BOUNDARY OF EQUISYMMETRIC LOCI OF RIEMANN SURFACES WITH ABELIAN SYMMETRY

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ABSTRACT. Let \mathcal{M}_g be the moduli space of compact connected Riemann surfaces of genus $g \ge 2$ and let $\widehat{\mathcal{M}}_g$ be its Deligne-Mumford compactification, which is stratified by the topological type of the stable Riemann surfaces. We consider the equisymmetric loci in \mathcal{M}_g corresponding to Riemann surfaces whose automorphism group is abelian and determine the topological type of the maximal dimension strata at their boundary. For the particular cases of the hyperelliptic and the cyclic *p*-gonal actions, we describe all the topological strata at the boundary in terms of trees with a fixed number of edges.

1. INTRODUCTION

Let \mathcal{M}_g be the moduli space of compact connected Riemann surfaces of genus $g \ge 2$ up to biholomorphism or, equivalently, the space of hyperbolic surfaces of genus g up to isometry, or the space of smooth projective curves of genus $g \ge 2$ defined over \mathbb{C} . As a space, it is easier to consider the Teichmüller space \mathcal{T}_g , i.e., the space of marked hyperbolic surfaces of genus g, since it is homoeomorphic to an open ball of \mathbb{C}^{3g-3} . The mapping class group Mod_g acts on \mathcal{T}_g by changing the marking of a surface, and therefore the quotient $\mathcal{T}_g/\operatorname{Mod}_g$ can be identified to \mathcal{M}_{g} . This action is properly discontinuously and thus \mathcal{M}_{g} is an orbifold whose singular points (when g > 2) are those Riemann surfaces which admit non-trivial automorphisms. This subset is called the *branch locus* and is denoted \mathcal{B}_g . It can be stratified in a natural way according to the topological action of the isometry group of the points in \mathcal{B}_{g} , the strata are called *equisymmetric strata* or *loci* (see Section 2.1). This stratification is the object of intense study, both determining all the strata and studying topological properties as the connectivity of \mathcal{B}_{g} . For instance Broughton [6] provides the classification of topological actions in genus 2 and 3 and Kimura [16] that in genus 4. In a series of papers, Costa, Izquierdo and Bartolini (see for instance [9], [2], [3]) study the orbifold structure of moduli space and the connectivity of its branched locus.

On the other hand, Teichmüller space can be augmented by adding marked stable Riemann surfaces ([1], [4], Section 2.3). The resulting space $\widehat{\mathcal{T}_g}$ is not compact, but its quotient

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 $\widehat{\mathcal{M}_g}$ by the mapping class group is compact and coincides with the Deligne-Mumford compactification of moduli space, that is, the set of stable curves, i.e. curves which have only nodes as singularities and with finite group of automorphisms ([12], see [15] for a detailed exposition on this). Intuitively, this compactification is saying that the only way of going to the infinity at moduli space is that the length of a multicurve (a subset of disjoint simple closed curves) is going to zero.

The previous construction provides a stratification to the boundary of moduli space $\partial M_g = \widehat{\mathcal{M}_g} \setminus \mathcal{M}_g$, each strata consisting on a subset of stable Riemann surfaces with the same topological type. We will refer to this stratification as the topological stratification of $\partial \mathcal{M}_g$ and to its strata as *topological strata*. It can be noticed that the topological type of a stable Riemann surface is captured by its associated weighted stable graph.

A natural question to ask is which topological strata of ∂M_g appear at the boundary of a given equisymmetric locus \mathcal{E} . If X_n is a sequence in \mathcal{E} going to the boundary, there is a curve whose length along the sequence is going to 0. Since X_n has non-trivial isometries, there is actually an equivariant multicurve whose length is going to zero. Quotienting by the isometry group, we obtain a curve going to zero along the sequence of quotient orbifolds O_n . Now, this process can be reversed: starting with a curve in the quotient orbifold and making its length going to zero produces an equivariant multicurve in X_n with length going to zero. Decreasing to 0 the length of a multicurve is always possible provided that all the components of the complement of the multicurve are hyperbolic. These multicurves are called *admissible*. Summing up, to find the topological strata in the boundary of an equisymmetric stratum \mathcal{E} reduces to find the preimage of admissible multicurves under a branched covering. In [13], we addressed this question and described a procedure to solve it. Also in that paper and in [14], we applied this method to find all the topological strata at the boundary of the pyramidal equisymmetric locus. The case of the 1-complex dimensional equisymmetric strata for a family uniformized by a Fuchsian group of signature $s = (2, 2, 2, m), m \ge 3$ is studied in [11]. In [10] it is shown that the completion of the 3-gonal equisymmetric space in $\widehat{\mathcal{M}_g}$ is connected for $g \ge 5$. In recent papers ([7], [8]) Broughton, Costa and Izquierdo have studied the topology of the one-dimensional equisymmetric loci, which are punctured Riemann surfaces. Some of these punctures correspond to limiting stable, nodal surfaces in \mathcal{M}_{g} , i.e., the topological strata at the boundary of the equisymmetric locus. An algebraic stack treatment of boundary of loci of curves can be seen in [17].

In this paper we focus on the equisymmetric strata corresponding to abelian actions. For the hyperelliptic and the *p*-gonal actions, we provide easy descriptions of the topological strata at the boundary of the corresponding equisymmetric loci in terms of trees with a bounded number of edges. For any abelian action, we determine the topological strata of maximal dimension at the boundary of the corresponding equisymmetric locus.

The structure of this paper is the following: In Section 3 we recall the main result in [13]: given a topological action (G, O, Φ) on a surface S and an admissible multicurve Σ in O = S/G, the dual graph of its preimage $p^{-1}(\Sigma)$ is determined. A first important

simplification of this result occurs when the group G is abelian, since we can work with homology instead homotopy. This will be seen in Section 4.2. A second simplification occurs when the orbifold O is topologically a sphere, this case is studied in Section 4.1. Both simplifications are present in the p-gonal and hyperelliptic equisymmetric strata. We will study these cases in sections 5 and 6. In particular, we find all the topological strata at the boundary of the equisymmetric loci corresponding to all the cyclic p-gonal actions up to genus 4. Finally, in section 7, we will study the case of G any abelian group and the multicurve Σ consisting on a single curve. This corresponds to finding the maximal dimension topological strata in the boundary of the equisymmetric locus.

2. Preliminaries

2.1. Equisymmetric loci of moduli space. We recall definitions and fix notation. For a general reference in this subject see [5]. Consider an action of a finite group G on a closed surface S, that is, a monomorphism $\iota: G \to \text{Hom}^+(S)$ into the group of preserving orientation homeomorphisms of S. It has an associated regular branched covering $p: S \to O$ over the set of orbits O, and an associated epimorphism $\Phi: \pi_1(O, *) \to G$ that assigns to a loop α in O the deck transformation that sends one end of a lift of α to the other one. The space O is a closed orbifold whose singular locus consists on a finite number of cone points.

The Euler characteristics of *S* and *O* are related by the Riemann-Hurwitz formula $\chi(S) = |G|\chi(O)$. We recall that Euler characteristic of an orbifold of signature $(g, c; m_1, ..., m_r)$ is $\chi(O) = 2 - 2g - c - \sum_{i=1}^r (1 - \frac{1}{m_i})$, where *g* is the genus of *O*, *c* the number of its boundary components and $m_1, ..., m_r$ are the orders of its cone points. We remark that in the above situation, that is, if *O* is the quotient of a closed surface by the action of a group, the orbifold *O* does not have boundary components. Nevertheless, along the paper we will need to use orbifolds with boundary and to compute their Euler characteristic.

Any epimorphism $\Phi: \pi_1(O, *) \to G$ whose kernel is the fundamental group of a surface *S* (*surface kernel epimorphism*) determines an action of *G* on *S*. We denote this action by (G, O, Φ) .

Two actions $\iota: G \to \text{Hom}^+(S)$ and $, \iota': G \to \text{Hom}^+(S')$, are *topologically equivalent* if there is an automorphism $\omega \in Aut(G)$ and a homeomorphism $h: S \to S'$ such that $\iota'(\omega(g)) = h \circ \iota(g) \circ h^{-1}$ for all $g \in G$.

The *equisymmetric stratum* or *locus* determined by the topological class of action represented by (G, O, Φ) is the subset $\mathcal{M}_g(G, O, \Phi)$ of points $X \in \mathcal{M}_g$ such that the action of $Aut^+(X)$ is topologically equivalent to the action (G, O, Φ) .

Broughton ([6]) determined all the topological actions in surfaces S of genera g = 2, 3 and Kimura ([16]) determined those of genus g = 4. We will use some of these topological actions to exemplify our results.



FIGURE 1. Admissible multicurve and its dual graph

2.2. Admissible multicurves and weighted graphs. Let O be an orientable 2-orbifold, i.e., one of signature $(g, c; m_1, \ldots, m_r)$, where g is the genus, c is the number of connected components and m_1, \ldots, m_r are the orders of the cone points.

Definition. Let *O* be an orbifold with $\chi(O) < 0$. An *(admissible) multicurve* in *O* is a collection $\Sigma = {\gamma_1, \ldots, \gamma_k}$ of simple closed curves or simple arcs joining pairs of cone points of order 2, all of them disjoint, and such that each component of $O \setminus \Sigma$ has negative Euler characteristic.

We will represent multicurves in orbifolds by weighted graphs, defined next.

Definition. (Weighted graph dual to a multicurve) Let Σ be a multicurve in a closed orbifold O. Its dual graph \mathcal{D}_{Σ} is defined as follows:

- (1) its vertices are the connected components of $O \setminus \Sigma$;
- (2) its edges are the closed curves in Σ ;
- (3) its semiedges are the arcs in Σ ;
- (4) an edge γ ∈ Σ connects two vertices O_{j1}, O_{j2} if and only if γ is in the boundary of O_{j1} and O_{j2}; a semiedge γ' ∈ Σ is connected to the vertex O_j if the arc γ' is in the boundary of O_j.
- (5) if $V = O_j$ is a vertex, its weight is $w(V) = (g(O_j); m_{i_1}, \dots, m_{i_j})$ where $g(O_j)$ is the genus of O_j and m_{i_k} are the orders of the cone points of O_j .

Figure 1 shows a multicurve in an orbifold and its dual graph.

It is clear from the definition that the graph \mathcal{D}_{Σ} determines the topological information of the pair (O, Σ) .

When the orbifold is a surface (there are no cone points), a multicurve is just a finite union of simple closed curves, and its dual graph is a weighted graph with no semiedges and where the weight of each vertex is the genus of the corresponding subsurface.

2.3. Augmented moduli space M_g . Let Σ be an admissible multicurve on a surface S. The graph dual to Σ is called *stable graph*. The space S/Σ where each component of Σ has been collapsed to a point is called a *stable surface*. Each collapsed curve is called a *node*, and the complement of the nodes are called *parts*. A *hyperbolic stable surface* is a stable surface with a hyperbolic structure on each part.

The augmented moduli space of genus g is the space $\widehat{\mathcal{M}}_g$ of hyperbolic stable surfaces of genus g up to isometry. This space provides a compactification of moduli space, once given a topology that intuitively works as follows. Consider a point $X \in \widehat{\mathcal{M}}_g$ with nodes N_1, \ldots, N_r . A sequence $X_n \in \mathcal{M}_g$ converges to $X \in \widehat{\mathcal{M}}_g$ if there is a family of geodesics $\mathcal{F}_n = \{\alpha_1^n, \ldots, \alpha_r^n\}$ in X_n so that:

- (i) the stable surfaces X_n/\mathcal{F}_n are homeomorphic to *X*;
- (ii) the length of each α_i^n , i = 1, ..., r, tends to zero when $n \to \infty$; and
- (iii) away from the curves α_i of X_n and from the nodes of X, the hyperbolic surfaces X_n are close to X when $n \to \infty$.

A way of formalizing this is by first considering marked surfaces and the augmented Teichmüller space. See, for instance, [15] for a detailed exposition on this.

The augmented moduli space is stratified in the following way. For each stable graph \mathcal{G} , we consider the stratum $\mathcal{E}(\mathcal{G})$, defined as the space of stable hyperbolic surfaces X whose associated graph \mathcal{G}_X is isomorphic to \mathcal{G} . This space is homeomorphic to the product of the moduli spaces of the parts of X, which are punctured surfaces. Since there is a finite number of non-isomorphic stable graphs of a fixed genus g, then $\widehat{\mathcal{M}}_g$ decomposes into a finite number of strata, one for each stable graph. This is called the *topological stratification* of $\widehat{\mathcal{M}}_g$.

Certainly, we can consider the closure of an equisymmetric stratum $\mathcal{M}_g(G, O, \Phi)$ in $\widehat{\mathcal{M}_g}$, and its boundary, which is the center of attention of this paper.

3. STATEMENT OF MAIN THEOREM IN [13]

Consider a topological action on a surface *S* given by a (surface kernel) epimorphism $\Phi: \pi_1(O, *) \to G$. Let Σ be an admissible multicurve in *O* and let $\tilde{\Sigma} = p^{-1}(\Sigma)$ be its preimage under the branched covering. The main theorem in [13] gives a criterion to determine the dual graph of $\tilde{\Sigma}$ from Σ and the epimorphism Φ . We recall here that theorem.

There are some new concepts and notation needed in the theorem. For each suborbifold O_j of $O \setminus \Sigma$, we consider the restriction $\Phi_j : \pi_1(O_j) \to G$. For each $\gamma \in \Sigma$, we consider similarly the restriction $\Phi_\gamma : \pi_1(\gamma) \to G$, and we also consider a dual curve c_γ . We will give details of these concepts after the statement of the theorem. For technical reasons in order to define the dual curves, we give an orientation to the graph \mathcal{D}_{Σ} . Finally, Σ^1 denotes the set of loops or semiedges of \mathcal{D}_{Σ} , Σ^2 denotes the set of remaining edges and Σ_j denotes the set of edges and semiedges adjacent to the vertex O_j . We warn that, in order to slightly simplify the reading, we have adopted here the term "dual curves" which was not explicit in the original statement in [13].

Theorem 3.1. Consider a topological action on a surface given by an epimorphism $\Phi: \pi_1(O, *) \to G$ and let Σ be an admissible multicurve in O. Then the dual graph $\mathcal{D}_{\tilde{\Sigma}}$ of the multicurve $\tilde{\Sigma} = p^{-1}(\Sigma)$ and a map $p_*: \mathcal{D}_{\tilde{\Sigma}} \to \mathcal{D}_{\Sigma}$ are determined as follows.

- (1) The preimages under p_* of a vertex O_j of \mathcal{D}_{Σ} (i.e., a component of $O \setminus \Sigma$) are denoted $V_{j,g \operatorname{Im} \Phi_j}, g \in G$; thus the vertex O_j has $\frac{|G|}{|\operatorname{Im} \Phi_j|}$ preimages.
- (2) The preimages of an edge γ are denoted $E_{\gamma,gIm \Phi_{\gamma}}, g \in G$; thus the edge γ has $\frac{|G|}{|Im \Phi_{\gamma}|}$ preimages.
- (3) (Edges connecting vertices.) If γ goes from O_{j_1} to O_{j_2} , then the edge $E_{\gamma,g \operatorname{Im} \Phi_{\gamma}}$ joins the vertices $V_{j_1,g \operatorname{Im} \Phi_{j_1}}$ and $V_{j_2,g \Phi(c_{\gamma}) \operatorname{Im} \Phi_{j_2}}$.
- (4) (Degrees of vertices.) The degree of the vertex $V_{j,gIm \Phi_i}$ is equal to

$$D_j = |\mathrm{Im}\,\Phi_j| \left(\sum_{\gamma \in \Sigma_j \cap \Sigma^2} \frac{1}{|\mathrm{Im}\,\Phi_\gamma|} + 2 \sum_{\gamma \in \Sigma_j \cap \Sigma^1} \frac{1}{|\mathrm{Im}\,\Phi_\gamma|} \right).$$

(5) (Weights of vertices.) The weight w^j of the vertex $V_{j,gIm\Phi_i}$ is equal to

$$w^{j} = 1 - \frac{1}{2} \left(|\operatorname{Im} \Phi_{j}| \chi(O_{j}) + D_{j} \right)$$

Remarks. (a) The theorem gives an action of G on $\mathcal{D}_{\tilde{\Sigma}}$ whose quotient map is p_* . Indeed, we can define

$$g(V_{j,h\operatorname{Im}\Phi_{j}}) = V_{j,gh\operatorname{Im}\Phi_{j}}, \ g(E_{\gamma,h\operatorname{Im}\Phi_{j}}) = V_{\gamma,gh\operatorname{Im}\Phi_{j}}$$

Notice that if $E_{\gamma,h\text{Im}\,\Phi_j}$ joins $V_{j_1,h\text{Im}\,\Phi_{j_1}}$ and $V_{j_2,h\Phi(c_\gamma)\text{Im}\,\Phi_{j_2}}$, then $g(E_{\gamma,h\text{Im}\,\Phi_j}) = E_{\gamma,gh\text{Im}\,\Phi_j}$ joins $V_{j_1,gh\text{Im}\,\Phi_{j_1}} = g(V_{j_1,h\text{Im}\,\Phi_{j_1}})$ and $V_{j_2,g\Phi(c_\gamma)h\text{Im}\,\Phi_{j_2}} = g(V_{j_2,h\Phi(c_\gamma)\text{Im}\,\Phi_{j_2}})$.

(b) The items (4) and (5) say that all the preimages by p_* of a vertex of \mathcal{D}_{Σ} have the same degree and the same weight.

Explanations. In order to define the restrictions Φ_j , Φ_γ we need to carefully choose base points and auxiliary paths, namely:

- (1) choose a base point $*_j$ in each suborbifold O_j (for simplicity, choose $*_1$ equal to the initial base point *) and a base point $*_\gamma$ in each curve $\gamma \in \Sigma$;
- (2) choose simple paths $\beta_{j,\gamma}$ contained in O_j going from $*_j$ to $*_{\gamma}$ for all the edges γ adjacent to O_j . If γ is a loop or semiedge of \mathcal{D}_{Σ} , we choose two paths $\beta^a_{j,\gamma}, \beta^b_{j,\gamma}$ from $*_j$ to $*_{\gamma}$ such that $\beta^a_{j,\gamma}(\beta^b_{j,\gamma})^{-1}$ intersects γ exactly once and, in the case that γ is a semiedge (arc in O_j joining two cone points of order 2), $\beta^a_{j,\gamma}(\beta^b_{j,\gamma})^{-1}$ bounds a disc which contains just one of the endpoints of γ and no other cone point. Moreover we choose all these paths so that they are disjoint except at their endpoints.

We finally choose an spanning tree \mathcal{T} of \mathcal{D}_{Σ} . All these choices allow to consider paths β_j going from * to $*_j$ following the unique path in \mathcal{T} from O_1 to O_j and using the auxiliary paths $\beta_{j,\gamma}$. Now we can define the restriction Φ_j as $\Phi_j(\alpha) = \Phi(\beta_j \alpha \beta_j^{-1})$. Similarly, we can consider paths β_{γ} for each $\gamma \in \Sigma$ and define the restrictions Φ_{γ} to the fundamental group of the curve γ . We remark that if γ is an arc, it is considered as a 1-orbifold with its endpoints cone points of order 2. Thus, its fundamental group is $\mathbb{Z}_2 * \mathbb{Z}_2$ generated by loops around the cone points. See [13] for more details.

Finally, the dual curves are as follows (see Figure 2):



FIGURE 2. Dual curves c_{γ}

- (1) if the oriented edge γ of \mathcal{D}_{Σ} goes from O_{i_1} to O_{i_2} , different vertices, then c_{γ} =
- $\beta_{j_1}\beta_{j_1,\gamma}\beta_{j_2,\gamma}^{-1}\beta_{j_2}^{-1};$ (2) if γ is a loop or a semiedge adjacent to the vertex O_j , then $c_{\gamma} = \beta_j \beta_{j,\gamma}^a (\beta_{j,\gamma}^b)^{-1} \beta_j^{-1}.$ Notice that, if γ is a semiedge, $\Phi(c_{\gamma})$ is one of the generators of Im Φ_{γ} .

Notation. To the weighted dual graph \mathcal{D}_{Σ} of a multicurve Σ we can add the information given by the action: to each vertex O_i we add the label Im Φ_i and to each edge or semiedge γ we add the labels Im Φ_{γ} and $\Phi(c_{\gamma})$. Knowing this graph allows to obtain all the information (1)-(5) of Theorem 3.1. We will call this graph the *extra decorated graph* associated to Σ (with respect to the action Φ).

4. FIRST SIMPLIFICATIONS

Theorem 3.1 admits two important simplifications in the cases where the orbifold O has genus 0 and no cone points of order 2 and in the case where the group G is abelian.

4.1. Genus of O equals zero and no cone points of order 2. Since the genus of O is zero, the dual graph \mathcal{D}_{Σ} of any multicurve Σ on Σ is a tree. Moreover, since there are no cone points of order 2, this tree has no semiedges. As a consequence, all the dual curves c_{γ} are homotopically trivial and $\Phi(c_{\gamma})$ is the identity. Item (3) in Theorem 3.1 thus says that if γ is an edge of \mathcal{D}_{Σ} going from O_{j_1} to O_{j_2} , then for each $g \in G$ the edge $E_{\gamma,gIm\Phi_{\gamma}}$ joins the vertices $V_{j_1,g\mathrm{Im}\Phi_{j_1}}$ and $V_{j_2,g\mathrm{Im}\Phi_{j_2}}$. Then we can interpret $\mathcal{D}_{\tilde{\Sigma}}$ as $(\sqcup_{g\in G}\mathcal{D}_{\tilde{\Sigma}}^g)/\sim$, where $\mathcal{D}_{\tilde{\Sigma}}^g$ is a copy of \mathcal{D}_{Σ} and where two vertices $V_{j,g\mathrm{Im}\Phi_j}$, $V_{j,g'\mathrm{Im}\Phi_j}$ are identified if and only if $g\mathrm{Im}\Phi_j = g'\mathrm{Im}\Phi_j$ and similarly with the edges, by (1) and (2) of that theorem. Thus we have:

Proposition 4.1. Consider a topological action given by an epimorphism $\Phi: \pi_1(O, *) \rightarrow \Phi$ G where the signature of O is equal to $(0; m_1, \ldots, m_k)$, $m_i \ge 3$. Let Σ be an admissible multicurve in \mathcal{O} . Then $\mathcal{D}_{\tilde{\Sigma}} = (\sqcup_{g \in G} \mathcal{D}_{\tilde{\Sigma}}^g) / \sim$, where $\mathcal{D}_{\tilde{\Sigma}}^g$ is a copy of \mathcal{D}_{Σ} , $V_{j,g \operatorname{Im} \Phi_j} \sim V_{j,g' \operatorname{Im} \Phi_j}$ if and only if $g(g')^{-1} \in \operatorname{Im} \Phi_j$ and $E_{\gamma,g \operatorname{Im} \Phi_\gamma} \sim E_{\gamma,g' \operatorname{Im} \Phi_\gamma}$ if and only if $g(g')^{-1} \in \operatorname{Im} \Phi_\gamma$. 4.2. Abelian case reduces to homology. Suppose that *G* is abelian. Then the homomorphism $\Phi: \pi_1(O, *) \to G$ factors through the abelianized group $H_1(O)$ of $\pi_1(O, *)$. By abuse of notation, we will still call Φ to the homomorphism from the homology group. If the signature of *O* is $(\sigma; m_1, \ldots, m_k)$, then $H_1(O)$ is generated by $\alpha_i, \beta_i, \delta_l, 1 \le i \le \sigma, 1 \le l \le k$, where the α_i, β_i are a homology basis of the underline space of *O* and the δ_l are meridians about the cone points. This is a great simplification, since we do not have to take care of the base points, in particular we do not need the choices of any base point, nor the choices of the paths $\beta_{j,\gamma}$. The restrictions Φ_j and Φ_γ are easily computed: it is needed to express the homology generators of O_j or γ in terms of the homology generators of *O*. For example, if O_j has genus zero, Φ_j only depends on the meridians about the cone points contained in O_j .

The homology classes of the dual curves c_{γ} are easily chosen when γ is contained in the boundary of just one suborbifold O_j : if γ is an arc, c_{γ} is a meridian about one of if its endpoints, and if γ is a simple closed curve, c_{γ} is a simple closed curve in the closure of O_j intersecting γ exactly one. If γ is in the boundary of two different suborbifolds, and more precisely, if γ is an oriented edge in \mathcal{D}_{Σ} from O_{j_1} to O_{j_2} , consider the oriented closed path ω in \mathcal{D}_{Σ} formed by the oriented edge γ followed by the path in the spanning tree from O_{j_2} to O_{j_1} . Now, take a simple oriented closed curve c_{γ} in O traveling through the suborbifolds indicated by ω .

To appreciate these simplifications, let us compare with an example of a non-abelian action.

Example. The pyramidal action is a certain action of the dihedral group D_n on a surface S_n with quotient the orbifold O with signature (0; 2, 2, 2, 2, n). In [13] we considered examples of multicurves $\Sigma = \{\gamma\}$ and $\Sigma' = \{\gamma'\}$ where both γ, γ' are arcs joining the same cone points of O although not homotopic. It was shown that the subgroup Im Φ_{γ} has order 2, while Im $\Phi_{\gamma'}$ is the whole group if n is odd, and has order n if n is even. For n > 2 this implies that the number of edges of \mathcal{D}_{Σ} and $\mathcal{D}_{\Sigma'}$ are different, so they are different graphs and provide different topological strata in the boundary.

This phenomenon occurs because the dihedral group is not abelian. Indeed, if we have any abelian action on S_n with quotient an orbifold of signature (0; 2, 2, 2, 2, n), and we consider the multicurves $\Sigma = \{\gamma\}$ and $\Sigma' = \{\gamma'\}$ as before, then $O \setminus \gamma$ and $O \setminus \gamma'$ have the same homology generators, and so do γ and γ' , and we can choose the dual curves $c_{\gamma}, c_{\gamma'}$ to be equal. Thus, the graphs $\mathcal{D}_{\Sigma}, \mathcal{D}_{\Sigma'}$ are equal and provide the same topological stratum in the boundary.

5. Cyclic *p*-gonal locus, $p \ge 3$

Definition. A cyclic *p*-gonal action, for *p* a prime number, is an action of \mathbb{Z}_p on a surface S_g of genus *g* whose quotient orbifold *O* has signature $(0; p, \dots^M, p)$. Thus, when $p \ge 3$, these actions satisfy both simplifications explained in Section 4, so we will study this case in this section.

The homology group of O with the above signature is generated by oriented loops x_1, \ldots, x_M around the cone points q_1, \ldots, q_M of O. We take all these loops with the same orientation so that its product $x_1 \ldots x_M$ is trivial. If a is the generator of \mathbb{Z}_p , the epimorphism $\Phi: H_1(O) \to G$ with $\Phi(x_i) = a^{m_i}$ is denoted briefly by its generating vector $(a^{m_1}, \ldots, a^{m_M})$. We say that m_i is the exponent of the cone point q_i .

The Riemann-Hurwitz formula and the Riemann existence theorem for topological actions permit obtain all the *p*-gonal actions, as follows.

Proposition 5.1. The epimorphism $\Phi: H_1(O) \to \mathbb{Z}_p$ with generating vector $(a^{m_1}, \ldots, a^{m_M})$ determines a topological action on a surface S with quotient an orbifold with signature $(0; p, \ldots^M, p)$ if and only if $m_i \in \{1, \ldots, p-1\}$ for all i and $\sum_{i=1}^M m_i$ is multiple of p. The genus of the surface S is and $g = \frac{(p-1)(M-2)}{2}$.

Remark. The above proposition does not say anything about the topological equivalence of the actions. See [6] and [16] to find all the topological inequivalent p-gonal actions on a surface of genus 2, 3 and 4. For instance, in genus 4 there are two inequivalent 3-gonal actions and three inequivalent 5-gonal actions.

Let Σ be a multicurve in O. Since \mathbb{Z}_p has only the two trivial subgroups, items (1) and (2) of Theorem 3.1 say that the preimage of any vertex of \mathcal{D}_{Σ} is either 1 vertex or p vertices, and the same for any edge of \mathcal{D}_{Σ} . We will say that a vertex O_j (resp. an edge γ) of \mathcal{D}_{Σ} is *multiple of* p if its preimage is a set of p vertices (resp. edges) of \mathcal{D}_{Σ} or, equivalently, if Im $\Phi_j = \{0\}$ (resp. Im $\Phi_{\gamma} = \{0\}$).

It is easy to see whether a vertex or an edge is multiple of p, this is done in the following lemma.

Lemma 5.1. Let $\Phi: H_1(O) \to \mathbb{Z}_p$ be a cyclic *p*-gonal action with generating vector $(a^{m_1}, \ldots, a^{m_M})$. Let $\Sigma \subset O$ be an admissible multicurve. Then

- (i) An edge γ of \mathcal{D}_{Σ} is multiple of p if and only if the sum of the exponents of the cone points contained in a disc bounded by γ is multiple of p.
- (ii) A vertex O_j of \mathcal{D}_{Σ} is multiple of p if and only if the suborbifold O_j does not contain any cone point and all the edges γ incident to the vertex O_j are multiple of p. In other words, if $c_j + n_j = 0$, where c_j is the number of cone points contained in O_j and n_j is the number of edges not multiple of p that are incident to O_j .
- (iii) If O_j is a vertex not multiple of p then $c_j + n_j \ge 2$.

Proof. (i) According to Theorem 3.1(2), the edge γ has $|G/\operatorname{Im} \Phi_{\gamma}|$ preimages. Now, Φ_{γ} is the restriction of Φ to the homology group of γ (since $G = \mathbb{Z}_p$ is abelian). Suppose the curve γ bounds a disc with containing the cone points q_{i_1}, \ldots, q_{i_k} . Since the loops x_i have been chosen with the same orientation, the generator of the homology of γ is $x_{i_1} \ldots x_{i_k}$. Therefore, $\operatorname{Im} \Phi_{\gamma}$ is generated by $a^{m_{i_1}+\dots+m_{i_k}}$. If $m_{i_1}+\dots+m_{i_k}$ is multiple of p then $\operatorname{Im} \Phi_{\gamma}$ is trivial, and then γ lifts to p edges. Otherwise, $\operatorname{Im} \Phi_{\gamma}$ is the total G and γ lifts to just one edge.

(ii) According to Theorem 3.1(1) and in a similar way as before, we have to determine when the subgroup Im Φ_i is trivial or the total group. Because O_i has genus 0, the subgroup

 $H_1(O_i)$ is generated by loops around the cone-points contained in O_i and by the boundary components of O_j , which correspond to the edges of \mathcal{D}_{Σ} incident to the vertex O_j . Therefore, Im Φ_i is trivial if and only if the condition written in the statement holds.

(iii) By the previous item, $n_i + c_i \neq 0$. Suppose that $n_i = 1$. By item (i) and since $n_i = 1$, the sum of exponents of the cone points outside O_i is not multiple of p. Since the sum of all the exponents is multiple of p (Proposition 5.1), there must be some cone point in O_i , thus $c_i \ge 1$. Finally, $n_i = 0$, $c_i = 1$ also contradicts Proposition 5.1 since no exponent is multiple of *p*.

We can now completely describe the graph $\mathcal{D}_{\tilde{\Sigma}}$ for any multicurve Σ in O, therefore the strata in the boundary of the *p*-gonal equisymmetric stratum.

Theorem 5.1. Let $\Phi: H_1(O, *) \to \mathbb{Z}_p$ be a cyclic *p*-gonal action. Let $\Sigma \subset O$ be an admissible multicurve and let \mathcal{D}_{Σ} its dual graph. Then the dual graph $\mathcal{D}_{\tilde{\Sigma}}$ of $\tilde{\Sigma}$ is obtained as $(\sqcup_{t=1}^{p} \mathcal{D}^{t})/\sim$ where \mathcal{D}^{t} are copies of \mathcal{D}_{Σ} and \sim is as follows. We denote V_{i}^{t} the vertex of \mathcal{D}^{t} corresponding to the vertex V_i of \mathcal{D}_{Σ} , and similarly with the edges. Then:

- V^t_j ~ V^s_j if and only if V_j is not multiple of p.
 Similarly, E^t_j ~ E^s_j if and only if E_j is not multiple of p.
- (3) If a vertex $\dot{V} = \dot{O}_i$ is multiple of p, then the degree D_i of any of its preimages is equal to the degree of V and its weight w^j is equal to zero.
- (4) If a vertex $V = O_i$ is not multiple of p then the degree of any of its preimages is equal to $D_i = p m_i + n_i$, where m_i is the number of edges multiple of p that are incident to O_i and n_i is the number of edges not multiple of p that are incident to O_i . Its weight is $w^j = \frac{p-1}{2}(n_j + c_j - 2)$, where c_j is the number of cone points contained in O_j .

We remark that the weight w^{j} obtained in (iv) is non-negative by Lemma 5.1(iii).

Proof. Since O has genus 0, the dual graph $\mathcal{D}_{\tilde{\Sigma}}$ is quotient of the union of p copies of \mathcal{D}_{Σ} , by Proposition 4.1. The precise identifications given by \sim come from that proposition and from Lemma 5.1. Thus we have (1) and (2).

(3) If $V = O_j$ is multiple of p, then O_j does not contain any cone point and all the edges incident to V are multiple of p, by Lemma 5.1. Thus, $|\text{Im} \Phi_i| = 1$ and $|\text{Im} \Phi_{\gamma}| = 1$ for all γ in the boundary of O_j . Then Theorem 3.1(4) gives that D_j is equal to the degree of O_j , and item (5) of that theorem says that

$$w^{j} = 1 - \frac{1}{2}(\chi(O_{j}) + \deg(O_{j})) = 0$$

because O_i has genus 0 and no cone points, so $\chi(O_i) + \deg(O_i) = 2$.

(4) If O_i is not multiple of p, then $|\text{Im} \Phi_i| = p$. Denote by \mathcal{M}_i the set of (multiple of p)edges adjacent to O_i and by N_i the set of (non-multiple of p)-edges adjacent to O_i . Then,



FIGURE 3. Cyclic 3-gonal action in genus 2. The red numbers, both in O and in \mathcal{D}_{Σ} , indicate exponents of the generating vector. The subgroups Im Φ_j and Im Φ_{γ} are also written at \mathcal{D}_{Σ} .

by Theorem 3.1(4), we have

$$D_j = |\mathrm{Im}\,\Phi_j| \left(\sum_{\gamma \in \Sigma_j \cap \Sigma^2} \frac{1}{|\mathrm{Im}\,\Phi_\gamma|} \right) = p(\sum_{\gamma \in \mathcal{M}_j} \frac{1}{1} + \sum_{\gamma \in \mathcal{N}_j} \frac{1}{p}) = p|\mathcal{M}_j| + |\mathcal{N}_j| = p \, m_j + n_j$$

For the weight, since O_j has genus 0, c_j cone points of order p and degree $m_j + n_j$, its Euler characteristic is $\chi(O_j) = 2 - n_j - m_j - c_j \left(1 - \frac{1}{p}\right)$. Thus, by Theorem 3.1(5), we have

$$w^{j} = 1 - \frac{1}{2} \left(p \left(2 - n_{j} - m_{j} - c_{j} \frac{p-1}{p} \right) + p m_{j} + n_{j} \right) = \frac{p-1}{2} (n_{j} + c_{j} - 2)$$

5.1. **Examples.** From the lists of topological actions in genus up to 4, we extract the *non-rigid* cyclic *p*-gonal actions, that is, those cyclic *p*-gonal actions whose equisymmetric strata is not a single point. For each of them, we find all the topological strata at the boundary.

5.1.1. Non-rigid p-gonal actions in genus g = 2. According to [6], on a surface of genus 2 there is one cyclic 3-gonal action and one cyclic 5-gonal action, which is rigid, and no other *p*-gonal actions. The 3-gonal action is given by the epimorphism $\Phi: H_1(O) \to \mathbb{Z}_3$ such that *O* has signature (0; 3, 3, 3, 3) and Φ has generating vector (a, a, a^2, a^2). An admissible multicurve on an orbifold of genus 0 and with 4 cone points consists just on one curve separating two cone points from the other two. Taking into account the exponents of generating vector, we have two cases for Σ , shown in figure 3. All the suborbifolds involved contain cone points, so none of them is multiple of 3. Thus each vertex of \mathcal{D}_{Σ} lifts to a vertex in \mathcal{D}_{Σ} . In the first case, the sum of exponents in a disc bounded by γ is not multiple of 3, thus γ is not multiple of 3 so it lifts to a single edge in \mathcal{D}_{Σ} . In the second case γ is multiple of 3 and so lifts to three edges.



FIGURE 4. Cyclic 3-gonal action in genus 3. The boundary of the equisymmetric locus intersects 4 topological strata.

5.1.2. Non-rigid p-gonal actions in genus g = 3. There is only one non-rigid p-gonal action, for p = 3, signature of O equal to $(0; 3^5)$ and with generating vector (a, a, a, a, a^2) . Now a multicurve may have 1 or 2 components. Figure 4 shows the four possible situations. We conclude that the boundary of the 3-gonal equisymmetric locus in \mathcal{M}_3 intersects 4 different topological strata.

5.1.3. Non-rigid p-gonal actions in genus g = 4. Using Kimura notation, there are the following non-rigid p-gonal actions in genus g = 4.

• $\Phi_7: H_1(0; 3^6) \to \mathbb{Z}_3, (a, a, a, a, a, a)$ • $\Phi_8: H_1(0; 3^6) \to \mathbb{Z}_3, (a, a, a, a^2, a^2, a^2)$ • $\Phi_{12}: H_1(0; 5, 5, 5, 5) \to \mathbb{Z}_5, (a, a, a, a^2)$ • $\Phi_{13}: H_1(0; 5, 5, 5, 5) \to \mathbb{Z}_5, (a, a, a^4, a^4)$ • $\Phi_{14}: H_1(0; 5, 5, 5, 5) \to \mathbb{Z}_5, (a, a^2, a^3, a^4)$

Table 1 shows all possible topological strata at the boundary of the equisymmetric locus determined by the action Φ_7 . Tables 2 and 3 do the same for the equisymmetric locus determined by Φ_8 . In both actions the orbifold *O* is a sphere with 6 cone points. There are four possible admissible multicurves Σ , two of them pants decompositions (maximal number of components). Pants decompositions give rise to minimal strata in the boundary of the equisymmetric space, in the sense that their closure do not contain any other strata. Table 2 shows all the non-minimal topological strata and Table 3 shows the minimal strata. Finally, Table 4 shows the same for the actions Φ_{12} , Φ_{13} and Φ_{14} .



TABLE 1. Action Φ_7 on genus g = 4: topological strata at the boundary of the corresponding equisymmetric locus. The group acting is \mathbb{Z}_3 and the orbifold O is a sphere with 6 cone points of order 3. The first columns show the trees \mathcal{D}_{Σ} , dual to all the admissible multicurves Σ in O. The left parts of the second columns show the same trees with all possible distribution of the 6 cone points assigned to the vertices; notice that a terminal vertex must have at least two cone points assigned, while a degree 2 vertex must have at least one. Since all the exponents of the generating vector are 1, no numbers are assigned to the vertices of \mathcal{D}_{Σ} . Finally, the right parts show the graphs $\mathcal{D}_{\tilde{\Sigma}}$, dual to the preimages $\tilde{\Sigma}$ of the multicurves Σ . The numbers are the genera of the corresponding vertex.

In all these tables, the graphs \mathcal{D}_{Σ} are shown at the left and the graphs $\mathcal{D}_{\tilde{\Sigma}}$ are shown at the rightmost column. The numbers assigned to the vertices of \mathcal{D}_{Σ} indicate the exponents of the generating vector.

In general, to find all the topological strata at the boundary of the equisymmetric locus $\mathcal{M}(\mathbb{Z}_p, O, \Phi)$ determined by a cyclic *p*-gonal action, one can proceed in the following way.

- (1) Find all the trees \mathcal{T} with at most r 3 edges, where r is the number of cone points of O (first column in Table 2).
- (2) Distribute *r* points among the vertices of the tree \mathcal{T} , taking into account that for each vertex, its degree plus the number of cone points assigned to it must be at least 3 (second column in Table 2).
- (3) Distribute the *r* exponents of the generating vector among the *r* points assigned to the vertices in step (2) (third column in Table 2). After these three steps we have determined the graphs \mathcal{D}_{Σ} dual to all the possible admissible multicurves Σ on *O*.
- (4) Use Theorem 5.1 to find $\mathcal{D}_{\tilde{\Sigma}}$ (fourth column in Table 2).



TABLE 2. Action Φ_8 on genus g = 4, (I). The group acting is \mathbb{Z}_3 and the orbifold O is a sphere with 6 cone points of order 3. The two first columns (of each subtable) are as in Table 1. The third column shows the trees \mathcal{D}_{Σ} with the distribution of the exponents of the generating vector. Finally, the last column shows the corresponding weighted graph $\mathcal{D}_{\tilde{\Sigma}}$.

5.1.4. *Cyclotomic action*. This action is defined for any prime p > 3 as $\Phi: H_1(O) \to \mathbb{Z}_p$ with generating vector $(a, a^2, a^3, \dots, a^{p-1})$. In particular, O_j has p-1 cone points q_i and the



TABLE 3. Action Φ_8 on genus g = 4, (II). Minimal strata at the boundary of the corresponding equisymmetric locus. The last graph is the complete bipartite graph $K_{3,3}$.



TABLE 4. Topological strata at the boundary of the equisymmetric loci determined by the actions Φ_{12} , Φ_{13} and Φ_{14} .



FIGURE 5. Example of graphs \mathcal{D}_{Σ} and $\mathcal{D}_{\tilde{\Sigma}}$ for the cyclotomic action with p = 7.

covering surface has genus $g = \frac{(p-1)(p-3)}{2}$. For p = 5, the cyclotomic action is the action Φ_{14} studied in the previous section.

Consider a multicurve $\Sigma = \{\gamma_1, \ldots, \gamma_r\}$ with $\frac{p-1}{2} \le r \le p-4$ where the curve γ_1 surrounds the cone points q_1, q_{p-1}, γ_2 surrounds q_2, q_{p-2} , and so on until $\gamma_{\frac{p-1}{2}}$ surrounds $q_{\frac{p-1}{2}}, q_{\frac{p+1}{2}}$. Notice that p-4 is the maximal number of components of an admissible multicurve in O. Applying Lemma 5.1 and Theorem 5.1, we see that all the edges of \mathcal{D}_{Σ} are multiple of p and all the vertices are multiple of p except the terminal ones (those with degree 1). The weights of all the vertices in \mathcal{D}_{Σ} are equal to 0. As a consequence, the graph \mathcal{D}_{Σ} is constructed by taking p copies of \mathcal{D}_{Σ} and identifying corresponding terminal vertices. Figure 5 shows an example of \mathcal{D}_{Σ} and its corresponding \mathcal{D}_{Σ} for p = 7.

In this way, when Σ is a pants decomposition (i.e., when r = p - 4), the multicurve $\hat{\Sigma}$ is an example of multicurve in S equivariant under G with the maximum number of connected components.

6. Hyperelliptic locus

The hyperelliptic action is defined in the same way as the cyclic *p*-action but with p = 2. That is, is an involution on a surface S_g , of genus *g* whose quotient orbifold *O* has genus 0, and therefore 2 + 2g cone points of order 2. It is well-known that if a Riemann surface admits a hyperelliptic involution, this is unique, and that a Riemann surface of genus 2 always admits the hyperelliptic involution. Hence in this section $g \ge 3$.

The hyperelliptic equisymmetric stratum is determined by the epimorphism $\Phi: H_1(O) \rightarrow \mathbb{Z}_2$ defined as $\Phi(x_i) = a$ for all *i*, where, as in the *p*-gonal case, x_i is a small loop around the cone point q_i .

Let $\Sigma \subset O$ a multicurve. Because O is a sphere, its dual graph \mathcal{D}_{Σ} is also a tree, as in the *p*-gonal case, but now it can have semiedges, one for each arc of Σ . The weight of each

vertex has the form $(0; 2^k)$, which we will abbreviate as k, i.e., the weight of any vertex will be the number of cone points the corresponding orbifold contains. The definitions of *even* vertex or edge is the analog to "multiple of p" in the p-gonal case.

We obtain a lemma totally analogous to Lemma 5.1 just by changing p by 2. The only new part appears for the semiedges.

Lemma 6.1. Let Φ : $H_1(O) \to \mathbb{Z}_2$ be the hyperelliptic action and let $\Sigma \subset O$ be an admissible *multicurve. Then*

- (i) An edge γ of \mathcal{D}_{Σ} is even if and only if γ bounds a disc in O containing an even number of cone points.
- (ii) A vertex O_j of \mathcal{D}_{Σ} is even if and only if the suborbifold O_j does not contain any cone point and all the edges γ incident to the vertex O_j are even.
- (iii) A semiedge is always odd. A semiedge incident to an odd vertex lifts to a loop. A semiedge incident to an even vertex lifts to an edge joining the two lifts of the vertex.

Proof. (i) If an edge γ bounds a disc containing the cone points $q_{i_1}, \ldots, q_{i_{2k}}$, then γ is homotopic to $x_{i_1} \ldots x_{i_{2k}}$ and thus $\Phi(\gamma) = a^{2k} = 1$. Hence γ is even. The converse is equal.

(ii) Equal as in Lemma 5.1.

(iii) Let γ be a semiedge of \mathcal{D}_{Σ} starting at a vertex O_j , which corresponds to an arc γ in the boundary of the suborbifold O_j joining two cone points q_i, q_j . Then $\pi_1(\gamma)$ is generated by the loops x_i, x_j and the dual curve c_{γ} is equal to one of these loops, so $\Phi(c_{\gamma}) = a$. Thus Im $\Phi_{\gamma} = \mathbb{Z}_2$ and so a semiedge is always odd. By Theorem 3.1(3), the edge $E_{\gamma,g\mathbb{Z}_2}$ joins the vertices $V_{j,g\mathrm{Im}\Phi_j}$ and $V_{j,g\Phi(c_{\gamma})\mathrm{Im}\Phi_j} = V_{j,ga\mathrm{Im}\Phi_j}$. Thus, γ lifts to a loop if O_j is odd and γ lifts to an edge joining the two lifts of O_j if O_j is even.

Finally, to describe the graph $\mathcal{D}_{\tilde{\Sigma}}$ we have the analog result to Theorem 5.1. See Figure 6 for an example of the dual graphs of a multicurve Σ and of its preimage $\tilde{\Sigma}$ under the hyperelliptic action.

Theorem 6.1. Let $\Phi: H_1(O, *) \to \mathbb{Z}_p$ be the hyperelliptic action on genus g. Let $\Sigma \subset O$ be an admissible multicurve and let \mathcal{D}_{Σ} its dual graph. Then the dual graph \mathcal{D}_{Σ} of $\tilde{\Sigma}$ is obtained as $(\mathcal{D}^1 \sqcup \mathcal{D}^2)/\sim$ where $\mathcal{D}^1, \mathcal{D}^2$ are copies of \mathcal{D}_{Σ} and \sim is as follows. We call V_j^t to the vertex of \mathcal{D}^t corresponding to the vertex V_j of \mathcal{D}_{Σ} , and similarly with the edges. Then:

- (1) $V_j^1 \sim V_j^2$ if and only if V_j is odd.
- (2) An edge $E_j^1 \sim E_j^2$ if and only if E_j is odd.
- (3) A semiedge E_{γ}^1 is joined to E_{γ}^2 through their "free end", giving rise in $\mathcal{D}_{\tilde{\Sigma}}$ to a loop or to an edge joining two different vertices as indicated in Lemma 6.1(iii).
- (4) If a vertex $V = O_j$ is even, then the degree D_j of any of its preimages is equal to the degree of V and its weight w^j is equal to zero.
- (5) If a vertex $V = O_j$ is odd then the degree of any of its preimages is equal to $D_j = n_j + 2(m_j + s_j)$, where n_j, m_j, s_j are, respectively the number of odd edges, even edges



FIGURE 6. Dual graphs \mathcal{D}_{Σ} , $\mathcal{D}_{\bar{\Sigma}}$ of a multicurve Σ and of its preimage $\bar{\Sigma}$ under the hyperelliptic action on genus 12. The numbers at the vertices of \mathcal{D}_{Σ} indicate the number of cone points in the corresponding suborbifold (that are not endpoints of any semiedge). Green vertices and edges are odd, hence they lift to one copy each; purple vertices and edges are even, hence lift to two copies each. Semiedges are not colored, they are always odd. Numbers at $\mathcal{D}_{\bar{\Sigma}}$ indicate genera.

and semiedges incident to O_j . Its weight is $w^j = \frac{1}{2}(n_j + c_j - 2)$, where c_j is the number of cone points contained in O_j .

Proof. (1) and (2) are as in Theorem 5.1 and (3) is Lemma 6.1(iii).

(4) Let O_j be an even vertex, so $|\text{Im }\Phi_j| = 1$. By Lemma 6.1(ii), each edge γ incident to O_j is even, so $|\text{Im }\Phi_{\gamma}| = 1$, and by part (iii) of that lemma, any semiedge γ is odd so $2\frac{1}{|\text{Im }\Phi_{\gamma}|} = 1$. Thus, by Theorem 3.1(4), D_j is equal to the degree of O_j .

To compute the weight, by Lemma 6.1(ii), O_j does not contain any cone point, so that $\chi(O_j) = 2 - deg(O_j)$. Then using Theorem 3.1(5), we obtain that $w^j = 0$.

(5) Let $V_j = O_j$ be an odd vertex, so $|\text{Im }\Phi_j| = 2$. By Theorem 3.1, we have that

$$D_j = 2(n_j \frac{1}{2} + m_j \frac{1}{1} + 2s_j \frac{1}{2}) = n_j + 2(m_j + s_j).$$

The weight of any preimage of V_i is equal to

$$w^{j} = 1 - \frac{1}{2} \left(2 \left(2 - n_{j} - m_{j} - s_{j} - c_{j} \frac{1}{2} \right) + n_{j} + 2(m_{j} + s_{j}) \right) = \frac{1}{2} (n_{j} + c_{j} - 2).$$

7. ACTION BY AN ABELIAN GROUP

In this section we consider any action by an abelian group and study the preimage of a multicurve consisting on just one curve $\Sigma = \{\gamma\}$. This will provide the topological strata of maximal dimension in the boundary of the corresponding equisymmetric loci. We study

three cases, depending on whether γ is a separating or non-separating simple closed curve or is an arc.

7.1. Case 1: separating simple closed curve. In this case we will show in Proposition 7.1 that the dual graph of the preimage is a complete bipartite multigraph.

Notation. We denote by $K_{m,n}^d$ the complete bipartite graph of order *d* between a subset of *m* vertices and a subset of *n* vertices, i.e., each vertex of the first subset is joined to each vertex of the second subset with *d* edges.

In the proof of Proposition 7.1 we will need the following result about abelian groups which says that if an abelian group is generated by two subgroups, then any class with respect to the first subgroup intersects any class with the second subgroup in a class with respect the intersection of both subgroups.

Lemma 7.1. Let G be an abelian group. Let H_1, H_2 be subgroups such that $G = H_1H_2$. Then for all $a, b \in G$ there is $c \in G$ such that $aH_1 \cap bH_2 = c(H_1 \cap H_2)$. As a consequence, $|G| = \frac{|H_1||H_2|}{|H_1 \cap H_2|}$.

Proof. Suppose first that b = 1. Since $G = H_1H_2$, $a = a_1a_2$ with $a_i \in H_i$. Let us check that $aH_1 \cap H_2 = a_2(H_1 \cap H_2)$. Indeed, on the one hand, if $h \in H_1 \cap H_2$, we have that $a_2h \in H_2$ and $a_2h = aa_1^{-1}h \in aH_1$. On the other hand take $y \in aH_1 \cap H_2$, i.e., $y = ak_1$ for some $k_1 \in H_1$ and $y = k_2$ for some $k_2 \in H_2$. We want to show that $y \in a_2(H_1 \cap H_2)$ or equivalently that $a_2^{-1}y \in (H_1 \cap H_2)$. But $a_2^{-1}y = a_2^{-1}k_2 \in H_2$, and

$$a_2^{-1}y = a_2^{-1}ak_1 = a_2^{-1}a_1a_2k_1 = a_1k_1 \in H_1,$$

so we are done.

For the general case, applying the previous argument to $a' = ab^{-1}$ we have that $a'H_1 \cap H_2 = a''(H_1 \cap H_2)$ for some $a'' \in G$. Thus $b(a'H_1 \cap H_2) = ba''(H_1 \cap H_2)$, i.e., $aH_1 \cap bH_2 = ba''(H_1 \cap H_2)$.

As a consequence, any class aH_1 is a union of $[G : H_2]$ classes with respect to $H_1 \cap H_2$, and so

$$[G: (H_1 \cap H_2)] = [G: H_1][G: H_2]$$

and we easily obtain the result.

Proposition 7.1. Consider a topological action on a surface described by an epimorphism $\Phi: \pi_1(O) \to G$, with G an abelian group. Let $\gamma \subset O$ be a closed curve separating O in two suborbifolds O_1, O_2 and consider the multicurve $\Sigma = \{\gamma\}$. Then $\mathcal{D}_{\tilde{\Sigma}}$ is isomorphic to the graph $K_{m,n}^d$, where $m = \frac{|G|}{|\operatorname{Im} \Phi_2|}$ and $d = \frac{r}{mn}$, with $r = \frac{|G|}{|\operatorname{Im} \Phi_\gamma|}$.

Proof. Since *G* is abelian, we can work either with fundamental groups or with homology groups.

By Van-Kampen theorem (or Mayer-Vietoris sequence), we have that $H_1(O)$ is generated by the subgroups $H_1(O_1), H_1(O_2)$. Also notice that $H_1(O_1) \cap H_1(O_2) = H_1(\gamma)$. Since Φ is

surjective, these properties pass to G, precisely, G is generated by Im Φ_1 , Im Φ_2 and Im $\Phi_1 \cap$ Im $\Phi_2 \supset$ Im Φ_{γ} .

By Theorem 3.1, the number of preimages of the vertex O_j is $|\frac{G}{\operatorname{Im} \Phi_j}|$, j = 1, 2, and the number of preimages of γ is $|\frac{G}{\operatorname{Im} \Phi_{\gamma}}|$. Since the edge γ in \mathcal{D}_{Σ} joins O_1, O_2 , each preimage of γ joins a preimage of O_1 with a preimage of O_2 . Thus \mathcal{D}_{Σ} is bipartite.

Given two vertices $V_{1,aIm\Phi_1}$, $V_{2,bIm\Phi_2}$, we will show that there is some edge $E_{\gamma,cIm\Phi_\gamma}$ joining them. Indeed, by Theorem 3.1(3), an edge $E_{\gamma,cIm\Phi_\gamma}$ joins $V_{1,aIm\Phi_1}$ and $V_{2,bIm\Phi_2}$ if and only if $c \in aIm\Phi_1$ and $c\Phi(c_\gamma) \in bIm\Phi_2$. Notice that c_γ is trivial (because \mathcal{D}_{Σ} is a tree), so the above is equivalent to

 $c \in a \operatorname{Im} \Phi_1 \cap b \operatorname{Im} \Phi_2 = c' (\operatorname{Im} \Phi_1 \cap \operatorname{Im} \Phi_2),$

for some $c' \in G$, by Lemma 7.1. Since $\operatorname{Im} \Phi_{\gamma}$ is a subgroup of $\operatorname{Im} \Phi_1 \cap \operatorname{Im} \Phi_2$, the class $c'(\operatorname{Im} \Phi_1 \cap \operatorname{Im} \Phi_2)$ is divided into $d' = \frac{|\operatorname{Im} \Phi_1 \cap \operatorname{Im} \Phi_2|}{|\operatorname{Im} \Phi_{\gamma}|}$ classes with respect to $\operatorname{Im} \Phi_{\gamma}$. Thus there are d' edges joining $V_{1,a\operatorname{Im} \Phi_1}, V_{2,b\operatorname{Im} \Phi_2}$. Finally, by Lemma 7.1, we have

$$d' = \frac{|\mathrm{Im}\,\Phi_1 \cap \mathrm{Im}\,\Phi_2|}{|\mathrm{Im}\,\Phi_\gamma|} = \frac{|\mathrm{Im}\,\Phi_1|\,|\mathrm{Im}\,\Phi_2|}{|G||\mathrm{Im}\,\Phi_\gamma|} = \frac{|\mathrm{Im}\,\Phi_1|\,|\mathrm{Im}\,\Phi_2|}{|G|}\frac{|G|}{|\mathrm{Im}\,\Phi_\gamma|} = \frac{r}{mn} = d$$

Remark. In the situation of Proposition 7.1 but with *G* non-abelian, the graph $\mathcal{D}_{\tilde{\Sigma}}$ is also contained in a graph of the form $K_{m,n}^{d'}$, but is not clear to be complete.

7.2. Case II: non-separating simple closed curve. In this case we will show in Proposition 7.2 that if Σ is a multicurve consisting on a non-separating simple closed curve, then the dual graph of its preimage is a multiple cycle C_n^d , that is, a cycle of length *n* and order *d*, i.e., each edge is multiple of order *d*.

Proposition 7.2. Consider a topological action on a surface described by an epimorphism $\Phi: \pi_1(O) \to G$, with G an abelian group. Let $\gamma \subset O$ be a non-separating simple closed curve and consider the multicurve $\Sigma = \{\gamma\}$ and the suborbifold $O_1 = O \setminus \gamma$. Then \mathcal{D}_{Σ} is isomorphic to the graph C_m^d , where $m = \frac{|G|}{|\operatorname{Im} \Phi_1|}$, $d = \frac{|\operatorname{Im} \Phi_1|}{|\operatorname{Im} \Phi_2|}$.

Proof. The graph \mathcal{D}_{Σ} consists on one vertex and one loop. By Theorem 3.1, the graph $\mathcal{D}_{\tilde{\Sigma}}$ has *m* vertices and $\frac{|G|}{|\operatorname{Im} \Phi_{\gamma}|}$ edges. Let $h = \Phi(c_{\gamma})$. Because *G* is finite, some power of *h* is contained in $\operatorname{Im} \Phi_{\gamma}$. Let *r* be the minimum positive number such that $h^r \in \operatorname{Im} \Phi_1$.

The fundamental group $\pi_1(O)$ is generated by $\pi_1(O_1)$ and by c_{γ} , and hence G is generated by Im Φ_1 and by $\langle h \rangle$. Because G is abelian, the above means that any $g \in G$ can be written as $g = h^t h_1$, for some integer t and some $h_1 \in \text{Im } \Phi_1$. In other words,

$$G = \operatorname{IIm} \Phi_1 \cup h\operatorname{Im} \Phi_1 \cup \cdots \cup h^{r-1}\operatorname{Im} \Phi_1.$$

Notice that any two of the above classes are different, since otherwise it would contradict the definition of r. In particular r = m.

Now, the edge $E_{\gamma,1\mathrm{Im}\,\Phi_{\gamma}}$ joins $V_{1,1\mathrm{Im}\,\Phi_{1}}$ with $V_{1,h\mathrm{Im}\,\Phi_{1}}$; the edge $E_{\gamma,h\mathrm{Im}\,\Phi_{\gamma}}$ joins $V_{1,h\mathrm{Im}\,\Phi_{1}}$ with $V_{1,h^2 \operatorname{Im} \Phi_1}$; ... the edge $E_{\gamma,h^{r-1} \operatorname{Im} \Phi_{\gamma}}$ joins $V_{1,h^{r-1} \operatorname{Im} \Phi_1}$ with $V_{1,h^r \operatorname{Im} \Phi_1} = V_{1,1 \operatorname{Im} \Phi_1}$. In this way we obtain a cycle of r different vertices, so all the vertices are in a cycle.

Finally, let us see that the above cycle is multiple of order d. First, notice that if $E_{\gamma,cIm\Phi_{\gamma}}$ joins $V_{1,a \text{Im} \Phi_1}$ with $V_{1,b \text{Im} \Phi_1}$, then $E_{\gamma,cz \text{Im} \Phi_\gamma}$ joins the same vertices, for all $z \in \text{Im} \Phi_1$ (indeed, if $c \in a \operatorname{Im} \Phi_1$ and $ch \in b \operatorname{Im} \Phi_1$, then $cz \in a \operatorname{Im} \Phi_1$ and $czh \in b \operatorname{Im} \Phi_1$, since G is abelian). Thus, each pair of vertices joined by an edge are actually joined by $\frac{|Im \Phi_1|}{|Im \Phi_2|}$ different edges. Since the total number of edges of $\mathcal{D}_{\tilde{\Sigma}}$ is $\frac{|G|}{|\operatorname{Im} \Phi_1|} \frac{|\operatorname{Im} \Phi_1|}{|\operatorname{Im} \Phi_2|} = md$, we have the result.

7.3. Case III: multicurve is an arc. Finally in this section we study the case where the multicurve consists in just one arc.

Proposition 7.3. Consider a topological action on a surface described by an epimorphism $\Phi: \pi_1(O) \to G$, with G an abelian group. Let $\gamma \subset O$ be an arc and consider the multicurve $\Sigma = \{\gamma\}$. Then $\mathcal{D}_{\tilde{\Sigma}}$ consists on

- (i) either a single vertex with $\frac{|G|}{2}$ or $\frac{|G|}{4}$ loops; or (ii) two different vertices with $\frac{|G|}{2}$ or $\frac{|G|}{4}$ joining them.

Proof. The curve γ is an arc joining two cone points of order 2, and therefore Im Φ_{γ} is generated by two elements of order 2. Since G is abelian, the only possibilities is that Im $\Phi_{\gamma} \sim \mathbb{Z}_2$ or Im $\Phi_{\gamma} \sim \mathbb{Z}_2 \times \mathbb{Z}_2$.

Notice that $\pi_1(O)$ is generated by $\pi_1(O_1)$ and by a loop α surrounding one of the endpoints of γ . This is so because the loop surrounding γ , which is the product of the 2 generators of $\pi_1(\gamma)$, also belongs to $\pi_1(O_1)$. Taking images by Φ we have that G is generated by Im Φ_1 and by $a = \Phi(\alpha)$. As a consequence we have:

- (i) If $a \in \text{Im}\,\Phi_1$, i.e., $\text{Im}\,\Phi_1 = G$, then $\mathcal{D}_{\tilde{\Sigma}}$ has 1 vertex. The number of edges is $\frac{|G|}{2}$ or $\frac{|G|}{4}$ depending on whether Im Φ_{γ} is isomorphic, respectively, to \mathbb{Z}_2 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since there is only one vertex, all the edges are loops.
- (ii) If $a \notin \operatorname{Im} \Phi_1$, then $G = \operatorname{Im} \Phi_1 \cup a \operatorname{Im} \Phi_1$ and so $\mathcal{D}_{\tilde{\Sigma}}$ has 2 vertices. The number of edges is exactly as before. It is left to see whether or not each edge joins the two different vertices. To see this, notice that we can take $c_{\gamma} = \alpha$, and so $\Phi(c_{\gamma}) = a$. The edge $E_{\gamma,cIm \Phi_{\gamma}}$ joins $V_{1,cIm \Phi_{1}}$ and $V_{1,caIm \Phi_{1}}$. The lateral classes $cIm \Phi_{1}, caIm \Phi_{1}$ are always different because $c(ca)^{-1} = a^{-1} \notin \operatorname{Im} \Phi_1$. Thus, any edge joins the 2 different vertices.

Remark. In the situation of Proposition 7.3 but with G non-abelian, more options can appear. For instance, in [13] it is shown that in the pyramidal action with $G = D_n$ the dihedral group of order 2n, any graph with a single vertex and k loops with k dividing n appears as one of the graphs $\mathcal{D}_{\tilde{\Sigma}}$.

7.4. **Examples.** We show examples proving that all the graphs appearing in Propositions 7.1, 7.2 and 7.3 actually occur, that is, each of them is at the boundary of the equisymmetric locus of some abelian action.

Examples 1. Let *O* be an orbifold of signature $(0; 2^4, 3^2, 5^2)$. Denote by $q_i, i = 1, ..., 8$ its singular points with order, respectively, 2,2,5,5,2,2,3,3, and by $x_i, i = 1, ..., 8$ small loops around the q_i , all of them with the same orientation. Consider the epimorphism Φ defined by

Φ : $H_1(O)$	\rightarrow	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
x_1	\mapsto	(1, 0, 0)
x_2	\mapsto	(1, 0, 0)
x_3	\mapsto	(0, 0, 1)
x_4	\mapsto	(0, 0, 4)
x_5	\mapsto	(1, 0, 0)
x_6	\mapsto	(1, 0, 0)
x_7	\mapsto	(0, 1, 0)
x_8	\mapsto	(0, 2, 0)

It is clear that Φ determines a topological action since (in additive notation) $\sum_{i=1}^{8} \Phi(x_i) = (0, 0, 0)$. Now, consider a closed curve γ separating O in two suborbifolds O_1, O_2 , one of them containing the points x_1, x_2, x_3, x_4 and the other the remaining ones. Let $\Sigma = \{\gamma\}$. Then we have that Im Φ_1 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_5$; Im Φ_2 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$ and Im Φ_{γ} is trivial. Thus, by Proposition 7.1, \mathcal{D}_{Σ} is the graph $K_{3,5}^2$. We remark that the genus of the covering surface is g = 45.

For the general case consider an orbifold O with signature $(0; d^{2d}, m^{2m}, n^{2n})$. Let x_1, \ldots, x_{2d} be loops around the singular points of order d, y_1, \ldots, y_{2m} loops around the singular points of order m, and z_1, \ldots, z_{2n} loops around the singular points of order n, all of them with the same orientation. Let $\Phi: H_1(O) \to \mathbb{Z}_d \times \mathbb{Z}_m \times \mathbb{Z}_n$ the epimorphism defined by

 $\Phi(x_i) = (1, 0, 0), \quad \Phi(y_i) = (0, 1, 0), \quad \Phi(z_i) = (0, 0, 1).$

The order of $\Phi(x_i)$ is *d*, the order of $\Phi(y_i)$ is *m* and order of $\Phi(z_i)$ is *n*; also the product of all $\Phi(x_i)$, $\Phi(y_i)$, $\Phi(z_i)$ is trivial. So Φ determines a topological action with the covering surface of genus g = 1 + dmn(d + m + n - 4).

Now, consider a closed curve γ separating O in two suborbifolds O_1, O_2 , one of them containing the points $x_1, \ldots, x_d, y_1, \ldots, y_{2m}$ and the other the remaining ones. Let $\Sigma = \{\gamma\}$. Then we have that Im Φ_1 is isomorphic to $\mathbb{Z}_d \times \mathbb{Z}_m$, Im Φ_2 is isomorphic to $\mathbb{Z}_d \times \mathbb{Z}_n$ and Im Φ_γ is trivial. Thus, by Proposition 7.1, $\mathcal{D}_{\tilde{\Sigma}}$ is the graph $K_{m,n}^d$.

Examples 2. (i) For d > 1, let O be an orbifold of signature (1; d, d) and let α, β, x_1, x_2 be a basis for its homology, where x_1, x_2 are loops with the same orientation surrounding the cone points. Consider the epimorphism $\Phi: H_1(O) \to \mathbb{Z}_d \times \mathbb{Z}_m$ defined as

$$\Phi(\alpha) = (0,0), \quad \Phi(\beta) = (0,1), \quad \Phi(x_1) = (1,0), \quad \Phi(x_2) = (d-1,0),$$

Then Φ is a surface kernel epimorphism which determines a topological action with branching data (1; *d*, *d*) and covering surface of genus g = 1 + m(d - 1). Let $\gamma = \alpha$ and consider the multicurve $\Sigma = \{\gamma\}$. Then Im Φ_1 is isomorphic to \mathbb{Z}_d , and Im Φ_γ is trivial. Hence, by Proposition 7.2, \mathcal{D}_{Σ} is the graph C_m^d .

(ii) For d = 1, take an orbifold O of signature (g;) with $g \ge 2$, and let α_i, β_i be a homology basis. Consider the epimorphism $\Phi: H_1(O) \to \mathbb{Z}_m \times \mathbb{Z}_a$ defined as

 $\Phi(\alpha_1) = (1, 0), \quad \Phi(\alpha_i) = \Phi(\beta_i) = (0, 1), \text{ for all } i > 1 \text{ and for all } j.$

Then Φ is a surface kernel epimorphism. Taking $\gamma = \beta_1$, we have that $\text{Im } \Phi_1 = \text{Im } \Phi_{\gamma} = \langle (0, 1) \rangle$. Then, by Proposition 7.2, \mathcal{D}_{Σ} is the graph C_m^1 .

Examples 3. (i) Let *O* be an orbifold of signature $(1; 2^4)$ and let $\alpha, \beta, x_1, x_2, x_3, x_4$ a homology basis, where the x_i are the loops surrounding the cone points, all with the same orientation. Let $\Phi: H_1(O) \to \mathbb{Z}_2 \times \mathbb{Z}_a \times \mathbb{Z}_b$ be the epimorphism defined as

$$\Phi(\alpha) = (0, 1, 0), \quad \Phi(\beta) = (0, 0, 1), \quad \Phi(x_1) = \Phi(x_2) = \Phi(x_3) = \Phi(x_4) = (1, 0, 0).$$

Let γ be an arc joining the cone points q_1, q_2 . Then Im $\Phi_1 = G$, Im $\Phi_{\gamma} \approx \mathbb{Z}_2$, so that $\mathcal{D}_{\tilde{\Sigma}}$ has one vertex and |G|/2 loops. In this case the covering surface has genus g = 1 + 2ab.

(ii) Let *O* be as before. Let $\Phi: H_1(O) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a \times \mathbb{Z}_b$ the epimorphism defined as

$$\begin{aligned} \Phi(\alpha) &= (0, 0, 1, 0), & \Phi(\beta) &= (0, 0, 0, 1), \\ \Phi(x_1) &= \Phi(x_2) &= (1, 0, 0, 0), & \Phi(x_3) &= \Phi(x_4) &= (0, 1, 0, 0) \end{aligned}$$

Let γ be an arc joining the cone points q_1, q_3 . Then Im $\Phi_1 = G$, Im $\Phi_{\gamma} \approx \mathbb{Z}_2 \times \mathbb{Z}_2$, so that $\mathcal{D}_{\tilde{\Sigma}}$ has one vertex and |G|/4 loops. In this case the covering surface has genus g = 1 + 4ab.

(iii) Let *O* be an orbifold of signature $(1; 2^2)$ and let α, β, x_1, x_2 a basis of homology. Let $\Phi: H_1(O) \to \mathbb{Z}_2 \times \mathbb{Z}_a \times \mathbb{Z}_b$ the epimorphism defined as

$$\Phi(\alpha) = (0, 1, 0), \ \Phi(\beta) = (0, 0, 1), \ \Phi(x_1) = (1, 0, 0), \ \Phi(x_2) = (1, 0, 0).$$

Let γ be an arc joining the two cone points and let $\Sigma = \{\gamma\}$. Then Im $\Phi_1 \approx \mathbb{Z}_a \times \mathbb{Z}_b$ and Im $\Phi_{\gamma} \approx \mathbb{Z}_2$. Then $\mathcal{D}_{\bar{\Sigma}}$ has 2 vertices and |G|/2 edges joining them. In this case the covering surface has genus g = 1 + ab.

(iv) Let *O* be an orbifold with signature $(0; 2^5)$ and let x_i loops around the cone points q_i , all with the same orientation. Let $\Phi: H_1(O) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ the epimorphism defined as

$$\Phi(x_1) = (1, 1, 0), \quad \Phi(x_2) = (1, 0, 1), \quad \Phi(x_3) = (0, 1, 1), \quad \Phi(x_4) = \Phi(x_5) = (0, 0, 1).$$

Let γ be an arc joining the cone points q_1, q_2 and let $\Sigma = \{\gamma\}$. Then Im $\Phi_{\gamma} = \langle (0, 1, 1), (1, 0, 1) \rangle$ and Im $\Phi_1 = \langle (0, 1, 0), (0, 0, 1) \rangle$. Hence $\mathcal{D}_{\bar{\Sigma}}$ has two vertices and $2 = \frac{|G|}{4}$ edges joining them. One can construct more examples from this one, for instance, by adding genus g to the orbifold O, taking $G = \mathbb{Z}_2^3 \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_g}$ and defining Φ for the extra generators α_i, β_i as $\Phi(\alpha_i) = \Phi(\beta_i) = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the (3 + i)-th place. We see that $|\text{Im } \Phi_1|$ contains all the elements $(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the *i*-th place for any i > 1. Thus, $\frac{|G|}{|\text{Im } \Phi_1|} = 2$ and $\mathcal{D}_{\bar{\Sigma}}$ has 2 vertices. On the other hand,

Im
$$\Phi_{\gamma} = \langle (1, 1, 0, \dots, 0), (1, 0, 1, 0, \dots, 0) \rangle$$
,

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so it has 4 elements. Hence $\mathcal{D}_{\tilde{\Sigma}}$ has $\frac{|G|}{4} = 2a_1 \dots a_g$ edges joining the 2 vertices.

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