,h) \geq \frac{1}{4}\sigma(f,h)^2$ for all $f, h \in S$ to ensure positivity. From μ and σ, one defines the bilinear form $\omega_2 = \mu + \frac{\mathrm{i}}{2}\sigma$, which becomes the scalar product on the one-particle Hilbert space $\mathcal{H}$. Furthermore, the representation π is regular such that $\pi(W(f)) = \mathrm{e}^{\mathrm{i}\phi(f)}$ with a self-adjoint operator $\phi(f)$ defined on a dense subset of $\mathcal{F}$ [32, Sec. 5.2.3], the state is represented by the cyclic vector $|\Omega\rangle$ through the relation

$$\omega(W(f)) = \langle\Omega|\pi(W(f))|\Omega\rangle = \langle\Omega|\mathrm{e}^{\mathrm{i}\phi(f)}|\Omega\rangle \,, \tag{2.22}$$

and the von Neumann algebra $\mathfrak{A}$ is obtained as the weak closure $\mathfrak{A} = \pi(\mathcal{A})''$. If in addition the state ω is faithful for $\mathcal{A}$, the vector $|\Omega\rangle$ is also separating for $\mathfrak{A}$, and we will assume this in the following. Therefore, the modular objects can be constructed as before. Since in the following we will only work with this representation, for ease of notation we do not write π explicitly.

At this point, we can derive a more explicit expression for the Petz–Rényi relative entropy between the state ω and a state ψ obtained as a coherent excitation of ω. Indeed, the state ψ is implemented by a vector $|\Psi\rangle = W(f)|\Omega\rangle$, and since $W(f)$ is unitary we can employ equation (2.19) with $U' = V' = V = \mathbb{1}$ and $U = W(f)$. Therefore we have

$$\mathcal{S}_\alpha(W(f)\Omega\|\Omega) = \frac{1}{\alpha - 1} \ln \int_0^\infty \lambda^{1-\alpha} \, \mathrm{d}\langle\Omega|W(-f)E_\lambda W(f)|\Omega\rangle \,, \tag{2.23}$$

and as explained before we will compute this by analytically continuing the result for the modular flow. For free theories, it has been shown [33, 34] that the modular objects are second-quantized operators on Fock space, which in particular means that the modular group acts on the Weyl generators according to

$$\Delta_\Omega^{\mathrm{i}t} W(f) \Delta_\Omega^{-\mathrm{i}t} = W(f_t) \quad \text{with} \quad f_t \in \mathcal{L} \;\forall t \in \mathbb{R} \,. \tag{2.24}$$

This action holds even more generally, namely when the modular group acts in a local geometric way [35, 36] (for example, in the cases covered by the Bisognano–Wichmann theorem). We then obtain

$$\begin{aligned}\langle\Omega|W(-f)\Delta_\Omega^{\mathrm{i}t}W(f)|\Omega\rangle &= \langle\Omega|\Delta_\Omega^{\frac{-\mathrm{i}t}{2}}W(-f)\Delta_\Omega^{\frac{\mathrm{i}t}{2}}\Delta_\Omega^{\frac{\mathrm{i}t}{2}}W(f)\Delta_\Omega^{\frac{-\mathrm{i}t}{2}}|\Omega\rangle \\ &= \langle\Omega|W\left(-f_{-\frac{t}{2}}\right)W\left(f_{\frac{t}{2}}\right)|\Omega\rangle = \langle\Omega|\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right)}W\left(-f_{-\frac{t}{2}}+f_{\frac{t}{2}}\right)|\Omega\rangle \\ &= \mathrm{e}^{-\frac{1}{2}\omega_2\left(-f_{-\frac{t}{2}}+f_{\frac{t}{2}},-f_{-\frac{t}{2}}+f_{\frac{t}{2}}\right)}\mathrm{e}^{\frac{\mathrm{i}}{2}\sigma\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right)},\end{aligned} \tag{2.25}$$

where we used that the action of Δ_Ω leaves the state $|\Omega\rangle$ invariant. Performing the analytic continuation and taking the logarithm, we obtain that the Petz–Rényi relative entropy is given by

$$\mathcal{S}_\alpha(W(f)\Omega\|\Omega) = \frac{1}{\alpha-1}\ln M(\mathrm{i}(\alpha-1))\,, \tag{2.26}$$

where M is the function defined for real t by

$$M(t) = \exp\left[-\frac{1}{2}\omega_2\left(f_{\frac{t}{2}}-f_{-\frac{t}{2}},f_{\frac{t}{2}}-f_{-\frac{t}{2}}\right)+\frac{\mathrm{i}}{2}\sigma\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right)\right], \tag{2.27}$$

and the analytic continuation is well-defined by our previous arguments. Using the relation $\mathrm{i}\sigma(f,g) = \omega_2(f,g)-\omega_2(g,f)$ between the antisymmetric part of the two-point function ω_2 and the symplectic form as well as $\omega_2(f_t,f_t) = \omega_2(f,f)$, we can also express $M(t)$ as

$$M(t) = \exp\left[\omega_2\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right)-\omega_2(f,f)\right]. \tag{2.28}$$

Since we have shown that $\mathcal{S}_\alpha(W(f)\Omega\|\Omega)\geq 0$, it follows that the analytic continuation of $M(t)$ to $M(\mathrm{i}(\alpha-1))$ is a real function satisfying $0\leq M(\mathrm{i}(\alpha-1))\leq 1$. Performing the analytic continuation through $\mathfrak{Re}\,z = \mathfrak{Re}\,(t-\mathrm{i}r) = t\geq 0$ and employing the principal branch of the logarithm, we can interchange the analytic continuation with taking the logarithm, and obtain

Proposition 2. *Under the assumption* (2.24) *of geometric modular action, the Petz–Rényi relative entropy between the vacuum* $|\Omega\rangle$ *and the unitary excitation* $W(f)|\Omega\rangle$ *is given by*

$$\mathcal{S}_\alpha(W(f)\Omega\|\Omega) = \frac{1}{\alpha-1}F(\mathrm{i}(\alpha-1))\,, \tag{2.29}$$

where F *is the function defined for real* t *by*

$$\begin{aligned}F(t) &= -\frac{1}{2}\omega_2\left(f_{\frac{t}{2}}-f_{-\frac{t}{2}},f_{\frac{t}{2}}-f_{-\frac{t}{2}}\right)+\frac{\mathrm{i}}{2}\sigma\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right) \\ &= \omega_2\left(f_{-\frac{t}{2}},f_{\frac{t}{2}}\right)-\omega_2(f,f)\,,\end{aligned} \tag{2.30}$$

and the analytic continuation is well-defined.

We see that in the limit $\alpha \to 1$, where the Petz–Rényi relative entropy reduces to the Araki–Uhlmann relative entropy, only the linear expansion of $F(t)$ around $t = 0$ is relevant since $\omega_2\Big(f_{\frac{t}{2}} - f_{-\frac{t}{2}}, f_{\frac{t}{2}} - f_{-\frac{t}{2}}\Big)$ vanishes to first order in t. From (2.30), we see that in this limit only the sympletic form σ makes a contribution. Because the symplectic form already enters the classical analysis, while the symmetric part of the two-point function ω_2 only becomes important in the quantum theory, in contrast to the Araki–Uhlmann relative entropy which can be reduced to the classical entropy of a wave packet [37], the Petz–Rényi relative entropy is a genuinely quantum measure of entropy.

2.3 Relation to standard subspaces

For free theories, both the modular data and relative entropy can be nicely determined using standard subspaces [7, 37, 38]. A standard subspace $\mathcal{L}$ of a complex Hilbert space $\mathcal{H}$ is a closed real-linear subspace such that $\mathcal{L} \cap \mathrm{i}\mathcal{L} = \{0\}$ and $\overline{\mathcal{L} + \mathrm{i}\mathcal{L}} = \mathcal{H}$. For such subspaces, one can define the modular objects, starting from the Tomita operator which is obtained as the closure of the map s acting as $s(h + \mathrm{i}k) = h - \mathrm{i}k$ for $h, k \in \mathcal{L}$. The polar decomposition $s = j\delta^{\frac{1}{2}}$ of the (completion of) s then defined the modular conjugation j and the modular operator δ. For simplicity we only treat the bosonic case and further assume that $\mathcal{L}$ is factorial, that is $\mathcal{L} \cap \mathcal{L}' = 0$ where $\mathcal{L}' = (\mathrm{i}\mathcal{L})^{\perp_\mathbb{R}}$, with $\perp_\mathbb{R}$ denoting the orthogonal complement with respect to the real part of the scalar product on $\mathcal{H}$. It has been shown [33, 34] that this condition is equivalent to the fact that 1 is not in the point spectrum of the modular operator Δ, and it will be fulfilled in our examples. Moreover, the Fock space modular operators J_Ω and Δ_Ω for the Fock vacuum state $|\Omega\rangle$ are obtained as the second-quantized versions of j and δ on the single-particle Hilbert space [33, 34]. This means in particular that the modular group action on the Weyl generators (2.24) on Fock space has an analogue in $\mathcal{H}$, namely for $f \in \mathcal{L}$ it holds that

$$\delta^{\mathrm{i}t} f = f_t \quad \text{with} \quad f_t \in \mathcal{L} \ \forall t \in \mathbb{R}\,, \tag{2.31}$$

and with the same f_t that appears in equation (2.24).

With $f \in \mathcal{L}$, the Petz–Rényi relative entropy between the vacuum $|\Omega\rangle$ and the unitary excitation $W(f)|\Omega\rangle$ is given by Prop. 2. Using that the two-point function ω_2, evaluated on functions $f \in \mathcal{H}$, is equal to the scalar product on the one-particle Hilbert space $\mathcal{H}$, we can rewrite the result (2.30) as

$$F(t) = \Big(f_{-\frac{t}{2}}, f_{\frac{t}{2}}\Big)_\mathcal{H} - (f, f)_\mathcal{H} = \Big(\delta^{-\frac{\mathrm{i}t}{2}} f, \delta^{\frac{\mathrm{i}t}{2}} f\Big)_\mathcal{H} - (f, f)_\mathcal{H}\,, \tag{2.32}$$

where we used the action (2.31) of the single-particle modular operator. The analytic continuation $t \to \mathrm{i}(\alpha - 1)$ leads to

$$\mathcal{S}_\alpha(W(f)\Omega\|\Omega) = \frac{1}{\alpha - 1}\Big[\Big(\delta^{-\frac{1-\alpha}{2}} f, \delta^{\frac{1-\alpha}{2}} f\Big)_\mathcal{H} - (f, f)_\mathcal{H}\Big]\,. \tag{2.33}$$

Employing the spectral resolution e_λ of δ, we thus arrive at

Definition 1. *The Rényi entropy of a vector $f \in \mathcal{L}$ with respect to the standard subspace $\mathcal{L} \subset \mathcal{H}$ is defined as*

$$\mathcal{S}_\alpha(f) = \int_0^\infty \frac{\lambda^{1-\alpha} - 1}{\alpha - 1}\, \mathrm{d}(f, e_\lambda f)_\mathcal{H}\,. \tag{2.34}$$

Completely analogous to Prop. 1, one shows that $\mathcal{S}_\alpha(f)$ is continuous in α, and in the limit $\alpha \to 1$ it follows that

$$\mathcal{S}_\alpha(f) = -\int_0^\infty \ln\lambda \, \mathrm{d}(f, e_\lambda f)_{\mathcal{H}} \,, \tag{2.35}$$

which is nothing else but the entropy of the vector $f \in \mathcal{L}$ as defined in [7]. In analogy to Prop. 1, one further shows that

Proposition 3. *The Rényi entropy of a vector $f \in \mathcal{L}$ is well-defined for $\alpha \in [0,1]$, and a differentiable function of α for all $\alpha \in [0,1)$. It vanishes for $\alpha = 0$ and increases monotously with α, and is thus non-negative for all $\alpha \in [0,1]$ (but may become infinite). In the limit $\alpha \to 1^-$, it reduces to the entropy of the vector $f \in \mathcal{L}$, which is thus an upper bound for the Rényi entropy of this vector.*

3 Free scalar fields in a wedge

We can now compute Petz–Rényi relative entropy for a free scalar massive real field in the Bisognano–Wichmann situation. More specifically, we consider the abstract Weyl algebra $\mathcal{A}^r$ generated by the Weyl operators localized in the right wedge $\mathcal{W}_r = \{x \in \mathbb{M}^{d+1} \colon x^1 > |x^0|\}$. On this algebra we define the state ω^r as the restriction to $\mathcal{A}^r$ of the vacuum state ω defined on the Weyl algebra $\mathcal{A}$ generated by Weyl operators localized in $\mathbb{M}^{d+1}$. Thanks to the Bisognano–Wichmann theorem, we know that in the GNS representation $\pi(\mathcal{A}^r)$ induced by ω^r the state ω^r is implemented by a cyclic and separating vector $|\Omega\rangle$ and the modular operator for the couple $(\pi(\mathcal{A}^r)'', |\Omega\rangle)$ coincides with the boost operator in the positive x^1 direction. In particular, we have the geometric modular action (2.24) with

$$f_t(x) = f(\Lambda_{-t} x) \,, \tag{3.1}$$

where we denoted with Λ_{-t} the boosted coordinates in the x^1 direction:

$$(\Lambda_t x)^0 = \cosh(2\pi t) x^0 + \sinh(2\pi t) x^1 \,, \tag{3.2a}$$
$$(\Lambda_t x)^1 = \sinh(2\pi t) x^0 + \cosh(2\pi t) x^1 \,, \tag{3.2b}$$
$$(\Lambda_t x)^i = x^i \quad \text{for} \quad i = 2, \ldots, d+1 \,. \tag{3.2c}$$

The integral kernel of the two point function of the state ω (and, therefore, of the state ω^r) is given by

$$\omega_2(x,y) = \int \frac{\mathrm{d}^d \mathbf{p}}{2\omega_{\mathbf{p}}} \mathrm{e}^{\mathrm{i}\omega_{\mathbf{p}}(x_0 - y_0) - \mathrm{i}p_1(x_1 - y_1) - \mathrm{i}p_j(x_j - y_j)} \,, \tag{3.3}$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $j = 2, \ldots, d+1$. We also used the mostly plus signature for the metric and the convention

$$f(x) = \int \hat{f}(p) \mathrm{e}^{-\mathrm{i}px} \, \mathrm{d}^{d+1} p \tag{3.4}$$

for the inverse Fourier transform. To obtain an explicit expression for the Petz–Rényi relative entropy, we need to find the analytic extension of the function of real variable

$$t \in \mathbb{R} \to \omega_2\Big(f_{-\frac{t}{2}}, f_{\frac{t}{2}}\Big) \tag{3.5}$$

in the complex strip $\mathscr{I}$. The existence of such an analytic extension has been proven on a general ground in section 2. We start noticing that, since any orthochronous Lorentz transformation has determinant one, it holds that

$$\begin{aligned}\omega_2(f_s, g_{s'}) &= \iint \omega_2(x,y) f(\Lambda_{-s}x) g(\Lambda_{-s'}y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint \omega_2(\Lambda_s x, \Lambda_{s'} y) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y\,.\end{aligned} \tag{3.6}$$

In particular, we have

$$\omega_2\Big(f_{-\frac{t}{2}}, f_{\frac{t}{2}}\Big) = \iint \omega_2\Big(\Lambda_{-\frac{t}{2}} x, \Lambda_{\frac{t}{2}} y\Big) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y\,. \tag{3.7}$$

We can now extend the distributional kernel $\omega_2(x,y)$ to a bounded function of complex variables in the following way. Let $z = a + \mathrm{i}b$ with $a, b \in \mathbb{R}$ and $w = c + \mathrm{i}d$ with $c, d \in \mathbb{R}$ be two complex numbers. By direct computation, we obtain

$$\begin{aligned}\omega_2(\Lambda_z x, \Lambda_w y) = \int \frac{\mathrm{d}^d\mathbf{p}}{2\omega_{\mathbf{p}}} &\exp(\mathrm{i}\omega_{\mathbf{p}}(\cosh(2\pi z)x_0 + \sinh(2\pi z)x_1 - \cosh(2\pi w)y_0 - \sinh(2\pi w)y_1) \\ &- \mathrm{i}p_1(\sinh(2\pi z)x_0 + \cosh(2\pi z)x_1 - \sinh(2\pi w)y_0 - \cosh(2\pi w)y_1))\mathrm{e}^{-\mathrm{i}p_j(x_j - y_j)} \\ = \int \frac{\mathrm{d}^d\mathbf{p}}{2\omega_{\mathbf{p}}} &\exp(\mathrm{i}\cosh(2\pi z)(\omega_{\mathbf{p}}x_0 - p_1x_1)) \exp(\mathrm{i}\sinh(2\pi z)(\omega_{\mathbf{p}}x_1 - p_1x_0)) \\ &\exp(-\mathrm{i}\cosh(2\pi w)(\omega_{\mathbf{p}}y_0 - p_1y_1)) \exp(-\mathrm{i}\sinh(2\pi w)(\omega_{\mathbf{p}}y_1 - p_1y_0))\mathrm{e}^{-\mathrm{i}p_j(x_j - y_j)}\,.\end{aligned} \tag{3.8}$$

Using well-known hyperbolic function identities, we obtain the following formula for the real part of the exponents

$$\begin{aligned}&- \sin(2\pi b)(\omega_{\mathbf{p}}(\sinh(2\pi a)x_0 + \cosh(2\pi a)x_1) - p_1(\sinh(2\pi a)x_1 + \cosh(2\pi a)x_0)) \\ &+ \sin(2\pi d)(\omega_{\mathbf{p}}(\sinh(2\pi c)y_0 + \cosh(2\pi c)y_1) - p_1(\sinh(2\pi c)y_1 + \cosh(2\pi c)y_0)) \\ &= \sin(2\pi b)(\omega_{\mathbf{p}}, p_1)\cdot(v_1, v_2) - \sin(2\pi d)(\omega_{\mathbf{p}}, p_1)\cdot(w_1, w_2)\,,\end{aligned} \tag{3.9}$$

where we defined the bidimensional vectors v, w with components

$$\begin{aligned}v_1 &= \sinh(2\pi a)x_0 + \cosh(2\pi a)x_1 \quad v_2 = \sinh(2\pi a)x_1 + \cosh(2\pi a)x_0 \\ w_1 &= \sinh(2\pi c)y_0 + \cosh(2\pi c)y_1 \quad w_2 = \sinh(2\pi c)y_1 + \cosh(2\pi c)y_0\end{aligned} \tag{3.10}$$

We now notice that, for $x, y \in \mathcal{W}^r$, the vectors v, w are both future pointing and time-like. Indeed, since $\cosh(l) > 0\ \forall l \in \mathbb{R}, x_1, y_1 > 0$ and $x_1 \geq |x_0|$, $y_1 \geq |y_0|$ we have

$$\begin{aligned}&\sinh(2\pi m)z_0 + \cosh(2\pi m)z_1 > \cosh(2\pi m)z_1 - |z_0||\sinh(2\pi m)| > \cosh(2\pi m)z_1 - z_1\cosh(2\pi m) = 0\,, \\ &-(\sinh(2\pi m)z_0 + \cosh(2\pi m)z_1)^2 + (\sinh(2\pi m)z_1 + \cosh(2\pi m)z_0)^2 = -z_1^2 + z_0^2 < 0\,,\end{aligned} \tag{3.11}$$

where the vector (z_0, z_1) denotes either the vector (x_0, x_1) or the vector (y_0, y_1) and $m \in \mathbb{R}$. Since the vector $(\omega_{\mathbf{p}}, p_1)$ is future pointing and time-like, we see that the real part of the exponents has a negative sign if $\sin(2\pi b) > 0$ and $\sin(2\pi d) < 0$. In particular, the presence of fast decaying factors implies that the function of real variable $\omega_2(f_{-\frac{t}{2}}, f_{\frac{t}{2}})$ can be extended to a well defined bounded function of complex variable $\omega_2(f_{-\frac{z}{2}}, f_{\frac{z}{2}})$ for $\Im\mathfrak{m}\,(z) \in \mathscr{I}$. It is actually possible to prove that this extension is analytic in the interior of the strip and continuous on the boundaries. In conclusion, using equation (2.30) we obtain

$$\begin{aligned}&\mathcal{S}_\alpha(W(f)\Omega|\Omega)\\ &= \frac{1}{\alpha-1}\left(-\iint \omega_2(x,y)f(x)f(x)\,\mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y + \iint \omega_2(\Lambda_{-\frac{\mathrm{i}(\alpha-1)}{2}}x, \Lambda_{\frac{\mathrm{i}(\alpha-1)}{2}})f(x)f(y)\,\mathrm{d}^{d+1}x\,\mathrm{d}^{d+1}y\right).\end{aligned} \tag{3.12}$$

In the limit $\alpha \to 1$, this expression reproduces the known result for the relative entropy [6].

4 Outlook

We have studied the Petz–Rényi relative entropy $\mathcal{S}_\alpha(\Psi|\Phi)$ between two states $|\Psi\rangle$ and $|\Phi\rangle$ of a von Neumann algebra $\mathfrak{A}$ using modular theory. We have shown that it is positive, monotonous, and bounded from above by the Araki–Uhlmann relative entropy $\mathcal{S}(\Psi|\Phi)$. We have further shown that for unitary excitations of a common state $|\Omega\rangle$, it can be computed using only the modular Hamiltonian of $|\Omega\rangle$, and that the computation can be done by analytically continuing the modular flow generated by this Hamiltonian. For free bosonic QFTs and under the assumption of geometric modular action, we have then shown that the $\mathcal{S}_\alpha(\Psi|\Phi)$ can be obtained using only the two-point function ω_2 of $|\Omega\rangle$. In contrast to the Araki–Uhlmann relative entropy, apart from the antisymmetric part of the two-point function (the sympletic form), here also the symmetric part contributes. In particular, this entails that the Petz–Rényi relative entropy can not be expressed using just the classical entropy of a wave packet, such that it contains genuine quantum effects.

As an example, we have computed $\mathcal{S}_\alpha(\Psi|\Phi)$ for a free massive scalar field in the Minkowski vacuum state, with $\mathfrak{A}$ being the algebra of Weyl operators in the right wedge. We have shown explicitly how the Petz–Rényi entropy is computed by the analytic continuation of the modular flow, and we have verified that in the limit $\alpha \to 1$ it reduces to the known results for the relative entropy [6]. In the future, it would be very interesting to extend our work also to fermions, where results for the relative entropy are also available [39].

Acknowledgments

This work has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — project no. 396692871 within the Emmy Noether grant CA1850/1-1.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data availability statement

This manuscript has no associated data.

References

[1] M. Tomita, *On canonical forms of von Neumann algebras (in Japanese)*, in *Fifth Functional Analysis Symposium, Tôhoku Univ., Sendai*, (Sendai, Japan), pp. 101–102, Tôhoku Univ., Math. Inst., 1967.

[2] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, vol. 128 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, Heidelberg, Germany (1970), 10.1007/BFb0065832.

[3] H. Araki, *Inequalities in von Neumann algebras*, *Rencontr. phys.-math. de Strasbourg -RCP25* **22** (1975) 1.

[4] H. Araki, *Relative Entropy of States of von Neumann Algebras*, *Publ. RIMS, Kyoto Univ.* **11** (1976) 809.

[5] A. Uhlmann, *Relative Entropy and the Wigner-Yanase-Dyson-Lieb Concavity in an Interpolation Theory*, *Commun. Math. Phys.* **54** (1977) 21.

[6] H. Casini, S. Grillo and D. Pontello, *Relative entropy for coherent states from Araki formula*, *Phys. Rev. D* **99** (2019) 125020 [1903.00109].

[7] R. Longo, *Entropy of coherent excitations*, *Lett. Math. Phys.* **109** (2019) 2587 [1901.02366].

[8] M.B. Fröb, A. Much and K. Papadopoulos, *Relative entropy in de Sitter spacetime is a Noether charge*, *Phys. Rev. D* **108** (2023) 105004 [2310.12185].

[9] E. D'Angelo, M.B. Fröb, S. Galanda, P. Meda, A. Much and K. Papadopoulos, *Entropy-Area Law and Temperature of de Sitter Horizons from Modular Theory*, *PTEP* **2024** (2024) 021A01 [2311.13990].

[10] H. Casini, E. Teste and G. Torroba, *Modular Hamiltonians on the null plane and the Markov property of the vacuum state*, *J. Phys. A* **50** (2017) 364001 [1703.10656].

[11] N. Dowling, S. Floerchinger and T. Haas, *Second law of thermodynamics for relativistic fluids formulated with relative entropy*, *Phys. Rev. D* **102** (2020) 105002 [2008.02706].

[12] V. Morinelli, Y. Tanimoto and B. Wegener, *Modular Operator for Null Plane Algebras in Free Fields*, *Commun. Math. Phys.* **395** (2022) 331 [2107.00039].

[13] H. Casini and M. Huerta, *Lectures on entanglement in quantum field theory*, *PoS* **TASI2021** (2023) 002 [2201.13310].

[14] H. Casini, I. Salazar Landea and G. Torroba, *Entropic g Theorem in General Spacetime Dimensions*, *Phys. Rev. Lett.* **130** (2023) 111603 [2212.10575].

[15] M. Schröfl and S. Floerchinger, *Relative Entropy and Mutual Information in Gaussian Statistical Field Theory*, 2307.15548.

[16] D. Katsinis and G. Pastras, *Entanglement in (1+1)-dimensional free scalar field theory: Tiptoeing between continuum and discrete formulations*, *Phys. Rev. D* **110** (2024) 085015 [2406.11031].

[17] N. Abate and G. Torroba, *Quantum information and the C-theorem in de Sitter*, 2409.18186.

[18] R. Longo and V. Morinelli, *An entropy bound due to symmetries*, 2401.02345.

[19] E. Witten, *APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*, *Rev. Mod. Phys.* **90** (2018) 045003 [1803.04993].

[20] F. Hiai and D. Petz, *The proper formula for relative entropy and its asymptotics in quantum probability*, Commun. Math. Phys. **143** (1991) 99.

[21] A. Rényi, *On measures of entropy and information*, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, vol. 4, (Berkeley, California, U.S.A.), pp. 547–562, University of California Press, 1961.

[22] D. Petz, *Quasi-entropies for States of a von Neumann Algebra*, Publ. Res. Inst. Math. Sci. **21** (1985) 787–800.

[23] D. Petz, *Quasi-entropies for finite quantum systems*, Rept. Math. Phys. **23** (1986) 57.

[24] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr and M. Tomamichel, *On quantum Rényi entropies: A new generalization and some properties*, J. Math. Phys. **54** (2013) 122203 [1306.3142].

[25] M.M. Wilde, A. Winter and D. Yang, *Strong Converse for the Classical Capacity of Entanglement-Breaking and Hadamard Channels via a Sandwiched Renyi Relative Entropy*, Commun. Math. Phys. **331** (2014) 593 [1306.1586].

[26] A. Jenčová, *Rényi Relative Entropies and Noncommutative L_p-Spaces*, Ann. H. Poincaré **19** (2018) 2513–2542 [1609.08462].

[27] M. Berta, V.B. Scholz and M. Tomamichel, *Rényi Divergences as Weighted Non-commutative Vector-Valued L_p-Spaces*, Ann. H. Poincaré **19** (2018) 1843 [1608.05317].

[28] F. Hiai, *Quantum f-divergences in von Neumann algebras. II. Maximal f-divergences*, J. Math. Phys. **60** (2019) 012203 [1807.03118].

[29] S. Kato, *On α-z-Rényi divergence in the von Neumann algebra setting*, J. Math. Phys. **65** (2024) 042202 [2311.01748].

[30] R. Kadison and J. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. II, American Mathematical Society, Providence, Rhode Island, U.S.A. (1997).

[31] B.S. Kay and R.M. Wald, *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon*, Phys. Rept. **207** (1991) 49.

[32] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. 2 of *Theoretical and Mathematical Physics*, Springer-Verlag, Berlin, Heidelberg, Germany (1996), 10.1007/978-3-662-03444-6.

[33] F. Figliolini and D. Guido, *The Tomita operator for the free scalar field*, Ann. Inst. H. Poincaré **51** (1989) 419.

[34] F. Figliolini and D. Guido, *On the type of second quantization factors*, J. Oper. Theory **31** (1994) 229.

[35] D. Buchholz and S.J. Summers, *An Algebraic Characterization of Vacuum States in Minkowski Space*, Commun. Math. Phys. **155** (1993) 449.

[36] D. Buchholz, O. Dreyer, M. Florig and S.J. Summers, *Geometric modular action and space-time symmetry groups*, Rev. Math. Phys. **12** (2000) 475 [math-ph/9805026].

[37] F. Ciolli, R. Longo and G. Ruzzi, *The Information in a Wave*, Commun. Math. Phys. **379** (2020) 979 [1906.01707].

[38] H. Bostelmann, D. Cadamuro and S. Del Vecchio, *Relative Entropy of Coherent States on General CCR Algebras*, *Commun. Math. Phys.* **389** (2022) 661 [2012.14401].

[39] S. Galanda, A. Much and R. Verch, *Relative Entropy of Fermion Excitation States on the CAR Algebra*, *Math. Phys. Anal. Geom.* **26** (2023) 21 [2305.02788].