SPECTRAL SEQUENCES IN UNSTABLE HIGHER HOMOTOPY THEORY AND APPLICATIONS TO THE CONIVEAU FILTRATION

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Abstract. With the aim of understanding Morel's result on the $\mathbb{A}^1$-homotopy sheaves over a field, one extends the theory of unstable spectral sequences of Bousfield and Kan in the ∞-categorical setting. With this natural extension, parallel to the classical formalism of *cohomology theory with supports*, we introduce the notion of *cohomotopy theory with supports*. We extend the Bloch-Ogus-Gabber theorem for Cohomology theory with supports to that of unstable setting, to obtain unstable Gersten (or Cousin) resolutions associated to the coniveau filtration, under suitable assumptions. We apply this theory to motivic homotopy, Nisnevich-local torsors and Artin-Mazur étale homotopy type.

Contents

1. Introduction

The notion of spectral sequence for homotopy groups is classical in algebraic topology — the first occurrence seems to be in [Fed56]. It has been popularized by Bousfield and Kan in [BK72, IX, §4] as a tool to study towers of fibrations and obstruction theory for CW-complexes (equivalently, simplicial sets).

Next, in the stable case, it was put into a very general setup by Jacob Lurie, via stable ∞-categories and t-structures ([Lur17]). The first aim of these notes is to extend these constructions to the *unstable setting*, or rather that of not necessarily stable ∞-categories. This is based on a new notion of *(co)homotopy functors* which plays the role of (co)homology functors for triangulated categories (or in fact stable ∞-categories). See Section 2.2. Note in particular that we have

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chosen to work internally in a given 1-topos to get a flexible enough category which receives "homotopy pointed objects and groups". We also seized the opportunity to show that in any pointed ∞-category with finite limits/colimits, fiber/cofiber sequences satisfy the exact analog of the properties of a triangulated category, including the octahedron axiom. See Section 2.1. This is crucial in our study of the unstable spectral sequences.

Once this theory has been developed, one can define the notion of unstable exact couples following [Fed56, BK72] and of the associated unstable spectral sequence. Here, we follow the path of Bousfield and Kan, but we also introduce the notion of unstable exact couples (internally within some fixed topos) and moreover, we introduce a useful notion of *augmentation* which is critical to relate terms between successive pages. It also plays a crucial role when we derive "resolutions" out of unstable spectral sequences, as it provides canonical augmentations of the given resolutions (see Definition 2.25 and Definition 2.27).

One of our main motivations to develop the above machinery is to get coniveau spectral sequences in an unstable setting, strongly motivated by the pioneering work of Morel in [Mor12]. We get two types of results here.

First we set up a general framework of coniveau spectral sequences in the homotopical/unstable setting, extending the framework from [CTHK97]. This leads us to extend Grothendieck's definition of Cohen-Macaulay sheaves and their associated Cousin complexes to the non-abelian setting. We introduce an appropriate notion of being *(homotopically) Cohen-Macaulay* for sheaves of groups on the Zariski site of a scheme X (and also a Nisnevich variant) in Definition 3.35, and we draw out a kind of Cousin resolution to approximate the associated set of torsors (see Proposition 3.33). We prove that any separated algebraic group scheme over a scheme X regular in codimension less than 3 is homotopically Cohen-Macaulay (Theorem 3.38).

We also extend the *trick of Gabber* to our unstable setting. This allows us to establish Gersten resolutions of certain homotopy groups, pushing further the approach of Morel — by allowing to drop perfectness assumptions in some cases (see Corollary 4.7). We also get that certain $\mathbb{A}^1$-homotopy sheaves are Cohen-Macaulay in the sense of [Har66], up to some truncation. This extends recent works of [DFJ22] and [DKØ24] on Cousin resolutions to the unstable setting. Finally, we apply our theory to Artin-Mazur étale homotopy types, to obtain Gersten resolution in the homotopical setting without appealing to $\mathbb{A}^1$-homotopy. This requires proving an effaceability property in étale homotopy. See Proposition 4.12.

In a second work [DP24], we will extend the comparison between the coniveau spectral sequence and the cohomological spectral sequence associated with the homotopy t-structure [Dég14], [Bon10] to the unstable setting.

Plan of this work

The first section is the technical heart of the paper, and is of independent interest as it applies to ∞-categories with appropriate finite limits or colimits.

We first show that Verdier's axioms for triangulated categories are incarnated in unstable homotopy, that now are properties of pointed ∞-categories with appropriate limits (resp. colimits). We also highlight properties specific to the unstable setting such as the interaction between (homotopy) fiber and cofiber sequences, and an ∞-categorical description of the monodromy of a homotopy fiber sequence. We then introduce a new notion of homotopy functors playing the role of homological functors for ∞-categories. This is based on the notion of π_*-structure, and leads to the notion of homotopy complexes. This allows us to set up the machinery of unstable exact couples and unstable spectral sequences, drawing significant inspiration from [BK72]. The main result of this section is the proof of a degeneracy criterion Theorem 2.47 for unstable spectral sequences (which also applies to the stable setting and extend an original formulation of Quillen).

The second section applies the previous machinery to define coniveau spectral sequences in the unstable setting, inspired by Grothendieck's theory of Cousin (residual) complexes as exposed in [Har66]. We then study the induced complexes, called either Gersten or Cousin homotopical complexes depending on the context. We introduce relevant properties of these complexes, called Gersten and Cohen-Macaulay conditions according to the terminology of the domain. We study these conditions in the particular case of Eilenberg-MacLane complexes, leading to a homotopical analogue of the theory of Cohen-Macaulay sheaves of groups and Cousin complexes. We prove that separated algebraic groups G give examples of such (see Theorem 3.38) and deduce an adelic computation of the set of G-torsors (see Corollary 3.39).

The third section focuses on the trick of Gabber to deduce Gersten properties in various context. Notably, one applies this theory in Section 4.2 to extend the results of [CTHK97] to Artin-Mazur étale homotopy types.

Notation

One says that a pointed map is trivial if it is equal to the composite of the projection to the final object followed by the map to the base point. A terminal object in an (∞-)category (if it exists) is generically denoted by $*$. We usually just says colimit/limit (resp. commutative) for homotopy colimit/limit (homotopically commutative) in an ∞-category. Sometimes we identify a 1-category with its nerve when it is clearly intended by the context.

2. Unstable exact couples and spectral sequences

2.1. Fiber sequences and monodromy action.

Notation 2.1. Given a pointed ∞-category $\mathscr{C}$,[1] we will denote conventionally by $*$ an initial object of $\mathscr{C}$. Products (resp. coproducts) in $\mathscr{C}$ will be denoted by the symbol $\times$ (resp. $\vee$).

Recall the following classical definition ([Qui67, I, §3], [Lur17, 1.1.4]).

Definition 2.2. Let $\mathscr{C}$ be a pointed ∞-category. A *triangle* in $\mathscr{C}$ is commutative square of the form:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \Delta & \downarrow g \\ * & \longrightarrow & Z. \end{array}$$

Let $\square$ be the nerve of the finite category associated with the poset $(0,0),(1,0),(0,1),(1,1)$ with lexicographic order. The ∞-category of triangles is the sub-∞-category $\mathrm{Tri}(\mathscr{C})$ of $\mathrm{Fun}(\square,\mathscr{C})$ made of objects Δ such that $\Delta(1,0)$ is the zero object $*$. We will usually abusively denote triangles as a sequence $X \to Y \to Z$, the homotopy $gf \simeq *$ being implied.

Such a triangle is called a *fiber* (resp. *cofiber*) *sequence* in $\mathscr{C}$ if Δ is a (homotopy) pullback (resp. *pushout*) square. In this case X (resp. Z) is uniquely determined by the pullback (resp. pushout) diagram Δ, and one says that X (or (X,f)) (resp. Z (or (Z,g)) is the *(homotopy) fiber* (resp. *cofiber*) of g (resp. f).

In particular, fiber and cofiber sequences are exactly dual: a cofiber sequence in $\mathscr{C}$ is a fiber sequence in $\mathscr{C}^{op}$. Therefore, we will restrict to fiber sequences below.

Notation 2.3. Let again $\mathscr{C}$ be a pointed ∞-category which admits finite limits (resp. colimits). Then one defines the loop space (resp. suspension) of an object X of $\mathscr{C}$ by the following pullback (resp. pushout) square:

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

This defines an ∞-functor $\Omega : \mathscr{C} \to \mathscr{C}$ (resp. $\Sigma : \mathscr{C} \to \mathscr{C}$). By symmetry of the diagram, the functor which exchanges the lower left and upper right corner induces a natural auto-equivalence $\Omega \to \Omega$ ($\Sigma \to \Sigma$), and we will denote by $-\Omega$ (resp. $-\Sigma$) the ∞-functor obtained by composing the original one with the latter auto-equivalence.

Remark 2.4. Recall finally that, when $\mathscr{C}$ admits finite limits and colimits, Ω is left adjoint to Σ.

[1] An ∞-category which admits an object which is both final and initial.

Notation 2.5. Let us fix a pointed ∞-category $\mathscr{C}$ with finite limits. Then fiber sequences satisfy properties that precisely correspond to the axioms of a triangulated category, except one is not allowed to suspend. This has been observed in [Qui67, I, §3] for model categories, but not formalized to our knowledge. So we state the exact properties as they will be critical for our results.

Proposition 2.6. *Let $\mathscr{C}$ be a pointed ∞-category admitting all finite limits. The following properties of fiber sequences hold:*

(Fib1) (a) *Given any morphism $f : Y \to X$, there exists a fiber sequence $F \xrightarrow{i} X \xrightarrow{f} Y$ in $\mathscr{C}$, unique up to a contractible space of choices.*
(b) *If a triangle $\Delta : F \to Y \to X$ is isomorphic to a fiber sequence (in the ∞-category $\mathrm{Tri}(\mathscr{C})$), then Δ is a fiber sequence.*
(c) *The homotopy fiber of the identity $\mathrm{Id}_X : X \to X$ is the zero object $*$ (with canonical map $* \to X$).*

(Fib2) A fiber sequence $F \xrightarrow{i} Y \xrightarrow{f} X$ can be extended on the left

$$\dots \xrightarrow{\Omega^2 f} \Omega^2 X \xrightarrow{-\Omega\partial_f} \Omega F \xrightarrow{-\Omega g} \Omega Y \xrightarrow{-\Omega f} \Omega X \xrightarrow{\partial_f} F \xrightarrow{i} Y \xrightarrow{f} X$$

in such a way that any couple of consecutive maps is a fiber sequence. This extension is unique up to a contractible space of choices. The resulting sequence is called a long fiber sequence*.*

(Fib3) Given a commutative diagram of solid arrows in $\mathscr{C}$

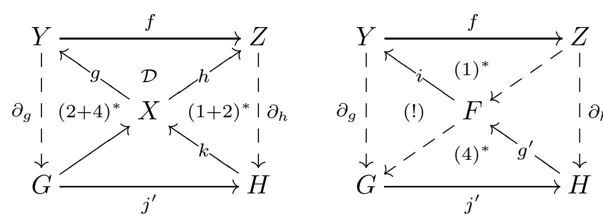
$$\begin{array}{ccccccc}
\Omega X & \xrightarrow{d} & F & \xrightarrow{i} & Y & \xrightarrow{f} & X \\
{\scriptstyle \Omega a}\downarrow & & \downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} \\
\Omega X' & \xrightarrow{d'} & F' & \xrightarrow{i'} & Y' & \xrightarrow{f'} & X'
\end{array}$$

such that the horizontal lines are the end of a long exact fiber sequence, there exists a dotted arrow making the resulting squares commutative.

(Fib4) Consider a commutative diagram $\mathcal{D}$ in $\mathscr{C}$: $X \xrightarrow{h} Z$, $X \xrightarrow{g} Y \xrightarrow{f} Z$. Then there exists an essentially unique commutative diagram in $\mathscr{C}$ made of pullback squares and of the following form:

$$\begin{array}{ccccccccccc}
Z & \xleftarrow{f} & Y & \xleftarrow{g} & X & \longleftarrow & * & & & & \\
\uparrow & (1) & \uparrow{\scriptstyle i} & (2) & \uparrow{\scriptstyle k} & (3) & \uparrow & & & & \\
* & \longleftarrow & F & \xleftarrow{g'} & H & \xleftarrow{\partial_h} & \Omega Z & \longleftarrow & * & & \\
 & & \uparrow & (4) & \uparrow{\scriptstyle j'} & (5) & \uparrow{\scriptstyle -\Omega f} & (6) & \uparrow & & \\
 & & * & \longleftarrow & G & \xleftarrow{\partial_g} & \Omega Y & \xleftarrow{-\Omega i} & \Omega F & \longleftarrow & * \\
 & & & & \uparrow & (7) & \uparrow{\scriptstyle -\Omega g} & (8) & \uparrow{\scriptstyle -\Omega g'} & (9) & \uparrow \\
 & & & & * & \longleftarrow & \Omega X & \xleftarrow{-\Omega k} & \Omega H & \xleftarrow{-\Omega j'} & \Omega G
\end{array}$$

This diagram can be organized in an (unstable) octahedral diagram:



in such a way that dotted arrows means a map such that the source is composed with the functor Ω, the diagrams indicated by a symbol $(-)^*$ represent cofiber sequences corresponding to the pullback diagram obtained by eventually pasting the pullback diagrams coming from the first diagram in $\mathscr{C}$. The other parts of the diagram are commutative diagram, except for diagram (!) which is *anti-commutative.*[2] *Finally, one has the two homotopies:*

$$(2) : ig' \simeq gk \qquad (5) : j'\partial_g \simeq \partial_h(-\Omega f).$$

Proof. The property (Fib1) follows from the properties of pushouts in $\mathscr{C}$. The subtlety of (Fib2) comes from the needed signs. They follow from the definition of fiber sequences, as explained in [Lur17, Lem. 1.1.2.9 and proof of (TR2) in section 1.1.2]. (Fib3) follows from the functoriality of pushouts. The existence and uniqueness of the first diagram in (Fib4) is a consequence of the existence and uniqueness of pullback squares, through an obvious iterative construction, which starts by adding the pullback squares (1), (2) and so on. Signs appear for the same reason that for axiom (Fib2). The translation into an octahedron diagram as in the final assertion follows from the existence of the first diagram. □

Remark 2.7. (1) All the above properties hold dually for cofiber sequences. One change the direction of all maps and replace Ω by Σ (signs remain).
(2) These axioms are more or less classical in homotopy theory, with the exception of the octahedron axioms.[3] It is remarkable that the axioms of ∞-categories turn all of Verdier's axioms into properties.
(3) The octahedron axiom is usually not fully stated. In particular, the relations (2) and (5) are often forgotten. However, we will use all these relations in our analysis of the unstable spectral sequence associated with a tower (see Theorem 2.47). We refer the reader to [BBD82, 1.1.5] for the complete statement in a triangulated category.[4]

Notation 2.8. Further, there is a property which is specific to the unstable case and concerns the compatibility of fiber and cofiber sequences. It was underlined by Quillen in [Qui67, I.3, prop. 6].

Let $\mathscr{C}$ be a pointed ∞-category with finite limits and colimits. In this case, the functor Σ is left adjoint to Ω.[5] Quillen's result can be stated in $\mathscr{C}$ as follows.

Proposition 2.9. *Let $\mathscr{C}$ be a pointed ∞-category with finite limits and colimits. Consider a diagram: $A \xrightarrow{g} B \xrightarrow{\tau} Y \xrightarrow{f} X$ in $\mathscr{C}$. Then it can be essentially uniquely completed into the following commutative diagram in $\mathscr{C}$*

$$\begin{array}{ccccccc} A & \xrightarrow{g} & B & \longrightarrow & Q & \longrightarrow & \Sigma A \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & \searrow^{\tau} & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ \Omega X & \longrightarrow & F & \longrightarrow & Y & \xrightarrow{f} & X \end{array}$$

in such a way that the upper (resp. lower) line is the beginning of a long cofiber (resp. fiber) sequence. Moreover, there exists an equivalence: $\delta = -\hat{\alpha}$ where $\hat{\alpha}$ is the map adjoint to α with respect to the adjunction (Ω, Σ), and the minus sign correspond to the canonical automorphism of Ω (see Notation 2.3).

Proof. The existence of the left (resp. right) part of the commutative diagram follows as F and ΩX (resp. Q and ΣA) are computed as fibers (resp. cofibers). The last assertion follows from functoriality of pullbacks, paying attention to the orientations of the resulting squares and use [Lur17, Lem. 1.1.2.9] to get the correct sign. □

We will use this proposition in the following corollary.

Corollary 2.10. *Let $g : A \to B$ and $f : Y \to X$ be morphisms, fitting in the following commutative square in $\mathscr{C}$:*

$$\begin{array}{ccc} \Sigma A & \xrightarrow{-\Sigma g} & \Sigma B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{f} & Y \end{array}$$

[2] The sign comes from the sign in the bottom map of diagram (6).

[3] We refer the reader to [Qui67, Rem. p. 3.10] for an amusing commentary about this axiom.

[4] Beware that we have rotated the octahedron diagram.

[5] This follows essentially as Σ (resp. Ω) is computed as a finite colimit (resp. finite limit), therefore left (resp. right) adjoint of a constant diagram functor.

Then there exists an essentially unique commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccccccccc} A & \xrightarrow{g} & B & \longrightarrow & Q & \longrightarrow & \Sigma A & \xrightarrow{-\Sigma g} & \Sigma B \\ {\scriptstyle\hat{\phi}}\downarrow & & \downarrow{\scriptstyle\hat{\psi}} & & \downarrow{\scriptstyle\eta} & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Omega Y & \xrightarrow{-\Omega f} & \Omega X & \longrightarrow & F & \longrightarrow & Y & \xrightarrow{f} & X \end{array}$$

in such a way that the upper (resp. lower) line is the beginning of a long cofiber (resp. fiber) sequence.

Notation 2.11. Let again $\mathscr{C}$ be an ∞-category with finite limits. Recall from [Lur09, 6.1.2.7] that a *groupoid object* of $\mathscr{C}$ is a simplicial object $G_\bullet : \Delta^{op} \to \mathscr{C}$ satisfying the *Segal condition* (see *loc. cit.* 6.1.2.6). Examples are provided by the Čech simplicial object $\check{S}_\bullet(Y/X)$ associated with any morphism $p : Y \to X$ in $\mathscr{C}$.

One says (*loc. cit.* 7.2.2.1) that $G_\bullet$ is a *group object* if $G_0 \simeq *$. Note that $G_\bullet$ can also be seen as an A_∞-algebra in $\mathscr{C}$ according to [Lur17, 5.1.3.3]. One can see $G_\bullet$ as an object G_1 of $\mathscr{C}$ equipped with a multiplication map $G_1 \times G_1 \simeq G_2 \to G_1$,[6] the simplicial structure encoding the associativity and unity properties.

Usually, one just says that $G = G_1$ is a group object in $\mathscr{C}$, referring to the underlying simplicial object $G_\bullet$ as the structure of G.

Example 2.12. Consider a pointed object $x : * \to X$ in $\mathscr{C}$. Then the loop space $\Omega_x X = * \times_X *$ (see Notation 2.3) admits a canonical group object structure $\Omega_x^\bullet(X)$ in $\mathscr{C}$, given by the Čech construction $\check{S}_\bullet(x : * \to X)$. In particular, the n-th component of this group object structure is the iterated loop space $\Omega_x^n(X)$. Moreover, the automorphism indicated by the minus sign in Notation 2.3 does in fact correspond to the inverse morphism with respect to that group structure.

Notation 2.13. Consider again an ∞-category with finite limits. Let G be a group object in $\mathscr{C}$. Following [NSS15, Def. 3.1], an action of G on an object X in $\mathscr{C}$ is a groupoid object $(X//G)_\bullet$ such that $(X//G)_0 = X$ and with a morphism of groupoid objects $(X//G)_\bullet \to G_\bullet$. By looking at the first two stages of the simplicial object $(X//G)_\bullet$ this determines a diagram

$$d_1^0, d_1^1 : G \times X \rightrightarrows X$$

such that d_1^1 corresponds to the projection on the second factor. The other map $d_1^0 : G \times X \to X$ corresponds to the bare action of G on X, while the rest of the data can be seen as coherences.

Let us now assume $\mathscr{C}$ is pointed, and consider a fiber sequence:

$$F \xrightarrow{i} Y \xrightarrow{f} X \tag{2.1.1}$$

Then one deduces an action of the loop space ΩX with its canonical group structure on the fiber F as the map of Čech simplicial objects

$$\mu_\bullet : (F//\Omega X)_\bullet := \check{S}_\bullet(i : F \to Y) \to \check{S}_\bullet(* \to X) = \Omega X_\bullet$$

associated with the underlying pullback square. As explained previously, this determines a canonical map

$$\mu : \Omega X \times F \to F.$$

Definition 2.14. Consider the above notation. Then the action $\mu_\bullet$ of ΩX on the fiber F of p will be called the *monodromy action* associated with the fiber sequence (2.1.1).

Remark 2.15. The dual monodromy operation exists for cofiber sequences (as a coaction of a cogroup ΣA on the cofiber).

[6]The first equivalence exists because of the Segal condition.

2.2. Homotopy functors in higher homotopy theory.

Notation 2.16. We will work inside a topos $\mathscr{E}$, using the internal notion of (abelian) groups in $\mathscr{E}$ called $\mathscr{E}$-groups.

We also consider the category of pointed objects of $\mathscr{E}$, $\mathscr{E}_* = */\mathscr{E}$. An $\mathscr{E}$-group object will always be assumed to be pointed by its neutral element. Note that it makes sense to speak about the kernel of maps in $\mathscr{E}_*$, as well as image and cokernel in $\mathscr{E}_*$ or in the category of $\mathscr{E}$-groups. Given a morphism $f : X \to Y$ of pointed objects (resp. $u : G \to H$ of $\mathscr{E}$-group objects), one has the following equivalent conditions:

(T1) f is an epimorphism $\Leftrightarrow \mathrm{Coker}(f) = * \Leftrightarrow \mathrm{Im}(f) \to H$ iso.
(T2) u is a monomorphism $\Leftrightarrow \mathrm{Ker}(u) = *$.

A group G, being a monoid for the cartesian structure, can act on the left on an object (resp. pointed objet) X of $\mathscr{E}$ as in usual set theory. One says that X is a G-object (aka G-equivariant object). The notion of morphism of G-objects, aka G-equivariant morphism, is clear. The *orbit object* X/G of a G-object X is the coequalizer of $G \times X \underset{p}{\overset{\gamma}{\rightrightarrows}} X$ where γ is the G-action map and p the projection. The action of G on X is transitive if $X/G = *$. We will also use the following equivalent conditions on a G-equivariant pointed map $f : X \to Y$ in the sequel:

(T3) f is a monomorphism $\Leftrightarrow \mathrm{Ker}(f) = *$.

Definition 2.17. A π_*-*structure* in $\mathscr{E}$ is an $\mathbb{N}$-graded object $G_* = (G_n)_{n\in\mathbb{N}} \in (\mathscr{E}_*)^{\mathbb{N}}$ such that G_1 is an $\mathscr{E}$-group, and for all $n > 1$, G_n is an abelian $\mathscr{E}$-group, with an action of G_1 by $\mathscr{E}$-group automorphisms.

A morphism of π_*-structures is a morphism $f : G_* \to H_*$ which is a homogeneous morphism of degree 0 of $\mathbb{N}$-graded objects, such that each $f_n : G_n \to H_n$ respects the relevant algebraic structures. The corresponding category is denoted by $\mathscr{E}_{\pi_*}$.

Remark 2.18. The category $\mathscr{E}_{\pi_*}$ admits finite products and filtered colimits. They both are computed term-wise, in the categories of pointed $\mathscr{E}$-objects, $\mathscr{E}$-groups, abelian $\mathscr{E}$-groups in the appropriate degrees.

Example 2.19. (1) Let $\mathscr{T}$ be an ∞-topos ([Lur09, Chap. 6]), $\mathscr{T}_*$ the ∞-category of pointed objects of $\mathscr{T}$. Let X be an object of $\mathscr{T}_*$. We let $\mathscr{T}^{\mathrm{disc}}$ be the sub-1-category of $\mathscr{T}$ made of discrete objects, *i.e.* the underlying topos, and let $\pi_0 : \mathscr{T}_* \to \mathscr{T}_*^{\mathrm{disc}}$ be the natural projection (in other words, $\mathscr{T}^{\mathrm{disc}}$ is the 0-th truncation of the ∞-category $\mathscr{T}$, [Lur09, Not. 5.5.6.2]).

For any integer $n \geq 0$, we put $\pi_n(X) := \pi_0(\Omega^n X)$. It now readily follows from the canonical group object structure on the loop space ΩX (see Example 2.12) that $\pi_*(X)$ is a π_*-structure of $\mathscr{T}^{\mathrm{disc}}$. Note that our definition coincides with [Lur09, 6.5.1].

(2) An important example for us comes from the $\mathbb{A}^1$-homotopy category $\mathscr{H}_{\mathbb{A}^1}(S)$. As it is the $\mathbb{A}^1$-localization of the Nisnevich ∞-topos $\mathrm{Sh}^{\infty}(\mathrm{Sm}_S)$ on the smooth site over S, one has a forgetful functor $\mathcal{O} : \mathscr{H}_{\mathbb{A}^1}(S) \to \mathrm{Sh}^{\infty}(\mathrm{Sm}_S)$. Given a pointed object $\mathcal{X}$ of $\mathscr{H}_{\mathbb{A}^1}(S)$, one deduces a π_*-structure $\boldsymbol{\pi}_*^{\mathbb{A}^1}(\mathcal{X}) := \pi_*(\mathcal{O}(X))$ in the topos of Nisnevich sheaves $\mathrm{Sh}(\mathrm{Sm}_S)^{\mathrm{disc}}$ over S, using the construction of the preceding point.

The following definition is an abstraction of the properties of long exact sequences of homotopy groups.

Definition 2.20. A *long homotopy sequence* in $\mathscr{E}$ is a triangle
$$\begin{array}{ccc} F_* & \xrightarrow{f} & G_* \\ & \partial \nwarrow\!\!\nwarrow \quad \swarrow g & \\ & H_* & \end{array}$$
such that:

1. F_*, G_* and H_* are π_*-structures in $\mathscr{E}$, f and g are morphisms of π_*-structures in $\mathscr{E}$.
2. $\partial : H_* \to F_*$ is a homogeneous morphism of degree -1 of $(\mathscr{E}_*)^{\mathbb{N}}$ such that for all $n > 1$, $\partial_n : H_n \to F_{n-1}$ is a morphism of groups.
3. F_0 is an H_1-object[7] such that $\partial_1 : H_1 \to F_0$ (resp. $f_0 : F_0 \to G_0$) is H_1-equivariant where H_1 acts by left multiplication on the source (resp. trivially on the target).

[7] beware it is not necessarily a pointed H_1-object: the orbit of the base point of F_0 can be non-trivial.

(4) The image of $\partial_2 : H_2 \to F_1$ lands in the center of F_1.
(5) The composite of any two consecutive maps is trivial.

One says that this long homotopy sequence is *exact* if for all $n \geq 0$, one has an equality of sub-objects:

$$\mathrm{Im}(f_n) = \mathrm{Ker}(g_n), \mathrm{Im}(g_{n+1}) = \mathrm{Ker}(\partial_{n+1}), \mathrm{Im}(\partial_{n+1}) = \mathrm{Ker}(f_n),$$

and moreover, the map $\widetilde{f}_0 : F_0/H_1 \to G_0$ induced by f_0 according to point (3) is a monomorphism.

Following Morel (see [Mor12, §2.2]), we have used the symbol $\Rightarrow$ to indicate the boundary map ∂, meaning this map is actually extended to an action of the left hand-side on the right hand-side. We will also use the following notation to indicate such a sequence:

$$\dots H_2 \xRightarrow{\partial_2} F_1 \xrightarrow{f_1} G_1 \xrightarrow{g_1} H_1 \xRightarrow{\partial_1} F_0 \xrightarrow{f_0} G_0 \xrightarrow{g_0} H_0$$

Remark 2.21. (1) Axiom (2) can also be formulated by saying that for all $n > 1$, F_{n-1} is an H_n-object such that $\partial_n : H_n \to F_{n-1}$ is H_n-equivariant where H_n acts on the source by left multiplication.
(2) An important consequence of the axiom (3) is that the map ∂_1 is equal to the composite $H_1 \times * \xrightarrow{\mathrm{Id} \times x} H_1 \times F_0 \xrightarrow{\gamma} F_0$ where x is the base point of F_0 and γ is the action of H_1 on F_0.

Example 2.22. Consider the setting of Example 2.19(1). Let $f : Y \to X$ be a morphism in $\mathscr{T}_*$. Let $i : F = \mathrm{Ker}(f) \to Y$ be the homotopy fiber of f. On deduces from the long fiber sequence (Fib2) of Proposition 2.6 a homotopy exact sequence in $\mathscr{T}^{\mathrm{disc}}$:

$$\begin{array}{ccc} \pi_*(F) & \xrightarrow{\quad i_* \quad} & \pi_*(Y) \\ \partial \Uparrow\!\!\!\!\nwarrow & & \swarrow f_* \\ & \pi_*(X) & \end{array}$$

where the π_*-structures come from Example 2.19 and the action of $\pi_1(X)$ on $\pi_0(F)$ is given by the monodromy action of Definition 2.14.

The preceding definition allows us to extend the classical definition of a homological functor:

Definition 2.23. Let $\mathscr{C}_*$ be as above and $\mathscr{E}$ be a 1-topos.

A *homotopy functor* is an ∞-functor $\Pi_* : \mathscr{C}_* \to \mathrm{N}\,\mathscr{E}_{\pi_*}$, which turns fiber sequences (Definition 2.2) into long homotopy exact sequences. Dually, a cohomotopy functor on $\mathscr{C}$ will be a homotopy functor starting from the opposite of $\mathscr{C}_*$: so that it turns cofiber sequences into long homotopy exact sequences.

In addition, we will say that the homotopy functor Π_* is *additive* if it commutes with products. As a result of this definition, a cohomotopy functor is additive if it sends coproducts (of pointed $\mathscr{E}$-objects) to products in the category of π_*-structures (see Remark 2.18).

Example 2.24. (1) Example 2.22 implies that $\pi_* : \mathscr{T}_* \to \mathscr{T}^{\mathrm{disc}}_{\pi_*}$ is a homotopy functor.
(2) Consider a pointed ∞-category $\mathscr{C}_*$. Then $\mathscr{C}_*$ admits a canonical (unique in a suitable sense) enrichment in pointed spaces. Then for objects A and X of $\mathscr{C}_*$, the π_*-structure in sets $\pi_*(\mathrm{Map}_{\mathscr{C}_*}(A, X))$ defines a cohomotopy functor in A (for X fixed) and a homotopy functor in X (for A fixed).

The following definition is motivated by the unstable Bousfield-Kan spectral sequence that will be reviewed in the next section (see there for examples).

Definition 2.25. A *homotopical complex* with coefficients in $\mathscr{E}$ is an $\mathbb{N}$-graded object E_* of $\mathscr{E}_*$ with a homogeneous morphism $d : E_* \to E_*$ in $\mathscr{E}_*^{\mathbb{N}}$ of degree -1 such that

- $d \circ d = *$.
- For all $n > 1$, E_n is an abelian $\mathscr{E}$-group, E_1 is an $\mathscr{E}$-group and E_0 is a pointed $\mathscr{E}$-object, and an $\mathscr{E}_1$-object (after forgetting the base point).

The homotopy groups of the complex E_* are defined for $n > 0$ as:

$$\pi_n(E_*) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1}).$$

One says that the complex is *exact* if for all $n > 0$, $\pi_n(E_*) = *$.[8]

[8]Equivalently, kernel=image at all possible places.

An *augmentation* of E_* is an E_1-equivariant map $\epsilon : E_0 \to F$ of pointed objects in $\mathscr{E}$ where F has been given the trivial E_1-action. One also says that (E_*, ϵ) is an augmented complex. Then one defines the homotopy groups of (E_*, ϵ) by the formula $\pi_n(E_*, \epsilon) = \pi_n(E_*)$ for $n > 0$, and $\pi_0(E_*, \epsilon) = \mathrm{Ker}(\epsilon)/E_1$ (orbit space). We will say that the augmented homotopy complex (E_*, ϵ) is *exact* if for all integers $n \geq 0$, $\pi_n(E_*, \epsilon) = *$. One further says that (E_*, ϵ) is *strongly exact* if it is exact and the induced map $\widetilde{\epsilon} : E_0/E_1 \to F$ is a monomorphism.

In other words, the exactness condition concerning the augmentation means that E_1 acts transitively on the kernel $\mathrm{Ker}(\epsilon)$.

Notation 2.26. We will further consider *bounded* homotopical complexes: such a homotopical complex E_* as above is said to be *d-truncated* if for all $n > d$, $E_n = *$.

To relate to the classical literature on the coniveau spectral sequence, we will use a "cohomotopical" indexing for truncated homotopical complexes: a d-truncated homotopical complex C^* will be denoted as:

$$* \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \Rightarrow C^d$$

with the previous notation, one has: $C^n = E_{d-n}$. An augmentation map $\epsilon : E_0 \to F$ of E_* will be called a *coaugmentation* $\epsilon : C^d \to F$ of the associated cohomotopical complex C^*. Then the notions of exactness and strong exactness of (C^*, ϵ) are the same.

Such a complex can also be co-augmented (we say "bi-augmented") by a map $X \xrightarrow{\tau} C^0$ which is a morphism of pointed sets, groups or abelian groups if $d = 0, 1$ or $d \geq 2$, such that $d^0 \circ \tau = *$. The co-augmented complex is exact if $\mathrm{Ker}(\tau) = *$ and $\mathrm{Im}(\tau) = \mathrm{Ker}(d^0)$. Strong exactness is defined as for augmented complexes.

Following Morel [Mor12, Def. 2.20, 6.1], we further adopt the following notation:

Definition 2.27. Consider a d-truncated cohomotopical complex C^* as above. An *augmentation* of C^* will be a map $X \xrightarrow{\tau} C^0$ which is respectively a morphism of pointed sets, groups or abelian groups if $d = 0, 1$ or $d \geq 2$, and such that $d^0 \circ \tau = *$. One says that (C^*, τ) is augmented. One says that (C^*, τ) is exact if $\mathrm{Ker}(\tau) = *$ and $\mathrm{Im}(\tau) = \mathrm{Ker}(d^0)$.

When C^* admits an augmentation τ and a co-augmentation ϵ, one says that (C^*, τ, ϵ) is bi-augmented. Notions of exactness (resp. strong exactness) are to be taken for the augmented and the co-augmented cohomotopical complex.

Remark 2.28. The above definition contains three important and rather distinct cases:

(1) When $d = 0$, an augmented complex is given by a diagram $* \to X \xrightarrow{\tau} C^0$. Exactness simply means that $\mathrm{Ker}(\tau) = *$.

(2) The case $d = 1$ is possibly the most intricate, and corresponds to Morel's Definition 2.20 in [Mor12] (with slightly more flexibility). A bi-augmented 1-truncated cohomotopical complex is given by a diagram:

$$* \to X \xrightarrow{\tau} C^0 \xRightarrow{d} C^1 \xrightarrow{\epsilon} F$$

where X and C^0 are just $\mathscr{E}$-groups, while C^1 and F a pointed $\mathscr{E}$-objects. The actual complex C^* is only given by one map d, which is in fact an action of the $\mathscr{E}$-group C^0 on the pointed $\mathscr{E}$-object C^1.

Recall that exactness on the right means that C^0 acts transitively on $\mathrm{Ker}(\epsilon)$ while strong exactness implies in addition that the induced (pointed) map $\widetilde{\epsilon} : C^1/C^0 \to F$ is a monomorphism (of pointed objects of $\mathscr{E}$). Here, C^1/C^0 means the orbit "space" of C^0 acting on C^1.

(3) Finally for $d > 1$, we can see a bi-augmented d-truncated complex (C^*, τ, ϵ) as a co-augmented $(d+1)$-truncated complex, by interpreting the augmentation τ as a differential starting from degree -1.

We end up this section with an important technical point that will be used for defining and studying the unstable coniveau spectral sequence (see Notation 3.1).

Proposition 2.29. *Let $\mathscr{E}$ be a topos. The category of long homotopy sequences (resp. homotopy complexes, augmented homotopy complexes, bi-augmented complexes) in $\mathscr{E}$ admits filtered colimits.*

These colimits are term-wise computed by filtered colimits of abelian $\mathscr{E}$-groups in degree bigger than 2*, filtered colimits of $\mathscr{E}$-groups in degree* 1*, and by filtered colimit of pointed $\mathscr{E}$-objects in degree* 0*. Finally, filtered colimits preserve exactness in all cases.*

Proof. We consider only the case of long homotopy sequences as the other cases are similar. As the associated sheaf functor commutes with colimits, it is sufficient to consider the case of discrete toposes, and therefore one is reduced to the case $\mathscr{E} = \mathscr{S}et$.

Let $\{(F^i_*, G^i_*, H^i_*)\}_{i\in I}$ be an I-filtered diagram of *long homotopy sequences* in sets:

$$\dots H^i_2 \xRightarrow{\partial_2} F^i_1 \xrightarrow{f_1} G^i_1 \xrightarrow{g_1} H^i_1 \xRightarrow{\partial_1} F^i_0 \xrightarrow{f_0} G^i_0 \xrightarrow{g_0} H^i_0$$

The proposition states that in the category of exact long homotopy sequences, $\operatorname{colim}_{i\in I}(F^i_*, G^i_*, H^i_*) = (F_*, G_*, H_*)$ where for $\square = F, G, H$

$$\square_* = \begin{cases} \operatorname{colim}_i \square^i_* & \text{if } * > 1 \\ \operatorname{colim}'_i \square^i_1 & \text{if } * = 1 \\ \operatorname{colim}'' \square^i_0 & \text{if } * = 0 \end{cases} \tag{2.2.1}$$

Here $\operatorname{colim}_i \square^i_*$ for $* > 1$ (resp. $\operatorname{colim}'_i \square^i_1$, and resp. $\operatorname{colim}''_i \square^i_0$) denotes respectively the colimit in the category of pointed sets (resp. groups, and resp. abelian groups). In all three cases, the colimit is computed by taking the appropriate quotient of the corresponding coproduct (obtained after forgetting maps in I), that is respectively the wedge sum, free product and direct sums. In the case of groups, $\operatorname{colim}'_i \square^i_1$ is given by the quotient of the free product $\bigstar_{i\in I}\square^i_1$ of groups $\square^i_1$ by the sub-group generated by the relations in $\square^k_1$ and $(s_j \circ \mu_{ij}(g_i)) \cdot s_i(g_i)^{-1}$ where for $i \leq j$, $\mu_{ij} : \square^i_1 \to \square^j_1$ and $s_i : \square^i_1 \to \bigstar_{i\in I}\square^i_1$.

We first prove that the sequence

$$\dots H_2 \xRightarrow{\partial_2} F_1 \xrightarrow{f_1} G_1 \xrightarrow{g_1} H_1 \xRightarrow{\partial_1} F_0 \xrightarrow{f_0} G_0 \xrightarrow{g_0} H_0 \tag{2.2.2}$$

is a long homotopy sequence. The action of H_1 on F_0 is obtained by taking the colimit of the action maps:

$$H^i_1 \times F^i_0 \to F^i_0$$

and using the fact filtered colimits preserve finite products. The rest of the axioms of Definition 2.20, *i.e.* points (3), (4) and (5), are clear. It is then easy to check that the long exact sequence (2.2.2) satisfies the universal properties of colimits.

We finally prove that, if all long homotopy sequences (F^i_*, G^i_*, H^i_*) are exact, then (2.2.2) is exact.

Exactness at F_i, H_i and G_i, for $i > 1$: clear as filtered colimits of abelian groups are exact.

Exactness at F_1: we need to show that the sequence

$$H_2 \xRightarrow{\partial_2} F_1 \xrightarrow{f_1} G_1$$

is exact. We have exact sequence

$$\operatorname{colim}'_{i\in I} H^i_2 \xRightarrow{\partial_2} \operatorname{colim}'_{i\in I} F^i_1 \xrightarrow{f_1} \operatorname{colim}'_{i\in I} G^i_1$$

of $\mathscr{E}$-group objects. Now since the image of $H^i_2 \to F^i_1$ is in the center of F^i_1 (axiom (4) of Definition 2.20), it follows that the map $\operatorname{colim}'_{i\in I} H^i_2 \xRightarrow{\partial_2} \operatorname{colim}'_{i\in I} F^i_1$ factors through the abelianization $(\operatorname{colim}'_{i\in I} H^i_2)^{ab} = \operatorname{colim}_{i\in I} H^i_2$, where $\operatorname{colim}_{i\in I} H^i_2$ is the colimit in the category of abelian groups, and this concludes.

Exactness at G_1: as in the abelian case.

Exactness at H_1: to prove the exactness at H_1, we need to show that given $h \in H_1$ such that $h \cdot * = *$, then h is in the image of $G_1 \to H_1$. Since $h \in \operatorname{colim}'_{i\in I} H_1^i$ and I is filtered, there exists an index $i \in I$ and an element $h_i \in H_1^i$ such that $h = s_i(h_i)$ for $s_i : H_1^i \to \operatorname{colim}'_{i\in I} H_1^i$. Hence $h \cdot * = *$ in $F_0 = \operatorname{colim}''_{i\in I} F_0^i$ implies that $\mu_{ij}(h_i) \cdot * = *$ in F_0^j for some $j \geq i$. By exactness of the sequence $G_1^j \to H_1^j \xRightarrow{\partial_2} F_0^j$, the element $\mu_{ij}(h_i)$ is in the image of the map $G_1^j \to H_1^j$. Hence $h = s_i(h_i) = s_j(\mu_{ij}(h_i))$ is in the image of the map $G_1 \to H_1$. This implies the exactness at H_1.
Exactness at F_0: we need to show $f_0^{-1}(*) = \operatorname{Im}(\partial_1)$. For $x \in f_0^{-1}(*)$, there is $i \in I$ such that $x \in F_0^i$ and $f_0^i(x) = *$ in G_0^i. Hence by exactness of $H_1^i \to F_0^i \to G_0^i$, there is $h_i \in H_1^i$ such that $\partial_1^i(h_i) = x$. Hence, $\partial_1(s_i(h_i)) = x$ in H_1. This implies the exactness at F_1.
Exactness at G_0: This is clear. □

Remark 2.30. In particular, one deduces that the category of left (resp. right) unstable exact couples admits filtered colimits. Moreover, one can check that the derived exact couple functor, as well as the associated spectral sequence functor, commutes with filtered colimits.

2.3. **Unstable exact couples and spectral sequences.** The notion of exact couples in unstable homotopy is very old (see [Fed56] but also [BK72]). It is a delicate subject, because one needs to take care of π_1-actions (*i.e.* monodromy as explained in Definition 2.14). The next definition is a synthesis of the approaches via exact couples (and Rees system, see Notation 2.42) and the direct one from [BK72, IX, §4], with the aim to apply it to ∞-categories and the previously introduced homotopy exact functors.

Definition 2.31. Let $\mathscr{E}$ be a topos. A *right (*resp. *left) unstable exact couple of degree* 1 is a triangle

$$\begin{array}{ccc} D^{**} & \xrightarrow[\alpha]{(-1,-1)} & D^{**} \\ & {\scriptstyle (0,0)}\,\nwarrow^{\gamma} \quad {}^{\beta}\Swarrow\,{\scriptstyle (1,0)} & \\ & E^{**} & \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} \bar{D}^{**} & \xrightarrow[\alpha]{(-1,-1)} & D^{**} \\ & {\scriptstyle (1,0)}\,\Nwarrow^{\bar{\gamma}} \quad {}^{\bar{\beta}}\swarrow\,{\scriptstyle (0,0)} & \\ & E^{**} & \end{array}$$

of object of $\mathscr{E}^{\Upsilon}_{\pi_*}$, $\Upsilon = \{(p,q) \in \mathbb{Z}^2 \mid q - p \geq 0\}$, such that the three maps are bi-homogeneous with indicated bidegree and for all q, the sequence

$$\cdots \to D^{p,q+1} \xrightarrow{\alpha} D^{p-1,q} \xRightarrow{\beta} E^{p,q} \xrightarrow{\gamma} D^{p,q} \xrightarrow{\alpha} \cdots D^{q,q+1} \xRightarrow{\beta} E^{q+1,q+1} \xrightarrow{\gamma} D^{q+1,q+1} \xrightarrow{\alpha} D^{q,q}$$

$$\text{resp.} \cdots \to \bar{D}^{p+1,q+1} \xrightarrow{\bar{\alpha}} \bar{D}^{p,q} \xRightarrow{\bar{\beta}} E^{p,q} \xrightarrow{\bar{\gamma}} \bar{D}^{p+1,q} \xrightarrow{\bar{\alpha}} \cdots \bar{D}^{q,q+1} \xRightarrow{\bar{\beta}} E^{q,q+1} \xrightarrow{\bar{\gamma}} \bar{D}^{q+1,q+1} \xrightarrow{\bar{\alpha}} \bar{D}^{q,q}$$

is an exact long homotopy sequence in $\mathscr{E}$, in the sense of Definition 2.20.

Given any integer $r \geq 1$, a right (resp. left) unstable exact couple of degree r is defined similarly as above except that γ (resp. $\bar{\beta}$) has degree $(r-1, r-1)$.

The situation of right and left unstable exact couples is completely analogous. In the following, without precision, an unstable exact couple will be a right one.

Example 2.32. (1) Let $\mathscr{T}$ be an ∞-topos, and consider a (decreasing) tower in $\mathscr{T}$:

$$\cdots \to X_n \xrightarrow{f_n} X_{n-1} \to X_1 \to X_0$$

i.e. an object of the ∞-category $\mathscr{T}^{\overleftarrow{\mathbb{N}}}$, where $\overleftarrow{\mathbb{N}}$ is the category associated with the opposite ordered set of non-negative integers. Then one gets a right unstable exact couple of degree 1 by considering the homotopy fibre $i_p : F_p = \operatorname{Ker}(f_p) \to X_p$ and putting:

$$\cdots \xrightarrow{\alpha = f_{p*}} D^{p-1,q} = \pi_{q-p+1}(X_{p-1}) \xRightarrow{\beta=\partial} E^{p,q} = \pi_{q-p}(F_p) \xrightarrow{\gamma = i_{p*}} D^{p,q} = \pi_{q-p}(X_p)$$

according to Example 2.22(1).

Dually, if one considers the homotoy cofiber $\pi_p : X_p \to \operatorname{Coker}(f_p) =: C_p$, one gets a left unstable exact couple of degree 1, $E^{p,q} = \pi_{q-p}(C_p)$, $\bar{D}^{p,q} = \pi_{q-p}(X_p)$.

(2) Let $\Pi_* : \mathscr{C}_* \to \mathscr{E}_{\pi_*}$ be a homotopy functor in the sense of Definition 2.20. Then one associates to any tower $X_\bullet$ in $\mathscr{C}_*^{\overleftarrow{\mathbb{N}}}$ a right unstable exact couple of degree 1 using the above recipe: $D^{p,q} = \Pi_{q-p}(X_p)$, $E^{p,q} = \Pi_{q-p}(F_p)$, where F_p is the homotopy fibre of $X_p \to X_{p-1}$.
(3) The preceding considerations dualize in the obvious way: given a cohomotopy functor $\Pi_* : \mathscr{C}_*^{op} \to \mathscr{E}_{\pi_*}$, one associates to any (increasing) tower[9] $X^\bullet$ in $\mathscr{C}_*^{\overrightarrow{\mathbb{N}}}$ a right unstable exact couple of degree 1 by considering the homotopy cofiber $C^p = \mathrm{Coker}(X^{p-1} \to X^p)$, and using the formulas: $D^{p,q} = \Pi_{q-p}(X^p)$, $E^{p,q} = \Pi_{q-p}(C^p)$.

Remark 2.33. The distinction between right and left exact couples is special to the unstable case. Up to our knowledge, it has not been highlighted in the literature yet, because one usually restricts to tower of fibrations. We refer the reader to the Notation 2.42 for the relevance of considering both forms of exact couples.

Notation 2.34. Let $(D, E, \alpha, \beta, \gamma)$ be an exact couple of degree r.

One associates to it a co-augmented cohomotopical complex (Notation 2.26) by defining the boundary operator on E using the formula $d_r = \beta \circ \gamma$:

$$\dots E^{q-(n+1)r,q-(n+1)r+n+1} \to E^{q-nr,q-nr+n} \to \dots \to E^{q-r,q-r+1} \xrightarrow{d_r^{q-r,q-r+1}} E^{q,q} \xrightarrow{\epsilon} \widetilde{D}^{q,q}$$

where $\widetilde{D}^{q,q}$ is the cokernel of the pointed map $D^{q+r,q+r} \xrightarrow{\alpha} D^{q,q}$ and the co-augmentation map is the composite: $\epsilon : E^{q,q} \xrightarrow{\gamma} D^{q,q} \twoheadrightarrow \widetilde{D}^{q,q}$. Note that d_r is homogeneous of bidegree $(r, r-1)$.[10]

Using the classical procedure, one can now define a *derived exact couple* by the following formulas:

$$D' = \mathrm{Im}(\alpha), (E')^{p,q} = \pi^p(E_r^{*,q}, \epsilon).$$

The bi-grading on D' is obtained by the bi-grading on D, according to the inclusion $D'^{p,q} \subset D^{p,q}$. Similarly, one defines the bi-grading on E' such that $E'^{p,q}$ is a sub-quotient of $E^{p,q}$. Then one obtains the maps α', β', γ' as the one induced respectively by α, β, γ by the universal properties of the image and of the homotopy of an augmented unstable complex. Note the resulting homotopy exact couple $(D', E', \alpha', \beta', \gamma')$ has degree $r+1$.

Definition 2.35. We call $(D', E', \alpha', \beta', \gamma')$ the derived unstable exact couple associated with $(D, E, \alpha, \beta, \gamma)$. Iterating this procedure, for any $s > 0$, one defines the s-th derived unstable exact couple $(D^{(s)}, E^{(s)}, \alpha^{(s)}, \beta^{(s)}, \gamma^{(s)})$ which has degree $r+s$.

We define the *unstable spectral sequence* associated with the exact couple $(D, E, \dots)$ by putting, for $n \geq r$, $s = n - r$:

$$E_n^{p,q} = (E^{(s)})^{p,q},$$

with differential $d_n^{p,q} = \gamma^{(s)} \circ \beta^{(s)} : E_n^{p,q} \to E_n^{p+n+1,q+n}$. We view $E_n^{*,q}$ as an augmented unstable complex with augmentation $\epsilon^{(s)} = E_n^{q,q} \to \widetilde{D}_n^{q+n,q+n}$.

One will remember that the $(n+1)$-th term $E_{n+1}^{*,q}$ is obtained as the (co)homotopy of the augmented unstable complex $E_n^{*,q}$. Moreover, we view it as an *augmented* unstable complex (Definition 2.25).

Remark 2.36. The consideration of augmented complexes seems to be new in this context. While not changing the definitions, we find it very enlightening. Note in particular, that one gets the following formulas to directly compute the r-th derived exact couple (as in [BK72, IX, 4.1]):

$$D_{r+1}^{p,q} = \mathrm{Im}\left(\alpha^r : D^{p+r,q+r} \to D^{p,q}\right),$$
$$E_{r+1}^{p,q} = \mathrm{Ker}\left(E^{p,q} \to D^{p,q}/D_{r+1}^{p,q}\right) / \mathrm{Ker}\left(D^{p-1,q} \xrightarrow{\alpha^r} D^{p-r-1,q-r}\right),$$

where the quotient when $p = q$ means the set of orbits for the relevant group action. On the other hand, the notion of derived unstable exact couple is useful for inductive arguments.

[9]this notation is designed to fit in with the case of the coniveau tower in Notation 3.1;

[10]We follow here the indexing of Bousfield and Kan, though it is unusual in the general theory of spectral sequences.

Example 2.37. (1) Consider the notation of Example 2.32(1). We let $X_\infty = \lim_n X_n$. Then the spectral sequence associated with the exact couple of *loc. cit.* starts from the E_1-term and abuts to $\pi_{q-p}(X_\infty)$:

$$E_1^{p,q} = \pi_{q-p}(F_p) \Rightarrow \pi_{q-p}(X_\infty).$$

When $\mathscr{T}$ is the ∞-topos of spaces, and one starts from a tower of fibrations $X_\bullet$, this is precisely the Bousfield-Kan spectral sequence ([BK72, IX, 4.2]). We refer to *loc. cit.*, 5.3 and 5.4 for conditions of convergence of this spectral sequence. In fact, strong convergence will always be fulfilled in our examples.
(2) The same construction is valid starting from a homotopy functor $\Pi_* : \mathscr{C}_* \to \mathscr{E}_{\pi_*}$ as in Example 2.32(2), such that $\mathscr{C}_*$ admits sequential limits. The same observation holds dually for a cohomotopy functor as in Example 2.32(3). The abutment is given by the cohomotopy of the colimit (topologically, this compares to a tower of cofibrations).

Remark 2.38. The above examples extend the discussion of [Lur17, 1.2.2] to the unstable case.

In fact, if we work in a stable ∞-category $\mathcal{C}_*$, and Π_* is the restriction of a homology functor to non-negative degrees, then the spectral sequence in point (2) of the above example is half of the spectral sequence one naturally derives from the given tower by considering the whole homology functor. One the other hand, one can recover the whole spectral sequence by using the suspended tower $X_\bullet[n]$ for arbitrary $n \geq 0$. Thus, all the results that we will get in the unstable case will imply the analogous result for the complete spectral sequences in a stable situation.

2.4. **The degeneracy criterion.**

Notation 2.39. We will consider a situation similar to Example 2.37(2). Let $\mathscr{C}_*$ be a pointed ∞-category and $\Pi_* : \mathscr{C}_* \to \mathscr{E}_{\pi_*}$ be a homotopy functor.

We let X be an object of $\mathscr{C}_*$ and consider a tower of objects under X,

$$\cdots \to X_n \xrightarrow{f_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{f_1} X_0$$

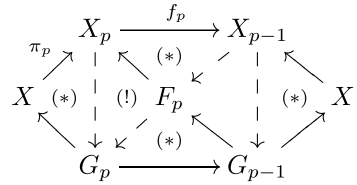
with projection maps $\pi_n : X \to X_n$. In other words, $X \to X_\bullet$ is an object of $(X/\mathscr{C}_*)^{\overleftarrow{\mathbb{N}}}$. We usually extend this tower slightly by letting $f_0 : X_0 \to X_{-1} = *$ be the (essentially) unique map.

Definition 2.40. We will say that the tower $X/X_\bullet$ is bounded if there exists an integer $d \in \mathbb{N}$ such that for all $n > d$, $p_n : X \to X_n$ is an isomorphism in the ∞-category $\mathscr{C}_*$. We say that $X/X_\bullet$ is d-bounded (∞-bounded for no condition).

Remark 2.41. One can say that $X/X_\bullet$ is a coaugmented tower. This is the (pointed) ∞-categorical generalization of a cofiltered object.

Notation 2.42. Consider again the situation of an arbitrary coaugmented tower $X/X_\bullet$, as in Notation 2.39. By applying the octahedron property (Fib4) of Proposition 2.6, one deduces for any integer $p \geq 0$ an octahedron diagram[11]:

(2.4.1)

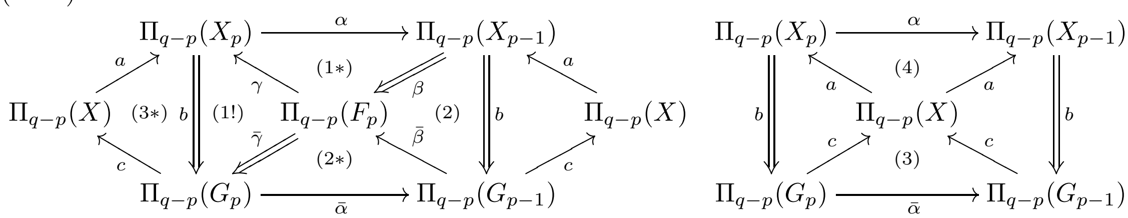


where a symbol (*) refers to a fiber sequence, the dotted arrows indicate a boundary map of the form $\Omega A \to B$, the other parts of the diagram are commutative except for (!) which is anti-commutative. Note in particular that F_p (resp. G_p) is the fiber of f_p (resp. π_p).

[11] Apply the axiom to the composable maps $X \xrightarrow{\pi_p} X_p \xrightarrow{f_p} X_{p-1}$, use the right-hand square in (Fib4), and use the left-hand square to complete the diagram

If one applies the homotopy functor Π_* to this diagram, one gets the following diagrams for $p \leq q$:

(2.4.2)



where we put $\alpha = f_{p*}$, $a = g_{p*}$. All diagrams (?∗) (resp. (?), (1!)) are homotopy long exact sequences (resp. commtutative, anti-commutative) in the 1-topos $\mathscr{E}$. There are two abuses of notation. First, all double arrows are boundary maps, and therefore are of the form $\Pi_{q-p}(-) \to \Pi_{q-p-1}(-)$. Second, we have used the notation two times for the maps a, b, c. This abuse can be solved by precising the degrees in the notation. Finally, using the last two relations ((2) and (5)) of Proposition 2.6(Fib4), one gets:

(5) $\gamma \circ \bar{\beta} = a \circ c$
(6) $b \circ \alpha = -\bar{\alpha} \circ b$.

Consider the left commutative diagram above. Then the upper (resp. lower) triangle defines a right (resp. left) unstable exact couple of degree $+1$ in the sense of Definition 2.31. Further, the above diagram is the unstable analog of a *Rees system* in the terminology of Eilenberg and Moore (see [McC01, 3.1]). In particular, both exact couples induce the same spectral sequence, up to the sign of the differential,[12] by using the procedure of Notation 2.34 and Definition 2.35:

$$E_1^{p,q} = \Pi_{q-p}(F_p) \Rightarrow \Pi_{q-p}(X). \tag{2.4.3}$$

Indeed, one computes the two possible differentials on E_1^{**} as follows:

$$d_1 := \beta \circ \gamma \overset{(2)}{=} \bar{\beta} \circ b \circ \gamma \overset{(1!)}{=} -\bar{\beta} \circ \bar{\gamma} =: -d'_1 \tag{2.4.4}$$

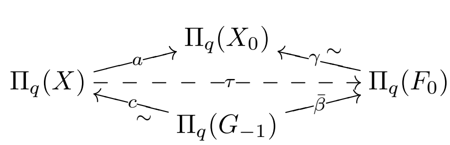
We still put $D_1^{p,q} = \Pi_{q-p}(X_p)$ (resp. $\bar{D}_1^{p,q} = \Pi_{q-p}(G_{p-1})$), so that $(E_1, D_1, \alpha, \beta, \gamma)$ is the exact couple of Example 2.32(2).

Remark 2.43. Beware that the symbol $\Rightarrow$ does not imply any convergence assertion. However, if the tower $X/X_\bullet$ is bounded, one gets that the preceding spectral sequence is strongly convergent in the sense of [BK72, IX, 5.3].

Example 2.44. The following example can help the reader understand our construction. We let X be a pointed CW-complex (or a pointed simplicial sheaf), and X_p be the p-th stage of the associated Postnikov tower. Then with the above notation, F_p is the Eilenberg MacLane space $K(\pi_p(X), p)$ and G_p is the p-th stage of the Moore tower (in particular, G_p is p-connected). The above unstable spectral sequence is well-known in topology (the so-called *unstable Atiyah-Hirzebruch spectral sequence*).

Notation 2.45. Let us continue to introduce notation in the situation of the preceding paragraph. Fix an integer $q \geq 0$. We define a canonical map τ by the following commutative diagram:

(2.4.5)



which follows from the relation (5).

[12] One could have avoided the sign issue by changing $\bar{\gamma}$ for example. However, the sign of the differential does not change either its kernel or its cokernel, so that it will not interfere with our computations at any time.

We note finally that the q-truncated cohomotopical complex $E_1^{*,q}$ admits τ as an augmentation (see Definition 2.27). Going back to homotopical indexing, one gets the following co-augmented homotopical complex:

$$* \to \Pi_q(X) \xrightarrow{\tau} \Pi_q(F_0) \xrightarrow{d_1^{0,q}} \Pi_{q-1}(F_1) \to \cdots \to \Pi_1(F_{q-1}) \xRightarrow{d_1^{q-1,q}} \Pi_0(F_q) \tag{2.4.6}$$

Before stating our main result, we start with a lemma which shows that it is possible to directly compute all the terms at once on the diagonal $p = q$.

Lemma 2.46. *Consider the above notation, and assume that the tower $X/X_\bullet$ is d-bounded for $d \in \mathbb{N}$. Then one gets:*

$$E_r^{q,q} = \begin{cases} \mathrm{Im}\left(\Pi_0(X) \xrightarrow{\tau} \Pi_0(F_0)\right) & q = 0 \wedge r > d \\ \gamma^{-1}\big(\mathrm{Im}\left(\Pi_0(X) \xrightarrow{a} \Pi_0(X_q)\right)\big)/\Pi_1(X_{q-1}) & (0 < q \leq d) \wedge r > \max(d-q, q-1) \\ * & (q > d) \wedge r \geq 1 \end{cases}$$

where, on the second line, we used the morphism of pointed $\mathscr{E}$-objects $\gamma : \Pi_0(F_q) \to \Pi_0(X_q)$, and the action of $\Pi_1(X_{q-1})$ on $\Pi_0(F_q)$.

In particular, the abutment filtration on $\Pi_0(X)$ is finite, with graded pieces given by the pointed $\mathscr{E}$-objects $E_\infty^{q,q}$ for $0 \leq q \leq d$.

Proof. This follows from Remark 2.36 taking into account the d-boundedness condition. □

Our main technical result is the following degeneracy criterion:

Theorem 2.47. *Consider the above notation, and assume the tower $X/X_\bullet$ is d-bounded for $d \in \mathbb{N} \cup \{+\infty\}$. Fix an integer $q \geq 0$, and let I be the set of pairs of integers (p, i) such that $0 \leq p \leq \min(d, q)$, $i \in \{0, 1\}$, except that $i \neq 1$ if $p = 0$. Then the following conditions are equivalent.*

- *(i) $\forall (p, i) \in I$, the pointed map $\bar{\alpha} : \Pi_{q-p}(G_{p-i}) \to \Pi_{q-p}(G_{p-i-1})$ is trivial.*
- *(i') $\forall (p, i) \in I$, the pointed map $\bar{\beta} : \Pi_{q-p}(G_{p-i-1}) \to \Pi_{q-p}(F_{p-i})$ has trivial kernel, and is even a monomorphism if $q > 0$.*
- *(ii) $\forall (p, i) \in I$, the pointed maps $c : \Pi_{q-p}(G_{p-i}) \to \Pi_{q-p}(X)$ and $b \circ \alpha : \Pi_{q-p+1}(X_{p-i}) \to \Pi_{q-p+1}(X_{p-i-1}) \to \Pi_{q-p}(G_{p-i-1})$ are trivial.*
- *(iii) For the fixed integer q, the coaugmented homotopical complex (2.4.6) is exact in the sense of Definition 2.27.*

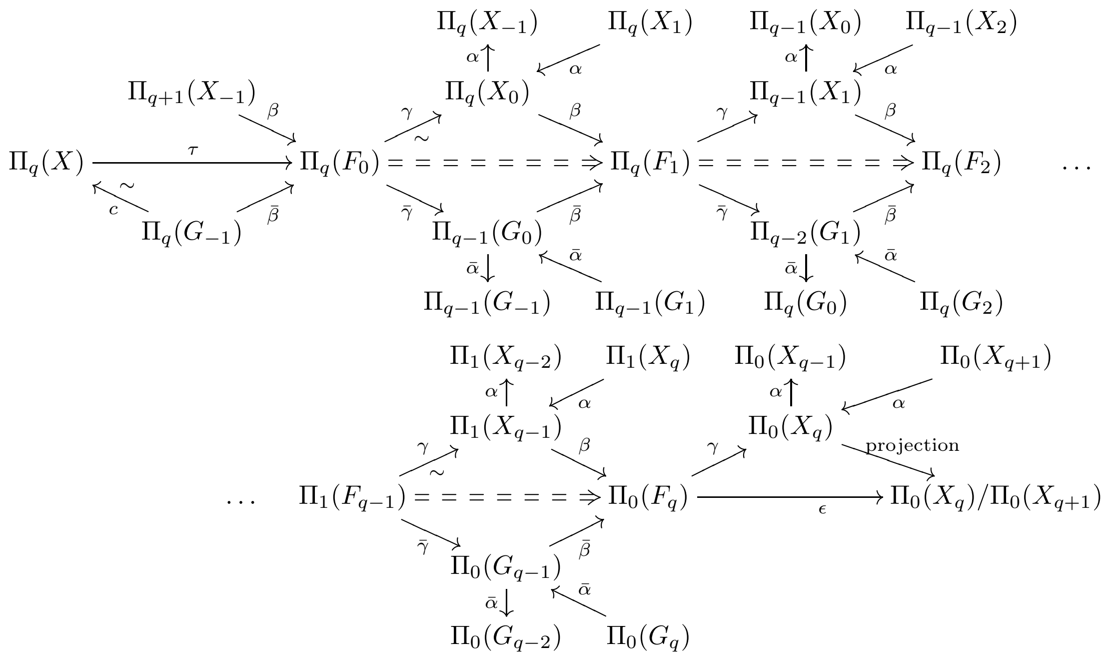
Note that point (iii) implies that:

$$E_2^{p,q} = \begin{cases} \Pi_q(X) & p = 0 \wedge q > 0, \\ * & 0 < p < q. \end{cases}$$

In particular, when $d < +\infty$ and under the above equivalent conditions, the spectral sequence degenerates at E_{d+1}^{**}, one has $E_\infty^{0,q} = \pi_q(X)$ for $q > 0$, and there is a finite number of non-necessarily trivial terms $E_\infty^{q,q}$ for $0 \leq q \leq d$ which describe the graded pieces of a finite filtration on $\Pi_0(X)$.

Proof. We heavily use notation from Notation 2.42. It can also be helpful for the reader to have the following "unfolded" diagram:

(2.4.7)



The dotted double arrows are the differentials of the unstable spectral sequence. Beware that the square part of this diagram is anti-commutative (see (2.4.4)). Finally, when one follows a path of two or more arrows in the diagram, one gets a part of a homotopy exact sequence.

With all these preparations, one can start our proof. The fact (i') implies (i) follows from (2∗). Reciprocally, (i) implies that $\ker(\bar\beta) = *$, which implies the result for $q = 0$. When $q > 0$, we need to prove that $\bar\beta : \Pi_0(G_{p-1}) \to \Pi_0(F_p)$ is a monomorphism, we apply property (T3) of Notation 2.16 as $\bar\beta$ is $\Pi_1(X_{p-1})$-equivariant according to (2), and the action of $\Pi_1(X_{p-1})$ on $\Pi_0(G_{p-1})$ is transitive because of (2∗) and the assumption (i).

The implication $(i) \Rightarrow (ii)$ follows by combining (3) and (6). To prove $(ii) \Rightarrow (i)$, we first remark that (ii) implies that b is an epimorphism according to (3∗). Thus we can conclude according to (6).

It remains to show that (iii) is equivalent to all the other conditions. Let us first remark that, when $q = 0$, (iii) just means that τ has trivial kernel. This is equivalent to (i') has $\bar\beta$ can be identified with τ thanks to the commutativity of (2.4.5). Thus, we assume $q > 0$ for the remaining of the proof.

Let us first prove that all the other conditions imply (iii). According to (i'), we know that $\bar\beta$ is a monomorphism. Thus the commutative diagram (2.4.5) shows that τ is a monomorphism. According to (2∗), we know that $\mathrm{Im}(\bar\beta) = \mathrm{Ker}(\bar\gamma)$. Applying again the fact that $\bar\beta$ is a monomorphism, one deduces that $\mathrm{Im}(\tau) = \mathrm{Ker}(d_1^{0,q})$. Next, condition (i) and (2∗) imply that $\bar\gamma$ is an epimorphism. Thus, one deduces $\mathrm{Im}(d_1^{0,q}) = \mathrm{Im}(\bar\beta) \overset{(2*)}{=} \mathrm{Ker}(\bar\gamma) = \mathrm{Ker}(d_1^{1,q})$. Repeating this argument allows one to conclude by an obvious induction.

We finally prove that $(iii) \Rightarrow (i)$. According to the commutativity of (2.4.5), one deduces that $\bar\beta : \Pi_q(G_{-1}) \to \Pi_{q+1}(F_0)$ is a monomorphism. Thus (2∗) implies that $\bar\alpha : \Pi_*(G_{-1}) \to \Pi_*(G_0)$ is trivial. The latter triviality together with (2∗) implies that $\bar\gamma : \Pi_{*+1}(F_0) \to \Pi_*(G_0)$ is an epimorphism. Now we apply the exactness of the unstable complex at $\Pi_q(F_0)$: one gets that $\mathrm{Ker}(\bar\gamma) = \mathrm{Im}(\tau) = \mathrm{Ker}(d_1^{0,q}) = \mathrm{Ker}(\bar\beta' \circ \bar\gamma)$. This implies that $\bar\beta' : \Pi_*(G_0) \to \Pi_*(F_1)$ has trivial kernel when restricted to the image of $\bar\gamma$. So $\mathrm{Ker}(\bar\beta) = *$ and applying (2∗) again, one deduces that $\bar\alpha : \Pi_*(G_0) \to \Pi_*(G_1)$ is trivial. Iterating this reasoning, one concludes by induction on p. □

Remark 2.48. Point (iii) in the case $q = 0$ states that the canonical map $\tau : \Pi_0(X) \to \Pi_0(F_0)$ has trivial kernel. One should be careful that it is not a monomorphism in general. The defect of injectivity is measured by the filtration on $\Pi_0(X)$, and therefore by the non-triviality of the graded pieces $E_\infty^{q,q}$ when $q > 0$ (computed in Lemma 2.46).

Definition 2.49. When the equivalent conditions of the preceding theorem are satisfied, we will say that the spectral sequence (2.4.3) essentially collapses at E_2^{**} on the column $* = 0$ and line q. If it collapses at all lines, we just say that it collapses at E_2^{**} on the column $* = 0$.

Notation 2.50. One can refine slightly the exactness part of the above *collapsing* property. Consider again the assumption of the preceding theorem, that the spectral sequence E_1^{**} essentially collapses and that $d < +\infty$.

Recall from Notation 2.34 that the coaugmented homotopical complex (2.4.6) also admits an augmentation ϵ:
(2.4.8)

$$* \to \Pi_q(X) \xrightarrow{\tau} \Pi_q(F_0) \xrightarrow{d_1^{0,q}} \Pi_{q-1}(F_1) \to \cdots \to \Pi_1(F_{q-1}) \xRightarrow{d_1^{q-1,q}} \Pi_0(F_q) \xrightarrow{\epsilon} \Pi_0(X_q)/\Pi_0(X_{q+1}).$$

where the target of ϵ is the cokernel of the map α.

Proposition 2.51. *Consider the above notation and assumptions. Then for any $q \geq d - 1$ the biaugmented homotopical complex (2.4.8) is exact in the sense of Definition 2.27 (up to reindexing).*

Moreover, for $q = 1$ and $d \leq 2$, the corresponding biaugmented 1-truncated homotopical complex is strongly exact:

$$* \to \Pi_1(X) \xrightarrow{\tau} \Pi_1(F_0) \xRightarrow{d_1^{0,1}} \Pi_0(F_1) \xrightarrow{\epsilon} \Pi_0(X_1)/\Pi_0(X_2). \tag{2.4.9}$$

Left to the reader.

Remark 2.52. The exactness of (2.4.9) can be surprising at first sight. Let us first mentioned that it appears crucially in Morel's study of the $\mathbb{A}^1$-fundamental sheaf: see in particular [Mor12, Def. 2.20].

For further links with Morel's theory, we refere the reader to Example 3.36 and Remark 3.37. We will also highlight examples in terms of torsors under algebraic groups for which the preceding proposition is crucial: see Corollary 3.39.

If we drop the equivalence of the three conditions in Theorem 2.47, we can give a refined version that will be useful in some of our examples (in particular in the relative case of Gersten conjecture).

Proposition 2.53. *We consider the notation of Notation 2.42. We assume in addition that the tower $X/X_\bullet$ under X is d-bounded for an integer $d > 0$ and consider two integers $0 \leq e \leq d$, and $q \geq 0$. We assume the following condition:*

(i) $\forall p \in [e, d]$ such that $p \leq q$, both pointed maps

$$\Pi_{q-p}(G_p) \xrightarrow{\bar{\alpha}^{p,q}} \Pi_{q-p}(G_{p-1}) \xrightarrow{\bar{\alpha}^{p-1,q-1}} \Pi_{q-p}(G_{p-1})$$

are trivial.

Then the $(q - e)$-truncated cohomotopical complex

$$\Pi_{q-e}(F_e) \xrightarrow{d_1^{e,q}} \Pi_{q-e-1}(F_{e+1}) \to \cdots \to \Pi_1(F_{q-1}) \xRightarrow{d_1^{q-1,q}} \Pi_0(F_q)$$

is exact in the sense of Definition 2.27.

Proof. The proof works as the one of Theorem 2.47: condition (i) implies that, for $e \leq p \leq d$ and $p \leq q$, $\bar{\gamma}^{p-1,q}$ is an epimorphism and $\bar{\beta}^{p-1,q-1}$ is a monomorphism (beware that for $p = q$, one needs the argument at the beginning of the proof of *loc. cit.* to deduce from $\mathrm{Ker}(\bar{\beta}^{p-1,p-1}) = *$ to $\bar{\beta}^{p-1,p-1}$ is a monomorphism). These two facts allow to conclude, as in *loc. cit.* □

3. Unstable coniveau spectral sequence

3.1. Unstable coniveau exact couple.

Notation 3.1. Let X a scheme, with topology $t = \mathrm{Zar}, \mathrm{Nis}$. Let $\mathrm{Sh}^\infty(X_t)$ be the associated ∞-topos, and consider an additive cohomotopy functor (Definition 2.23) on $\mathrm{Sh}^\infty(X_t)$ with values in an arbitrary topos $\mathscr{E}$:

$$\Pi_* : \mathrm{Sh}^\infty(X_t)^{op}_* \to \mathscr{E}_{\pi_*},$$

as in Definition 2.23. We will also consider $\mathscr{C} = \mathrm{pro}-\mathrm{Sh}^\infty(X_t)$ the ∞-category of pro-objects, and let $\Pi_* : \mathscr{C}^{op}_* \to \mathscr{E}_*$ be the homotopy functor obtained by Kan extension, according to the formula:

$$\Pi_*(\text{“}\varprojlim_{i\in I}\text{”} F_i) = \varinjlim_{i\in I} \Pi_*(F_i).$$

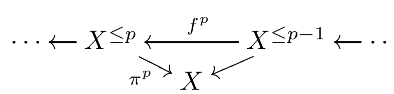
The filtered colimit is taken in the category of π_*-structures in $\mathscr{E}$ (see Remark 2.18). The fact that this formula does provide a homotopy functor follows from Proposition 2.29.

Recall that a flag on X is a decreasing sequence $Z^* = (Z^p)_{p\in\mathbb{N}}$ of closed subschemes of X such that $\mathrm{codim}_X(Z^p) \geq p$.[13] We let $\mathscr{F}(X)$ be the set of flags of X, ordered by inclusion. This is a filtered ordered set. For a non-negative integer p, we will consider the pro-object of X_t:

$$X^{\leq p} := \text{“}\varprojlim_{Z^*\in\mathscr{F}(X)}\text{”} (X - Z^{p+1}).$$

The notation reflects the fact that the limit of this pro-object in the underlying category of sets is equal to the set of points of codimension less or equal to p in X.

One deduces the so-called *coniveau tower of* X, which is an *increasing* tower[14] over the object X in the category $\mathrm{pro}-X_t$ of pro-objects in $X - t$:

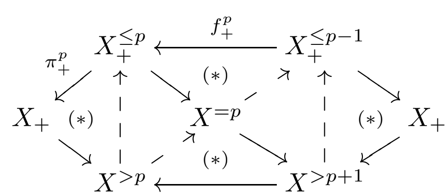


If X is of finite dimension d, this tower is bounded and $X^{\leq n} = X$ for $n \geq d$.

By considering the Yoneda embedding and adding a base point, we will view this as an increasing tower $X_+^{\leq *}$ over X_+ in the category $\mathscr{C}_*$. Equivalently, this is a tower over X_+ in $\mathscr{C}_*^{op}$ in the sense of Notation 2.39 (in particular, decreasing). One can therefore apply the constructions of Notation 2.42 to this latter tower.

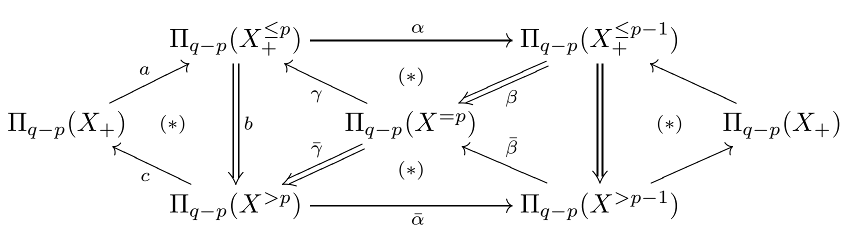
We do it explicitly for future references and for the comfort of the reader. One starts with a dual octahedron diagram in $\mathscr{C}_*$, dual to (2.4.1):

(3.1.1)



where $X^{=p}$ and $X^{>p}$ are respectively the cokernel of f_+^p and π_{p+}. Then one applies the contravariant functor Π_* and gets a diagram for $p \leq q$ satisfying the same properties as (2.4.2):

(3.1.2)



[13]Note that $Z^p = \varnothing$ if $p > \dim(X)$.

[14]In other words, with the notation of Example 2.32, this is a functor $X^{\leq \bullet} : \overrightarrow{\mathbb{N}} \to \mathrm{pro}-X_t/X$.

Therefore, according to *loc. cit.*, one gets an unstable spectral sequence in $\mathscr{E}$:

$$E_{1,c}^{p,q}(X,\Pi_*) := \Pi_{q-p}(X^{=p}) \Rightarrow \Pi_{q-p}(X_+) \tag{3.1.3}$$

which is associated with both exact couples $(E_{1,c}^{**}, D_1^{p,q} = \Pi_{q-p}(X_+^{\leq p}), \alpha, \beta, \gamma)$ and $(E_{1,c}^{**}, \bar{D}_1^{p,q} = \Pi_{q-p}(X^{>p-1}), \bar{\alpha}, \bar{\beta}, \bar{\gamma})$ respectively of type I and II.

Definition 3.2. With the above notation, we will call (3.1.3) the unstable coniveau spectral sequence of X with coefficients in Π_*.

The spectral sequence is strongly convergent whenever the scheme X is of finite dimension.

Example 3.3. Consider a simplicial sheaf $\mathcal{X}$ on X_t. It represents an additive cohomotopy functor with values in the punctual topos $\mathscr{S}et$:

$$\Pi_*^{\mathcal{X}} := \Pi_*(-,\mathcal{X}) : \mathrm{Sh}^\infty(\mathrm{Sm}_k)_*^{op} \to \mathscr{S}et_{\pi_*}, \mathcal{Y} \mapsto \Pi_*(\mathcal{Y},\mathcal{X}) := \pi_*\big(\mathrm{Map}_*(\mathcal{Y},\mathcal{X})\big).$$

The unstable coniveau spectral sequence associated with the latter cohomotopy functor and any scheme X has the form:

$$E_{1,c}^{p,q}(X,\mathcal{X}) := \pi_{q-p}\big(\mathrm{Map}(X^{=p},\mathcal{X})\big) = \varinjlim_{Z^* \in \mathscr{F}(X)} [S^{q-p} \wedge (X - Z^{p+1}/X - Z^p), \mathcal{X}] \Rightarrow [S^{q-p} \wedge X_+, \mathcal{X}]$$

where $[-,-]$ means homotopy classes of pointed t-sheaves on Sm_k.[15]

Notation 3.4. *Cohomotopy theory with supports.* To obtain the preceding spectral sequence, we can weaken the needed theory.

Let $\mathscr{S}$ be a category of schemes whose objects are stable under étale extensions. A closed $\mathscr{S}$-pair is a couple (X,Z) such X is a scheme in $\mathscr{S}$ and Z is a closed subscheme of X.

A *cohomotopy theory with supports* Π_* on $\mathscr{S}$ and coefficients in $\mathscr{E}$[16] is the data for any closed $\mathscr{S}$-pair (X,Z) of a π_*-structure $\Pi_*(X,Z)$ in $\mathscr{E}$ (Definition 2.17), and of the following additional structures:

- *contravariance*: for $f : Y \to X$ in $\mathscr{S}$, a pullback map $f^* : \Pi_n(X,Z) \mapsto \Pi_n(Y, f^{-1}(Z))$, morphism of groups for $n > 0$, of pointed sets for $n = 0$
- *covariance*: for a closed immersion $i : Z \to T$, X in $\mathscr{S}$, there are $i_* : \Pi_n(X,Z) \mapsto \Pi_n(X,T)$ morphism of groups for $n > 0$, of pointed sets for $n = 0$ and
- *long exact sequence*: for a closed immersion $i : Z \to T$, with open complement $j : X - Z \to X$, there exists a homotopy long exact sequence (Definition 2.20):

$$\cdots \Rightarrow \Pi_n(X,Z) \xrightarrow{i_*} \Pi_n(X,T) \xrightarrow{j^*} \Pi_n(X-Z, T-Z) \xRightarrow{\partial_{X,Z}} \Pi_{n-1}(X,Z) \to \cdots \tag{3.1.4}$$

Let again $t = \mathrm{Zar}, \mathrm{Nis}$. Then we say that Π_* is t-local if the following properties hold:

- *First additivity*: given closed pairs (X,Z) and (Y,T), the canonical map: $\Pi_*(X \sqcup Y, Z \sqcup T) \to \Pi_*(X,Z) \times \Pi_*(Y,T)$ is an isomorphism.
- *t-excision*: for any closed $\mathscr{S}$-pair (X,Z), any map $f : V \to X$ which is an open immersion in case $t = \mathrm{Zar}$ and an étale map if $t = \mathrm{Nis}$, and such that $T = f^{-1}(Z) \to Z$ is an isomorphism, the induced map $f^* : \Pi_n(X,Z) \to \Pi_n(V,T)$ is an isomorphism.[17]
- *Second additivity*: Given a scheme X and disjoint closed subschemes Z, T of X, the canonical map $\Pi_*(X, Z \sqcup T) \to \Pi_*(X-T, Z) \times \Pi_*(X-Z, T) \simeq \Pi_*(X,Z) \times \Pi_*(X,T)$, where the second map is defined using Zariski excision, is an isomorphism.

The reader can check that, to define the preceding unstable coniveau spectral sequence, we only need the axioms of a cohomotopy theory with support. The fact it is t-local will come into consideration solely for computations.

[15] Beware that the above inductive limit must be taken in the appropriate category.

[16] If the coefficients are not indicated, it is intended that $\mathscr{E} = \mathscr{S}et$.

[17] Recall that when $t = \mathrm{Nis}$, we say that $f : (V,T) \to (X,Z)$ is *excisive*.

Remark 3.5. On the one hand, it is useful to have the preceding down-to-earth axiomatic. On the other hand, it is always satisfied in the abstract context of Notation 3.1. Indeed, given an additive cohomotopy functor $\Pi_* : \mathrm{Sh}^\infty(X_t) \to \mathscr{E}_{\pi_*}$, one puts:

$$\Pi_n(X,Z) = \Pi_n^Z(X) := \Pi_n(X/X-Z) \tag{3.1.5}$$

where $X/X-Z$ is the (homotopy) cofiber of the open immersion $j : (X-Z) \to X$ computed in $\mathrm{Sh}^\infty(X_t)$. This defines a cohomotopy theory with supports on X_t with coefficients in $\mathscr{E}$, as follows from Proposition 2.6 (applied for cofiber sequences, see Remark 2.7(1)). Besides it is t-local as defined above.[18]

Notation 3.6. Let I be a set. In any category $\mathscr{C}$ with finite products and filtered colimits, one can define the *restricted product* of a family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$ by the following formula:

$$\prod\nolimits'_{i\in I} X_i = \mathrm{colim}_{S\in\mathcal{P}^f(I)}\left(\prod_{i\in S} X_i\right). \tag{3.1.6}$$

Note that in an additive category with filtered colimits, the restricted products coincide with coproducts. Note also that if we consider a family of pointed objects, then their restricted product is canonically pointed.

With this notation, we can state the following computation which gives the general shape of unstable coniveau spectral sequences.

Proposition 3.7. *Let Π_* be a t-local cohomotopy theory with support with coefficients in an arbitrary topos $\mathscr{E}$, as defined previously. Then for any couple of integers (p,q), there exists a canonical isomorphism:*

$$E_{1,c}^{p,q}(X,\Pi_*) \simeq \begin{cases} \bigoplus_{x\in X^{(p)}} \Pi^x_{q-p}(X^t_{(x)}) & p \le q-2, \\ \prod'_{x\in X^{(q-1)}} \Pi^x_1(X^t_{(x)}) & p = q-1, \\ \prod'_{x\in X^{(q)}} \Pi^x_0(X^t_{(x)}) & p = q, \end{cases}$$

where $\prod'$ denotes the restricted product (in the appropriate category), $X^{(p)}$ is the set of codimension p points $X^t_{(x)}$ denotes the pro-scheme of t-neighborhoods of x in X, and we have used notation (3.1.5) for $\Pi^x_*(X^t_{(x)})$, cohomotopy of $X^t_{(x)}$ with support in the closed point x.

Proof. Once the additivity property as been properly identified, the proof goes as in [Dég12, Lem. 1.14] taking into account formula (3.1.6). □

Corollary 3.8. *Consider the assumptions of the preceding proposition.*

Then for any integer $q \ge 0$, the q-truncated cohomotopical complex $E_{1,c}^{*,q}(X,\Pi_*)$ (Definition 2.27) associated with the coniveau unstable spectral sequence (3.1.3) has the following shape:

$$\bigoplus_{x\in X^{(0)}} \Pi_q(\kappa(x)) \to \bigoplus_{x\in X^{(1)}} \Pi^x_q(X^t_{(x)}) \to \cdots \to \prod_{x\in X^{(q-1)}}{}' \Pi^x_1(X^t_{(x)}) \Rightarrow \prod_{x\in X^{(q)}}{}' \Pi^x_0(X^t_{(x)}) \tag{3.1.7}$$

where we have assumed X is reduced to simplify the first term.

Definition 3.9. Consider the assumptions of the preceding corollary. The unstable q-truncated homotopical complex (3.1.7) will be called the *homotopical Gersten complex* of X with coefficients in Π_* and in degree q. We will use the notation:

$$C^p(X,\Pi_q) := E_{1,c}^{p,q}(X,\Pi_*) = \prod_{x\in X^{(q)}}{}' \Pi^x_{q-p}(X^t_{(x)}).$$

Crucially, the homotopical Gersten complex is bi-augmented: according to Notation 2.45, it admits an augmentation map

$$\tau_q^X : \Pi_q(X) \to C^0(X,\Pi_q) = \prod_{x\in X^{(0)}}{}' \Pi^x_q(X^t_{(x)})$$

[18]Use the isomorphism $X/(X-Z\sqcup T) \simeq X/(X-Z) \wedge X/(X-T)$ in $\mathrm{Sh}^\infty(X_t)$ for the second additivity.

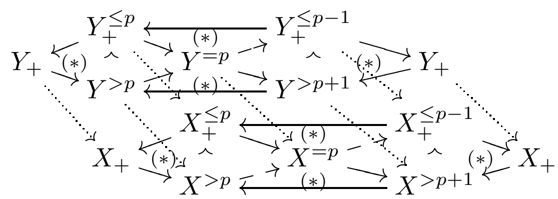
and, according to Notation 2.50, a co-augmentation map:

$$\gamma : C^q(X, \Pi_q) = \prod_{x \in X^{(q)}}{}' \Pi_0^x(X^t_{(x)}) \to \Pi_0(X_+^{\leq q}).$$

It is usually more accurate to consider the following co-augmentation:

$$\epsilon_q^X : C^q(X, \Pi_q) = \prod_{x \in X^{(q)}}{}' \Pi_0^x(X^t_{(x)}) \xrightarrow{\gamma} \Pi_0(X_+^{\leq q}) \to \Pi_0(X_+^{\leq q})/\Pi_0(X_+^{\leq q+1}).$$

Notation 3.10. It is well-known that the coniveau filtration is functorial with respect to flat pullbacks. In particular, given a flat morphism $f : Y \to X$ of schemes, as the pullback along f respects codimension, one gets a morphism of ordered sets: $f^{-1} : \mathscr{F}(X) \to \mathscr{F}(Y), Z^* \mapsto T^* = Z^* \times_X Y$. Moreover, f induces morphisms of closed pairs $(Y, T^p) \to (X, Z^p)$ and therefore one deduces that the diagram (3.1.1) is natural with respect to f, resulting in a commutative diagram in $\mathrm{PSh}^\infty(\mathscr{S})$:



Let us now come back to the situation of a small site X_t for one of the topologies: $t = \mathrm{Zar}, \mathrm{Nis}$. If $f : V \to W$ is a morphism of schemes in X_t, it is in particular flat (in fact étale). Thus the above diagram can be taken in the ∞-topos $\mathrm{Sh}^\infty(X_t)$.

In particular, given a homotopy functor $\Pi_* : \mathrm{Sh}(X_t) \to \mathscr{E}_{\pi_*}$, one deduces that diagram (3.1.2) and therefore the left and right unstable coniveau exact couples are functorial in X_t. Consequently, the same holds for the unstable coniveau spectral sequence (3.1.3) and for the biaugmented unstable Gersten complex (3.9). As a result, one gets pullback maps:

$$f^* : C^*(U, \Pi_*) \to C^*(V, \Pi_*) \tag{3.1.8}$$

in the category of (biaugmented) unstable complexes with coefficients in $\mathscr{E}$.

3.2. **Unstable Gersten complexes.**

Notation 3.11. Let again $t = \mathrm{Zar}, \mathrm{Nis}$, and X be a scheme. The classical framework of t-local cohomology with support can be extended to the non-abelian setting (see [Har66, §IV] for Zariski sheaves of abelian groups, and [DFJ22, §4.3] in the Nisnevich case). Given a pointed t-sheaf F of sets, in $\mathrm{Sh}(X_t)_*$, and $Z \subset X$ a closed subscheme, one puts:

$$\Gamma_Z(X, F) = \{\rho \in F(X) \mid \forall x \notin Z, \rho_x = *\}.$$

Similarly, $\underline{\Gamma}_Z(F)$ is the pointed t-sheaf $V \mapsto \underline{\Gamma}_{Z \times_X V}(F|_V)$. If $T \subset Z$, one let $\underline{\Gamma}_{Z/T}(F)$ be the cokernel, in the category of pointed t-sheaves, of the natural pointed map $\underline{\Gamma}_T(F) \to \underline{\Gamma}_Z(F)$.

We next use the notation of the coniveau tower laid down in Notation 3.1, and define:

$$\underline{\Gamma}_{X^{\geq p}}(F) = \mathrm{colim}_{Z^* \in \mathscr{F}(X)} \underline{\Gamma}_{Z^p}(F), \quad \underline{\Gamma}_{X^{=p}}(F) = \mathrm{colim}_{Z^* \in \mathscr{F}(X)} \underline{\Gamma}_{Z^p/Z^{p+1}}(F).$$

The following lemma is the non-abelian version (and t-local) version of [Har66, Prop. 2.1], based on the notion of restricted product (3.1.6).

Lemma 3.12. *Let F be a pointed t-sheaf of sets on a scheme X and consider the above notation. For any integer $p \geq 0$, the following conditions are equivalent:*

(i) *There exists an isomorphism $F \simeq \prod'_{x \in X^{(p)}} x_*(M_x)$ where for any point $x \in X^{(p)}$, M_x is a pointed set, and $i_x : \{x\} \to X$ is the canonical immersion.*
(ii) *The canonical pointed maps $F \leftarrow \underline{\Gamma}_{X^{\geq p}}(F) \to \underline{\Gamma}_{X^{=p}}(F)$ are isomorphisms.*

Moreover, when these conditions hold, the isomorphism of (i) induces an isomorphism $M_x \simeq F_x$.

Proof. The lemma easily follows once we get the computation, for any open $U \subset X$:

$$\underline{\Gamma}_{X^{=p}}(F)(U) \simeq \prod\nolimits'_{x \in U^{(p)}} F_x$$

which follows as in the proof of Proposition 3.7. □

We can now consider a non-abelian and t-local version of Cousin complexes (see [Har66, Def. p. 241]).

Definition 3.13. Consider the above assumption and notation. Let X be a scheme and $p \geq 0$ be an integer. A pointed t-sheaf F on X is said to be *supported in* $X^{(p)}$ if the equivalent conditions of the preceding proposition hold.[19]

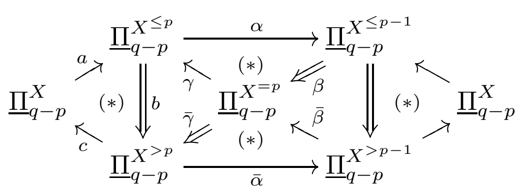
Let C^* be a q-truncated cohomotopical complex with coefficients in the topos $\mathrm{Sh}(X_t)$ (Notation 2.26). One says that C^* is a *q-truncated t-local unstable Cousin complex* if for any integer $0 \leq p \leq q$, the sheaf C^p is supported in $X^{(p)}$ in the above sense. If $q \geq \dim(X)$, one simply says *unstable t-local Cousin complex.*

In fact, if $q \geq \dim(X) + 2$, a q-truncated cohomotopical complex C^* with coefficients in the topos $\mathrm{Sh}(X_t)$ is simply a complex of abelian t-sheaves on X. In this case, C^* is an unstable Zar-local (resp. Nis-local) Cousin complex over X if and only if it is a Cousin complex over X in the sense of Hartshorne [Har66, Def. p. 241] (resp. [DFJ22, Def. 4.3.7]).

Notation 3.14. We consider a scheme X and one of the topologies $t = \mathrm{Zar}, \mathrm{Nis}$. We let Π_* be a t-local cohomotopy theory with support defined on X_t (for example a cohomotopy functor $\Pi_* : \mathrm{Sh}^\infty(X_t)^{op} \to \mathscr{S}et$).

We have associated to Π_* respectively in Equation (3.1.1), Equation (3.1.3), Definition 3.9 two unstable coniveau exact couples, a coniveau unstable spectral sequence and a Gersten complex, with respect to X, and even to any scheme V in X_t. We know from Notation 3.10 that they are in fact functorial in the scheme V in X_t. The associated t-sheaf functor being exact, it preserves homotopy long exact sequences and (co)homotopical biaugmented complexes. This implies that we can t-sheafify all the constructions of (3.1) meaning we obtain unstable exact couples with coefficients in the 1-topos $\mathrm{Sh}(X_t)$:

(3.2.1)



where $\underline{\Pi}_* = \underline{\Pi}_*^X$ (resp. $\underline{\Pi}_*^{X^{\leq p}}$, $\underline{\Pi}_*^{X^{>p}}$, $\underline{\Pi}_*^{X^{=p}}$) is the t-sheaf, on the small site X_t, associated with the presheaf $V \mapsto \Pi_*(V)$ (resp. $\Pi_*(V^{\leq p})$, $\Pi_*(V^{>p})$, $\Pi_*(V^{=p})$), seen as a $\mathrm{Sh}(X_t)_*$-structure (in the sense of Definition 2.17). Therefore, one deduces as in *loc. cit.* an unstable spectral sequence with coefficients in $\mathrm{Sh}(X_t)$:

$$\underline{E}_{1,c}^{p,q}(X_t, \Pi_*) := \underline{\Pi}_{q-p}^{X^{=p}} \Rightarrow \underline{\Pi}_{q-p}^X \tag{3.2.2}$$

which is part of both a left and a right unstable exact couple. The left one is given by the formula: $\underline{D}_c^{p,q}(X_t, \Pi_*) = \underline{\Pi}_{q-p}^{X^{\leq p}}$.

Definition 3.15. The unstable spectral sequence (3.2.2) will be called the *t-local unstable coniveau spectral sequence* associated with the cohomotopy theory with support Π_*.

For any integer $q \geq 0$, the cohomotopical complex $\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q) := \underline{E}_{1,c}^{*,q}(X_t, \Pi_*)$ corresponding to the q-th line of the preceding spectral sequence will be called the *t-local (unstable) Gersten complex* in degree q associated with Π_* on the site X_t. When t is clear we simply say *Gersten complex.*

When $\Pi_* = \Pi_*^{\mathcal{X}}$ is represented by a simplicial t-sheaf $\mathcal{X}$ as in Example 3.3, we will write: $\underline{\mathrm{Ge}}_t^*(\mathcal{X}, q) = \underline{\mathrm{Ge}}^*(X_t, \Pi_*^{\mathcal{X}}, q)$.

[19] This terminology seems preferable to "lies on the p-th skeleton" in [Har66, Def. p. 231]

The Gersten complex $\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q)$ associated with a cohomotopy theory Π_* on the small site X_t is a cohomotopical complex in the topos $\mathrm{Sh}(X_t)$, which, according to Corollary 3.8 can be computed by the following formula:

$$\underline{\mathrm{Ge}}^p(X,\Pi_*, q) = {\prod}'_{x\in X^{(p)}} x_*\Big(\Pi^x_{q-p}(X_{(x)})\Big) \tag{3.2.3}$$

where the restricted product is taken in the category of t-sheaves on X respectively of pointed sets if $q-p=0$, groups if $q-p=1$ and abelian groups if $q-p>1$ (in which case it is simply a direct sum). Moreover, it admits a canonical augmentation (as described in Notation 2.45):

$$\tau : \underline{\Pi}_q \xrightarrow{c^{-1}} \underline{\Pi}_q^{X>-1} \xrightarrow{\beta} \underline{\Pi}_q^{X=0} = \underline{\mathrm{Ge}}^0(X_t, \Pi_*, q).$$

Remark 3.16. We have restricted the above definition to homotopical functors Π_* with values in $\mathscr{S}et$, but everything will work with coefficients in an arbitrary topos. We will not use this generality here, but the case of the classifying topos BG associated with a discrete group G, or the pro-étale topos associated to a field, are interesting examples.

Lemma 3.17. *Consider the notation of the above definition. Then the Gersten complex $\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q)$ associated with Π_* is a q-truncated t-local Cousin complex in the sense of Definition 3.13.*

This is immediate according to Corollary 3.8.

Definition 3.18. Consider the setting of preceding definition. Given an integer $q \geq 0$, we will say that Π_* is *Gersten in degree* q on X_t if the q-truncated augmented homotopical complex of sheaves on X_t

$$\underline{\Pi}_q \to \underline{\mathrm{Ge}}^*(X_t, \Pi_*, q)$$

is exact. We simply say that Π_* is *Gersten on* X_t if it is so in all degree $q \geq 0$.

If $\Pi_* = \Pi_*^{\mathcal{X}}$ is represented by a simplicial sheaf $\mathcal{X}$ on X_t (Example 3.3), we simply say that $\mathcal{X}$ is *Gersten on* X_t (resp. *and in degree* q) if $\Pi_*^{\mathcal{X}}$ is so.

The remaining of the paper will give several situations where this property holds. According to our main result Theorem 2.47, we deduce the following characterisation of the above (unstable) Gersten property.

Proposition 3.19. *Let $t = \mathrm{Zar}, \mathrm{Nis}$, X be a scheme, and Π_* be a t-local cohomotopy with support defined on X_t. Then the following conditions are equivalent:*

(i) *Π_* is Gersten in degree q.*

(ii) *For any $x \in X$, the unstable coniveau spectral sequence*

$$E_{1,c}^{i,j}(X^t_{(x)}, \Pi_*) \Rightarrow \Pi_{j-i}(X^t_{(x)})$$

with coefficients in the t-local cohomotopy theory with support Π_* restricted to the small t-site of $X^t_{(x)}$, collapses on the column $j=0$ and line q (see Definition 2.49).

(iii) *for any point $x \in X$, $\mathfrak{X} = X^t_{(x)}$, any integer $0 \leq p \leq q$, any $n = p, p+1$, any closed subscheme $Z \subset \mathfrak{X}$ such that $\mathrm{codim}_{\mathfrak{X}}(Z) > n$, and for any element $\alpha \in \Pi_{q-p}(\mathfrak{X}, Z)$, there exists a closed subscheme T of $\mathfrak{X}$ such that $Z \overset{i}{\subset} T$, $\mathrm{codim}_{\mathfrak{X}}(T) \geq n$, and $i_*(\alpha) = *$ in $\Pi_{q-p}(\mathfrak{X}, T)$.*

Proof. The equivalence of (i) and (ii) is obvious. Then the equivalence between (ii) and (iii) follows[20] from Theorem 2.47 as property (iii) above is a simple translation of property (i) in *op. cit.* □

Remark 3.20. Property (iii) is a (slightly more precise) version of the *effaceability condition* first highlighted by Bloch and Ogus in order to axiomatize the proof of Quillen (see [BO74]). The interest of the preceding proposition is that condition (iii) is not only sufficient but also necessary.

[20]Use the following dictionnary: X_p, F_p, G_p replaced respectively by $X^{\leq p}$, $X^{=p}$, $X^{>p}$.

3.3. The abelian case and Eilenberg-MacLane sheaves.

Notation 3.21. In this section, we want to compare the unstable coniveau spectral sequence with the more classical framework of the (stable) coniveau spectral sequence. We consider again the conventions of the preceding sections. X is an arbitrary scheme, with topology $t = \mathrm{Zar}, \mathrm{Nis}$.

Let C be a complex of abelian sheaves on X_t. In [Har66, Def. p. 277], Hartshorne defines the Cousin complex associated with C, when C is bounded and $t = \mathrm{Zar}$. This construction has been, in an obvious way, extended to the case where C is unbounded and $t = \mathrm{Zar}, \mathrm{Nis}$ in [DFJ22, Def. 4.3.7, Th. 4.3.8], by taking as a function $\delta = -\operatorname{codim}_X$ on points of X.[21]

In fact, we can easily recall the definitions of *loc. cit.* by using the exact method of Notation 3.1, up to changing the conventions on indexation. One first uses the cohomological functor on the stable ∞-category $\mathrm{D}^\infty(X_t, \mathbb{Z})$ with coefficients in the abelian category $\mathrm{Sh}(X_t, \mathbb{Z})$:

$$\underline{\mathrm{H}}^C : \mathrm{D}^\infty(X_t, \mathbb{Z})^{\mathrm{op}} \to \mathrm{Sh}(X_t, \mathbb{Z}), D \mapsto \underline{\mathrm{H}}_t^0 \,\underline{\mathrm{Hom}}(D, C).$$

As in *loc. cit.*, it can be Kan-extended (by taking colimits) to pro-objects of $\mathrm{D}^\infty(X_t, \mathbb{Z})$. Then one defines a cohomological exact couple (see [Dég12, 1.1.1] for our conventions) in the abelian category $\mathrm{Sh}(X_t, \mathbb{Z})$

$$^{st}\underline{E}_{1,c}^{p,q}(X_t, C) = \underline{\mathrm{H}}_t^{p+q} \,\underline{\mathrm{Hom}}(\mathbb{Z}(X^{=p}), C),$$
$$^{st}\underline{D}_{1,c}^{p,q}(X_t, C) = \underline{\mathrm{H}}_t^{p+q} \,\underline{\mathrm{Hom}}(\mathbb{Z}(X^{\leq p-1}), C).$$

This gives the t-local (stable) coniveau spectral sequence:

$$^{st}\underline{E}_{1,c}^{p,q}(X_t, C) \Rightarrow \underline{\mathrm{H}}_t^{p+q}(C)$$

The next definitions were given in [DFJ22, Section 4.3], again following [Har66, Def. p. 247].

Definition 3.22. Consider the above notation. One defines the *t-local Gersten complex* of the complex C as the following complex of abelian sheaves on X_t:

$$\underline{\mathrm{Ge}}^*(X, C, 0) := {}^{st}\underline{E}_{1,c}^{*,0}(X_t, C).$$

One says that C is *Cohen-Macaulay* on X_t if the preceding t-local coniveau spectral sequence is concentrated on the line $q = 0$.

When $C = F[0]$ is concentrated in degree 0, we will put $\underline{\mathrm{Cz}}_t^*(F) = \underline{\mathrm{Ge}}_t^*(F[0])$. This is the *$t$-local Cousin complex* associated with the sheaf F on X_t (with respect to the codimension filtration of X).

Note that the complex $\underline{\mathrm{Ge}}^*(X_t, C, 0)$ is in fact a (non-truncated) t-local Cousin complex made of abelian components in the sense of Definition 3.13.

Remark 3.23. When $t = \mathrm{Zar}$ (resp. $t = \mathrm{Nis}$), and $C = F[0]$ is a single sheaf placed in degree 0, the above definition coincides with that of [Har66, Def. p. 247] (resp. [DFJ22, Def. 4.3.6]). Recall moreover that F is Cohen-Macaulay on X_t if and only if the canonical augmentation map

$$F \to \underline{\mathrm{Cz}}_t^*(F)$$

is a quasi-isomorphism of t-sheaves.

The preceding property has several remarkable consequences.

Proposition 3.24. *Let C be a Cohen-Macaulay complex of abelian sheaves on X_t.*

Then the t-local coniveau spectral sequence converges. The edge morphisms of this spectral sequence induce for any $p \geq 0$ an isomorphism $\underline{\mathrm{H}}^p(C) \simeq \underline{\mathrm{H}}^p(\underline{\mathrm{Ge}}^*(X, C, 0))$ of abelian sheaves on X_t. Morevoer, the edge morphisms of the usual coniveau spectral sequence $^{st}E_{1,c}^{**}(X, C)$ associated with the cohomology theory with coefficients in C induces for any integer $p \in \mathbb{Z}$ an isomorphism:

$$H^p\big(\Gamma(X, \underline{\mathrm{Ge}}^*(X, C, 0))\big) = {}^{st}E_{2,c}^{p,0}(X, C) \simeq H^p(X_t, C).$$

Finally, all these isomorphisms can be lifted to a quasi-isomorphism of complexes $C \to \underline{\mathrm{Ge}}^*(X, C, 0)$.

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[21] Beware that this is not in general a dimension function. Nevertheless, for the definitions and results of *loc. cit.*, Section 4.3, one does not use this assumption. In fact, the only necessary condition is that if x is a specialization of y, then $\delta(x) \leq \delta(y)$. In fact latter case, we say that δ is a weak dimension function.

The first three assertions are consequences of the definitions. The last one is actually the generalized version of a theorem of Hartshorne-Suominen extended to the Nisnevich (and the Zariski) topology in [DFJ22, Th. 4.3.8].

Notation 3.25. Consider the above notation, but assume in addition that C is concentrated in non-negative degree. In that case, one defines the Eilenberg-Mac Lane simplicial t-sheaf $K(C)$ on X_t, which an object of $\mathrm{Sh}^\infty(X_t)$, using the (toposic) Dold-Kan correspondence. The main case comes from an abelian sheaf F on X_t and an integer $n \geq 0$; one puts: $K(F,n) = K(F[n])$.

The next proposition shades light on the fact unstable spectral sequences are *fringed*.

Proposition 3.26. *Consider the above notation. Then for any integers $p \leq q$, there exists a canonical isomorphism:*

$$\underline{E}_{1,c}^{p,q}(X, K(C)) \simeq {}^{st}\underline{E}_{1,c}^{p,-q}(X, C).$$

In fact, for any $q \geq 0$, there exists an isomorphism of cohomotopical complexes:

$$\underline{E}_{1,c}^{*,q}(X, K(C)) \simeq \tau_{nv}^{\leq q}\left({}^{st}\underline{E}_{1,c}^{*,-q}(X, C)\right)$$

where $\tau_{nv}^{\leq q}$ denotes the naive truncation functor in cohomological degree less or equal to q.

The proof is now just a matter of checking definitions as, from the Dold-Kan correspondence, one gets: $\pi_n \operatorname{Map}(X, K_t(C)) \simeq H_t^{-n}(X, C)$ and this isomorphism is compatible with the long exact sequences of cohomotopy (resp. cohomology) with support.

Corollary 3.27. *Consider the above notation. Let F be an abelian sheaf on X_t, and $n \geq 0$ be an integer.*

Then for any integer $q \geq 0$, there exists an isomorphism of augmented (by F) cohomotopical complexes

$$\underline{\mathrm{Ge}}^*(X_t, K(F,n), q) \simeq \tau_{nv}^{\leq q}\left(\underline{\mathrm{Cz}}_t^*(X, F[n-q])\right).$$

In particular, if X is finite dimensional and $q \geq \dim(X)$, one gets an isomorphism of augmented cohomotopical complexes

$$\underline{\mathrm{Ge}}^*(X_t, K(F,n), q) \simeq \underline{\mathrm{Cz}}_t^*(F[n-q]).$$

Corollary 3.28. *Consider the above notation and the following conditions:*

1 F is Cohen-Macaulay on X_t.
2 $K(F,n)$ is unstably Cohen-Macaulay in degree n on X_t.

Then (i) implies (ii). Moreover in that case, the unstable coniveau spectral sequence $E_{1,c}^{**}(X, K(F,n))$ is concentrated on the line $q = n$.

Assume that $n \geq \dim(X)$. Then (ii) implies (i), the cohomotopical complex $\underline{\mathrm{Ge}}^*(X_t, K(F,n), n)$ is a complex of abelian sheaves and a cohomological resolution of F. In fact, there exists a canonical isomorphism of cohomological resolutions of F:

$$\underline{\mathrm{Ge}}^*(X_t, K(F,n), n) \simeq \underline{\mathrm{Cz}}_t^*(X, F).$$

The next proposition and its corollary are the abstract version for the extension of Bloch-Ogus theory in the unstable setting, taken into account Proposition 3.19. It also relates the *Gersten setting* with the more classical Cousin setting as formalized by Grothendieck (see again [Har66]).

Proposition 3.29. *Let X be a scheme, $t = \mathrm{Zar}, \mathrm{Nis}$ and Π_* be a t-local cohomotopy theory with supports. Assume that Π_* is t-locally Gersten in degree $q > 1$.*

Then $\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q)$ is a q-truncated t-Cousin complex augmented by the abelian sheaf $\underline{\Pi}_q$ on X_t. There exists a unique isomorphism of complex augmented by $\underline{\Pi}_q$:

$$\tau_{nv}^{\leq q}\,\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q) \simeq \tau_{nv}^{\leq q}\,\underline{\mathrm{Cz}}_t^*(\underline{\Pi}_q)$$

where the right hand-side is the t-Cousin complex associated with the abelian sheaf $\underline{\Pi}_q$. In particular, $\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q)$ is in fact a complex of abelian groups, and for any integer $0 \leq p \leq q$, and $x \in X^{(p)}$, there exists a canonical isomorphism of abelian groups for $p < q-1$, groups for $p = q-1$ and pointed sets for $q = p$:

$$\Pi_{q-p}^x(X_{(x)}^t) \simeq H_x^p(X_{(x)}^t, \underline{\Pi}_q).$$

Finally, the preceding isomorphism induces an isomorphism for all $0 \leq p < q$*:*

$$E_{2,c}^{p,q}(X, \Pi_*) \simeq H^p\Gamma\big(X, \underline{\mathrm{Cz}}_t^*(\underline{\Pi}_q)\big) \simeq H^p(X_t, \underline{\Pi}_q).$$

Proof. The first statement is Lemma 3.17. The second statement follows from the uniqueness of q-truncated cohomotopical Cousin complexes (see Appendix: Theorem 4.19). The isomorphism then follows, as both complexes are t-flasque. □

In particular, if q is bigger than the dimension of X, we deduce from the proposition that $\underline{\Pi}_q$ is Cohen-Macaulay. Using the finer sutdy of the unstable spectral sequence, we can slightly improve that resul.

Proposition 3.30. *Consider the assumptions of the previous proposition and assume in addition that* $q \geq \dim(X)$*. Then the abelian sheaf* $\underline{\Pi}_q$ *over* X_t *(obtained by sheafification of* Π_q*) is Cohen-Macaulay, there exists a unique* isomorphism *of resolutions of* $\underline{\Pi}_q$

$$\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q) \simeq \underline{\mathrm{Cz}}_t^*(\underline{\Pi}_q)$$

and for all $0 \leq p \leq q$*,*

$$E_{2,c}^{p,q}(X, \pi_*) \simeq H^p(X_t, \underline{\Pi}_q).$$

Proof. As explained previously, the case $q > \dim(X)$ follows from the preceding proposition. For the case $q = \dim(X)$, we need to prove the exactness of the Gersten complex coaugmented by the constant sheaf $*$. This follows from Proposition 2.51. □

3.4. **The case of homotopy groups and non-abelian cohomology.**

Notation 3.31. Based on our notion of homotopy complex, we can extend the classical definition of Cousin resolutions and Cohen-Macaulay sheaves to the non-abelian context.

As in the preceding sections, we let X be a scheme equipped, and consider one of the topologies $t = \mathrm{Zar}, \mathrm{Nis}$. Let $\mathcal{G}$ be a sheaf groups on X_t, and $B\mathcal{G}$ be its classifying space (see e.g. [MV99, §4.1]), as an object of $\mathrm{Sh}^\infty(X_t)$. Recall that:

$$\pi_n(\mathrm{Map}(X, B\mathcal{G})) = [S^n \wedge X_+, B\mathcal{G}]_* \simeq \begin{cases} H^1(X_t, \mathcal{G}) & n = 0, \\ \mathcal{G}(X) & n = 1 \\ * & n > 1 \end{cases}$$

where $H^1(X_t, \mathcal{G})$ denotes the pointed set of t-local $\mathcal{G}$-torsors on X, and $[-,-]_*$ denotes the pointed homotopy classes in $\mathrm{Sh}^\infty(X_t)$. More generally, for any closed subset $Z \subset X$, $p = 0, 1$, we put:

$$H_Z^p(X_t, \mathcal{G}) = [S^{1-p} \wedge X/X - Z, B\mathcal{G}]_*$$

where $X/X - Z$ is pointed in the obvious way. According to this definition, we get a homotopy exact sequence in the classical sense (or "lonh homotopy sequence" in the final topos according to Definition 2.20):

$$* \to H_Z^0(X_t, \mathcal{G}) \to \mathcal{G}(X) \xrightarrow{j^*} \mathcal{G}(X - Z) \overset{\partial}{\Longrightarrow} H_Z^1(X_t, \mathcal{G}) \to H^1(X_t, \mathcal{G}) \xrightarrow{j^*} H^1\big((X - Z)_t, \mathcal{G}\big).$$

In particular, the boundary map corresponds to an action of $\mathcal{G}(X - Z)$ on the (pointed) set $H_Z^1(X_t, \mathcal{G})$. In fact, $H^*(-_t, \mathcal{G})$ is the t-local cohomotopy theory with support in the sense of Notation 3.4 represented by $B\mathcal{G}$).

We can therefore apply the previous considerations to this theory. For once, by taking colimits (see Proposition 2.29), this definition extends to any t-localization of X: given a point $x \in X$, we get in particular a homotopy exact sequence:

$$* \to H_x^0(X_{(x)}^t, \mathcal{G}) \to \mathcal{G}(X_{(x)}^t) \xrightarrow{j^*} \mathcal{G}(X_{(x)}^t - \{x\}) \overset{\partial}{\Longrightarrow} H_x^1\big(X_{(x)}^t, \mathcal{G}\big) \to * \tag{3.4.1}$$

so that $H_x^0(X_{(x)}^t, \mathcal{G})$ is the kernel of the morphism of groups j^* and $H_x^1(X_{(x)}^t, \mathcal{G})$ is the homogeneous set $\mathcal{G}(X_{(x)}^t - \{x\})/\mathcal{G}(X_{(x)}^t)$. For readability of the notation, we will put:

$$H_x^p(X_{(x)}^t, \mathcal{G}) = H_x^p(X_t, \mathcal{G}).$$

We also get the unstable coniveau spectral sequence Definition 3.2 as well as its sheafified version Definition 3.15. Motivated by the abelian case, we adopt the following terminology.

Definition 3.32. Consider the preceding notation. We define the homotopy Cousin complex of the sheaf $\mathcal{G}$ over X_t as

$$\underline{\mathrm{Cz}}_t^*(\mathcal{G}) = \underline{E}_{1,c}^{*,1}(X_t, \Pi_*).$$

It is a 2-term homotopy complex, concentrated in degree 0 and 1, co-augmented by $\mathcal{G}$:

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{\tau} & \prod'_{\eta \in X^{(0)}} \eta_*(\mathcal{G}(\kappa_\eta)) & \Longrightarrow & \prod'_{x \in X^{(1)}} x_* H^1_x(X_t, \mathcal{G}) \\ & & \parallel & & \parallel \\ & & \underline{\mathrm{Cz}}_t^1(\mathcal{G}) & \Longrightarrow & \underline{\mathrm{Cz}}_t^0(\mathcal{G}). \end{array}$$

See Notation 3.6 for the notation.

Based on the theory developed so far, we get the following conditions, non-abelian analogous of the characterization of Cohen-Macaulay abelian sheaves:

Proposition 3.33. *Let X be a scheme and $\mathcal{G}$ be a sheaf of groups on X_t, $t = \mathrm{Zar}, \mathrm{Nis}$. The following conditions are equivalent:*

(i) *The co-augmented homotopical complex $\mathcal{G} \to \underline{\mathrm{Cz}}_t^*(\mathcal{G})$ is exact (in the sense of Definition 2.27).*

(ii) *For any open (resp. étale if $t = \mathrm{Nis}$) V/X, one gets an isomorphism:*

$$\mathcal{G}(V) \simeq \left\{ g \in \prod\nolimits'_{\eta \in V^{(0)}} \mathcal{G}(\kappa_\eta) \mid \forall x \in V^{(1)}, g \text{ acts trivially on } H^1_x(V_t, \mathcal{G}) \right\}$$

induced by the canonical (restriction) map τ defined above.

(iii) *For any open (resp. étale if $t = \mathrm{Nis}$) V/X, any closed subset $Z \subset V$, the restriction map*

$$j^* : \mathcal{G}(V) \to \mathcal{G}(V - Z)$$

is an isomorphism if $\mathrm{codim}_Z(V) > 1$ and a monomorphism if $\mathrm{codim}_Z(V) = 1$.

(iv) *For for any point $x \in V$, and any $i = 0, 1$,*

$$H^i_x(X, \mathcal{G}) = *$$

if $i \neq \mathrm{codim}_X(x)$.

(v) *The simplicial sheaf $B\mathcal{G}$ is Gersten on X_t in the sense of Definition 3.18.*

Moreover, when these conditions hold, the obvious map

$$H^1(X_t, \mathcal{G}) \to \left(\prod\nolimits'_{x \in X^{(1)}} H^1_x(X_t, \mathcal{G})\right) / \left(\prod\nolimits'_{\eta \in X^{(0)}} \mathcal{G}(\kappa_\eta)\right)$$

is an injection of pointed sets. Finally, if $\dim(X) \leq 1$, the sequence of pointed sheaves

$$* \to \mathcal{G} \xrightarrow{\tau} \underline{\mathrm{Cz}}_t^0(\mathcal{G}) \overset{d}{\Rightarrow} \underline{\mathrm{Cz}}_t^1(\mathcal{G}) \to *$$

is exact and the above map is a bijection.

Proof. It is clear that (i) implies (ii), and (ii) implies (iii). One obtains that (iii) implies (iv) by applying colimits and using the homotopy exact sequence (3.4.1). Condition (iv) implies that the t-local coniveau spectral sequence $\underline{E}_{1,c}^{**}(X_t, B\mathcal{G})$ is concentrated on line $q = 1$. Therefore it degenerates at E_2 and in fact, it collapses on the column $* = 0$ in the sense of Definition 2.49. In other words, (iv) implies (v). Finally, (v) obviously implies (i).

To get the remaining assertions, we consider the homotopy sequence, exact by (i):

$$* \to \mathcal{G} \xrightarrow{\tau} \underline{\mathrm{Cz}}_t^0(\mathcal{G}) \overset{d}{\Rightarrow} \underline{\mathrm{Cz}}_t^1(\mathcal{G})$$

We let $F \subset \underline{\mathrm{Cz}}_t^1(\mathcal{G})$ be the pointed t-sheaf which is the image of d, so that $F = \underline{\mathrm{Cz}}_t^0(\mathcal{G})/\mathcal{G}$ by exactness of the preceding sequence. Therefore one can apply [Gir71, Chapitre III, Prop. 3.2.2] to the monomorphism τ to get a homotopy exact sequence:

$$* \to \mathcal{G}(X) \to \Gamma(X, \underline{\mathrm{Cz}}_t^0(\mathcal{G})) \Rightarrow \Gamma(X, F) \to H^1(X_t, \mathcal{G}) \to H^1(X_t, \underline{\mathrm{Cz}}_t^0(\mathcal{G})).$$

One concludes by noticing that $\underline{\mathrm{Cz}}_t^0(\mathcal{G})$ is a t-flasque sheaf of groups (flasque when $t = \mathrm{Zar}$, satisfying the Brown-Gersten property when $t = \mathrm{Nis}$). Finally $\dim(X) = 1$, we get the stated exactness by appealing to Proposition 2.51. □

Remark 3.34. If one uses the étale topology, all the above conditions are again equivalent, but the last assertion does not necessarily holds.

Definition 3.35. When a sheaf of groups $\mathcal{G}$ on X_t satisfies the equivalent conditions above, we will say that $\mathcal{G}$ is *homotopy Cohen-Macaulay* on X_t.

Example 3.36.
(1) An abelian sheaf which is Cohen-Macaulay is obvious a homotopy Cohen-Macaulay as sheaf of groups.
(2) It follows from [Mor12, Th. 6.1] that over a perfect field k, for any pointed simplicial sheaf $\mathcal{X}$ on the Nisenvich site Sm_k, the sheaf of groups $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is homotopy Cohen-Macaulay in the above sense over X_t for any smooth k-scheme X. This fact was our inspiration to this section. We will give a direct proof in the next section.

Remark 3.37. When $\mathcal{G}$ is a sheaf of groups over the Nisnevich site Sm_k, given a smooth k-scheme X, our augmented Cousin complex $\mathcal{G} \to \underline{\mathrm{Cz}}_t(\mathcal{G})$ on X_{Nis} is precisely the restriction of the complex $1 \to \mathcal{G} \to \mathcal{G}^{(0)} \rightrightarrows \mathcal{G}^{(1)}$ restricted to X_t as considered by Morel in [Mor12, §2.2].

Thanks to a result of Česnavičius and Scholze, and according to our previous proposition, one deduces the following result.

Theorem 3.38. *Let X be a noetherian scheme which is regular in codimension less than 3. Let G be a separated algebraic group scheme over X, which we assume to be affine if $\dim(X) > 1$.*
Then G is a homotopy Cohen-Macaulay sheaf in the above sense.

Proof. In fact, one deduces property (iii) of Proposition 3.33 by applying [ČS24, Lem. 7.2.7] to $Y = G$, $X = V$ and $Z = Z$ — under the notation of (iii). □

Corollary 3.39. *Let X be a connected Dedekind scheme with function field K, and G be a separated X-group scheme. Then G is homotopy Cohen-Macaulay and the canonical pointed map:*

$$H^1(X_t, G) \to \left(\prod\nolimits'_{x \in X^{(1)}} H^1_x(X_t, G)\right) / G(K)$$

is a bijection.

We can compare this with [Gro17, Theorem 0.1, Theorem 0.3].

Corollary 3.40. *Let X be a connected Dedekind scheme with function field K, and G be a separated X-group scheme. Then*

$$H^1(X_t, \mathcal{G}) \simeq G(\mathbb{O}_X) \backslash G(\mathbb{A}_X) / G(K)$$

is a bijection, where

$$G(\mathbb{O}_X) := \prod\nolimits'_{x \in X^{(1)}} G(X^t_{(x)})$$

and

$$G(\mathbb{A}_X) := \prod\nolimits'_{x \in X^{(1)}} G(X^t_{(x)} - \{x\}).$$

Proof. As noted in (3.4.1), we have

$$H^1_x(X_t, G) \simeq G(X^t_{(x)}) \backslash G(X^t_{(x)} - \{x\})$$

as the quotient set. The claim then follows. □

4. Examples of unstable Gersten resolutions

4.1. An unstable Bloch-Ogus-Gabber theorem.

Notation 4.1. In all this section, we fix be a base scheme S, Sm_S be the category of smooth S-schemes. and a Nis-local cohomotopy theory with support Π_* on Sm_S with coefficients in an arbitrary topos $\mathscr{E}$ in the sense of Notation 3.4. By taking colimits as usual, we can extend this theory to the category $\overline{\mathrm{Sm}}_S$ of essentially smooth S-schemes.

We can therefore apply the constructions of the previous section to any essentially smooth S-scheme X, by restriction to X_{Nis}, and get the (resp. Zariski or Nisnevich-local) unstable coniveau spectral sequence of X with coefficients in Π_* of Definition 3.2 (resp. Definition 3.15).

As Π_* is defined over the smooth site, we can use the classical methods to get the Zariski-local Gersten property (and more). To this end one introduces the following key property on Π_*.

Definition 4.2. We will say that the cohomotopy theory Π_* satisfies the *Gabber property*, if for any closed pair (X, Z) such that X is a smooth affine S-scheme and any $q \geq 0$, the following diagram is commutative:

$$\begin{array}{ccccc} \Pi_q(\mathbb{P}^1_X, \mathbb{P}^1_Z) & & \xrightarrow{\quad j^* \quad} & & \Pi_q(\mathbb{A}^1_X, \mathbb{A}^1_Z) \\ & \searrow^{s_\infty^*} & & \nearrow_{\pi^*} & \\ & & \Pi_q(X, Z) & & \end{array}$$

where j (resp. s_∞, π) is the obvious open immersion (resp. section at ∞, canonical projection).

Note that using the homotopy long exact sequence (3.1.4), one deduces from the above that for any closed subscheme $Z \subset X$, it suffices to check the above property in the case $Z = \varnothing$.

Example 4.3. If Π_* is $\mathbb{A}^1$-invariant, one easily checks that it automatically satisfies the Gabber property. A key example will be the case of the homotopy functor $\Pi_* = \Pi_*^{\mathcal{X}}$ as in Example 3.3 for $\mathcal{X}$ an $\mathbb{A}^1$-local sheaf.

The following lemma follows classical lines in the unstable setting (see [CTHK97]).

Lemma 4.4. *Assume Π_* satisfies the Gabber property.*

Let V be an affine smooth S-scheme, $i : Z \subset \mathbb{A}^1_V$ be a closed subscheme such that the composite map $p : Z \xrightarrow{i} \mathbb{A}^1_V \xrightarrow{\pi} V$ is finite. Let $F := p(Z)_{red}$, so that one has a closed immersion $k : Z \to \mathbb{A}^1_F$ which factors i.

Then for any $q \geq 0$, the following pointed map is trivial:

$$k_* : \Pi_q(\mathbb{A}^1_V, Z) \to \Pi_q(\mathbb{A}^1_V, \mathbb{A}^1_F).$$

Proof. We embed all schemes in $\mathbb{P}^1_V$ via the open immersion $j : \mathbb{A}^1_V \subset \mathbb{P}^1_V$. As $s_\infty(V) \cap Z = \emptyset$, there is a factorization $s_\infty : V \xrightarrow{s'} (\mathbb{P}^1_V - Z) \xrightarrow{l} \mathbb{P}^1_V$. According to the Gabber property, one deduces a commutative diagram:

$$\begin{array}{ccccccc} \Pi_q(\mathbb{P}^1_V, Z) & \xrightarrow{\widetilde{k}_*} & \Pi_q(\mathbb{P}^1_V, \mathbb{P}^1_F) & & \xrightarrow{\quad l^* \quad} & & \Pi_q(\mathbb{P}^1_V - Z, \mathbb{P}^1_F - Z) \\ {\scriptstyle excision}\downarrow\simeq & & \downarrow j^* & \searrow^{s_\infty^*} & & \swarrow_{s'^*} & \\ & & & & \Pi_q(V, F) & & \\ & & & \swarrow^{\pi^*} & & & \\ \Pi_q(\mathbb{A}^1_V, Z) & \xrightarrow{k_*} & \Pi_q(\mathbb{A}^1_V, \mathbb{A}^1_F) & & & & \end{array}$$

According to the localization long exact sequence (3.1.4), one deduces that $l^* \circ \widetilde{k}_* = *$ and this concludes. ☐

The strategy of [CTHK97] to exploit the previous lemma works fine over an infinite (not necessarily perfect) field.

Lemma 4.5. *(Effaceability) Assume Π_* satisfies the Gabber property, and $S = \mathrm{Spec}(k)$ be the spectrum of an arbitrary field.*

Let X be a smooth affine k-scheme and $F \subset X$ be a finite subset. If k is finite, we assume that $F = \{x\}$ is a single point.

Then, for any closed subscheme $Z \subset X$ of codimension $p > 0$, there exists an open neighborhood U of F in X, a closed subscheme $T \subset U$, of codimension $p-1$, containing $Z_U := Z \cap U$, $i : Z_U \to T$, and such that for any integer $g \geq 0$, the following pointed map is trivial:

$$\Pi_q(U, Z_U) \xrightarrow{i_*} \Pi_q(U, T).$$

Proof. According to [CTHK97, 3.1.1, 3.1.2] if k is infinite, or [HK20] if k is finite, there exists a Zariski open neighbourhood U of S in X, an excisive morphism $f : (U, Z_U) \to (\mathbb{A}^1_V, W)$ for some closed pair $(\mathbb{A}^1_V, W)$ such that V is a smooth k-scheme V and the composite map $W \subset \mathbb{A}^1_V \xrightarrow{\pi} V$ is finite (take $f = \varphi|_U$ in the notation of [CTHK97, 3.1.1]). Consider the closed subscheme $F := \pi(W)_{red} \subset V$ and $T = f^{-1}(\mathbb{A}^1_F)$. Note that there is an inclusion $i : Z_U \to T$. Hence there is a commutative diagram of pointed sets

$$\begin{array}{ccc} \Pi_n(\mathbb{A}^1_V, T) & \longrightarrow & \Pi_n(\mathbb{A}^1_V, \mathbb{A}^1_F) \\ {\scriptstyle f^*}\downarrow\simeq & & \downarrow \\ \Pi_n(U, Z_U) & \xrightarrow{i_*} & \Pi_n(U, T) \end{array}$$

and excision for f as well as Lemma 4.4 allows us to conclude. □

The lemma gives exactly condition (i) of Theorem 2.47 for the unstable coniveau exact couple of the semi-localization $X_{(F)}$ with coefficients in Π_*. In other words, wa have obtained.

Proposition 4.6. *Assume Π_* is a cohomotopy theory with supports defined on Sm_k, where k is an arbitrary field, which satisfies the Gabber property.*

Then for any essentially smooth semi-local (resp. local if k is finite) k-scheme X, the unstable coniveau exact sequence $E^{**}_{1,c}(X, \Pi_*)$ of X with coefficients in Π_*, (3.2), collapses on the column $* = 0$ in the sense of Definition 2.49.

In other words, for any $q \geq 0$, the bi-augmented coniveau unstable complex:

$$* \to \Pi_q(X) \xrightarrow{\tau} \bigoplus_{\eta \in X^{(0)}} \Pi_q\big(\kappa(\eta)\big) \xrightarrow{d_1^{0,q}} \bigoplus_{x \in X^{(1)}} \Pi^x_{q-1}\big(X^h_{(x)}\big) \to \cdots$$
$$\to \bigoplus_{x \in X^{(q-1)}} \Pi^x_1\big(X^h_{(x)}\big) \xrightarrow{d_1^{q-1,q}} \bigoplus_{s \in X^{(q)}} \Pi^s_0\big(X^h_{(s)}\big) \xrightarrow{\gamma} \Pi_0(X^{\leq q}_+)$$

is exact in the sense of Definition 2.25. Moreover, for $q = dim(X) + 1$ (resp. $q \leq \dim(X)$), one can replace the augmentation γ by the augmentation (see Notation 2.45 for the first one):

$$\bigoplus_{s \in X^{(q)}} \Pi^s_0\big(X^h_{(s)}\big) \xrightarrow{\epsilon} \Pi_0(X^{\leq q}_+)/\Pi_0(X^{\leq q+1}_+) \quad \textit{resp.} \quad \bigoplus_{s \in X^{(q)}} \Pi^s_0\big(X^h_{(s)}\big) \xrightarrow{ct} *$$

The local case immediately implies that Π_* is both Zariski and Nisnevich locally Gersten on any essentially smooth k-scheme X (see Definition 3.18). Taking into account the results of the previous section, one further gets:

Corollary 4.7. *Consider the above assumptions, and let $q > 1$ be an integer.*

Let $\underline{\Pi}_q$ be the Zariski sheaf on Sm_k associated with Π_q. Then the following results hold:

(1) For any smooth k-scheme X, $\underline{\Pi}_q$ is Cohen-Macaulay up to degree q on X_{Zar}, and it is fully Cohen-Macaulay if $\dim(X) \leq q$.

(2) $\underline{\Pi}_q$ is a Nisnevich sheaf on Sm_k. For any $0 \leq p < q$, and any smooth k-scheme X, there exists isomorphisms which are natural with respect to flat pullbacks:

$$E^{p,q}_{2,c}(X, \Pi_*) \simeq H^p_{\mathrm{Zar}}(X, \underline{\Pi}_q) \simeq H^p_{\mathrm{Nis}}(X, \underline{\Pi}_q).$$

(3) For $t = \mathrm{Zar}, \mathrm{Nis}$, and any smooth k-scheme X, there exists a unique isomorphism of complex of sheaves on X_t:

$$\underline{\mathrm{Ge}}^*(X_t, \Pi_*, q) \simeq \tau^{\leq q}_{nv}\, \underline{\mathrm{Cz}}^*_t(\underline{\Pi}^X_q)$$

between the t-local Gersten complex in degree q of Π_* and the naively truncated t-local Cousin complex on the t-sheaf $\underline{\Pi}_q$ restricted to the small Nisnevich site X_t. This isomorphism extends to the site of smooth k-schemes with morphisms as the flat (equivalently syntomic) ones.

In particular, for any point $x \in X^{(p)}$, one deduces isomorphisms:

$$\Pi^x_{q-p}(X_{(x)}) \simeq \Pi^x_{q-p}(X^h_{(x)}) \simeq H^p_x(X_{\mathrm{Zar}}, \underline{\Pi}_q) \simeq H^p_x(X_{\mathrm{Nis}}, \underline{\Pi}_q).$$

Finally, given the current technology, one can improve the previous results and get a weaker Gersten property over positive dimensional bases.

Lemma 4.8. *(Effaceability) Assume Π_* satisfies the Gabber property over Sm_S, and let $e = \dim(S)$.*

Let X be a smooth affine S-scheme, $x \in X$ a point, and $Z \subset X$ be a closed subscheme of codimension $p > e$. Then there is an Nisnevich neighbourhood U of x in X, a closed subscheme $T \subset U$, of codimension $p-1$, containing $Z_U := Z \cap U$, $i : Z_U \to T$, and such that for any integer $q \geq 0$, the following pointed map is trivial:

$$\Pi_q(U, Z_U) \xrightarrow{i_*} \Pi_q(U, T).$$

Proof. This is the same proof as for Lemma 4.5, except that we appeal to the relative presentation lemma [DHKY21, Th. 1.1]. Indeed, under the stated hypothesis, we get that $\dim(Z_s) < \dim(X_s)$, so Z_s does not contain any irreducible component of X_s then we can apply the relative presentation lemma [DHKY21]. Indeed, one has: $\dim(X_s) = d - e$ because X/S is smooth. Indeed $\dim(Z_s) \leq \dim(Z) = d - p$ by definition of p so $p > e$ implies $d - p < d - e$.

As in the proof of Lemma 4.5, by the presentation lemma, there is an Nisnevich neighbourhood U of x in X and also replacing S by a Nisnevich neighbourhood of $s = \pi(x)$ in S, we have excisive pair $f : (U, Z_U) \to (\mathbb{A}^1_V, W)$ for some closed pair $(\mathbb{A}^1_V, W)$ such that V is a smooth S-scheme V and the composite map $W \subset \mathbb{A}^1_V \xrightarrow{\pi} V$ is finite (take $f = \varphi|_U$ in the notation of [CTHK97, 3.1.1]). Consider the closed subscheme $F := \pi(W)_{red} \subset V$ and $T = f^{-1}(\mathbb{A}^1_F)$. Note that there is an inclusion $i : Z_U \to T$. Hence there is a commutative diagram of pointed sets

$$\begin{array}{ccc} \Pi_n(\mathbb{A}^1_V, T) & \longrightarrow & \Pi_n(\mathbb{A}^1_V, \mathbb{A}^1_F) \\ {\scriptstyle f^*}\downarrow{\scriptstyle \simeq} & & \downarrow \\ \Pi_n(U, Z_U) & \xrightarrow{i_*} & \Pi_n(U, T) \end{array}$$

and excision for f as well as Lemma 4.4 allows us to conclude. □

As a corollary and using Proposition 2.53, we immediately get:

Corollary 4.9. *Assume Π_* is a cohomotopy theory with support defined on Sm_S. Assume Π_* satisfies the Gabber property.*

Then for any essentially smooth local k-scheme X of absolute dimension e, for any $q \geq e$, the augmented naively e-cotruncated coniveau unstable complex:

$$\bigoplus_{x \in X^{(e)}} \Pi^x_{q-e}\big(X^h_{(x)}\big) \xrightarrow{d_1^{e,q}} \cdots \to \bigoplus_{x \in X^{(q-1)}} \Pi^x_1\big(X^h_{(x)}\big) \overset{d_1^{q-1,q}}{\Longrightarrow} \bigoplus_{s \in X^{(q)}} \Pi^s_0\big(X^h_{(s)}\big) \xrightarrow{\gamma} \Pi_0(X^{\leq q}_+)$$

is exact in the sense of Definition 2.25.

4.2. Artin-Mazur étale homotopy.

Notation 4.10. In this section our goal is to study natural cohomotopy theories with supports associated to the *étale homotopy types* in the sense of Artin-Mazur, via the formalism developed in the previous section.

The étale homotopy type $\mathrm{Et}(X)$ of a scheme X has been introduced by Artin and Mazur in [AM86] — we follow the notation of Friedlander [Fri82]. More recently, the construction has been formulated in terms of ∞-categories after Toën and Vezzosi, and we recall the definition from [Hoy18] as we find it illuminating. Given a scheme X, we let $\widehat{\mathrm{Sh}}^{\infty}(X_{\mathrm{et}})$ the ∞-topos of hypercomplete sheaves.[22] The obvious (unique) morphism of sites $\varnothing \to X_{\mathrm{et}}$ induces the canonical geometric morphism of ∞-topos $p^* : \mathscr{S} = \mathrm{Sh}^\infty(\varnothing) \to \widehat{\mathrm{Sh}}^\infty(X_{\mathrm{et}})$, the constant sheaf functor. As a morphism of ∞-topos, it always admits a right adjoint, but as it commutes with finite limits, it also admits a pro-left adjoint $p_! : \mathrm{Sh}^\infty(X_{\mathrm{et}}) \to \mathrm{pro}-\mathscr{S}$, with values in the ∞-category of pro-spaces. With this notation, we put: $\mathrm{Et}(X) = p_!p^*(*) = p_!(X)$, where we have abusively denoted by X the constant étale sheaf on X with value $*$.[23] It is proved in [Hoy18, Cor. 5.6] that this definition coincides with the one of Artin-Mazur if X is locally connected (e.g. locally noetherian).

Note that $\mathrm{Et}(X)$ behaves like a homology theory in X: it is covariant with respect to arbitrary maps of schemes, and contravariant with respect to étale covers. For this reason, we will consider the associated cohomotopy theory with coefficients in an arbitrary pro-space.[24]

Definition 4.11. We consider a category of schemes $\mathscr{S}$ as in Notation 3.4. Let $\mathbb{G}$ be a pro-space, that we view as a pro-∞-groupoid of coefficients. We define the *étale cohomotopy theory with supports and coefficients in* $\mathbb{G}$ by associating to a closed $\mathscr{S}$-pair (X,Z) the π_*-structure in $\mathscr{S}et$ (see Definition 2.17):

$$\Pi_*^{\mathrm{et}}(X, Z; \mathbb{G}) := \pi_* \operatorname{Map}\big(\mathrm{Et}(X/X-Z), \mathbb{G}\big)$$

where $\mathrm{Et}(X/X-Z)$ denotes the homotopy cofiber of the map $\mathrm{Et}(X-Z) \to \mathrm{Et}(X)$ and Map denotes the mapping space in the ∞-category $\mathrm{pro}-\mathscr{S}$.

Indeed, it follows as in Remark 3.5 that the above definition does indeed define a cohomotopy theory with supports. Besides, it is Nisnevich-local as the étale homotopy type satisfies Nisnevich excision. In fact, one can define this theory in much simpler term. Let us write $\mathbb{G} = (\mathbb{G}_i)_{i \in I}$. Then, one gets:

$$\Pi_*^{\mathrm{et}}(X, Z; \mathbb{G}) \simeq \pi_* \Big[\varinjlim_{i \in I} \operatorname{Map}_{\widehat{\mathrm{Sh}}^\infty(X_{\mathrm{et}})} \big(X/X-Z, \mathbb{G}_{i,X}\big) \Big]$$

where $X/X-Z$ is the homotopy cofiber of the map $(X-Z) \to X$ computed on the associated represented sheaves in $\widehat{\mathrm{Sh}}^\infty(X_{\mathrm{et}})$, and $\mathbb{G}_{i,X} = p^*\mathbb{G}_i$ is the constant étale sheaf on X with values in the space $\mathbb{G}_i$.[25]

The main result of this section is the following theorem.

Proposition 4.12. *Let k be a separably closed field and $\mathbb{G}$ be a pro-space. Then the cohomotopy theory with supports $\Pi_*(-;\mathbb{G})$ restricted to the category Sm_k satisfies the Gabber property (Definition 4.2).*

This follows easily from the following observation which is of independent interest.

Lemma 4.13. *Let k be a separably closed field. Let X be a smooth affine k-scheme and (X,Z) be a closed pair. Consider the morphism of schemes*

$$\begin{array}{ccccc} \mathbb{A}^1_X & \xrightarrow{j} & \mathbb{P}^1_X & \xleftarrow{s_\infty} & X \\ & \searrow^{\pi} & \downarrow{\scriptstyle \overline{\pi}} & \swarrow_{=} & \\ & & X & & \end{array}$$

where j (resp. s_∞, π) is the obvious open immersion (resp. section at ∞, canonical projection). Then the induced diagram

$$\begin{array}{ccc} \mathrm{Et}(\mathbb{A}^1_X/\mathbb{A}^1_X - \mathbb{A}^1_Z) & \xrightarrow{\quad j_* \quad} & \mathrm{Et}(\mathbb{P}^1_X/\mathbb{P}^1_X - \mathbb{P}^1_Z) \\ & {}_{\pi_*}\searrow \quad \mathrm{Et}(X/X-Z) \quad \nearrow_{s_{\infty *}} & \end{array}$$

is commutative in the ∞-category $\mathrm{pro}-\mathscr{S}$.

[22] See [Lur09, p. 663]. We use this level of generality for the sake of presentation. As we restrict to essentially smooth schemes over a separably closed field, the reader can freely use the ∞-category of étale sheaves.

[23] After Toën-Vezzosi-Lurie, this is also called the *shape* of the ∞-topos $\widehat{\mathrm{Sh}}^\infty(X_{\mathrm{et}})$.

[24] Note also that according to a fundamental of Artin and Mazur ([AM86, Th. 11.1]), $\mathrm{Et}(X)$ is profinite whenever X is geometrically unibranch. So in this case, one can rather consider $\mathrm{Et}(X)$ as taking values in the ∞-category of profinite spaces as in [Qui08]. In particular, for our purpose, one could as well restrict to coefficients in a profinite space instead of an arbitrary pro-space.

[25] This easily follows from the definition of $\mathrm{Et}(X)$ and the fact $p_!$ is a pro-left adjoint to p^*.

Proof. Note that one needs only prove the diagram is commutative in the associated homotopy category, as this will produce a homotopy between j_* and $s_{\infty *} \circ \pi_*$. By definition of $\mathrm{Et}(X/X-Z)$ as a homotopy cofiber, one is reduced to the case $X = Z$. That is, we need to prove that the following diagram commutes in $\mathrm{Ho}\,\mathrm{pro}-\mathscr{S}$:

$$\begin{array}{ccc} \mathrm{Et}(\mathbb{A}^1_X) & \xrightarrow{\quad j_* \quad} & \mathrm{Et}(\mathbb{P}^1_X). \\ & \searrow^{\pi_*} \ \mathrm{Et}(X) \ \nearrow_{s_{\infty *}} & \end{array}$$

Let $p : \mathbb{P}^1_X \to \mathbb{P}^1_k$ be the canonical projection. Then we can apply the Künneth formula for the étale homotopy type proved in [Cho22, Th. 5.3] so that we get that the canonical map:

$$\mathrm{Et}(\mathbb{P}^1_X) \xrightarrow{(\overline{\pi}_*, p_*)} \mathrm{Et}(\mathbb{P}^1_k) \times \mathrm{Et}(X)$$

is an equivalence.[26] So to prove the above diagram is homotopy commutative, one can post-compose by the equivalence $(\overline{\pi}_*, p_*)$. This reduces to the case $X = \mathrm{Spec}(k)$. We prove this separately depending on the characteristic of the base field k.

Case 1: char $k = 0$. In this case we can assume $k \subset \mathbb{C}$. Then by [AM86, Corollary 12.10], we have that $\mathrm{Et}(\mathbb{A}^1_k) \simeq \widehat{\mathbb{A}^1(\mathbb{C})} = *$, hence the diagram commutes, since both the maps are constant.

Case 2: char $k = p > 0$. In this case k is a separably closed field of positive characteristic. This implies that $\mathrm{Et}(\mathbb{P}^1_k) \simeq K(\widehat{\mathbb{Z}}(1), 2)$ where the profinite group $\widehat{\mathbb{Z}}(1) = \mu(k)$ is the group of roots of unity. Note that $\mu(k) \simeq \varprojlim_n \mu_n(k)$.

Using the relation between continuous étale cohomology and étale homotopy we get:

$$[\mathrm{Et}(\mathbb{P}^1_k), \mathrm{Et}(\mathbb{P}^1_k)] \simeq [\mathrm{Et}(\mathbb{P}^1_k), K(\widehat{\mathbb{Z}}(1), 2)] \simeq H^2_{cts}(\mathbb{P}^1_k, \widehat{\mathbb{Z}}(1)) = \varprojlim_n H^2_{\mathrm{et}}((\mathbb{P}^1_k, \mu_n(k))$$

where the first two groups are Hom-groups taken in the homotopy category of $\mathrm{pro}-\mathscr{S}$. By [CTHK97, Equation (4.1)], we further have

$$H^2_{\mathrm{et}}((\mathbb{P}^1_k, \mathbb{Z}/n) = \begin{cases} 0 & p \mid n \\ H^0_{\mathrm{et}}((\mathrm{Spec}\,k, \mathbb{Z}/n(-1)) & p \nmid n. \end{cases} \tag{4.2.1}$$

In the case $p \nmid n$, as noted in [CTHK97, Lemma 4.1.3], the restriction of the maps j^*, s^*_∞ from $H^2_{\mathrm{et}}((\mathbb{P}^1_k, \mathbb{Z}/n)$ on the factor $H^0_{\mathrm{et}}((\mathrm{Spec}\,k, \mathbb{Z}/n(-1))$ is 0. As a consequence the maps $j^*, \pi^* \circ s^*_\infty$: $[\mathrm{Et}(\mathbb{P}^1_k), \mathrm{Et}(\mathbb{P}^1_k)] \to [\mathrm{Et}(\mathbb{A}^1_k), \mathrm{Et}(\mathbb{P}^1_k)]$ are equal. In particular $j_{\mathrm{et}} = j^*(\mathrm{Id}) = \pi^* \circ s^*_\infty(\mathrm{Id}) = (s_\infty)_{\mathrm{et}} \circ \pi_{\mathrm{et}}$. Hence the lemma. □

Appendix

Uniqueness of truncated Cousin complexes. The goal of this appendix is to give details on a proof of a result of [Har66], showing the uniqueness of Cousin resolutions. This is used to show that the result can be extended to the case of truncated Cousin resolutions that appeared naturally in our unstable context.

Throughout this section we recall the Notation 3.11. Let $t = \mathrm{Zar}, \mathrm{Nis}$, and X be a scheme. The classical framework of t-local cohomology with supports can be extended to the non-abelian setting (see [Har66, §IV] for Zariski sheaves of abelian groups, and [DFJ22, §4.3] in the Nisnevich case). Given a pointed t-sheaf F of sets, in $\mathrm{Sh}(X_t)_*$, and $Z \subset X$ a closed subscheme, one puts:

$$\Gamma_Z(X, F) = \{\rho \in F(X) \mid \forall x \notin Z, \rho_x = *\}.$$

Similarly, $\underline{\Gamma}_Z(F)$ is the pointed t-sheaf $V \mapsto \underline{\Gamma}_{Z \times_X V}(F|_V)$. If $T \subset Z$, one let $\underline{\Gamma}_{Z/T}(F)$ be the cokernel, in the category of pointed t-sheaves, of the natural pointed map $\underline{\Gamma}_T(F) \to \underline{\Gamma}_Z(F)$.

First we review the well-known results [Har66, Lemma 2.2, Proposition 2.3].

Lemma 4.14. *([Har66, Lemma 2.2]) Let $X \supset Z \supset Z'$ such that Z' is stable under specialization and such that every $x \in Z - Z'$ is maximal (ordered by the specialization) in Z.*

Given a sheaf of abelian groups F on X with supports in Z,

[26] Note that under our assumptions, both $\mathrm{Et}(\mathbb{P}^1_k)$ and $\mathrm{Et}(X)$ are profinite so that *loc. cit.* does apply.

(1) the sheaf $\underline{H}^0_{Z/Z'}(F)$ *lies on the* Z/Z'*-skeleton of* X *and the canonical map*

$$F \to \underline{H}^0_{Z/Z'}(F)$$

is Z'*-isomorphism, i.e., the kernel and cokernel are supported on* Z'*.*

(2) given a Z'*-isomorphism*

$$F \to G$$

into a sheaf G *which lies on the* Z/Z'*-skeleton, it factors uniquely (unique upto unique isomorphism) through the canonical map*

$$\begin{array}{ccc} F & \longrightarrow & G \\ & \searrow \quad \nearrow_{\simeq} & \\ & \underline{H}^0_{Z/Z'}(F) & \end{array}$$

Corollary 4.15. *Let* F *be a sheaf of abelian groups on* X *and* $X \supset Z \supset Z'$ *(as above). Then the map*

$$\underline{H}^0_{Z/Z'}(F) \to \underline{H}^0_{Z/Z'}(\underline{H}^0_{Z/Z'}(F))$$

is Z'*-isomorphism i.e., the kernel and cokernel is supported on* Z'*.*

Proof. This follows by noting that for a sheaf of abelian groups F on X, the sheaf $\underline{H}^0_{Z/Z'}(F)$ is with supports in Z. □

Theorem 4.16. *(assumptions as in* [Har66, Proposition 2.3]*) Let* F *be a sheaf of abelian groups. Then there is a unique (unique upto unique isomorphism) augmented complex*

$$F \to C^\bullet$$

such that

(1) for each $p \geq 0$*,* C^p *lies on the* Z^p/Z^{p+1}*-skeleton.*
(2) for each $p > 0$ $C^p/\operatorname{Im}(C^{p-1})$ *has supports in* Z^{p+1}*.*
(3) the map $F \to H^0(C^\bullet)$ *has kernel supported on* Z^1 *and* $C^0/\operatorname{Im}(F)$ *is supported on* Z^1*.*

Furthermore, the complex $C^\bullet$ *satisfies the conditions*

(i) *for each* $p \geq 0$*,* C^p *lies on the* Z^p/Z^{p+1}*-skeleton.*
(ii) *for each* $p > 0$ $H^p(C^\bullet)$ *has supports in* Z^{p+2}*.*
(iii) *the map* $F \to H^0(C^\bullet)$ *has kernel supported on* Z^1 *and cokernel is supported on* Z^2*.*

Remark 4.17. We will see in the proof below that the complex we construct satisfies the properties (i), (ii) and (iii).

Conversely, if we construct a complex $C^\bullet$ satisfying the properties (i), (ii) and (iii) then it satisfies the properties (1), (2) and (3).

More precisely, (ii)+ (i) $\Rightarrow$ (2) and (iii)+ [(i) for $p = 1$)] implies (3).

Proof of Theorem 4.16. We construct such a complex by induction on $p \geq 0$.

Step 1: ($p = 0$) Let $C^0 := \underline{H}^0_{Z/Z^1}(\underline{H}^0_{Z/Z^1}(F))$ with the canonical map

$$F \to \underline{H}^0_{Z/Z^1}(F) \to C^0$$

We claim that kernel $\ker(F \to C^0)$ and $C^0/\operatorname{Im} F$ is supported on Z^1.

Note that we have an exact sequence

$$0 \to \underline{\Gamma}_{Z^1} F \to \ker(F \to C^0) \to \ker(\underline{H}^0_{Z/Z^1}(F) \to C^0)$$

Hence by above Corollary 4.15 since $\ker(\underline{H}^0_{Z/Z^1}(F) \to C^0)$ is supported on Z^1 and by definition $\underline{\Gamma}_{Z^1} F$ is supported on Z^1, we get that $\ker(F \to C^0) = \ker(F \to H^0(C^\bullet))$ is supported on Z^1.

Similary there is an exact sequence

$$\frac{\underline{H}^0_{Z/Z^1}(F)}{\operatorname{Im}(F)} \to \frac{C^0}{\operatorname{Im}(F)} \to \frac{C^0}{\underline{H}^0_{Z/Z^1}(F)} \to 0$$

By Corollary 4.15, we have that $\dfrac{C^0}{\underline{H}^0_{Z/Z^1}(F)}$ is supported on Z^1.

To see that $\dfrac{\underline{H}^0_{Z/Z^1}(F)}{\operatorname{Im}(F)}$ is supported on Z^1, note that there is an exact sequence [Har66, Page 219, Motif B (sheafified version)]

$$0 \to \underline{\Gamma}_{Z^1} F \to F \to \underline{H}^0_{Z/Z^1}(F) \to \underline{H}^1_{Z^1}(F)$$

This implies that the cokernel $\dfrac{\underline{H}^0_{Z/Z^1}(F)}{\operatorname{Im}(F)}$ is a sub-object of $\underline{H}^1_{Z^1}(F)$, which is supported on Z^1. Hence the claim.

Thus we have a map

$$F \to C^0$$

such that the kernel $\ker(F \to C^0)$ and the cokernel $C^0/\operatorname{Im} F$ are supported on Z^1.

Step 2: (Induction) Assume that the complex $C^\bullet$ is determined up to degrees $\leq p$ satisfying the desired properties. So we have

$$F = C^{-1} \to C^0 \to C^1 \to \cdots \to C^{p-1} \to C^p.$$

Since by the inductive assumption $C^p/\operatorname{Im} C^{p-1}$ has support in Z^{p+1}, we can apply the above Lemma 4.14 to $C^p/\operatorname{Im} C^{p-1}$ with $Z = Z^{p+1}$ and $Z' = Z^{p+2}$. Let us define

$$C^{p+1} := \underline{H}^0_{Z^{p+1}/Z^{p+2}}\left(\frac{C^p}{\operatorname{Im} C^{p-1}}\right).$$

Then the canonical map

$$\frac{C^p}{\operatorname{Im} C^{p-1}} \to C^{p+1}$$

has

$$\ker = \begin{cases} H^0(C)/F & for\ p = 0 \\ H^p(C^\bullet) & for\ p > 0 \end{cases}$$

and its cokernel $\dfrac{C^{p+1}}{\operatorname{Im} C^p}$ both have supports in Z^{p+2}. Also C^{p+1} lies on the skeleton Z^{p+1}/Z^{p+2}.

Thus by induction we have constructed the augmented complex

$$F \to C^\bullet$$

satisfying the desired properties.

Step 3: (Uniqueness) Assume that there is an augmented complex

$$F \to C'^\bullet$$

satisfying the listed properties. We will again show this by induction on p.

Assume that the uniqueness is true for complexes upto degrees $\leq p$. We need to show that it induces uniqueness for complexes upto degrees $\leq p+1$.

By inductive assumption, there is a unique isomorphism of augmented complexes $\tau_{\leq p} C^\bullet \to \tau_{\leq p} C'^\bullet$.

$$\begin{array}{ccccccc} F & \longrightarrow \cdots \longrightarrow & C^p & \longrightarrow & C^{p+1} \\ \| & & \downarrow \simeq & & \\ F & \longrightarrow \cdots \longrightarrow & C'^p & \longrightarrow & C'^{p+1} \end{array}$$

Hence there is a unique isomorphism

$$C^p/\operatorname{Im} C^{p-1} \to C'^p/\operatorname{Im} C'^{p-1}.$$

Composing this we have a map

$$C^p/\operatorname{Im} C^{p-1} \to C'^p/\operatorname{Im} C'^{p-1} \to C'^{p+1}$$

where by assumption on $C'^{\bullet}$, C'^{p+1} lies on the skeleton Z^{p+1}/Z^{p+2} , hence by Lemma 4.14, there is a unique Z^{p+2}-isomorphism

$$C^{p+1} := \underline{H}^0_{Z^{p+1}/Z^{p+2}}\left(\frac{C^p}{\operatorname{Im} C^{p-1}}\right) \to C'^{p+1}.$$

It clearly commutes with previous differentials.

We note that the Z^{p+2}-isomorphism

$$C^{p+1} \to C'^{p+1}$$

is actually an isomorphism as

$$\begin{array}{ccc} C^{p+1} & \longrightarrow & C'^{p+1} \\ \downarrow \simeq & & \downarrow \simeq \\ \underline{H}^0_{Z^{p+1}/Z^{p+2}}(C^{p+1}) & \xrightarrow{\simeq} & \underline{H}^0_{Z^{p+1}/Z^{p+2}}(C'^{p+1}) \end{array}$$

□

Definition 4.18. The Cousin complex $\underline{\mathrm{Cz}}^*(F)$ of sheaf of abelian groups F is the complex satisfying the properties as in Theorem 4.16.

Let us recall from Notation 2.26, a q-truncated unstable augmented homotopical complex

$$F \to C^0 \to C^1 \to \cdots C^{q-1} \Rightarrow C^q$$

is such that C^p is an abelian group object for $0 \leq p \leq q-2$, C^{q-1} is a group object and C^q is a pointed set with the group C^{q-1} acting on C^q.

Theorem 4.19. *Let F be a sheaf of abelian groups on X_t. Let $q > 1$. Let $F \to C^{\bullet}$ be a q-truncated unstable augmented homotopical complex*

$$F \to C^0 \to C^1 \to \cdots C^{q-1} \Rightarrow C^q$$

such that

1. *for each $0 \leq p \leq q$, C^p is supported in $X^{(p)}$ (in the sense of Definition 3.13).*
2. *for each $0 < p < q$ $C^p/\operatorname{Im}(C^{p-1})$ has supports in $X^{\geq p+1}$ i.e., isomorphic to $\underline{\Gamma}_{X^{\geq p+1}}\Big(C^p/\operatorname{Im}(C^{p-1})\Big)$.*
3. *the map $F \to H^0(C^{\bullet})$ has kernel supported on $X^{\geq 1}$ and $C^0/\operatorname{Im}(F)$ is supported on $X^{\geq 1}$.*

Then there is a unique isomorphism

$$\tau^{\leq q}\, \underline{\mathrm{Cz}}^*(F) \to C^*$$

of augmented complexes.

Proof. The proof follows along the same lines as in the proof of Theorem 4.16 to give a map

$$\tau^{\leq q}\, \underline{\mathrm{Cz}}^*(F) \to C^*$$

upto degrees $q-1$ *i.e.*, we have

$$\begin{array}{ccccccc} F & \longrightarrow & \cdots & \longrightarrow & \underline{\mathrm{Cz}}^{q-1}(F) & \longrightarrow & \underline{\mathrm{Cz}}^{q}(F) \\ \| & & & & \downarrow \simeq & & \\ F & \longrightarrow & \cdots & \longrightarrow & C^{q-1} & \longrightarrow & C^q \end{array}$$

Hence there is a unique isomorphism

$$\underline{\mathrm{Cz}}^{q-1}/\operatorname{Im}\underline{\mathrm{Cz}}^{q-2} \to C^{q-1}/\operatorname{Im} C^{q-2}.$$

sheaves of pointed sets. Composing this we have a map

$$\underline{\mathrm{Cz}}^{q-1}/\operatorname{Im}\underline{\mathrm{Cz}}^{q-2} \to C^{q-1}/\operatorname{Im} C^{q-2} \to C^q$$

where by assumption on $C^\bullet$, C^q is supported in $X^{(q)}$, hence by Lemma 4.20, there is a unique $X^{\geq q+1}$-isomorphism

$$\underline{\mathrm{Cz}}^q = \underline{\Gamma}_{X^{\geq q}/X^{\geq q+1}}\left(\frac{\underline{\mathrm{Cz}}^{q-1}}{\mathrm{Im}\,\underline{\mathrm{Cz}}^{q-2}}\right) \to C^q.$$

It clearly commutes with previous differentials.

We note that the $X^{\geq q+1}$-isomorphism

$$\underline{\mathrm{Cz}}^q \to C^q$$

is actually an isomorphism as

$$\begin{array}{ccc} \underline{\mathrm{Cz}}^q & \longrightarrow & C^q \\ \simeq\uparrow & & \simeq\uparrow \\ \underline{\Gamma}_{X^{\geq q}}(\underline{\mathrm{Cz}}^q) & \xrightarrow{\simeq} & \underline{\Gamma}_{X^{\geq q}}(C^q) \end{array}$$

□

Lemma 4.20. *Let $X \supset Z \supset Z'$ such that Z' is stable under specialization and such that every $x \in Z - Z'$ is maximal (ordered by the specialization) in Z.*

Given a sheaf of pointed sets F on X with supports in Z,

(1) the sheaf $\underline{\Gamma}_{Z/Z'}(F)$ lies on the Z/Z'-skeleton of X and the canonical map

$$F \to \underline{\Gamma}_{Z/Z'}(F)$$

is Z'-isomorphism, i.e., isomorphism when restricted to the open complement $Z - Z'$.

(2) given a Z'-isomorphism

$$F \to G$$

into a sheaf G which lies on the Z/Z'-skeleton, it factors uniquely (unique upto unique isomorphism) through the canonical map

$$\begin{array}{ccc} F & \longrightarrow & G \\ & \searrow \quad \nearrow_{\simeq} & \\ & \underline{\Gamma}_{Z/Z'}(F) & \end{array}$$

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