Efficient Neural Hybrid System Learning and Transition System Abstraction for Dynamical Systems *

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Abstract: This paper proposes a neural network hybrid modeling framework for dynamics learning to promote an interpretable, computationally efficient way of dynamics learning and system identification. First, a low-level model will be trained to learn the system dynamics, which utilizes multiple simple neural networks to approximate the local dynamics generated from data-driven partitions. Then, based on the low-level model, a high-level model will be trained to abstract the low-level neural hybrid system model into a transition system that allows Computational Tree Logic Verification to promote the model's ability with human interaction and verification efficiency.

Keywords: Hybrid and Distributed System Modeling; Neural Networks; Nonlinear System Modeling; Maximum-Entropy Partitioning; Model Abstraction.

1. INTRODUCTION

In recent years, the development of neural networks has received particular attention in various fields, including natural language processing Wang et al. (2023b), computer vision Stefenon et al. (2022), etc. The applications of neural networks in system identification hold significant promise for they provide a precise approximation of the dynamics while requiring no prior knowledge of the system's mechanism. Neural networks serve as a predominant approach in machine learning, renowned for their exceptional ability to model complex phenomena with limited prior knowledge. Their proficiency in capturing intricate patterns in data offers valuable insights for dynamical system modeling, verification, and control.

However, neural networks are opaque, limiting our ability to validate them solely from an input-output perspective. This opacity also renders neural network models vulnerable to perturbations Zhang et al. (2021), Yang et al. (2022). When it comes to applications in safety-critical scenarios, it requires time-consuming reachability analysis of the specific trajectories for verification, which poses challenges to real-time applications. According to Brix et al. (2023), the computational efficiency is highly related to the scale of the neural network model.

This paper aims to promote the interpretability and computational efficiency of neural networks in dynamical system modeling by introducing a novel dual-level modeling framework. Specifically, our proposed approach will divide dynamical system modeling into two essential levels: the low-level neural hybrid system model and its high-level transition system abstraction. The low-level model is employed to precisely capture the system's local behavior and enhance the computational efficiency with a parallel set of shallow neural networks aimed at approximating the local dynamics. Then the high-level transition model, which is an abstraction based on neural hybrid systems, can be obtained based on reachability analysis designed to capture relationships and transition patterns among system subspaces.

The contributions of this paper are summarized as follows.

- Maximum Entropy partitioning is applied to partition the system state space into multiple local subspaces, which allows analysis of the dynamics within local subspaces.
- A concept of neural hybrid systems is proposed for distributed training and verification of a set of shallow neural networks, thereby enhancing computational efficiency.
- A novel transition system abstraction method is proposed to investigate the transition relationships between local partitions, which will further enhance model interpretability.

This paper is organized as follows: Preliminaries and problem formulations are given in Section II. The main result, the dual-level modeling framework, is given in Section III. In Section IV, modeling of the LASA data sets is given to illustrate the effectiveness of our proposed framework¹. Conclusions are given in Section V.

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¹ The developed modeling tool and code for experiments are publicly available online at: https://github.com/aicpslab/ Dual-Level-Dynamic-System-Modeling.

Notations. In the rest of the paper, \mathbb{N} denotes the natural number sets, where $\mathbb{N}^{\leq n}$ indicates $\{1, 2, \dots, n\}$, \mathbb{R} is the field of real numbers, \mathbb{B} is the set of the Boolean variables, \mathbb{R}^n stands for the vector space of *n*-tuples of real numbers, and \underline{X} and \overline{X} are the lower bound and upper bound of an interval X, respectively.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this paper, the modeling problems for the discrete-time system will be discussed, i.e., we aim to model the system in the form of

$$x(k+1) = f(x(k), u(k)),$$
(1)

in which $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the external input, and $f : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ is the ideal mapping that precisely describes the system patterns. Due to dimensions and nonlinearity, obtaining f could be challenging, therefore we aim to approximate f with neural network $\Phi : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ in

$$x(k+1) = \Phi(x(k), u(k)).$$
 (2)

In the training of Φ , approximating f means adjusting the weight and bias of Φ in order to minimize the error between its output and the given data set. In this paper, the given data set consisting of input-output pairs is in the form of

$$\mathcal{D} = \{ (z^{(i)}, y^{(i)}) \mid z^{(i)} \in \mathbb{R}^{n_x + n_u}, y^{(i)} \in \mathbb{R}^{n_x} \}.$$
(3)

However, neural networks face challenges as they typically require extensive data for training and often lack an intuitive understanding of the system's behavior. To unveil the black-box model usually requires a reachability analysis of the neural network dynamical system.

2.1 Reachability Analysis for Neural Network Dynamical System

The reachability analysis of neural networks is useful in neural network dynamical system verification for it can determine the range of outputs based on the interplay between the input sets and the structure of the neural network, and according to Tran et al. (2019); Wang et al. (2021) and Feng et al. (2018), simple neural network structure, i.e., Φ that contains fewer layers and neurons will have advantages in reachable computation.

Taking a *L*-layer feed-forward neural network $\Phi : \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}$ as an example, its inter-layer propagation can be denoted as follows.

$$x_{i,k+1} = \sigma(\sum_{j} w_{ij,k+1} x_{j,k} + b_{i,k+1}), \tag{4}$$

in which $x_{i,k+1}$ is the *i*th neuron output from k+1th layer computed by applying the activation function σ to the weighted sum of the activations from the previous layer, plus a bias $b_{i,k+1}$ and $w_{ij,k+1}$ is the *i*th line, *j*th row value of the weight bias $W_k \in \mathbb{R}^{n_k \times n_{k+1}}$.

Reachability analysis of neural networks will go through the inter-propagation of the neural network in (4), namely, for neural network model in (2) output reachable set computation when given kth time step state input set $\mathcal{X}_{(k)} \subset \mathbb{R}^{n_x}$ and external input set $\mathcal{U} \subset \mathbb{R}^{n_u}$ can be denoted as

$$\mathcal{X}_{(k+1)} = \Phi^*(\mathcal{X}_{(k)}, \mathcal{U}), \tag{5}$$

in which $\mathcal{X}_{(k+1)}$ is the reachable set output of Φ at k+1th time step computed by reachable set computation method Φ^* such as Lopez et al. (2023); Xiang et al. (2018); Vincent and Schwager (2021), etc. Reachable sets in given Kth time steps requires propagation of (5) in

$$\mathcal{R}_{(K)} = \bigcup_{k=0}^{K} \mathcal{X}_{(k)},\tag{6}$$

in which $\mathcal{R}_{(K)}$ is the reachable sets in K time steps.

Due to the opacity of neural networks, the verification of (2) usually necessitates reachability computations of different trajectories to verify specific properties and can be heavily influenced by the neural network structure, posing a computational burden that challenges its application.

2.2 Maximum Entropy Partitioning

Maximum Entropy (ME) partitioning proposed in Yang and Xiang (2023) utilizes the Shannon Entropy to partition the state space according to the data, which can be very useful in obtaining subspaces for distributed learning and prediction in neural networks.

Given a set of $N \in \mathbb{N}$ subspaces $\mathcal{P} = \{\mathcal{P}_i\}_{i=1}^N$, where $\mathcal{P}_i \subset \mathbb{R}^{n_x}$, the Shannon Entropy of \mathcal{P} can be denoted by

$$H(\mathcal{P}) = -\sum_{i=1}^{N} p(\mathcal{P}_i) \log p(\mathcal{P}_i), \tag{7}$$

in which $p(\mathcal{P})$ denotes the probability of \mathcal{P}_i occurrence in \mathcal{P}_i . In this data-driven process, $p(\mathcal{P}_i)$ is extrapolated by the sample set in the form of

$$p(\mathcal{P}_i) = \frac{|\mathcal{D}_i|}{|\mathcal{D}|},\tag{8}$$

in which $|\mathcal{D}|$ is the number of samples of \mathcal{D} while D_i is defined by

$$\mathcal{D}_{i} = \{ (z^{(j)}, y^{(j)}) \in \mathcal{D} \mid x \in \mathcal{P}_{i}, \forall [x^{\top}, u^{\top}]^{\top} = z^{(i)} \}.$$
(9)

The ME partitioning employs the variation in Shannon entropy from system partitions to ascertain if the current set of partitions maximized the system's entropy after a bisecting method. Explicitly, the variation of Shannon Entropy is in the form of

$$\Delta H = H(\mathcal{P}) - H(\mathcal{P}) \tag{10}$$

in which $\hat{\mathcal{P}}$ is the post-bisecting set of partitions.

By setting a threshold $\epsilon \geq 0$ as a stop condition, namely the bisection process will stop if $\Delta H < \epsilon$, a proper set of partitions can be obtained.

2.3 Problem Formulation

This paper aims to promote the efficiency of learning and prediction of the neural network dynamical system in solving the following problem.

Problem 1. Given the data set \mathcal{D} in the form of (3), how do we model the dynamical system distributively with multiple simple neural networks?

To promote the interpretability of the learning model, the following problem will be the main concern after a distributed neural network model is obtained. Problem 2. Given a neural-network-based approximation Φ of f, how do we abstract Φ into an interpretable model that avoids real-time reachable set computation in (5)?

Solving Problem 1 will allow parallel training and verification of multiple simple neural networks, which enhances the efficiency of neural network modeling while providing an accurate low-level model. Based on the low-level model, we are able to enhance the interpretability by abstracting the low-level model into a high-level model by solving Problem 2.

3. DUAL-LEVEL MODELING FRAMEWORK

Before presenting the dual-level modeling framework, we make the assumption that the system training set (3) provides adequate information in the working zone for dynamical learning as follows.

Assumption 1. The working zone of ideal system dynamical description f in (1) is within the localized state space $x \in \mathcal{X}$, given the external input bound where $u \in [\underline{u}, \overline{u}]$.

In most cases of neural network dynamical system modeling, Φ in (2) will have high accuracy in approximating the dynamics based on a sample set \mathcal{D} . Based on \mathcal{D} , we assume that the learning model applies only to a working zone with Assumption 1.

3.1 Neural Hybrid System Model and Transition System Abstraction

To solve Problem 1, we proposed the neural hybrid system model, which allows precise learning of the dynamical system through multiple small-scale neural networks. The neural hybrid system model is defined as

Definition 1. A neural hybrid system model is a tuple $\mathcal{H} = \langle \mathcal{P}, \Omega, \delta, \tilde{\Phi} \rangle$ where

- $\Omega \subset \mathbb{R}^d$: Working zone, with states $x(k) \in \Omega$.
- $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{N_p}\}$: Finite set of non-overlapping partitions in the working zone, where: 1) $\mathcal{P}_i \subseteq \Omega$; 2) $\bigcup_{i=1}^{N} \mathcal{P}_i = \Omega$; 3) $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, $\forall i \neq j$.
- $\bigcup_{i=1}^{N} \mathcal{P}_i = \Omega; \ 3) \ \mathcal{P}_i \cap \mathcal{P}_j = \emptyset, \ \forall i \neq j.$ • $\delta : \Omega \to \{1, 2, \dots, N_p\}$: Function mapping states to partitions $\delta(x(k)) = i$, implies $x(k) \in \mathcal{P}_i$.
- $\tilde{\Phi} = \{\Phi_1, \Phi_2, \dots, \Phi_{N_p}\}$: Set of neural networks, each Φ_i models dynamics in \mathcal{P}_i .

Definition 1 introduces a distributed structure of the neural networks that allows local approximations of the subspaces of state space called partitions. The dynamics of low-level model \mathcal{H} is denoted as

$$x(k+1) = \Phi_{\delta(x(k))}(x(k), u(k)).$$
(11)

This distributive structure will help reduce the scales of the neural network approximation and result in the enhancement of the computational efficiency in training and verification.

Compared with the conventional model, the neural hybrid system modeling will have the advantages of real-time computation and verification. However, to gain insights from the neural hybrid system modeling and enhance interactivity between the learning model and human users, we can abstract the neural hybrid system into a transition system defined as Definition 2. A transition system abstraction is a tuple $\mathcal{T} \triangleq \langle \Omega, \mathcal{Q}, \mathcal{E} \rangle$ where its elements are:

- $\Omega \subset \mathbb{R}^{n_x}$: Working zone, where this abstraction is applying to.
- $\mathcal{Q} = \{\mathcal{Q}_1, \cdots, \mathcal{Q}_{N_q}\}$: The finite set of subspaces called cells, where: 1) $\mathcal{Q}_i \subseteq \Omega$; 2) $\Omega = \bigcup_{i=1}^{N_q} \mathcal{Q}_i$; 3) $\mathcal{Q}_i \bigcap \mathcal{Q}_j = \emptyset$. With an index function $idx : \mathcal{Q} \to \mathbb{N}^{\leq N_q}$ for $idx(\mathcal{Q}_i) = i$.
- $\mathbb{N}^{\leq N_q} \text{ for } idx(\mathcal{Q}_i) = i.$ • $R : \mathbb{N}^{\leq N_q} \times \mathbb{N}^{\leq N_q} \to \mathbb{B}$: Transition rules, if there exist a probable transition from \mathcal{Q}_i to \mathcal{Q}_j , then R(i, j) = 1, else R(i, j) = 0.

A transition system abstraction will unveil the interconnection of subspaces with transition rules T through abstracting the neural hybrid system \mathcal{H} . In this process, the data in the form of traces will be generated by the neural hybrid system \mathcal{H} by giving it randomized initial states, and randomized or user-specified external input for the non-autonomous dynamical systems.

3.2 Efficient Dynamics Learning via Low-Level Modeling

In this paper, we will be achieving efficient dynamics learning via our proposed low-level model, namely, neural hybrid system modeling. To begin with, ME-partitioning proposed in Yang and Xiang (2023) will be applied to bisecting the working zone Ω based on the data set \mathcal{D} . In this process, Ω and \mathcal{P} will be in the form of the interval, e.g., $\mathcal{P}_i = [\underline{p}_{i,1}, \overline{p}_{i,1}] \times [\underline{p}_{i,2} \times \overline{p}_{i,2}] \dots$ in which $\overline{\mathcal{P}}_i = \{\overline{p}_{i,1}, \overline{p}_{i,2}, \cdots, \overline{p}_{i,n_x}\} \in \mathbb{R}^{n_x}$, etc. Specifically, we will locate the *j*th dimension of the *i*th partition to bisect via

$$(i,j) = \arg\max_{i,j} D_{i,j},\tag{12}$$

in which

$$D_{i,j} = \overline{p}_{i,j} - \underline{p}_{i,j}.$$
(13)

We will keep bisecting the \mathcal{P} until $\Delta H \leq \epsilon$. After the ME partitioning, the set of partitions \mathcal{P} with N_p partitions can be obtained, which will subsequently define the segmented data set $\{\mathcal{D}_1, \ldots, \mathcal{D}_{N_p}\}$. With the segmented data set, we are able to train the set of neural networks once given a neural network structure, namely, the layers, neurons, and the activation function, etc., of neural networks.

To further optimize the ME partitioning and simplify the learning model, we will merge the redundant partitions based on the training performance of the neural network. Merging redundant partitions will be based on the Mean Square Error (MSE) performance of the neural network. Given \mathcal{D} and a trained neural network Φ , the MSE performance of Φ is

$$MSE(\Phi, D) = \frac{1}{|D|} \sum_{i=1}^{|D|} \left\| \Phi(z^{(i)}) - y^{(i)} \right\|.$$
(14)

By setting a threshold based on MSE performance $\gamma \geq 0$, we are able to identify the redundant partitions that are considered to have similar performance under the same neural network structure, namely, if

$$MSE(\Phi, \mathcal{D}_i \cup \mathcal{D}_j) \le \gamma \tag{15}$$

for a trained Φ , the corresponding partitions \mathcal{P}_i and \mathcal{P}_j will be considered redundant partitions, and hence they will be merged.

Merging the redundant partitions will subsequently define the switching logic δ and the set of neural networks $\tilde{\Phi}$ for the neural hybrid system \mathcal{H} . The low-level neural hybrid system modeling can be summarized in pseudo-code given in the Algorithm 1.

Algorithm 1 Low-Level Neural Hybrid System Modeling

▷ Maximum Entropy partitioning 1: procedure ME PARTITIONING $(\Omega, \mathcal{D}, \epsilon)$ Input: $\Omega; \mathcal{D}; \epsilon$. **Output:** \mathcal{P} ; \cup { \mathcal{D}_i }. $\bar{P}_{save} \leftarrow \emptyset; \; \mathcal{D}_{save} \leftarrow \emptyset;$ 2: $P_1 \leftarrow \Omega;$ 3: while $\exists \Delta H_i \geq \epsilon, \ \forall \mathcal{P}_i \ \mathbf{do}$ 4: $[i, j, Distance] \leftarrow \max(D_{i,j})$ 5: Obtain \mathcal{P}_{temp1} and \mathcal{P}_{temp2} under (12) 6: 7:Obtain \mathcal{D}_{temp1} and \mathcal{D}_{temp2} 8: if $\Delta H_i \geq entropy$ then \triangleright Using (7) $P_i \leftarrow \{P_{temp1}, P_{temp2}\}$ 9: $\mathcal{D}_i \leftarrow \{\mathcal{D}_{temp1}, \mathcal{D}_{temp2}\}$ 10: 11: else 12:Add P_i to P_{save} and delete P_i Add \mathcal{D}_i to \mathcal{D}_{save} and delete \mathcal{D}_i 13:end if 14: 15:end while 16: return $\mathcal{P} \cup \mathcal{P}_{save}$; $\mathcal{D} \cup \mathcal{D}_{save}$. end procedure 17: \triangleright Merging and dynamics learning **procedure** MERGE AND LEARN $(\mathcal{P}, \cup \{\mathcal{D}_i\}, \Phi)$ 18:Input: $\mathcal{P}, \cup \{\mathcal{D}_i\}, \Phi$ Output: $\mathcal{P}, \tilde{\Phi}$ $\ell \leftarrow |\mathcal{P}|, N \leftarrow 1;$ 19: \triangleright Segmented partitions merge while $N < \ell$ do 20: 21: $n \leftarrow 1;$ while $n \leq \ell$ do 22: 23: $n \leftarrow n+1;$ $\Phi_{N,n} \leftarrow \Phi, \, \mathcal{D}_{N,n} \leftarrow \mathcal{D}_N \cup \mathcal{D}_n$ 24:
$$\begin{split} \Phi_{N,n} &\leftarrow \arg\min_{\Phi_{N,n}} MSE(\Phi_{N,n},\mathcal{D}_{N,n}) \\ \text{if } MSE(\Phi_{N,n},\mathcal{D}_{N,n}) \leq \gamma \text{ then} \end{split}$$
25:26: $\mathcal{P}_N \leftarrow \{\mathcal{P}_N \cup \mathcal{P}_n\}$ 27:Delete \mathcal{P}_n 28: $\ell \leftarrow \ell - 1$ 29:end if 30: end while 31: $N \leftarrow N + 1;$ 32: end while 33: ▷ Generate neural network approximations 34: $i \leftarrow 1;$ 35: while $i \leq N$ do 36: $\Phi_i \leftarrow \arg\min_{\Phi_i} MSE(\Phi_i, \mathcal{D}_i)$ 37: end while 38: 39: return $\mathcal{P} = \{\mathcal{P}_1 \dots, \mathcal{P}_N\}; \Phi = \{\Phi_1, \dots, \Phi_N\}$ 40: end procedure

The low-level neural hybrid system can model the dynamical system through a distributive and computationally efficient framework, which makes it possible for parallel training in the Merge and Learning procedure, and distributive verification in Wang et al. (2023a). To further exploit this distributive structure and promote the interpretability of the low-level learning model, we proposed a transition system abstraction method as the high-level model.

Algorithm 2 Transition Computation via \mathcal{H}
$\boxed{\text{Input: } \mathcal{P}, \mathcal{Q}, \Phi, \mathcal{U}}$
Output: R
1: $N_p \leftarrow \mathcal{P} , N_q \leftarrow \mathcal{Q} $
2: $i \leftarrow 1; j \leftarrow 1;$
3: while $i \leq n \operatorname{do}$
4: $\mathcal{Q}'_i \leftarrow \bigcup_{l=1}^{N_p} \Phi^*_l(\mathcal{Q}_i \cap P_l, \mathcal{U}) \qquad \triangleright \text{ Using (16)}$
5: $j \leftarrow 1$
6: while $j \leq n \operatorname{do}$
7: if $\hat{\mathcal{Q}}'_i \cap \mathcal{Q}_j \neq \emptyset$ then
8: $R(i,j) \leftarrow 1$
9: else
10: $R(i,j) \leftarrow 0$
11: end if
12: $j \leftarrow j + 1$
13: end while
14: $i \leftarrow i + 1$
15: end while
16: return R

3.3 Interpretable Abstraction via High-Level Model

In high-level model abstraction, we intend to abstract the neural hybrid system model in Definition 1 into a transition system in Definition 2 with the help of the data generated by \mathcal{H} called the set of samples, defined by

Definition 3. Set of samples $\mathcal{W} = \{w_1, w_2, \cdots, w_L\}$ of neural hybrid system (11) is a collection of sampled Ltraces obtained by given \mathcal{H} different initial condition and randomized external input $u \in \mathcal{U}$, where for each trace w_i , $i = 1, \ldots, L$, is a finite sequence of time steps and data $(k_{0,i}, d_{0,i}), (k_{1,i}, d_{1,i}), \cdots, (k_{M_i,i}, d_{M_i,i})$ in which

- $k_{0,i} \in (0,\infty)$ and $k_{\ell+1,i} = k_{\ell,i} + 1, \forall \ell \in \mathbb{N}^{\leq M_i}, \forall i \in \mathbb{N}^{\leq L}$.
- $z_{\ell,i} = [x_i^{\top}(k_{\ell,i}), u_i^{\top}(k_{\ell,i})]^{\top} \in \mathbb{R}^{n_x+n_u}, \forall \ell = 0, 1, \ldots, M_i, \forall i \in \mathbb{N}^{\leq L}$, where $x_i(k_{\ell,i}), u_i(k_{\ell,i})$ denote the state and input of the system at ℓ th step for *i*th trace, respectively.

Remark 1. It should be noted that the abstraction of the neural hybrid system is specific, meaning that different transition system abstractions can be obtained based on different control strategies. This specificity aids system designers in implementing and validating control strategies tailored to specific partitions.

After obtaining the set of samples, the set of cells Q will be obtained via the ME partitioning method as in procedure ME partitioning in Algorithm 1 based on W. Then, the transition relationships between cells will be computed via reachability analysis in

$$\mathcal{Q}'_{i} = \bigcup_{j=1}^{N_{p}} \Phi_{j}^{*}(\mathcal{Q}_{i} \cap \mathcal{P}_{j}, \mathcal{U}), \qquad (16)$$

in which Φ_j^* indicates a reachable set computation method using the sub-neural network. Intuitively, based on Definition 2 the transition rule R(i, j) is

$$R(i,j) = \begin{cases} 1, \ \mathcal{Q}'_i \cap \mathcal{Q}_j \neq \emptyset \\ 0, \ \mathcal{Q}'_i \cap \mathcal{Q}_j = \emptyset \end{cases}$$
(17)

The process of transition computation can be summarized in pseudo-code given in Algorithm 2. The proposed dual-level modeling framework can be summarized as follows.

- The localized working zone of Ω , i.e., \mathcal{P} can be obtained based on an ME partitioning process, which is completely data-driven and can be easily tuned by adjusting the threshold.
- Partitions can be further optimized based on the MSE performance of the trained neural network to simplify the low-level model.
- The low-level model has a distributive structure consisting of simple neural networks that allow parallel training and verification, which will be computationally efficient.
- The low-level model can be further abstracted into a high-level transition system, this process can be specifically designed and allow system designers to develop and test control strategies that are specifically tailored for each distinct localized cell.
- The transitions can be off-line computed by reachability analysis, and can be transferred into a transition graph which will enhance the learning model's interpretability, and enable the feasibility of verifications based on logical descriptions.

4. APPLICATIONS TO DYNAMICAL SYSTEM MODELING

Regarding the modeling of complex dynamical systems like human behaviors, learning-based methodologies have garnered significant attention for their efficacy in Reinhart and Steil (2011); Kanazawa et al. (2019), etc. However, while learning-based approaches offer advantages over mechanistic modeling, they present numerous challenges in practical applications. For instance, typical issues include:

- The limited availability of sample data may result in a deep neural network-based dynamical system model that is not adequately trained, thereby hindering its ability to capture the full spectrum of human behavioral complexities.
- The inherent nature of human demonstrations, characterized by sudden shifts, suggests that a trained neural network-based dynamical system model might exhibit discrepancies in its behavior, especially in localized regions of the operational space.
- The interpretability deficit in neural network-based dynamical system models poses a significant challenge in real-time applications for limited, and computationally intensive verification methods.

The above issues are exemplified in the LASA dataset Khansari-Zadeh and Billard (2011) modeling, which encompasses a diverse range of handwriting motions demonstrated by human users across 30 distinct shapes. This paper will attempt to address these issues through our proposed dual-level dynamical system modeling framework. The dual-level modeling process can be summarized as follows.

• Extreme Learning Machines (ELMs) are employed, each comprising 20 ReLU-activated neurons. These ELMs feature a randomized input weight matrix and bias vector, forming the core structure of the model. To highlight the efficacy of our modeling approach, an ELM with a solitary hidden layer containing 200 ReLU-activated neurons is trained to serve as a singleneural network reference model.

- A threshold of ε = 4×10⁻² is set for ME partitioning variation in Algorithm 1. This setting led to the generation of the set of partitions in ME partitioning for the low-level model of all 30 shapes some results are given in Tab. 1 and Tab. 2, an illustration of *MutiModels*₂² is given in Fig. 1 (a).
 By setting a threshold γ = 1.5×10⁻⁵ in Merging, we
- By setting a threshold $\gamma = 1.5 \times 10^{-5}$ in Merging, we manage to simplify the low-level model by training fewer neural networks while maintaining accuracy.
- We obtain the abstraction data from randomly generated trajectories in the working zone Ω , where $\forall M_i =$ 400, $\forall i \in \mathbb{N}^{\leq 400}$ under Definition 3. By applying the threshold $\epsilon = 4 \times 10^{-2}$, a set of cells is then generated, as shown in Fig. 1 (c).
- Based on the set of cells, we employ the Algorithm 2 to compute the transition relationships of the high-level model abstraction. The transition from Fig. 1 (c) to Fig. 1 (d) allows for the interpretation of transition relationships between local working zones.
- We verify the transition system abstraction via Computation Tree Logic (CTL) formulae Pan et al. (2016), in which \diamond or \Box denote the for *some* or *all* traces, and \bigcirc denotes the next step. The formulae and results of *MultiModel*₂ are given in Tab. 3 as examples, ϕ_1 indicates the possibility of the neural hybrid system model being in Q_2 , ϕ_2 specifies that for every possible next step, the system will be in Q_4 , and ϕ_3 signifies whether there exists a trajectory that reaches Q_7 immediately after passing through Q_6 , given the initial condition is Q_9 .

Table 1. Training Time and MSE of the Lowlevel Model

Shape Name	Training Time (ms)	MSE (10^{-5})
Khamesh	0.7147	0.3466
LShape	0.7784	0.3745
$MultiModels_1$	0.5603	0.2858
$MultiModels_2$	0.6225	0.2892
•••		

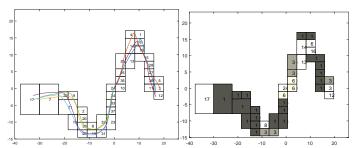
Table 2. Training Time and MSE for ELM Model

Shape Name	Training Time (ms)	MSE (10^{-5})
Khamesh	14.8098	0.0923
LShape	38.4118	0.0842
$MultiModels_1$	20.8079	0.0551
$MultiModels_2$	26.6117	0.0753

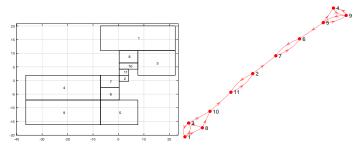
5. CONCLUSION

In this paper, a dual-level dynamical system learning framework is proposed to promote computational efficiency and interpretability in system identification. This framework utilizes a data-driven ME partitioning process to bisect the working zone, which makes it possible for parallel training and local analysis. In order to simplify

² Complete results includes modeling for all 30 shapes can be found on our GitHub repository on dual-level dynamical system modeling at https://github.com/aicpslab/ Dual-Level-Dynamic-System-Modeling/tree/main/Results



(a) *MultiModels*₂ Handwriting(b) 12 partitions obtained where Human Demonstration and 37 Par-redundant ones are given in the titions Obtained. same color.



(c) 11 Cells Abstraction for \mathcal{H} of(d) Transition Map based on \mathcal{T} of $MultiModels_2$. $MultiModels_2$.

- Fig. 1. Partitions, Cells, and Transition Map Abstraction of Dual-Level Models for $MultiModels_2$ from LASA data set.
 - Table 3. Verification results of CTL formula: $\mathcal{T}_{MultiModels_2}$ with \mathcal{Q}_9 as the initial cell.

CTL formula	$\mathcal{T}_{MultiModels_2}$
$\phi_1 = \exists \diamond \mathcal{Q}_2$	true
$\phi_2 = \forall \bigcirc \mathcal{Q}_4$	false
$\phi_3 = \exists (\mathcal{Q}_6) \land (\exists \bigcirc \mathcal{Q}_7)$	true

the learning model, a process called Merging is proposed to merge the partitions based on the training performance. The low-level model is then able to learn the dynamics precisely while only consisting of a set of simple neural networks. A high-level model is proposed to promote interpretability through the reachability analysis. This highlevel model will provide valuable insights into the transition relationship within the working zone with the transition map and allow user-specified verification through CTL formulae.

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