# A CONVERGENCE FRAMEWORK FOR AIRY $_\beta$ LINE ENSEMBLE VIA POLE EVOLUTION

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ABSTRACT. The Airy<sub> $\beta$ </sub> line ensemble is an infinite sequence of random curves. It is a natural extension of the Tracy-Widom<sub> $\beta$ </sub> distributions, and is expected to be the universal edge scaling limit of a range of models in random matrix theory and statistical mechanics. In this work, we provide a framework of proving convergence to the Airy<sub> $\beta$ </sub> line ensemble, via a characterization through the pole evolution of meromorphic functions satisfying certain stochastic differential equations. Our framework is then applied to prove the universality of the Airy<sub> $\beta$ </sub> line ensemble as the edge limit of various continuous time processes, including Dyson Brownian motions with general  $\beta$  and potentials, Laguerre processes and Jacobi processes.

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# 1. INTRODUCTION

During the 18th century, De Moivre established the Gaussian distribution for sums of independent binomial variables, a concept later generalized by Laplace. Gauss popularized the central limit theorem, used for error evaluation in systems characterized by independence. In recent decades, there has been increasing interest in understanding fluctuations in highly correlated systems, leading to the emergence of a different family of distributions known as Tracy-Widom<sub> $\beta$ </sub>, which are indexed by a positive parameter  $\beta$ . Historically, such Tracy-Widom<sub> $\beta$ </sub> distributions for  $\beta = 1, 2, 4$  were first observed in Random Matrix Theory, as the scaling limit of the extreme eigenvalues of the classical matrix ensembles [58, 82, 84, 86, 121, 122]. Later, such extreme eigenvalue statistics are proven to be universal, in the sense that Tracy-Widom<sub> $\beta$ </sub> limits

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hold for a wide range of random matrix models, including adjacency matrices of random graphs, which are usually sparse. See e.g., [16, 54, 71, 72, 78, 79, 90, 98, 118, 120]. Beyond matrix models, Tracy-Widom<sub> $\beta$ </sub> distributions also appear in lots of different random systems, such as random tilings, 1+1 dimensional random growth models, exclusion processes, planar random geometry such as first/last passage percolation models, and square ice models (six-vertex models). See e.g., [4,5,8,10–14,21,28,30,70,80,82–84,104,111,115,126,127]. Many of these models are related to each other through their underlying integrable structures, and are in the so called Kardar-Parisi-Zhang (KPZ) universality class, a topic in probability theory that has been intensively studied in recent years.

A motivation of the current paper is to better understand the mathematical structures behind Tracy-Widom<sub> $\beta$ </sub> distributions, and to develop new methods of proving convergence to them. The main object studied here is the Airy<sub> $\beta$ </sub> line ensemble (ALE<sub> $\beta$ </sub>) { $\mathcal{A}_{i}^{\beta}(t)$ }<sub> $i \in \mathbb{N}, t \in \mathbb{R}$ </sub>, which can be defined as a random process on  $\mathbb{N} \times \mathbb{R}$  or a family of continuous random processes on  $\mathbb{R}$ , with  $\beta > 0$ being a parameter, and is ordered, i.e.,  $\mathcal{A}_{1}^{\beta}(t) \geq \mathcal{A}_{2}^{\beta}(t) \geq \mathcal{A}_{3}^{\beta}(t) \geq \cdots$  for any t. They are natural generalizations of the Tracy-Widom<sub> $\beta$ </sub> distributions, akin to Brownian motions being generalizations of Gaussian distributions, and have Tracy-Widom<sub> $\beta$ </sub> as the one-point distribution of the top line  $\mathcal{A}_{1}^{\beta}$ . These ALE<sub> $\beta$ </sub> are believed to be universal objects, in the sense of being the scaling limit of many random matrix models and interacting particle systems. However, basic properties of these processes as well as these convergences remain quite mysterious so far, except for the special setting of  $\beta = 2$ , where a determinantal structure is present and has been largely exploited using algebraic methods.

In this paper, we take a new perspective, and our main result is a characterization of  $ALE_{\beta}$  in terms of its Stieltjes transform and a system of stochastic differential equations (SDE). Our result provides a new framework to prove convergence. As some examples, we prove convergence to  $ALE_{\beta}$ from various random processes, including the classical  $\beta$ -Hermite/Laguerre/Jacobi processes and their generalizations. We note that some of these were previously unknown even in the  $\beta = 1, 4$ cases, where such convergences can be interpreted as the joint convergence of extreme eigenvalues of correlated real/quaternion random matrices. Beyond these, our framework should also be applicable to prove convergence to  $ALE_{\beta}$  for many other models; and for some of them, even the Tracy-Widom<sub> $\beta$ </sub> limits were previously unknown. Moreover, our characterization also reveals some useful information for  $ALE_{\beta}$ , such as Hölder properties and collision of adjacent lines.

# 1.1. Background. We next provide some setup, starting with some more classical processes.

1.1.1. Edge limit of general  $\beta$ -ensembles. Tracy-Widom<sub> $\beta$ </sub> distributions for general  $\beta > 0$  was introduced and studied in [53, 119] by Edelman and Sutton, and [113] by Ramirez, Rider and Virág, as the scaling limit of the extreme eigenvalue of Gaussian  $\beta$ -ensembles. More generally,  $\beta$ -ensemble is a probability distribution on n particle system  $x_1 \ge x_2 \ge \cdots \ge x_n$ , with probability density:

(1.1) 
$$\frac{1}{Z_{n,\beta,W}} \prod_{i< j} |x_i - x_j|^{\beta} \prod_{i=1}^n W(x_i),$$

where  $Z_{n,\beta,W}$  is a renormalization constant, and  $W \ge 0$  is the weight function. There are three special cases referred to as the classical ensembles, which are defined by

(1.2) 
$$W(x) = \begin{cases} e^{-\beta x^2/4}, & \text{Hermite/Gaussian ensemble} \\ x^{\beta(m-n+1)/2-1}e^{-\beta x/2}, & \text{Laguerre ensemble} \\ x^{\beta(p-n+1)/2-1}(1-x)^{\beta(q-n+1)/2-1}, & \text{Jacobi ensemble} \end{cases}$$

These classical ensembles for  $\beta = 1, 2, 4$  originated from the study of eigenvalue distributions of random matrices. They represent the joint distributions of the eigenvalues of size *n* Gaussian, Wishart, and Jacobi matrices. These matrices and their extreme eigenvalues, with  $\beta = 1$  corresponding to the real case, have been widely used in high-dimensional statistical inference (see the survey by Johnstone [87]). Beyond random matrix theory,  $\beta$ -ensembles also describe the one-dimensional Coulomb gas in physics [58], and are connected to orthogonal polynomial systems [91].

As mentioned earlier, as  $n \to \infty$ , the distribution of the largest eigenvalues converges to the Tracy-Widom<sub> $\beta$ </sub> distribution for  $\beta = 1, 2, 4$ , respectively. More generally, one can consider the edge limit, i.e., the joint scaling limit of the top k eigenvalues for any arbitrary k, as a point process. It has been shown [41–43] that this edge limit does not depend on the potential function W, but it varies for each of  $\beta = 1, 2, 4$ . The  $\beta = 2$  edge limit is also known as the Airy point process.

For  $\beta$  other than 1, 2, 4, obtaining such edge limit was a challenging problem, partly due to the relative lack of exact-solvable structures. Based on a tri-diagonal random matrix model discovered by Dumitriu and Edelman [49], this was resolved in [113], where the edge limit of Gaussian and Laguerre  $\beta$ -ensemble is shown to be the eigenvalues of the  $\beta$  stochastic Airy operator (SAO<sub> $\beta$ </sub>), which is also called the Airy<sub> $\beta$ </sub> point process. In particular, for each  $\beta > 0$ , the law of the largest eigenvalue of SAO<sub> $\beta$ </sub>, i.e., the top particle in the Airy<sub> $\beta$ </sub> point process, is then defined as the Tracy-Widom<sub> $\beta$ </sub> distribution. Later, such Airy<sub> $\beta$ </sub> point process limit is also extended to more general W [17,18,25,95]. Analogous edge limits for discrete  $\beta$ -ensembles have been derived in [67].

1.1.2. Airy line ensemble. In another direction, the Tracy-Widom<sub>2</sub> distribution and the ( $\beta = 2$ ) Airy point process are extended to the Airy line ensemble (ALE), an ordered family of random processes { $\mathcal{A}_i(t)$ }<sub> $i \in \mathbb{N}, t \in \mathbb{R}$ </sub>, where each  $\mathcal{A}_i$  is continuous, and they are jointly stationary in the  $\mathbb{R}$ direction (see Figure 1). ALE was introduced by Prahöfer and Spohn [109], as the scaling limit for the multi-layer polynuclear growth (PNG) model from the KPZ universality class. The top line  $\mathcal{A}_1$  is known as the stationary Airy<sub>2</sub> process, whose one-point marginal is the Tracy-Widom<sub>2</sub> distribution; and for any  $t \in \mathbb{R}$ , { $\mathcal{A}_i(t)$ }<sub> $i \in \mathbb{N}$ </sub> is the Airy point process. ALE plays a central role in KPZ, in particular through the construction of the directed landscape [37]. (See also [38] for computing passage times in the directed landscape from ALE.)

ALE is particularly useful in KPZ, partly due to its Brownian Gibbs property, which was recognized by Corwin and Hammond [35]. Specifically, for ALE minus a parabola, it inside any domain, conditional on the boundary, is given by non-intersecting Brownian bridges. This fact is later widely used as a powerful tool to study ALE and many KPZ class models. Aggarwal and the first-named author provided a strong characterization of ALE, demonstrating that ALE (minus a parabola) is the only random process on  $\mathbb{N} \times \mathbb{R}$  with the Brownian Gibbs property as well as approximating a parabola [6]. Such a strong characterization would be a powerful tool to prove convergence to ALE and establishing KPZ universality for various models; see e.g. [4].

1.2. Airy<sub> $\beta$ </sub> line ensemble. From the success of ALE, the next question is to construct a time dependent evolution for Tracy-Widom<sub> $\beta$ </sub> (and more generally Airy<sub> $\beta$ </sub> point process) for any  $\beta > 0$ . Following [64] where this is formally introduced, we call it the Airy<sub> $\beta$ </sub> line ensemble (ALE<sub> $\beta$ </sub>). There are several problems in this program:

- *Construction* What should it be? How to construct it?
- Description What are its properties? Ideally, can some precise information be given?
- Universality Why is  $ALE_{\beta}$  natural and interesting? Can it be shown to be the universal scaling limit of many natural random processes, as the  $\beta = 2$  case?



FIGURE 1. An illustration of ALE

Towards these questions, there have been many results focusing on different aspects of  $ALE_{\beta}$  (some tracing back to the studies of  $Airy_{\beta}$  point process, or for special  $\beta$ ): infinite-dimensional SDE [88,97,106,107], correlation function [11,85,104,109], and Laplace transform [64,81,102,103,117,118]. In this paper, we provide a different perspective using Stieltjes transform, tailored to the universality problem. Moreover, we view results presented here and in the concurrent paper [64] (by Gorin, Xu, and the second-named author, to be discussed shortly) complement to each other.

We now give a more detailed account on the state of the art, and further motivate our results.

1.2.1. The edge limit construction and convergence. Beyond ALE where  $\beta = 2$ , ALE<sub> $\beta$ </sub> was also constructed and studied for several other special values: in [117] for  $\beta = 1$  (and the arguments there should also go through for  $\beta = 4$ ), and in [60] for  $\beta = \infty$ . For general  $\beta > 0$ , even its construction is relatively recent. One potential way, as inspired by the fact (from [35]) that ALE is the edge scaling limit of the  $\beta = 2$  Dyson Brownian motion (DBM), is to consider general  $\beta$  DBM:

(1.3) 
$$d\lambda_i(t) = \sqrt{\frac{2}{\beta}} dB_i(t) + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{dt}{\lambda_i(t) - \lambda_j(t)} - \frac{1}{2} \lambda_i(t) dt,$$

where  $n \in \mathbb{N}$  and  $\{B_i\}_{i=1}^n$  are independent two-sided Brownian motions. There is a solution to this SDE, such that  $\{\lambda_i(t)\}_{i=1}^n$  for any t is a Gaussian  $\beta$ -ensemble; and this is known as the (stationary) DBM of size n with parameter  $\beta$ . (See e.g., [9] for some more backgrounds on DBM.) One can then define  $ALE_{\beta} \{\mathcal{A}_i^{\beta}(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}$  as its edge limit, i.e., the limit of  $(i, t) \mapsto n^{1/6}\lambda_i(tn^{-1/3}) - 2n^{2/3}$  as  $n \to \infty$ . Note that for any fixed  $t, \{\mathcal{A}_i^{\beta}(t)\}_{i \in \mathbb{N}}$  should be the  $Airy_{\beta}$  point process, and  $\mathcal{A}_1^{\beta}(t)$  should follow the Tracy-Widom<sub> $\beta$ </sub> distribution.

Beyond DBM, another potential way of constructing  $ALE_{\beta}$  is via the edge limit of Gaussian corners processes, which are random *Gelfand-Tsetlin patterns*, and can be viewed as multi-level generalizations of Gaussian  $\beta$  ensembles. For  $\beta = 1, 2, 4$ , they can be defined as eigenvalues of the top-left corners of different sizes, for Wigner matrices with Gaussian real, or complex, or quaternionic entries (see e.g. [15, 59, 101]).

To justify the above definition of  $ALE_{\beta}$ , the above stated convergence should be proved. In [97], this is achieved for stationary DBM and any  $\beta \geq 1$  via a coupling argument. In [64] the convergences for both stationary DBM and Gaussian corners process are established, for any  $\beta > 0$ . The proofs are via explicit formulas, also showing that both limits are the same. Very recently, in [52] a tridiagonal model for DBM is proposed, which may also be used to demonstrate convergence to  $ALE_{\beta}$ . Additionally, the arguments in this paper provide an alternative proof of DBM convergence. 1.2.2. Description via explicit expressions. Historically, the theory of Tracy-Widom<sub> $\beta$ </sub> distributions used to largely rely on explicit formulas, based on determinantal/Pfaffian structures or matrix models (see e.g., [63, 113, 116, 118, 121, 123]). As for ALE<sub> $\beta$ </sub>, formulas used to be available only for  $\beta = 2$ , i.e., ALE, in the construction by Prahöfer and Spohn [109]; and for  $\beta = \infty$  in [60]. There are various challenges in obtaining precise formulas for general  $\beta$ , primarily due to the lack of structures in this generality. As above mentioned, the first construction of ALE<sub> $\beta$ </sub> for  $\beta \ge 1$  in [97] was via the edge limit of DBM, using a more abstract convergence argument. Then in [64], precise formulas for the Laplace transform of ALE<sub> $\beta$ </sub> are obtained. Such Laplace transforms also determine the law of ALE<sub> $\beta$ </sub>, thereby [64] gives a direct and explicit definition of ALE<sub> $\beta$ </sub> for any  $\beta > 0$ .

1.2.3. Uniqueness and universality. As indicated by the convergence of both DBM and Gaussian corners process to the same limit, i.e.,  $ALE_{\beta}$ , in [64], it is natural to expect that  $ALE_{\beta}$  is also the scaling limit of many other well-known processes. Some examples include DBM with general potentials, non-intersecting random walks [61,74,93], various other  $\beta$ -corners processes [22,62,65] and measures on Gelfand-Tsetlin patterns [29,108], Macdonald processes [20], and  $(q, \kappa)$ -distributions on lozenge tilings [23,46].

A robust and general approach to establishing convergence involves a suitable characterization of  $ALE_{\beta}$ . Specifically, this means identifying easily verifiable properties of  $ALE_{\beta}$  and demonstrating that these properties uniquely determine  $ALE_{\beta}$ . To prove convergence, it would then suffice to establish tightness and confirm that any subsequential limit satisfies these properties.

For  $ALE_{\beta}$  with  $\beta = 2$ , an elegant characterization is the Brownian Gibbs property [6] as mentioned above. However, this does not hold for any  $\beta \neq 2$ . The next natural candidate of characterization would be an 'infinite dimensional DBM', by taking  $n \to \infty$  in (1.3). For example, [97] shows that  $ALE_{\beta}$  (for  $\beta \geq 1$ ), i.e., the edge limit of finite dimension DBM, is a solution to such an infinite dimensional DBM in a weak sense.

However, there are several technical challenges in developing a characterization and convergence framework along this direction. First, particles (i.e., those  $\lambda_i$  in (1.3)) may collide or adjacent particles may get too close, leading to singularities in the drift  $1/(\lambda_i(t) - \lambda_j(t))$  term. In [97], for  $\beta \geq 1$  such singularities are ruled out using existing level replusion estimates (see [97, Theorem 2.2] and [25, Theorem 3.2]), which are known only for stationary DBM whose laws are given by  $\beta$ -ensembles. Deriving such estimates for other models could be difficult. Moreover, for  $\beta < 1$ collisions are inevitable. Second, the long-range interactions introduce additional complications when analyzing infinitely many particles. In fact, even the well-posedness of the infinite-dimensional DBM starting from a fixed initial condition appears to be absent from the literature, except for the cases  $\beta = 1, 2, 4$  [88, 106, 107] where the underlying algebraic structure is used<sup>1</sup>. As a result, for the infinite-dimensional DBM, both proving the uniqueness of solution and verifying it for any subsequential limit face various barriers.

To overcome these difficulties, in this paper we take an alternative approach, and characterize  $ALE_{\beta}$  as the pole dynamics of meromorphic functions, satisfying a function-valued SDE. In other words, we characterize  $ALE_{\beta}$  via the dynamics of its Stieltjes transform. This method completely circumvent the issue of long-range interactions and collisions, and is applicable for any  $\beta > 0$ .

<sup>&</sup>lt;sup>1</sup>Note that for an analogous infinite-dimensional SDE corresponding to the bulk limit of DBM, such well-posedness has been achieved for  $\beta \geq 1$  (see [89, 105, 106, 124]). A key property used in the bulk setting is the cancellation of particle interactions from left and right, and that is absent at the edge.

1.3. Main characterization result. In the rest of this paper, we fix  $\beta > 0$ . We study (infinite) line ensembles, by which we mean ordered families of continuous random processes, denoted by  $\{\boldsymbol{x}(t)\}_{t\in I} = \{x_i(t)\}_{i\in\mathbb{N},t\in I}$ , for  $I = \mathbb{R}$  or any interval, satisfying  $x_1(t) \ge x_2(t) \ge x_3(t) \ge \cdots$ .

As already alluded to, we shall characterize  $ALE_{\beta}$  using its Stieltjes transform. If  $\{x(t)\}_{t\in\mathbb{N}}$  (were  $ALE_{\beta}$ , it would be imperative that (for each  $t \in \mathbb{R}$ ) the Stieltjes transform of  $\{x_i(t)\}_{i\in\mathbb{N}}$  (with complex variable w) asymptotically behaves like  $\sqrt{w^2}$  as  $w \to \infty$  in the complex plane. This is because we expect  $\{x_i(t)\}_{i\in\mathbb{N}}$ , as a one time slice of  $ALE_{\beta}$ , to be the  $Airy_{\beta}$  point process; hence, the particles should be close to the zeros of the Airy function Ai(w). Therefore, the Stieltjes transform of  $\{x_i(t)\}_{i\in\mathbb{N}}$  should exhibit similar asymptotic behaviors as -Ai'(w)/Ai(w), which is known to behave like  $\sqrt{w}$  as  $w \to \infty$ .

We next give a more precise formulation of such asymptotic behaviors. Below we use  $\mathbb{H}$  to denote the open upper half complex plane. For Stieltjes transforms, we shall use the notion of Nevanlinna functions, i.e., functions from  $\mathbb{H}$  to  $\mathbb{H} \cup \mathbb{R}$  that are holomorphic.

**Definition 1.1.** A measure  $\mu$  on  $\mathbb{R}$  is *particle-generated*, if it is locally finite, and is in the form of  $\sum_{x \in P} \delta_x$ , where P (the particles) is an at most countable multiset of real numbers. A Nevanlinna function Y has a Nevanlinna representation of the form

(1.4) 
$$Y(w) = b + cw + \int_{\mathbb{R}} \left( \frac{1}{x - w} - \frac{x}{1 + x^2} \right) d\mu(x),$$

where  $b, c \in \mathbb{R}$ ,  $c \geq 0$ . We say that Y is *particle-generated*, if the measure  $\mu$  in its Nevanlinna representation is particle-generated, namely  $\mu = \sum_{x \in P} \delta_x$ .

We note that such Y can be extended to a meromorphic function on  $\mathbb{C}$  with  $Y(\overline{w}) = \overline{Y(w)}$ , with P being all the poles, and each residue equals  $1^3$ .

**Definition 1.2.** For  $\mathfrak{d}, C_* > 0$ , a Nevanlinna function Y is  $(\mathfrak{d}, C_*)$ -Airy-like, or simply Airy-like, if

- (1) it is particle-generated, and all the poles are  $\leq C_*$ ;
- (2) for all w with  $\operatorname{Im}[w] \ge C_* \sqrt{\operatorname{Re}[w] \lor 0 + 1}$ ,

(1.5) 
$$|Y(w) - \sqrt{w}| \le \frac{C_* \operatorname{Im}[\sqrt{w}]^{1-\mathfrak{d}}}{\operatorname{Im}[w]}.$$

Remark 1.3. The condition (2) implies the existence of infinitely many poles  $x_1 \ge x_2 \ge \cdots$ . As we will see shortly (Lemma 2.3), bounds similar to (1) and (2) (but may with a different domain for w) would imply that the density of these poles would be close to  $\sqrt{-x}$  in  $\mathbb{R}_-$ . Such density closeness can be phrased as quantitative bounds on the distances between the poles and zeros of the Airy function  $\mathfrak{a}_1 > \mathfrak{a}_2 > \cdots$ , as stated in Lemma 2.3, and will be frequently used in our proofs. Therefore, as a slight misuse of notions, we will refer to such density closeness as *Airy-zero approximation*.

As another comment on (2): again note that the domain of w is different from that in Lemma 2.3, since the domain here is taken to be easily verifiable for the sub-sequential limit of various models, as will be evident in Section 7. In fact, bounds on  $|Y(w) - \sqrt{w}|$  for w closer to  $\mathbb{R}_+$  can be readily deduced for Airy-like Nevanlinna functions (see Lemma 2.5).

Take a family of random Nevanlinna functions  $\{Y_t\}_{t\in\mathbb{R}}$ . We next state two assumptions.

<sup>&</sup>lt;sup>2</sup>Here and throughout this paper,  $\sqrt{w}$ , or any rational power of  $w \in \mathbb{C}$ , is taken to be the branch on  $\mathbb{C} \setminus \mathbb{R}_{-}$ <sup>3</sup>More precisely, for each  $x \in \mathbb{R}$ , the residue at x equals the multiplicity of x in P.

Assumption 1.4. For any  $t \in \mathbb{R}$ ,  $Y_t$  is particle-generated. Moreover, there exists a (deterministic)  $\mathfrak{d} > 0$ , a sequence  $t_1, t_2, \dots \to -\infty$ , and a tight family of random numbers  $\{C_{*,j}\}_{j \in \mathbb{N}}$ , so that each  $Y_{t_j}$  is  $(\mathfrak{d}, C_{*,j})$ -Airy-like.

Assumption 1.5. Such  $Y_t$  is continuous in t, and satisfies the following SDE:

(1.6) 
$$dY_t(w) = dM_t(w) + \left(\frac{2-\beta}{2\beta}\partial_w^2 Y_t(w) + \frac{1}{2}\partial_w Y_t(w)^2 - \frac{1}{2}\right)dt$$

where  $M_t(w)$  are complex valued Martingales, with quadratic variation given by

(1.7) 
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle M_t(w)\rangle = \frac{1}{3\beta}\,\partial_w^3\,Y_t(w)$$

and

(1.8) 
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle M_t(w), M_t(w')\rangle = \frac{2}{\beta} \,\partial_w \,\partial_{w'} \,\frac{Y_t(w) - Y_t(w')}{w - w'},$$

for  $w \neq w'$ .

We now explain where this SDE comes from. Take the *n* dimensional stationary DBM  $\{\lambda_i\}_{i=1}^n$  with parameter  $\beta$ , i.e., the stationary solution to (1.3). Let  $m_t$  be the Stieltjes transform of  $\{\lambda_i(t)\}_{i=1}^n$ , i.e.,

$$m_t(z) = \sum_{i=1}^n \frac{1}{\lambda_i(t) - z}, \quad z \in \mathbb{C}.$$

Then one can use Ito's formula to write out an SDE satisfied by  $m_t$ ; by taking an appropriate scaling limit from there, one gets the SDE (1.6). More details on this derivation can be found in Section 7.

Our main result states that these two assumptions are sufficient to determine  $ALE_{\beta}$  uniquely.

**Theorem 1.6.** For any  $\{Y_t\}_{t\in\mathbb{R}}$  satisfying Assumption 1.4 and Assumption 1.5, its poles give a line ensemble, which has the same law as  $ALE_\beta$ .

Several remarks are in line.

(i) Essentiality of Assumption 1.4 (in characterizing  $ALE_{\beta}$ ). Both (1) and (2) in Definition 1.2 are necessary: without (1), the line ensemble may be  $ALE_{\beta}$  plus some additional lines (see [2, 45] for an example in the  $\beta = 2$  setting); while (2) rules out the possibility that the line ensemble is  $ALE_{\beta}$  shifted by a (deterministic or random) constant. As already mentioned in Remark 1.3, we've aimed to make Assumption 1.4 as minimal as possible to ensure broad applicability of our convergence framework. As will be seen in Section 7, Assumption 1.4 is straightforward to verify in these examples.

(ii) DBM convergence. Our proof of Theorem 1.6 does not a priori assume the convergence at the edge of stationary DBM. Instead, in Section 7, we show the tightness at the edge of stationary DBM, and that any subsequential limit satisfies Assumption 1.4 and Assumption 1.5. Therefore, we essentially provide an alternative construction of  $ALE_{\beta}$ .

(iii) Stationarity. We also note that in Theorem 1.6, we do not assume that  $Y_t$  is stationary. Rather, it is a consequence of the theorem that the poles of  $Y_t$  converge to  $ALE_\beta$ , which is stationary, and hence  $Y_t$  is stationary as well. Our proof of Theorem 1.6 in fact establishes a natural relaxation for the SDE (1.6). Specifically, for a family of random particle-generated Nevanlinna functions  $\{Y_t\}_{t\geq 0}$ , if there is a (deterministic)  $\mathfrak{d} > 0$  and a random number C such that  $Y_0$  is  $(\mathfrak{d}, C)$ -Airy-like, and  $\{Y_t\}_{t\geq 0}$  satisfies Assumption 1.5, then for  $T \to \infty$ , the poles of  $\{Y_t\}_{t\geq T}$  converge to  $ALE_\beta$ , under the uniform in compact topology.

(*iv*) Stieltjes transform and poles dynamics. Stieltjes transforms and Nevanlinna functions have been widely used to investigate and characterize eigenvalue distributions of random matrix ensembles. See [57] for studies on eigenvalue rigidity, [7, 56] for bulk limits, and [31, 90, 98] for edge limits.

The concept of characterizing the evolution of interacting particle systems through the pole dynamics of meromorphic functions has been explored previously. In integrable systems, it has been demonstrated that the movement of poles in certain solutions of various nonlinear PDEs can be formally linked to the dynamics of particle systems interacting through simple two-body potentials. This discovery was initially made in [34] for equations such as the Korteweg-de Vries and Burgers-Hopf equations, and in [99] for specific integrable Hamiltonian systems. Subsequently, these observations were extended to include elliptic solutions of equations such as the Kadomtsev-Petviashvili equation [94], the Korteweg–de Vries equation [40], the Kadomtsev-Petviashvili hierarchy [125], and the Toda lattice hierarchy [110]. Our results can be interpreted as a stochastic counterpart to these findings, wherein  $ALE_{\beta}$  is characterized as the pole evolution of the SDE (1.6).

1.4. Convergence framework. Given the characterization presented in Theorem 1.6, to prove convergence to  $ALE_{\beta}$ , it suffices to

- (1) establish the tightness of the Stieltjes transforms of the empirical particle density at the microscopic scale, and
- (2) verify that the scaling limit is Airy-like, and satisfies the SDE (1.6).

As a demonstration of this approach, we prove the convergence to  $ALE_{\beta}$  for several continuous interacting particle systems. We next give the formal statement of our result.

We use a strong topology of uniform in compact convergence for line ensembles. More precisely, for a sequence of ordered families of functions  $\{f_i^{(1)}(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}, \{f_i^{(2)}(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}, \ldots$ , they converge to  $\{f_i(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}$  under the uniform in compact topology, if for each  $i \in \mathbb{N}$ ,  $\lim_{n \to \infty} f_i^{(n)} = f_i$  uniformly in any compact interval.

**Theorem 1.7.**  $ALE_{\beta}$  is the edge scaling limit of stationary DBM with certain general potentials (satisfying Assumption 7.1 below), stationary Laguerre process, and stationary Jacobi process, all with parameter  $\beta$ , under the uniform in compact topology. We refer to Theorem 7.2 for a more detailed statement.

The definitions and background of these processes, as well as the precise statement and proof, will be given in Section 7. We emphasize that these convergence results are new even for the classical cases of  $\beta = 1, 4$  (except for the DBM with  $\beta = 1$ ), which can be viewed as the joint convergence of eigenvalues of time-evolved classical ensembles with real or quaternion entries.

We remark that the developed framework can also be applied to prove convergence to  $ALE_{\beta}$  for the other mentioned models. The main remaining task is to establish the desired tightness given in (1). While such tightness are not available from [26], a plausible way is to utilize the dynamical loop equation, as in [75] where local laws down to any mesoscopic scale have been proven for random tilings. We leave this for future works.

1.5. Other properties. In addition to proving convergence to  $ALE_{\beta}$ , our new characterization can be leveraged to further investigate its properties. First, we can study the regularity of  $ALE_{\beta}$ . The Brownian regularity for the ALE has been intensively studied in [35,36,68,69]. For  $ALE_{\beta}$  with  $\beta \geq 1$ , it has been established in [97] that the lines of  $ALE_{\beta}$  are locally Brownian. In Section 4 we show that the lines of  $ALE_{\beta}$  are Hölder continuous with an exponent 1/2 for any  $\beta > 0$ . The second property we study is the collision of lines. For  $\beta \ge 1$ , it has been established in [97] that the lines of  $ALE_{\beta}$  do not collide. Conversely, for  $\beta \in (0, 1)$ , collisions among lines are anticipated. We prove in Section 5 that the occurrence of collisions is almost surely of measure zero.

1.6. **Proof ideas.** We give an outline of our proofs, highlighting the main difficulties and ideas.

To prove the uniqueness in law as stated in Theorem 1.6, the overall strategy is to establish a certain sense of 'mixing in time' of the dynamics (given by Assumption 1.5). More precisely, we take two families of random particle-generated Nevanlinna functions, both satisfying the two assumptions. Using the SDE (1.6) we reconstruct the dynamics of the poles, which are 'infinite dimensional DBM' in a certain sense. We couple the two 'infinite dimensional DBM' obtained from both functions, by coupling the driven Brownian motions. Then we show that the poles get closer in time under this coupling. Thus since both dynamics start from time  $-\infty$ , necessarily they are the same.

For both the reconstruction of DBM and the coupling, an essential input is that the poles have Airy-zero approximation, uniformly in time. This is implied by the uniform in time Airy-like property, as explained in Remark 1.3. Then under Assumption 1.4, it remains to show that such an approximation propagates in time, for which we again resort to the SDE (1.6).

In summary, three tasks are inline: propagation of Airy-zero approximation, reconstruction of DBM, and coupling. We next explain each of them in more details.

1.6.1. Propagation of Airy-zero approximation. Our Assumption 1.4 concerns specific times, implying that the *i*-th pole remains constant away from the *i*-th zero of the Airy function. Utilizing the SDE (1.6), we manage to get refined estimates: the *i*-th pole approximates the *i*-th zero with a polynomially small error over arbitrarily long time intervals with high probability, as demanded for later steps. To achieve this, in Section 3, we analyze (1.6) along certain characteristics which offset the singularity of the nonlinear term. This idea has previously been used (see e.g., [1,24,76]) to study DBM down to any mesoscopic scale, where the distance from the spectral parameter wto the particles is much bigger than particle fluctuations. However, in our analysis of (1.6), we operate at a microscopic scale, where the distance from w to the poles is of the same order as their fluctuation size. While a straightforward union bound over characteristic flows from polynomially many points suffices at the mesoscopic scale, our case demands careful selection of characteristic flows and precise estimation of error terms in the SDE, tailored to their initial positions.

1.6.2. *DBM reconstruction.* As already mentioned, there are significant challenges in analyzing DBM due to the singular repulsive interaction and possible particle collisions, in particular when  $\beta \in (0, 1)$ . Even for finite dimensional DBM, establishing the existence and uniqueness of a strong solution requires the theory of multivalued SDE, see [32,33]. Our approach through pole evolutions circumvents these issues entirely. Notably, there are no singularities even when poles collide.

On the other hand, a key challenge of our method lies in reconstructing the dynamics of each pole, which requires ruling out the possibility of poles adhering to each other for prolonged periods. To address these, in Section 4 we first establish that the trajectory of each pole is Hölder continuous solely utilizing (1.6). Together with the Airy-zero approximation, for any short time interval, we can identify a large index k such that the k-th and (k+1)-th poles remain bounded away from each other. This enables us to localize the system and study the evolution of the first k poles, treating the remaining poles' influence as an additional potential. For this k poles system, in Section 5, by employing classical Itô calculus on certain elementary symmetric polynomials, we show that the time of collisions almost surely has measure zero. We note that similar ideas have been employed

to show the instant diffraction of the particles for DBM [66]. Subsequently, the evolution of each pole can be reconstructed by performing a contour integral of (1.6). As a result, the k poles system can be interpreted as a k-dimensional DBM with a time-dependent random drift, which exhibits a monotonicity property.

1.6.3. Uniqueness via coupling. In Section 6, we take two solutions to (1.6), and design a coupling where the poles get closer in time. Consider the k-dimensional DBMs with random drifts reconstructed in the previous step, for these two solutions respectively. Our coupling is by using the same set of driven Brownian motions for both. Note that such k-dimensional DBMs with random drifts are constructed with random k, and only for a short time interval; but we need a coupling for a long time (to let the poles get closer). A trick here is to concatenate these short intervals, and allow for different k in each of them, as long as k is always large enough.

There is a monotonicity property: if the *i*-th pole of the initial data for the first solution dominates that of the second solution for each i, then at any time after the i-th pole of the first solution dominates that of the second solution for each *i*. Then we can sandwich one solution between affine shifts of the other, while keeping the error arising from the affine shifts arbitrarily small. Such sandwiching forces the poles of the two solutions to get closer in time. By taking long enough time intervals, they must coincide, establishing the uniqueness as desired. Such coupling and sandwiching strategies have been used to establish local statistics universality in random lozenge tilings [3,5,77].

**Notations.** In the rest of this paper, for any  $a \leq b \in \mathbb{R}$ , we let  $[a, b] = [a, b] \cap \mathbb{Z}$ . For any  $w \in \mathbb{C}$ , we use  $\mathcal{O}(w)$  to denote some  $w' \in \mathbb{C}$ , satisfying |w'| < C|w| for some universal constant C > 0. We also write  $w' \leq w$  for  $w' = \mathcal{O}(w)$ .

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## 2. Preliminaries and decomposition

In this section, we set up some preliminaries of our arguments.

We start with an explicit expression for any Airy-like Nevanlinna Y from Definition 1.2. The expression involves the Airy function, which is usually denoted by Ai, and is a special function that appears in various areas of mathematics and physics. It can be defined as an entire function, and the solution to the Airy equation:  $\operatorname{Ai}''(w) - w \operatorname{Ai}(w) = 0$  with  $\operatorname{Ai}(w) \to 0$  as  $w \to \infty$  along  $\mathbb{R}_+$ . All the zeros of Ai are on the real line, and are all negative, and we denote them as  $0 > a_1 > a_2 > a_3 > \cdots$ .

We now give the expression.

**Proposition 2.1.** For any particle-generated Nevanlinna function  $Y: \mathbb{H} \to \mathbb{H} \cup \mathbb{R}$  with infinitely many poles  $x_1 \ge x_2 \ge \cdots$ , if i)  $\sup_{i \in \mathbb{N}} |x_i - \mathfrak{a}_i| < \infty$ ; and ii) there exists a sequence of complex numbers  $w_n \to \infty$  along any direction in  $(0, 3\pi/4)$ , such that  $Y(w_n) - \sqrt{w_n} \to 0$ , then

(2.1) 
$$Y(w) = \sum_{i=1}^{\infty} \frac{1}{x_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}.$$

Moreover, (2.1) holds if Y is Airy-like.

The Nevanlinna representation (1.4), for any particle-generated Nevanlinna function Y with poles P (a multiset), can be written as

(2.2) 
$$Y(w) = b + cw + \sum_{x \in P} \frac{1}{x - w} - \frac{x}{1 + x^2},$$

where  $b, c \in \mathbb{R}, c \geq 0$ . We remark that it is possible that P contains only finitely many numbers, and the summation in (2.2) is finite. Then to prove Proposition 2.1, it remains to determine b and c in (2.2) for  $Y_t$ , and establish that the sum  $\sum_{i=1}^{\infty} \frac{1}{\mathfrak{a}_i} - \frac{x_i}{1+x_i^2}$  converges.

To start with, we first collect some basic estimates on the Airy function Ai and Nevanlinna functions, which will also be used repeatedly in the rest of this paper.

2.1. Airy function. The Airy function has the following asymptotic formula. For  $|\arg(w)| < \pi$ ,

$$\begin{aligned} \operatorname{Ai}(w) &\sim \frac{\exp(-\zeta)}{w^{1/4}} \sum_{n=0}^{\infty} \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{4\pi^{3/2}n!(-2\zeta)^n}, \\ \operatorname{Ai}'(w) &\sim -w^{1/4} \exp(-\zeta) \sum_{n=0}^{\infty} \frac{1+6n}{1-6n} \cdot \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{4\pi^{3/2}n!(-2\zeta)^n} \end{aligned}$$

where  $\zeta = \frac{2}{3}w^{3/2}$ . In particular, there is

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$$\left| w^{1/4} \operatorname{Ai}(w) - \frac{\exp(-2w^{3/2}/3)}{2\sqrt{\pi}} \left( 1 - \frac{5}{48w^{3/2}} \right) \right| \le D(|\arg(w)|) \frac{|\exp(-2w^{3/2}/3)|}{|w^3|},$$
$$\left| w^{-1/4} \operatorname{Ai}'(w) + \frac{\exp(-2w^{3/2}/3)}{2\sqrt{\pi}} \left( 1 + \frac{7}{48w^{3/2}} \right) \right| \le D(|\arg(w)|) \frac{|\exp(-2w^{3/2}/3)|}{|w^3|},$$

for any  $w \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ , where  $D : [0, \pi) \to \mathbb{R}$  is a continuous function. See e.g., [47, Subsection 9.7.iv] and [100, Appendix B]. It follows that

(2.3) 
$$\left|\frac{\operatorname{Ai}'(w)}{\operatorname{Ai}(w)} + \sqrt{w}\right| \lesssim |w|^{-1},$$

for any  $w \in \mathbb{C}$  with |w| large enough and  $|\arg(w)| < 3\pi/4$ . The Weierstrass representation gives

(2.4) 
$$-\frac{\operatorname{Ai}'(w)}{\operatorname{Ai}(w)} = \sum_{i=1}^{\infty} \frac{1}{\mathfrak{a}_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}$$

It is also known that  $\mathfrak{a}_i$  is around  $-(3i\pi/2)^{2/3}$ . More precisely, for any  $i \in \mathbb{N}$  we have

(2.5) 
$$\left|\mathfrak{a}_{k} + \left(\frac{3\pi i}{2}\right)^{2/3}\right| \lesssim i^{-1/3}.$$

2.2. Estimates on Nevanlinna functions. We now present some estimates on (particle-generated) Nevanlinna functions, which will be used in the Airy-like function part of Proposition 2.1. We note that some of them are also used repeatedly in subsequent sections.

For any particle-generated Nevanlinna function  $Y : \mathbb{H} \to \mathbb{H} \cup \mathbb{R}$ , from (2.2) we have

(2.6) 
$$\operatorname{Im}[Y(w)] = c\operatorname{Im}[w] + \sum_{x \in P} \frac{\operatorname{Im}[w]}{|x - w|^2}$$

**Lemma 2.2.** The quantity Im[w]Im[Y(w)] is monotone in Im[w]; the derivatives of Y satisfy

(2.7) 
$$|Y^{(k)}(w)| \le \frac{k! \mathrm{Im}[Y(w)]}{\mathrm{Im}[w]^k} \le \frac{k! |Y(w)|}{\mathrm{Im}[w]^k}$$

where  $Y^{(k)}$  is the k-th derivative of Y, for any integer  $k \geq 2$ .

*Proof.* We first consider k = 1. Denote  $w = E + i\eta$ , then (2.6) gives

(2.8) 
$$\operatorname{Im}[w]\operatorname{Im}[Y(w)] = c\eta^2 + \sum_{x \in P} \frac{\eta^2}{|E - x|^2 + \eta^2},$$

which is increasing in  $\eta \ge 0$ . Using (2.2), the derivative of Y(w) satisfies

(2.9) 
$$|Y'(w)| = \left| c + \sum_{x \in P} \frac{1}{(x-w)^2} \right| \le c + \sum_{x \in P} \frac{1}{|x-w|^2} = \frac{\operatorname{Im}[Y(w)]}{\operatorname{Im}[w]}.$$

And by

$$|Y^{(k)}(w)| \le \sum_{x \in P} \frac{k!}{|x - w|^{k+1}} \le \frac{k! |Y'(w)|}{\operatorname{Im}[w]^{k-1}}$$

the  $k \geq 2$  case follows.

As already alluded to, if a particle-generated Nevanlinna function Y is close to the function  $\sqrt{w}$ , its poles would be close to the Airy function zeros. More precisely, we have the following estimate.

**Lemma 2.3.** Take any parameters K > 100 and  $0 \le \delta < 1$ . Suppose that a particle-generated Nevanlinna function Y satisfies the following conditions:

- there is no pole of Y in  $(K, \infty)$ ;
- for any w = x + iy with  $x \le K^2$  and  $y \ge 4K^2/(1 + |x|^{\delta/2})$ , we have

$$|Y(w) - \sqrt{w}| \le \frac{\operatorname{Im}[\sqrt{w}]^{1-\delta}}{\operatorname{Im}[w]}.$$

Then Y has infinitely many poles  $x_1 \ge x_2 \ge \cdots$ , and  $|x_i - \mathfrak{a}_i| < CK^4 i^{-\delta/6}$  for any  $i \in \mathbb{N}$ , where C > 0 is a universal constant.

In particular, these conditions are satisfied by Airy-like Nevanlinna functions (with K large and  $\delta = \mathfrak{d}$ ).

**Corollary 2.4.** For any  $(\mathfrak{d}, C_*)$ -Airy-like Nevanlinna function Y (with poles  $x_1 \ge x_2 \ge \cdots$ ), there exists B > 0 depending only on  $C_*$ , such that  $|x_i - \mathfrak{a}_i| \le B$  for each  $i \in \mathbb{N}$ .

The proof of Lemma 2.3 relies on the Helffer-Sjöstrand formula, which has become standard in random matrix theory. Therefore, we defer it to Appendix A.

2.3. Proof of Proposition 2.1. Thanks to (2.3) and (2.4), we have

(2.10) 
$$\sum_{i=1}^{\infty} \frac{1}{\mathfrak{a}_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} - \sqrt{w} \to 0,$$

when  $w \to \infty$  along any direction in  $(-3\pi/4, 3\pi/4)$ . By  $|\mathfrak{a}_i| \sim (3\pi i/2)^{2/3}$  from (2.5), we have (2.11)  $|\mathfrak{a}_i + (3\pi i/2)^{2/3}|, |x_i + (3\pi i/2)^{2/3}| \le B$  for a large B > 0; in particular  $x_1 \leq B$ . If we take w with  $\arg(w) \in (-3\pi/4, 3\pi/4)$  and |w| > 2B,

(2.12)  
$$\begin{aligned} \left| \sum_{i=1}^{\infty} \frac{1}{x_i - w} - \frac{1}{\mathfrak{a}_i} - \sum_{i=1}^{\infty} \frac{1}{\mathfrak{a}_i - w} - \frac{1}{\mathfrak{a}_i} \right| \\ &\leq \sum_{i=1}^{\infty} \frac{|\mathfrak{a}_i - x_i|}{|x_i - w||\mathfrak{a}_i - w|} \leq \sum_{i=1}^{\infty} \frac{B}{|x_i - w||\mathfrak{a}_i - w|} \\ &= \sum_{i>|w|^{3/2}} \frac{B}{|x_i - w||\mathfrak{a}_i - w|} + \sum_{i=1}^{\lfloor |w|^{3/2} \rfloor} \frac{B}{|x_i - w||\mathfrak{a}_i - w|} \\ &\lesssim \sum_{i>|w|^{3/2}} \frac{B}{i^{4/3}} + \sum_{i=1}^{\lfloor |w|^{3/2} \rfloor} \frac{B}{|w|^2} \lesssim \frac{1}{|w|^{1/2}}, \end{aligned}$$

where in the last line we used that, when  $i > |w|^{3/2}$ ,  $|x_i - w| |\mathfrak{a}_i - w| \gtrsim i^{4/3}$  for the first term, and  $|x_i - w| |\mathfrak{a}_i - w| \gtrsim |w|^2$  for the second term (since |w| > 2B).

Therefore, (2.10) and (2.12) together give that

(2.13) 
$$\sum_{i=1}^{\infty} \frac{1}{x_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} - \sqrt{w} \to 0,$$

for  $w \to \infty$  along any direction in  $(-3\pi/4, 3\pi/4)$ . Also note that, thanks to (2.11), we have

(2.14) 
$$\left|\sum_{i=1}^{\infty} \frac{1}{\mathfrak{a}_{i}} - \frac{x_{i}}{1+x_{i}^{2}}\right| \leq \sum_{i=1}^{\infty} \left|\frac{1+x_{i}(x_{i}-\mathfrak{a}_{i})}{\mathfrak{a}_{i}(1+x_{i}^{2})}\right| \leq \sum_{i=1}^{\infty} \frac{1+|\mathfrak{a}_{i}|B+B^{2}}{|\mathfrak{a}_{i}|(1+((\mathfrak{a}_{i}+B)\wedge 0)^{2})} < \infty.$$

By plugging (2.14) into the representation (2.2) for Y, we can rewrite Y as (for some  $b' \in \mathbb{R}$ )

(2.15) 
$$Y(w) = b + cw + \sum_{i=1}^{\infty} \frac{1}{x_i - w} - \frac{x_i}{1 + x_i^2} = b' + cw + \sum_{i=1}^{\infty} \frac{1}{x_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}$$

By our assumption,  $Y(w_n) - \sqrt{w_n} \to 0$  as  $n \to \infty$ . Taking  $w = w_n$  in (2.15), comparing with (2.13), we conclude that  $b' + cw_n \to 0$  when  $n \to \infty$ . It follows that b' = c = 0, and (2.1) holds.

Finally, if Y is Airy-like, then Corollary 2.4 implies that  $\sup_{i \in \mathbb{N}} |x_i - \mathfrak{a}_i| < \infty$ , and  $|Y(ni) - \sqrt{ni}| \rightarrow 0$  as  $n \rightarrow \infty$ . These verify the assumptions in Proposition 2.1, and (2.1) holds.

2.4. **Domain extension.** As stated in Remark 1.3, for an Airy-like Nevanlinna function Y, we also provide a bound of  $|Y(w) - \sqrt{w}|$  for w close to  $\mathbb{R}_+$ , which will be useful later.

**Lemma 2.5.** For  $Y : \mathbb{H} \to \mathbb{H} \cup \mathbb{R}$  being any  $(\mathfrak{d}, C_*)$ -Airy-like Nevanlinna function, we have

$$|Y(w) - \sqrt{w}| \le B|w|^{-1/2}, \quad \forall \arg(w) \in (0, 3\pi/4), |w| > B$$

for B > 0 depending only on  $\mathfrak{d}$  and  $C_*$ .

*Proof.* By Corollary 2.4, there is B' > 0 with each  $|x_i - \mathfrak{a}_i| \leq B'$ . Then, for any  $w \in \mathbb{H}$  with  $\arg(w) \in (0, 3\pi/4), |w| > 2B'$ , we have  $|\mathfrak{a}_i - w| \leq 2|x_i - w|$ .

By Proposition 2.1 and (2.4), we have

$$\left|Y(w) + \frac{\operatorname{Ai}'(w)}{\operatorname{Ai}(w)}\right| \le \sum_{i=1}^{\infty} \left|\frac{1}{\mathfrak{a}_i - w} - \frac{1}{x_i - w}\right| = \sum_{i=1}^{\infty} \left|\frac{x_i - \mathfrak{a}_i}{(x_i - w)(\mathfrak{a}_i - w)}\right| \le \sum_{i=1}^{\infty} \frac{2B'}{|\mathfrak{a}_i - w|^2}.$$

Using (2.5), we have that

$$\sum_{i=1}^{\infty} \frac{1}{|\mathfrak{a}_i - w|^2} = \sum_{i=1}^{\lfloor |w|^{3/2} \rfloor} \frac{1}{|\mathfrak{a}_i - w|^2} + \sum_{i=\lfloor |w|^{3/2} \rfloor + 1}^{\infty} \frac{1}{|\mathfrak{a}_i - w|^2} \lesssim \frac{|w|^{3/2}}{|w|^2} + \sum_{i=\lfloor |w|^{3/2} \rfloor + 1}^{\infty} \frac{1}{i^{4/3}} \lesssim |w|^{-1/2}.$$

Thus with (2.3), the conclusion follows.

2.5. Topological statements. As we shall derive convergence to  $ALE_{\beta}$  from Stieltjes transforms, we will need several statements on the functional spaces, which we provide here.

**Definition 2.6.** For any locally finite measures  $\mu_1, \mu_2, \cdots$  on  $\mathbb{R}$ , we say that they converge in the vague topology to another locally finite  $\mu$ , if  $\mu_n(f) \to \mu(f)$ , for any  $f : \mathbb{R} \to \mathbb{R}$  that is compactly supported and smooth.

Such vague topology arises naturally from Nevanlinna function convergence.

**Lemma 2.7.** Take Nevanlinna functions  $Y_1, Y_2, \cdots$  and Y such that  $Y_n \to Y$  as  $n \to \infty$ , uniformly in any compact subset of  $\mathbb{H}$ . Suppose the corresponding measures (in their Nevanlinna representation) are  $\mu_1, \mu_2, \cdots$  and  $\mu$ , respectively, then  $\mu_n \to \mu$  in the vague topology.

*Proof.* Take any f that is compactly supported and smooth, and let K be a large enough number such that f = 0 outside [-K, K]. Take a smooth function  $\chi : \mathbb{R}_+ \to \mathbb{R}$ , such that  $\chi = 1$  on (0, 1), and  $\chi = 0$  on  $(2, \infty)$ . By Lemma A.1, we have

$$\mu_n(f) = \frac{1}{\pi} \int_{x+iy \in \mathbb{H}} -\operatorname{Re}[Y_n(x+iy)]yf'(x)\chi'(y) - \operatorname{Im}[Y_n(x+iy)](yf''(x)\chi(y) + f(x)\chi'(y))dxdy.$$

We note that  $yf'(x)\chi'(y) = f(x)\chi'(y) = 0$  whenever  $(x, y) \notin [-K, K] \times [1, 2]$ . Also,  $yf''(x)\chi(y) = 0$ whenever  $(x, y) \notin [-K, K] \times (0, 2]$ ; and for  $y \leq 2$ , from Nevalinna representation we have that  $\operatorname{Im}[Y_n(x + iy)] \leq \frac{2}{y}\operatorname{Im}[Y_n(x + 2i)]$ . Therefore, we have that the integrand in the above integral is non-zero only in  $[-K, K] \times (0, 2]$ ; and it is bounded by a constant, which is independent of n by the uniform convergence of  $Y_n$  in  $[-K, K] \times [1, 2]$ . Thus we can apply dominated convergence theorem to deduce that the above integral converges to

$$\mu(f) = \frac{1}{\pi} \int_{x+iy \in \mathbb{H}} -\operatorname{Re}[Y(x+iy)]yf'(x)\chi'(y) - \operatorname{Im}[Y(x+iy)](yf''(x)\chi(y) + f(x)\chi'(y))dxdy.$$

So the conclusion follows.

On the other hand, in the setting of particle-generated measures, vague topology convergence can imply pole convergence.

**Lemma 2.8.** For a sequence of particle-generated measures  $\mu_1, \mu_2, \ldots$ , such that  $\mu_k \to \mu$  as  $k \to \infty$ in the vague topology, the limit  $\mu$  must also be particle-generated. Moreover, if there is some K > 0such that  $\mu_k([K,\infty)) = 0$  for each k, then the followings are true. We denote by  $x_i^k$  the *i*-th largest pole of  $\mu_k$  (with the convention that  $x_i^k = -\infty$  if there are less than *i* poles). For each  $i \in \mathbb{N}$ , either  $x_i^k \to -\infty$  as  $k \to \infty$ , or  $\lim_{k\to\infty} x_i^k$  exists and is a pole of  $\mu$ . Also, all the poles of  $\mu$  are given by such limits.

*Proof.* By vague topology convergence, for any a < b, we have that  $\limsup_{k\to\infty} \mu_k([a,b]) \le \mu([a,b])$ , and  $\liminf_{k\to\infty} \mu_k((a,b)) \ge \mu((a,b))$ . Then for any a < b with  $\mu([a,b]) < 1$ , we must have  $\mu_k([a,b]) = 0$  for k large enough, since each  $\mu_k([a,b])$  must be an integer. Therefore,  $\mu_k((a,b)) = 0$  for k large enough, and  $\mu((a, b)) = 0$ . These imply that  $\mu$  in any open interval is either  $\geq 1$  or zero. Therefore, in any compact interval,  $\mu$  is supported on finitely many points.

Now take any x where  $\mu(\{x\}) > 0$ . Take any a < x < b such that  $\mu((a, b)) = \mu([a, b]) = \mu(\{x\})$ . Then for k large enough,  $\mu_k((a, b)) = \mu_k([a, b]) = \mu(\{x\})$ ; so  $\mu(\{x\}) \in \mathbb{N}$ . These imply that  $\mu$  is particle-generated.

Under the additional assumption, there is  $\mu((K, \infty)) = 0$ . So we can write the poles of  $\mu$  as  $x_1 \ge x_2 \ge \cdots$  (with the convention that  $x_i = -\infty$  if there are less than *i* poles). Then we can show  $x_i^k \to x_i$  via induction in  $i \in \mathbb{N}$ , using that for each a < b with  $a, b \notin \{x_i\}_{i=1}^{\infty}$ ,  $\mu_k((a, b)) = \mu((a, b))$  for any *k* large enough.

## 3. Pole evolution: Uniform rigidity in time

In this section, we prove a uniform in time estimate for the poles. More precisely, the following proposition states that for the SDE (1.6), with high probability, its pole evolution gives a line ensemble (i.e., all the poles are bounded from above, and the trajectories are continuous), and the poles are close to the zeros of the Airy function, uniformly in time.

**Proposition 3.1.** For any  $\mathfrak{d}, C_* > 0$ , there exist small  $\delta, c > 0$  and large C > 0, such that the following holds. Take any particle-generated  $\{Y_t\}_{t \in \mathbb{R}}$  satisfying Assumption 1.5, and large T > 0. Conditional on the event that  $Y_0$  is  $(\mathfrak{d}, C_*)$ -Airy-like, with probability at least  $1 - e^{-c(\log T)^2}$ , we have

(1) The poles of  $\{Y_t\}_{t \in [T,2T]}$  give a line ensemble  $\{x_i(t)\}_{i \in \mathbb{N}, t \in [T,2T]}$ , and for each  $t \in [T,2T]$ and  $w \in \mathbb{H}$ ,

(3.1) 
$$Y_t(w) = \sum_{i=1}^{\infty} \frac{1}{x_i(t) - w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}$$

(2) For each  $t \in [T, 2T]$ ,  $i \in \mathbb{N}$ ,  $y \leq C$ , we have that

(3.2) 
$$|x_i(t) - \mathfrak{a}_i| \le \frac{C(\log T)^{40}}{i^{\delta}},$$

and

(3.3) 
$$|\{i \in \mathbb{N} : x_i(t) \in [y-1, y+1]\}| \le C\sqrt{|y|+1}.$$

This immediately implies the first part of Theorem 1.6.

**Corollary 3.2.** For any  $\{Y_t\}_{t\in\mathbb{R}}$  satisfying Assumption 1.4 and Assumption 1.5, its poles give a line ensemble  $\{x_i(t)\}_{i\in\mathbb{N},t\in\mathbb{R}}$ , and  $Y_t(w) = \sum_{i=1}^{\infty} \frac{1}{x_i(t)-w} - \frac{1}{\mathfrak{a}_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}$  for any  $t\in\mathbb{R}$ .

We note that (3.3) (which is a Wegner estimate) follows easily from (3.2), plus

$$|\{i \in \mathbb{N} : \mathfrak{a}_i(t) \in [y-1, y+1]\}| \le C\sqrt{|y|+1},$$

which directly follows from (2.5).

Our general strategy is to obtain uniform in time estimates for  $Y_t - \sqrt{w}$ , and to apply Lemma 2.3. The main tasks are (1) to estimate bulk pole densities, via bounding  $|Y_t(w) - \sqrt{w}|$  for w in a reasonable domain contained in  $\mathbb{H}$  (in particular, allowing for polynomially close to the real axis, when  $\operatorname{Re}[w] \to -\infty$ ); (2) to bound the first pole  $x_1$ . Both these are to be achieved through analyzing the SDE (1.6). 3.1. Characteristic flow. We consider the characteristic flow,

(3.4) 
$$\partial_t w_t = -\sqrt{w_t}, \quad w_0 \in \mathbb{H}$$

which can be solved as

(3.5) 
$$\partial_t \sqrt{w_t} = -\frac{1}{2}, \quad w_t = (w_0 - t/2)^2.$$

By plugging this characteristic flow into (1.6), itô's formula gives the following semi-martingale decomposition for  $Y_t(w_t) - \sqrt{w_t}$ 

(3.6) 
$$d(Y_t(w_t) - \sqrt{w_t}) = (dM_t)(w_t) + \left(\frac{2-\beta}{2\beta}\partial_w^2 Y_t(w_t) + (Y_t(w_t) - \sqrt{w_t})\partial_w Y_t(w_t)\right)dt,$$

with the Martingale term  $(dM_t)(w_t) = d(M_t(w_t)) - (\partial_w M_t)(w_t)dt$ , whose quadratic variations are given by (using (1.7)):

(3.7) 
$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \int_0^t (\mathrm{d}M_s)(w_s) \right\rangle = \frac{1}{3\beta} \partial_w^3 Y_t(w) \bigg|_{w=w_t}$$

In the rest of this section, we fix  $\mathfrak{d}, C_* > 0$ , and take  $\{Y_t\}_{t \in \mathbb{R}}$  satisfying Assumption 1.5, and (unless otherwise noted) conditional on  $Y_0$  which is  $(\mathfrak{d}, C_*)$ -Airy-like. All the constants below (including those in  $\leq$  and  $\mathcal{O}(\cdot)$ ) can depend on  $\mathfrak{d}$  and  $C_*$ . We take T to be a large number, and set  $K = (\log T)^8$ .

3.2. Estimates for the bulk. We next prove the following bound of  $|Y_t(w) - \sqrt{w}|$  for w in a domain contained in  $\mathbb{H}$ . It will be used to bound the bulk pole densities.

**Proposition 3.3.** There exist small  $\delta, c > 0$  such that the following holds. Define the spectral domain

(3.8) 
$$\mathcal{D} = \{ \kappa + \mathrm{i}\eta : \eta \ge K, \eta^{-1-\delta}K \le \kappa \le \sqrt{K^2 + \eta^2} \}.$$

With probability  $1 - e^{-c(\log T)^2}$ , for any  $t \in [T, 2T]$ , and  $\sqrt{w} \in \mathcal{D}$ , it holds

(3.9) 
$$\left|Y_t(w) - \sqrt{w}\right| \le \frac{\operatorname{Im}[\sqrt{w}]^{1-\delta}}{\operatorname{Im}[w]}.$$

Thanks to the (away from the real axis) Lipschitz property of  $Y_t(w) - \sqrt{w}$  in Lemma 2.2, we only need to prove (3.9) for a set of carefully chosen mesh points. Namely, we consider the following mesh of points in the upper half plane:

(3.10) 
$$\mathcal{L} = \left\{ \kappa + i\eta : \eta^3 \in \mathbb{Z}, \eta \ge K, \kappa = \frac{\mathbb{Z}}{\eta^2}, K \le \kappa \le T + K + \eta \right\}$$

**Lemma 3.4.** For any  $\kappa' + i\eta' \in \mathcal{D}$  as defined in (3.8) and  $t \in [T, 2T]$ , there exists  $\kappa + i\eta \in \mathcal{L}$  such that  $t \leq 2\kappa - 2\eta^{-1-\delta}K$ , and

(3.11) 
$$|\kappa' - \kappa + t/2|, |\eta' - \eta| \le \frac{1}{\eta^2}$$

Proof. Suppose that  $\eta' \in [i^{1/3}, (i+1)^{1/3}]$  for some integer  $i \ge K^3$ , we can take  $\eta = i^{1/3}$ , therefore  $|\eta - \eta'| \le (i+1)^{1/3} - i^{1/3} \le i^{-2/3}/3 = \eta^{-2}/3$ . Suppose that  $\kappa' + t/2 \in [j\eta^{-2}, (j+1)\eta^{-2}]$  for some  $j \in \mathbb{Z}$ , we can take  $\kappa = (j+1)\eta^{-2}$ . Then  $\kappa \ge \kappa' + T/2 > K$ , and  $\kappa \le \kappa' + T + \eta^{-2} \le T + K + \eta$ , and  $|\kappa' - \kappa + t/2| \le \eta^{-2}$ . Also  $\kappa - t/2 \ge \kappa' \ge \eta^{-1-\delta}K$ , thus  $t \le 2\kappa - 2\eta^{-1-\delta}K$ .

Now Proposition 3.3 follows from the following estimate on one point.

**Lemma 3.5.** The following holds true for small enough  $\delta, c > 0$ . Take any  $u = \kappa_0 + \eta i \in \mathcal{L}$ , and let  $\sqrt{w_t} = u - t/2 =: \kappa_t + \eta i$  for each  $t < 2\kappa_0$ . Conditional on  $Y_0$  with  $|Y_0(w_0) - \sqrt{w_0}| \leq \text{Im}[\sqrt{w_0}]^{1-\mathfrak{d}}/\text{Im}[w_0]$ , with probability  $1 - e^{-c\sqrt{\eta}}$ ,

(3.12) 
$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{1}{\kappa_t \eta^{\delta}}, \quad \forall \ 0 \le t \le 2T \land (2\kappa_0 - 2\eta^{-1-\delta}K).$$

Proof of Proposition 3.3. By a union bound over all the points in  $\mathcal{L}$ , Lemma 3.5 implies that

(3.13) 
$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{1}{\kappa_t \eta^{\delta}}, \quad \forall \ 0 \le t \le 2T \land (2\kappa_0 - 2\eta^{-1-\delta}K), \ \sqrt{w_0} \in \mathcal{L},$$

with probability at least

(3.14) 
$$1 - \sum_{\eta \ge K, \eta^3 \in \mathbb{Z}} C e^{-c\sqrt{\eta}} \eta^2 (2T + 2\eta) \ge 1 - e^{-c(\log T)^2},$$

where we used that  $K = (\log T)^8$  and T is large enough, and C > 0 is a large constant.

For any  $t \in [T, 2T]$  and  $\sqrt{w'} = \kappa' + i\eta' \in \mathcal{D}$ , thanks to Lemma 3.4, there exists  $\sqrt{w_0} = \kappa + i\eta \in \mathcal{L}$  such that  $|\kappa' - \kappa_t|, |\eta' - \eta| \leq \eta^{-2}$ . Then we have  $\eta^{-1-\delta}K/2 < \kappa_t < 2\eta$ . It also follows that

(3.15) 
$$|\sqrt{w_t} - \sqrt{w'}| \le \sqrt{|\kappa' - \kappa_t|^2 + |\eta' - \eta|^2} \le \sqrt{2\eta^{-2}},$$

and

$$(3.16) |w_t - w'| \le |\sqrt{w_t} + \sqrt{w'}| |\sqrt{w_t} - \sqrt{w'}| \le (2\sqrt{\kappa'^2 + \eta^2} + 2\eta^{-2})\sqrt{2\eta^{-2}} \lesssim \eta^{-1}.$$

In particular, this implies that  $|w_t - w'|$  is much smaller than  $2\kappa_t \eta = \text{Im}[w_t]$  (which is at least  $\eta^{-\delta}K$ ). It then follows from Lemma 2.2 that

(3.17) 
$$|Y_t(w_t) - Y_t(w')| \lesssim \frac{\text{Im}\sqrt{w_t}}{\text{Im}[w_t]} |w_t - w'| = \frac{|w_t - w'|}{2\kappa_t} \lesssim \frac{1}{\kappa_t \eta}$$

Combining this with (3.14) and (3.15), and using that  $\eta^{-2}$  is much smaller than  $\frac{1}{\kappa_i \eta^{\delta}}$  (which is at least  $\eta^{-1-\delta}/2$ ), we conclude that

$$|Y_t(w') - \sqrt{w'}| \lesssim \frac{1}{\kappa_t \eta^{\delta}} \lesssim \frac{1}{2\kappa' {\eta'}^{\delta}} = \frac{\mathrm{Im}[\sqrt{w}]^{1-\delta}}{\mathrm{Im}[w]}.$$

Then the proof finishes by taking a slightly smaller  $\delta$  (to remove the constant factor).

We now prove the one point estimate, using (3.6).

Proof of Lemma 3.5. We introduce the following stopping time,

(3.18) 
$$\sigma = \inf\left\{0 \le t \le 2T : |Y_t(w_t) - \sqrt{w_t}| \ge \frac{1}{\kappa_t \eta^\delta}, \text{ or } \kappa_t \le \eta^{-1-\delta} K, \text{ or } t = 2T\right\}.$$

We now bound the terms in the RHS of (3.6). For  $0 \le t \le \sigma$ , we have  $\text{Im}[Y_t(w_t)] \le 2\text{Im}[\sqrt{w_t}] = 2\eta$ , since

(3.19) 
$$|\operatorname{Im}[Y_t(w_t)] - \eta| \le \frac{1}{\kappa_t \eta^\delta} \le \frac{\eta}{K}.$$

Therefore (using Lemma 2.2) we get

(3.20) 
$$|\partial_w^2 Y_t(w_t)| \le \frac{2\mathrm{Im}[Y_t(w_t)]}{\mathrm{Im}[w_t]^2} \le \frac{4\mathrm{Im}[\sqrt{w_t}]}{\mathrm{Im}[w_t]^2} = \frac{1}{\kappa_t^2 \eta}.$$

The quadratic variation of the martingale term is given by (3.7); then (again using Lemma 2.2)

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\int_{0}^{t}(\mathrm{d}M_{s})(w_{s})\right\rangle\right| \leq \frac{2\mathrm{Im}[Y_{t}(w_{t})]}{\beta\mathrm{Im}[w_{t}]^{3}} \leq \frac{4\mathrm{Im}[\sqrt{w_{t}}]}{\beta\mathrm{Im}[w_{t}]^{3}} = \frac{1}{2\beta\kappa_{t}^{3}\eta^{2}}.$$

By integrating in time, we have

$$\int_0^{t\wedge\sigma} \frac{\mathrm{d}s}{2\beta\kappa_t^3\eta^2} \le \frac{1}{2\beta\kappa_t^2\eta^2}.$$

Take  $\theta = \sqrt{\eta}$ . By the Burkholder-Davis-Gundy inequality, for any  $0 \le t \le 2T$  the following holds with probability<sup>4</sup>  $\ge 1 - Ce^{-c\theta}$ :

$$\sup_{0 \le s \le t \land \sigma} \left| \int_0^s (\mathrm{d}M_u)(w_u) \right| \le \frac{\theta}{2\kappa_t \eta}$$

Then we can take a union bound over times

$$t = \left(2 - \frac{1}{2^{i-1}}\right)\kappa_0, \quad \kappa_t = \frac{\kappa_0}{2^i},$$

for  $i \in [[1, 3\log_2(T)]]$ , and get that with probability  $1 - C\log(T)e^{-c\theta}$ ,

(3.21) 
$$\left| \int_0^t (\mathrm{d}M_s)(w_s) \right| \le \frac{\theta}{\kappa_t \eta}, \quad \forall \ 0 \le t \le \sigma.$$

Now by (3.6), and using (3.20) and (3.21), we have that for any  $0 \le t \le \sigma$ ,

(3.22) 
$$|Y_t(w_t) - \sqrt{w_t}|| \le \int_0^t |Y_s(w_s) - \sqrt{w_s}| |\partial_w Y_s(w_s)| \, \mathrm{d}s + \frac{2\theta}{\kappa_t \eta} + |Y_0(w_0) - \sqrt{w_0}| \, .$$

Here we used that  $\int_0^t \frac{2}{\kappa_s^2 \eta} ds < \frac{4}{\kappa_t \eta}$ , which is much smaller than  $\frac{\theta}{\kappa_t \eta}$ . For  $0 \le t \le \sigma$ , (using Lemma 2.2) we have

$$|\partial_w Y_t(w_t)| \leq \frac{\operatorname{Im}[Y_t(w_t)]}{\operatorname{Im}[w_t]} \leq \frac{1}{2\kappa_t} + \frac{|Y_t(w_t) - \sqrt{w_t}|}{2\kappa_t \eta} \leq \frac{1}{2\kappa_t} + \frac{1}{2\kappa_t^2 \eta^{1+\delta}} =: \gamma(t) \leq \frac{1}{\kappa_t}.$$

Then for any  $0 \leq s < t \leq \sigma$ ,

(3.23) 
$$\int_{s}^{t} \gamma(u) \mathrm{d}u \leq \log(\kappa_{s}/\kappa_{t}) + \frac{1}{\kappa_{t}\eta^{1+\delta}} \leq \log(\kappa_{s}/\kappa_{t}) + \frac{1}{K}$$

By Grönwall's inequality and (3.22), for any  $0 \le t \le \sigma$ , we can bound  $|Y_t(w_t) - \sqrt{w_t}|$  by:

$$(3.24) \qquad \frac{2\theta}{\kappa_t \eta} + |Y_0(w_0) - \sqrt{w_0}| + \int_0^t \gamma(s) \left(\frac{2\theta}{\kappa_s \eta} + |Y_0(w_0) - \sqrt{w_0}|\right) \exp\left(\int_s^t \gamma(u) \mathrm{d}u\right) \mathrm{d}s.$$

We note that by (3.23),  $\exp(\int_s^t \gamma(u) du) \leq \frac{2\kappa_s}{\kappa_t}$ . Also

$$\int_0^t \gamma(s) \frac{\theta}{\kappa_s \eta} \cdot \frac{\kappa_s}{\kappa_t} \mathrm{d}s \le \frac{\theta}{\kappa_t \eta} \int_0^t \frac{1}{\kappa_s} \mathrm{d}s = \frac{2 \log(\kappa_0 / \kappa_t) \theta}{\kappa_t \eta}$$

and

$$\int_{0}^{t} \gamma(s) \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right| \frac{\kappa_{s}}{\kappa_{t}} \mathrm{d}s \leq \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right| \int_{0}^{t} \frac{1}{\kappa_{t}} \mathrm{d}s = \frac{t \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right|}{\kappa_{t}}$$

 $<sup>^{4}</sup>$ Here and in the rest of this proof, C is used to denote a large constant, whose value may change from line to line.

Therefore we have

$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{2(1 + 4\log(\kappa_0/\kappa_t))\theta}{\kappa_t \eta} + \left(1 + \frac{2t}{\kappa_t}\right)|Y_0(w_0) - \sqrt{w_0}| \le \frac{1}{2\kappa_t \eta^{\delta}},$$

where in the last inequality we used  $\log(\kappa_0/\kappa_t) \leq \log(T\eta^2)$ , which is much smaller than  $\eta^{1/2-\delta}$  provided K is large enough; and

$$(3.25) \quad \left(1 + \frac{2t}{\kappa_t}\right) |Y_0(w_0) - \sqrt{w_0}| \le \left(1 + \frac{2t}{\kappa_t}\right) \frac{\mathrm{Im}[\sqrt{w_0}]^{1-\mathfrak{d}}}{\mathrm{Im}[w_0]} = \left(1 + \frac{2t}{\kappa_t}\right) \frac{1}{(2\kappa_t + t)\eta^{\mathfrak{d}}} \le \frac{1}{3\kappa_t \eta^{\delta}}.$$

Therefore, since  $Y_t(w_t) - \sqrt{w_t}$  is continuous in t, we conclude that  $\sigma = 2T \wedge (2\kappa - 2\eta^{-1-\delta}K)$  with probability  $1 - Ce^{-c\sqrt{\eta}}$ .

3.3. Estimates for the first pole. Under the same setup as the previous subsection (in particular, T is taken to be any large number, and  $K = (\log T)^8$ ), we next prove the following proposition, which states that with high probability all the poles are bounded by K.

**Proposition 3.6.** There exists a small number c > 0 such that the following holds. With probability  $1 - e^{-c(\log T)^4}$ , for any  $t \in [0, 2T]$ ,  $Y_t$  has no pole in  $(K, \infty)$ .

We introduce the following stopping time, which is the first time the largest pole exceeds K,

(3.26) 
$$\sigma_0 = \inf \left\{ 0 \le t \le 2T : Y_t \text{ has a pole in } (K, \infty), \text{ or } t = 2T \right\}$$

We now denote  $K_j = jK$  for  $j \in \mathbb{N}$  (in particular  $K_1 = K$ ), and consider a mesh of points:

$$\mathcal{L} = \left\{ \kappa + \mathrm{i}\eta : \eta = (400K_j)^{-1/4}, \kappa = \mathbb{Z}\eta, \sqrt{K_j + \eta^2} \le \kappa \le \sqrt{K_{j+1} + \eta^2} + 2T, j \in \mathbb{N} \right\}.$$

Then for any  $\sqrt{w} = \kappa + i\eta \in \mathcal{L}$ , it holds  $\kappa \eta^2 \ge 1/4$ , and  $\operatorname{Im}[\sqrt{w}] = \eta \ge 1/\kappa \eta^{1-\delta}$ , provided K is large enough.

Now Proposition 3.6 follows from the following estimate on one point.

**Lemma 3.7.** For any B > 0, the following holds true for small enough  $\delta, c > 0$ . Take any  $j \ge 0$ ,  $u = \kappa_0 + \eta i \in \mathcal{L}$  with  $\eta = (400K_j)^{-1/4}$ , and let  $\sqrt{w_t} = u - t/2 =: \kappa_t + \eta i$  for each t. Conditional on  $Y_0$  with  $|Y_0(w_0) - \sqrt{w_0}| \le B|w_0|^{-1/2}$ , with probability  $\ge 1 - e^{-c\sqrt{K_j}}$ ,

(3.27) 
$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{1}{\kappa_t \eta^{1-\delta}}, \quad \forall \ 0 \le t \le \sigma_0 \land (2\kappa_0 - 2\sqrt{K_j + \eta^2}).$$

Proof of Proposition 3.6. As  $Y_0$  is  $(\mathfrak{d}, C_*)$ -Airy-like, Lemma 2.5 implies that (for some B > 0)

(3.28) 
$$|Y_0(w) - \sqrt{w}| \le B|w|^{-1/2} \text{ for all } \arg(w) \in (0, 3\pi/4), |w| > B$$

By an union bound over all the points in  $\mathcal{L}$ , Lemma 3.7 implies that conditional on (3.28), with probability at least (for some c' > 0, and taking c < c' and T large)

(3.29) 
$$1 - \sum_{j=1}^{\infty} 3T (400K_j)^{1/4} e^{-c'\sqrt{K_j}} = 1 - \sum_{j=1}^{\infty} 3T (400jK)^{1/4} e^{-c'\sqrt{jK}} \ge 1 - e^{-c\sqrt{K}},$$

it holds that for any  $j \in \mathbb{N}$  and  $\sqrt{w_0} = \kappa_0 + (400K_j)^{-1/4} \mathbf{i} \in \mathcal{L}$ ,

(3.30) 
$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{1}{\kappa_t \eta^{1-\delta}}, \quad \forall \ 0 \le t \le \sigma_0 \land (2\kappa_0 - 2\sqrt{K_j + \eta^2}).$$

Next we show that (3.30) implies that  $\sigma_0 = 2T$ . Take any  $t \in [0, 2T]$ , and  $x \geq K$ . Then  $K_j \leq x < K_{j+1}$  for some  $j \in \mathbb{N}$ . Take  $\eta = (400K_j)^{-1/4}$ . Then  $\sqrt{x+\eta^2}+t/2 \in [i\eta, (i+1)\eta]$  for some  $i \in \mathbb{N}$ . We can take  $\kappa_0 = (i+1)\eta$  and  $\sqrt{w_0} = \kappa_0 + i\eta \in \mathcal{L}$ , so that  $\kappa_t \in [\sqrt{x+\eta^2}, \sqrt{x+\eta^2}+\eta]$ . Then we have

$$\operatorname{Re}[w_t] = \kappa_t^2 - \eta^2 \in [x, x + 3\sqrt{x\eta}],$$
$$\operatorname{Im}[w_t] = 2\kappa_t \eta \ge 2\sqrt{x\eta}.$$

If there is a pole of  $Y_t$  at x, (by Nevanlinna representation (2.2)) necessarily,

$$\operatorname{Im}[Y_t(w_t)] \ge \frac{\operatorname{Im}[w_t]}{|x - w_t|^2} \ge \frac{\operatorname{Im}[w_t]}{(3\sqrt{x}\eta)^2 + \operatorname{Im}[w_t]^2} = \frac{2\kappa_t}{9x\eta + 4\kappa_t^2\eta^2} \ge \frac{1}{7\sqrt{x}\eta},$$

However, (3.30) (note that  $\kappa_t \ge \sqrt{K_j + \eta^2}$ , so  $t \le 2\kappa_0 - 2\sqrt{K_j + \eta^2}$ ) implies

$$\operatorname{Im}[Y_t(w_t)] \le \operatorname{Im}[\sqrt{w_t}] + \frac{1}{\kappa_t \eta^{1-\delta}} = \eta + \frac{1}{\kappa_t \eta^{1-\delta}} \le \eta + \frac{1}{\sqrt{x} \eta^{1-\delta}} < \frac{1}{7\sqrt{x}\eta}$$

This leads to a contradiction. Therefore (with probability  $\geq 1 - e^{-c\sqrt{K}}$ ) there is no pole in  $(K, \infty)$  at any time in  $[0, \sigma_0]$ . Recall the definition of  $\sigma_0$  from (3.26). Using that  $Y_t$  is continuous in t, and Lemma 2.7 and Lemma 2.8, we conclude that  $\sigma_0 = 2T$ , and Proposition 3.6 follows.

Proof of Lemma 3.7. We introduce the following stopping time

(3.31) 
$$\sigma = \inf\left\{t \le \sigma_0 : |Y_t(w_t) - \sqrt{w_t}| \ge \frac{1}{\kappa_t \eta^{1-\delta}}, \text{ or } \kappa_t^2 - \eta^2 \le K_j, \text{ or } t = \sigma_0\right\}.$$

For  $t \leq \sigma$ , it is necessary that  $t \leq \sigma_0$ , and  $Y_t$  has no pole in  $(K, \infty)$ . Moreover, we also have that  $\kappa_t \geq \sqrt{K_j + \eta^2} \geq \sqrt{K_j}$ , and  $\kappa_t \eta^{2-\delta} \geq 1$ , provided that K is large enough.

We now consider the terms in the RHS of (3.6). For  $0 \le t \le \sigma$ , using Nevanlinna representation (2.2) we have

$$|\partial_w^2 Y_t(w_t)| \le \sum_{x \in P} \frac{1}{|x - w_t|^3} \le \frac{1}{|w_t - K|} \sum_{x \in P} \frac{1}{|x - w_t|^2} \le \frac{\operatorname{Im}[Y_t(w_t)]}{|w_t - K|\operatorname{Im}[w_t]}$$

where the second inequality follows from that  $\operatorname{Re}[w_t] = \kappa_t^2 - \eta^2 \ge K_j \ge K \ge x$  for any  $x \in P$ . As  $|\operatorname{Im}[Y_t(w_t)] - \operatorname{Im}[\sqrt{w_t}]| \le 1/(\kappa_t \eta^{1-\delta}) \le \eta$ , we have  $\operatorname{Im}[Y_t(w_t)] \le 2\eta$ , so

$$\left|\partial_{w}^{2} Y_{t}(w_{t})\right| \leq \frac{2\eta}{|w_{t} - K| \operatorname{Im}[w_{t}]} \leq \frac{\sqrt{2}}{(\kappa_{t}^{2} - \eta^{2} - K + 2\kappa_{t}\eta)\kappa_{t}}$$

where we used  $\text{Im}[w_t] = 2\kappa_t \eta$ , and  $|w_t - K| = |\kappa_t^2 - \eta^2 - K + 2i\kappa_t \eta|$  for the second inequality. By integrating in time, we have

(3.32) 
$$\int_{0}^{t} |\partial_{w}^{2} Y_{s}(w_{s})| \mathrm{d}s \leq \int_{0}^{t} \frac{\sqrt{2}}{(\kappa_{t}\kappa_{s} - \eta^{2} - K + 2\kappa_{t}\eta)\kappa_{t}} \mathrm{d}s \\ \leq \frac{2\sqrt{2}(\log(\kappa_{t}\kappa_{0} - \eta^{2} - K + 2\kappa_{t}\eta) - \log(\kappa_{t}^{2} - \eta^{2} - K + 2\kappa_{t}\eta))}{\kappa_{t}^{2}} \leq \frac{2\sqrt{2}\log(1 + \kappa_{0}/(2\eta))}{\kappa_{t}^{2}}$$

The quadratic variation of the martingale term is given by (3.7). By using Nevanlinna representation (2.2) for  $Y_t$ , we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\int_{0}^{t}(\mathrm{d}M_{s})(w_{s})\right\rangle\right| \leq \sum_{x\in P}\frac{2}{\beta|x-w_{t}|^{4}}.$$

Then using  $\operatorname{Re}(w_t) \ge K_j \ge K \ge x$  for any  $x \in P$ , we can bound the above by

$$\frac{2}{\beta |w_t - K|^2} \sum_{x \in P} \frac{1}{|x - w_t|^2} \le \frac{2 \mathrm{Im}[Y_t(w_t)]}{\beta |w_t - K|^2 \mathrm{Im}[w_t]} \le \frac{4}{\beta (\kappa_t^2 - \eta^2 - K + 2\kappa_t \eta)^2 \kappa_t},$$

where we used  $\text{Im}[Y_t(w_t)] \leq 2\eta$ ,  $\text{Im}[w_t] = 2\kappa_t\eta$ , and  $|w_t - K| = |\kappa_t^2 - \eta^2 - K + 2i\kappa_t\eta|$  for the last inequality. By integrating in time, we have

$$\begin{split} \int_0^t \frac{4\mathrm{d}s}{\beta(\kappa_s^2 - \eta^2 - K + 2\kappa_s\eta)^2\kappa_s} &\leq \int_0^t \frac{4\mathrm{d}s}{\beta(\kappa_t\kappa_s - \eta^2 - K + 2\kappa_t\eta)^2\kappa_t} \\ &\leq \frac{8}{\beta(\kappa_t^2 - \eta^2 - K + 2\kappa_t\eta)\kappa_t^2} \leq \frac{4}{\beta\kappa_t^3\eta}. \end{split}$$

Similarly to (3.21), taking  $\theta = K_j^{1/4}$ , a union bound implies that with probability  $1 - \log(T)e^{-c\theta}$ ,

(3.33) 
$$\left| \int_0^t (\mathrm{d}M_s)(w_s) \right| \le \frac{\theta}{\sqrt{\kappa_t^3 \eta}}, \quad \forall \ 0 \le t \le \sigma.$$

Now by (3.6), and using (3.32) and (3.33), we have that for any  $0 \le t \le \sigma$ ,

(3.34) 
$$|Y_t(w_t) - \sqrt{w_t}|| \le \int_0^t |Y_s(w_s) - \sqrt{w_s}| |\partial_w Y_s(w_s)| \, \mathrm{d}s + \frac{2\theta}{\sqrt{\kappa_t^3 \eta}} + |Y_0(w_0) - \sqrt{w_0}|.$$

Here we used that  $\frac{2\sqrt{2}\log(1+\kappa_0/(2\eta))}{\kappa_t^2} < \frac{\theta}{\sqrt{\kappa_t^3\eta}}$ , provided K is large enough. As for  $\partial_w Y_t(w_t)$ , for  $0 \le t \le \sigma$  (using Lemma 2.2) we have

$$|\partial_w Y_t(w_t)| \le \frac{\operatorname{Im}[Y_t(w_t)]}{\operatorname{Im}[w_t]} \le \frac{1}{2\kappa_t} + \frac{|Y_t(w_t) - \sqrt{w_t}|}{2\kappa_t \eta} \le \frac{1}{2\kappa_t} + \frac{1}{2\kappa_t^2 \eta^{2-\delta}} =: \gamma(t) \le \frac{1}{\kappa_t}$$

Then for any  $0 \le s < t \le \sigma$  (noticing that  $\kappa_t \eta^2 \ge 1/20$ ),

(3.35) 
$$\int_{s}^{t} \gamma(u) \mathrm{d}u \leq \log(\kappa_{s}/\kappa_{t}) + \frac{1}{\kappa_{t}\eta^{2-\delta}} \leq \log(\kappa_{s}/\kappa_{t}) + 20\eta^{\delta}.$$

By Grönwall's inequality and (3.34), for any  $0 \le t \le \sigma$ , we can bound  $|Y_t(w_t) - \sqrt{w_t}|$  by:

$$\frac{2\theta}{\sqrt{\kappa_t^3\eta}} + |Y_0(w_0) - \sqrt{w_0}| + \int_0^t \gamma(s) \left(\frac{2\theta}{\sqrt{\kappa_s^3\eta}} + |Y_0(w_0) - \sqrt{w_0}|\right) \exp\left(\int_s^t \gamma(u) \mathrm{d}u\right) \mathrm{d}s.$$

By (3.35),  $\exp(\int_s^t \gamma(u) du) \le \frac{2\kappa_s}{\kappa_t}$ . Also

$$\int_0^t \gamma(s) \frac{\theta}{\sqrt{\kappa_s^3 \eta}} \cdot \frac{\kappa_s}{\kappa_t} \mathrm{d}s \le \frac{\theta}{\kappa_t} \int_0^t \frac{1}{\sqrt{\kappa_s^3 \eta}} \mathrm{d}s \le \frac{\theta}{\sqrt{\kappa_t^3 \eta}},$$

and

$$\int_{0}^{t} \gamma(s) \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right| \frac{\kappa_{s}}{\kappa_{t}} \mathrm{d}s \le \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right| \int_{0}^{t} \frac{1}{\kappa_{t}} \mathrm{d}s = \frac{t \left| Y_{0}(w_{0}) - \sqrt{w_{0}} \right|}{\kappa_{t}}.$$

Therefore for  $\theta = K_j^{1/4}$ , it holds

$$|Y_t(w_t) - \sqrt{w_t}| \le \frac{6\theta}{\sqrt{\kappa_t^3 \eta}} + \left(1 + \frac{2t}{\kappa_t}\right) |Y_0(w_0) - \sqrt{w_0}| \le \frac{1}{2\kappa_t \eta^{1-\delta}},$$

where in the last inequality we used that

$$\left(1 + \frac{2t}{\kappa_t}\right) |Y_0(w_0) - \sqrt{w_0}| \le B\left(1 + \frac{2t}{\kappa_t}\right) |w_0|^{-1/2} \le B\left(1 + \frac{2t}{\kappa_t}\right) \frac{1}{\kappa_t + t/2} \le \frac{1}{3\kappa_t \eta^{1-\delta}}.$$

Therefore, since  $Y_t(w_t) - \sqrt{w_t}$  is continuous in t, we conclude that  $\sigma = \sigma_0 \wedge (2\kappa_0 - 2\sqrt{K_j + \eta^2})$  with the desired probability.

3.4. **Proof of Proposition 3.1.** As before, take T large enough and  $K = (\log T)^8$ . Proposition 3.6 and Proposition 3.3 verify the two assumptions in Lemma 2.3, respectively. Thus with probability  $1 - e^{-c(\log T)^2}$ , for any  $t \in [T, 2T]$ ,  $Y_t$  has infinitely many poles  $x_1(t) \ge x_2(t) \ge \cdots$ , satisfying

$$(3.36) |x_i(t) - \mathfrak{a}_i| < CK^4 i^{-\delta/6}$$

This gives the second statement in Proposition 3.1, after replacing  $\delta/6$  by  $\delta$ .

The statement (3.36) also verifies the first assumption in Proposition 2.1. Moreover, in (3.9) we can take a sequence of complex numbers  $w_n = ni$ , so that  $|Y_t(w_n) - \sqrt{w_n}| \to 0$  as  $n \to \infty$ . This verifies the second assumption in Proposition 2.1, from which (3.1) holds.

Finally, for each  $i \in \mathbb{N}$ , the continuity of  $x_i(t)$  in  $t \in [T, 2T]$  follows from the continuity of  $Y_t$  in t, and Lemma 2.7, Lemma 2.8.

# 4. HÖLDER REGULARITY

In this section, we upgrade the trajectory continuity into Hölder regularity.

**Proposition 4.1.** For any B > 0, there exist large and small C, c > 0 such that the following holds. Take any  $\{Y_t\}_{t \in \mathbb{R}}$  satisfying Assumption 1.5, and that its poles are given by a line ensemble  $\{x_i(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}$ , and that  $Y_0(w) = \sum_{i=1}^{\infty} \frac{1}{x_i(0)-w} - \frac{1}{a_i} - \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)}$  for any  $w \in \mathbb{H}$ . Take  $k \in \mathbb{N}$  large enough and any  $0 < \xi < ck^{-4/3}$ . Then conditional on the event that

$$(4.1) |x_i(0) - \mathfrak{a}_i| \le B, \quad \forall i \in \llbracket k/2, 2k \rrbracket,$$

with probability  $> 1 - e^{-ck^{1/6}}$  we have

$$\max_{0 \le s \le \xi} |x_k(s) - x_k(0)| \le Ck^{2/3} \xi^{1/2}.$$

*Proof.* We first prove the (with conditional probability  $> 1 - e^{-ck^{1/6}}$ ) upper bound

(4.2) 
$$\max_{0 \le s \le \xi} x_k(s) \le x_k(0) + Ck^{2/3} \xi^{1/2}$$

The lower bound  $\min_{0 \le s \le \xi} x_k(s) \ge x_k(0) - Ck^{2/3}\xi^{1/2}$  can be proven in the same way.

By (4.1) with (2.5), we conclude that there exists a large constant  $C_1$  such that

(4.3) 
$$|\{i: x_i(0) \in [x_k(0) - 1, x_k(0) + 1]\}| \le C_1 k^{1/3}$$

Take small  $\delta \leq 1/(C_1 k^{1/3})$ . Then (4.3) implies that there exists some  $1 \leq \ell \leq C_1 k^{1/3}$ , such that  $x_{k-\ell-1}(0) - x_{k-\ell}(0) \geq \delta$ . We take the smallest such  $\ell$ , then  $|x_{k-\ell}(0) - x_k(0)| \leq \delta \ell$ .

Let  $E = x_{k-\ell}(0) + \delta/2$ , and take a small b > 0 and w = E + ib. Next we show that, provided  $\delta \ge 8bC_1^{1/2}k^{1/6}$ ,

(4.4) 
$$\operatorname{Im}[Y_0(w)] = \operatorname{Im}[Y_0(E+\mathrm{i}b)] = \sum_{i=1}^{\infty} \frac{b}{|E+\mathrm{i}b - x_i(0)|^2} \le \frac{1}{4b}.$$

For this, from (4.1) and (2.5), we have  $|\{i: x_i(0) \in [E - \delta/2 - m, E - \delta/2 - (m-1)]\}| \leq C_1 m^{1/2} k^{1/3}$  for each  $m \in \mathbb{N}$ . It follows that

$$\sum_{i:x_i(0) \le E - \delta/2} \frac{b}{|E + ib - x_i(0)|^2} \le \sum_{m=1}^{\infty} \sum_{i:x_i(0) \in [E - \delta/2 - m, E - \delta/2 - (m-1)]} \frac{b}{|E + ib - x_i(0)|^2} \le \sum_{m=1}^{\infty} \frac{C_1 b m^{1/2} k^{1/3}}{(\delta/2 + m - 1)^2} \le \frac{8C_1 b k^{1/3}}{\delta^2} \le \frac{1}{8b},$$

using that  $\delta \geq 8bC_1^{1/2}k^{1/6}$ . By a similar argument, we can also upper bound the summation over  $x_i(0) \geq E + \delta/2$ , and (4.4) follows.

We now introduce a stopping time  $\tau$ :

$$\tau = \inf\left\{s \ge 0 : \operatorname{Im}[Y_s(w)] \ge \frac{1}{2b}\right\} \land \xi$$

For  $0 \le s \le \tau$ , it follows from Lemma 2.2

$$|\partial_w^2 Y_s(w)| \le \frac{2\mathrm{Im}[Y_s(w)]}{b^2} \le \frac{1}{b^3}, \quad |\partial_w Y_s(w)^2| = 2|\partial_w Y_s(w)| \cdot |Y_s(w)| \le \frac{|Y_s(w)|}{b^2}$$

Thus by Assumption 1.5,

(4.5) 
$$|Y_s(w) - Y_0(w)| \le \left| \int_0^s \mathrm{d}M_u(w) \right| + \frac{\int_0^s |Y_u(w)| \mathrm{d}u}{2b^2} + \mathcal{O}\left(\frac{s}{b^3}\right).$$

For the Martingale term, using (1.7) and Lemma 2.2, it follows that (for  $0 \le s \le \tau$ )

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle M_s(w)\rangle\right| \leq \frac{2\mathrm{Im}[Y_s(w)]}{\beta b^3} \leq \frac{1}{\beta b^4}$$

Therefore, by the Burkholder-Davis-Gundy inequality, there exists a small constant  $c_1 > 0$ ,

(4.6) 
$$\mathbb{P}\left(\sup_{0\leq s\leq \tau}\left|\int_{0}^{s} \mathrm{d}M_{u}(w)\right| \geq \frac{k^{1/6}\xi^{1/2}}{b^{2}}\right) \leq e^{-c_{1}k^{1/6}}$$

By plugging (4.6) into (4.5), it follows that with probability  $e^{-c_1k^{1/6}}$ ,

(4.7) 
$$|Y_s(w) - Y_0(w)| = \mathcal{O}\left(\frac{k^{1/6}\xi^{1/2}}{b^2} + \frac{\xi}{b^3}\right), \quad \forall s \in [0, \tau],$$

provided that  $\xi \leq b^2$ . Now we take a large  $C_2 > 0$ , and assume that  $\xi \leq b^2/(C_2k^{1/3})$ . Then for any  $s \in [0, \tau]$ , and using (4.4), we have

$$\operatorname{Im}[Y_s(w)] \le |Y_s(w) - Y_0(w)| + \operatorname{Im}[Y_0(w)] < \frac{1}{2b},$$

which, in particular, implies that  $\tau = \xi$ . Since  $\operatorname{Im}[Y_s(w)] = \sum_{i=1}^{\infty} \frac{b}{|w-x_i(s)|^2}$ , this further implies that  $\{x_i(s)\}_{i\in\mathbb{N}}\cap [E-b, E+b] = \emptyset$ , for any  $s\in[0,\xi]$ .

In summary, we conclude that with probability  $1 - e^{-c_1 k^{1/6}}$ ,  $\{x_i(s)\}_{i \in \mathbb{N}} \cap [E - b, E + b] = \emptyset$ , for any  $s \in [0, \xi]$ . Since  $x_k(0) < E$ , it follows that (with probability  $1 - e^{-c_1 k^{1/6}}$ ) for any  $s \in [0, \xi]$ ,

$$x_k(s) \le E - b = x_{k-\ell}(0) + \delta/2 - b \le x_k(0) + \delta\ell + \delta/2 \le x_k(0) + 2C_1 k^{1/3} \delta.$$

Finally, we choose the parameter  $\delta$  and b, satisfying all the above constraints:

$$\delta \le 1/(C_1 k^{1/3}), \quad \delta \ge 8bC_1^{1/2} k^{1/6}, \quad \xi \le b^2/(C_2 k^{1/3}).$$

Then we can take  $b = C_2^{1/2} k^{1/6} \xi^{1/2}$  and  $\delta = 8C_1^{1/2} C_2^{1/2} k^{1/3} \xi^{1/2}$ . By taking C large enough and c small enough (depending on  $C_1$ ,  $C_2$ , and  $c_1$ ), the constraint  $\delta \leq 1/(C_1 k^{1/3})$  is also satisfied since  $\xi < ck^{-4/3}$ ; and (4.2) follows.

# 5. Recover Dyson Brownian Motion

In this section, we localize any line ensemble given by the poles of the SDE (1.6), by deriving another SDE satisfied by the first finitely many poles (in the sense of weak solution).

More precisely, for the line ensemble  $\{x_i(t)\}_{i\in\mathbb{N},t\in\mathbb{R}}$  of poles, we prove that, if for some large  $k\in\mathbb{N}$ ,  $x_k(t)$  and  $x_{k+1}(t)$  are bounded away from each other for certain amount of time, the evolution of  $x_1(t) \ge x_2(t) \ge \cdots \ge x_k(t)$  is then described by DBM plus a drift term, describing the effect of  $\{x_i(t)\}_{i=k+1}^{\infty}$ .

**Proposition 5.1.** For any C > 0 the following is true. Take any  $\{Y_t\}_{t \in \mathbb{R}}$  satisfying Assumption 1.5, and that its poles are given by a line ensemble  $\{x_i(t)\}_{i \in \mathbb{N}, t \in \mathbb{R}}$ . Fix a large  $k \in \mathbb{N}$ , and denote the stopping time

(5.1) 
$$\tau = \inf\{t \ge 0 : x_k(t) - x_{k+1}(t) \le 1/(Ck^{1/3})\} \cup \{1\}.$$

Conditional on the event  $\tau > 0$ , there exist independent Brownian motions  $B_1, \dots, B_k$  adapted to the filtration  $\mathcal{F}_t = \sigma(\{\boldsymbol{x}(u)\}_{u \leq t})$ , satisfying

(5.2) 
$$dx_i(t) = \sqrt{\frac{2}{\beta}} dB_i(t) + \sum_{\substack{1 \le j \le k \\ j \ne i}} \frac{dt}{x_i(t) - x_j(t)} + W_t(x_i(t)) dt, \quad \forall i \in [\![1,k]\!], \ t \in [\![0,\tau]\!],$$

where  $W_t$  is a random meromorphic function, defined as

$$W_t(w) = -Y_t(w) + \sum_{i=1}^k \frac{1}{x_i(t) - w}.$$

Moreover, almost surely the following holds:

(5.3) 
$$\int_0^\tau \mathbb{1}(\exists 1 \le i < j \le k : x_i(t) = x_j(t)) dt = 0.$$

The rest of this section is devoted to proving Proposition 5.1. Using that  $\{x_i(t)\}_{i\in\mathbb{N}}$  are poles of  $Y_t$ , we derive (5.2) from Assumption 1.5, using a contour integral. For this, we need first establish that the poles do not collide at almost every time (i.e., (5.3)). The idea to establish the collision time estimate is to consider the process  $(x_i(t) - x_j(t))^2$  for some i < j, showing that its level-0 local time equals 0. We mainly follow the standard argument used to study the Bessel process, see [114, Chapter XI, Section 1]. To analyze such processes we again resort to contour integrals, and therefore an induction will be used.

We next give the semi-martingale decomposition of a process, which is the sum of  $(x_i(t) - x_j(t))^2$ for i, j in an interval.

For any line ensemble  $\{x(t)\}_{t\in\mathbb{R}}$  satisfying Assumption 1.5, and any  $a, \alpha \in \mathbb{N}$  with  $\alpha \geq 2$ , denote

(5.4) 
$$W_t^{a,\alpha}(w) = -Y_t(w) + \sum_{i=a}^{a+\alpha-1} \frac{1}{x_i(t) - w}$$

and

$$Z^{a,\alpha}(t) = \sum_{a \le i < j \le a + \alpha - 1} (x_i(t) - x_j(t))^2 = \alpha \sum_{i=1}^{a + \alpha - 1} x_i(t)^2 - \left(\sum_{i=1}^{a + \alpha - 1} x_i(t)\right)^2.$$

**Lemma 5.2.** In the above setup, take any  $t_0 \in \mathbb{R}$ , and denote the stopping time

$$\sigma = \inf\{t \ge t_0 : x_{a-1}(t) = x_a(t) \text{ or } x_{a+\alpha-1}(t) = x_{a+\alpha}(t) \text{ or } t = t_0 + 1\}$$

where we use the convention that  $x_0(t) = \infty$ , when a = 1. Then  $Z^{a,\alpha}$  in  $[t_0, \sigma)$  satisfies  $dZ^{a,\alpha}(t) =$  $\mathrm{d}M^{a,\alpha}(t) + V^{a,\alpha}(t)\mathrm{d}t, where$ 

$$V^{a,\alpha}(t) = 2\sum_{1 \le i < j \le \alpha} (x_i(t) - x_j(t))(W_t^{a,\alpha}(x_i) - W_t^{a,\alpha}(x_j)) + \alpha^2(\alpha - 2) + \frac{2\alpha(\alpha - 1)}{\beta},$$

and  $dM^{a,\alpha}(t)$  is the Martingale term, with quadratic variation

(5.5) 
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle M^{a,\alpha}(t)\rangle = \frac{8\alpha}{\beta}Z^{a,\alpha}(t)$$

*Proof.* For simplicity of notations, we fix  $a, \alpha$ , and write  $Z(t) = Z(t)^{a,\alpha}$ ,  $V(t) = V(t)^{a,\alpha}$ , and  $dM(t) = dM(t)^{a,\alpha}$  within this proof.

For t with  $x_{a-1}(t) > x_a(t)$  and  $x_{a+\alpha-1}(t) > x_{a+\alpha}(t)$ , we take a contour  $\mathcal{C} = \mathcal{C}_t$  enclosing  $x_a(t), \dots, x_{a+\alpha-1}(t)$ , but not any  $x_i(t)$  for i < a or  $i \ge a + \alpha$ . Then by Assumption 1.5, we have

(5.6) 
$$d\sum_{i=a}^{a+\alpha-1} x_i(t) = -\frac{1}{2\pi i} \oint_{\mathcal{C}} w dY_t(w) dw$$
$$= -\frac{1}{2\pi i} \oint_{\mathcal{C}} w dw \left( dM_t(w) + \left( \frac{2-\beta}{2\beta} \partial_w^2 Y_t(w) + \frac{1}{2} \partial_w Y_t(w)^2 - \frac{1}{2} \right) dt \right)$$

Note that

$$\oint_{\mathcal{C}} w \mathrm{d}w = 0, \quad \oint_{\mathcal{C}} w \partial_w^2 Y_t(w) \mathrm{d}w = \oint_{\mathcal{C}} \sum_{i=1}^{\infty} \frac{2w \mathrm{d}w}{(x_i(t) - w)^3} = 0,$$

and

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{w}{2} \partial_w Y_t(w)^2 dw = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{Y_t(w)^2}{2} dw = \sum_{i=a}^{a+\alpha-1} W_t^{i,1}(x_i(t)).$$

Note that the poles of  $W_t^{i,1}$  are  $x_1(t), x_2(t), \ldots$ , except for  $x_i(t)$ . Then we have

(5.7) 
$$d\sum_{i=a}^{a+\alpha-1} x_i(t) = -\frac{1}{2\pi i} \oint_{\mathcal{C}} w dM_t(w) dw + \sum_{i=a}^{a+\alpha-1} W_t^{a,\alpha}(x_i(t)) dt,$$

using that  $\sum_{i=a}^{a+\alpha-1} W_t^{i,1}(x_i(t)) = \sum_{i=a}^{a+\alpha-1} W_t^{a,\alpha}(x_i(t)).$ Similarly, we have

$$d\sum_{i=a}^{a+\alpha-1} x_i^2(t) = -\frac{1}{2\pi i} \oint_{\mathcal{C}} w^2 dY_t(w) dw$$

$$= -\frac{1}{2\pi i} \oint_{\mathcal{C}} w^2 dw \left( dM_t(w) + \left( \frac{2-\beta}{2\beta} \partial_w^2 Y_t(w) + \frac{1}{2} \partial_w Y_t(w)^2 - \frac{1}{2} \right) dt \right)$$

$$= -\frac{1}{2\pi i} \oint_{\mathcal{C}} w^2 dM_t(w) dw + \left( 2\sum_{i=a}^{a+\alpha-1} W_t^{a,\alpha}(x_i(t)) x_i(t) + \alpha(\alpha-2) + \frac{2\alpha}{\beta} \right) dt.$$

Here for the last equality, we used that

$$\oint_{\mathcal{C}} w^2 \mathrm{d}w = 0, \quad -\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} w^2 \partial_w^2 Y_t(w) \mathrm{d}w = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \sum_{i=1}^{\infty} \frac{2w^2 \mathrm{d}w}{(x_i(t) - w)^3} = 2\alpha,$$

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{w^2}{2} \partial_w Y_t(w)^2 dw = \frac{1}{2\pi i} \oint_{\mathcal{C}} w Y_t(w)^2 dw = 2 \sum_{i=a}^{a+\alpha-1} W_t^{i,1}(x_i(t)) x_i(t),$$

and that  $\sum_{i=a}^{a+\alpha-1} W_t^{i,1}(x_i(t))x_i(t) = \sum_{i=a}^{a+\alpha-1} W_t^{a,\alpha}(x_i(t))x_i(t) + \frac{\alpha(\alpha-1)}{2}$ . Now using (5.6) and (5.8), and Ito's formula, we can write dZ(t) = dM(t) + V(t)dt, with

$$\begin{split} V(t) =& 2\alpha \sum_{i=a}^{a+\alpha-1} W_t^{a,\alpha}(x_i(t))x_i(t) + \alpha^2(\alpha-2) + \frac{2\alpha^2}{\beta} \\ &- 2\left(\sum_{i=a}^{a+\alpha-1} x_i(t)\right) \left(\sum_{i=a}^{a+\alpha-1} W_t^{i,1}(x_i(t))\right) - \frac{2}{(2\pi i)^2\beta} \oint_{\mathcal{C}^2} ww' \,\partial_w \,\partial_{w'} \frac{Y_t(w) - Y_t(w')}{w - w'} \mathrm{d}w \mathrm{d}w' \\ =& 2\sum_{1 \le i < j \le \alpha} (x_i(t) - x_j(t))(W_t^{a,\alpha}(x_i) - W_t^{a,\alpha}(x_j)) + \alpha^2(\alpha-2) + \frac{2\alpha(\alpha-1)}{\beta}. \end{split}$$

Here we used (1.8) in the first equality. For the second equality, it is by evaluating the contour integral in w and w', via integration by parts.

As for dM(t), we have

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$$dM(t) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left( 2w \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w^2 \right) dM_t(w) dw.$$

By (1.8), the quadratic variation  $d\langle M(t)\rangle/dt$  therefore equals

$$\frac{2}{(2\pi\mathrm{i})^2\beta} \oint_{\mathcal{C}^2} \left( 2w \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w^2 \right) \left( 2w' \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w'^2 \right) \partial_w \partial_{w'} \frac{Y_t(w) - Y_t(w')}{w - w'} \mathrm{d}w \mathrm{d}w'$$
$$= \frac{8}{(2\pi\mathrm{i})^2\beta} \oint_{\mathcal{C}^2} \left( \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w \right) \left( \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w' \right) \frac{Y_t(w) - Y_t(w')}{w - w'} \mathrm{d}w \mathrm{d}w'.$$

By taking the w' residues at  $x_a(t), \ldots, x_{a+\alpha-1}(t)$ , we get

$$\frac{8}{2\pi \mathrm{i}\beta} \oint_{\mathcal{C}} \left( \sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha w \right) \left( \sum_{j=a}^{a+\alpha-1} \frac{\sum_{i=a}^{a+\alpha-1} x_i(t) - \alpha x_j(t)}{w - x_j(t)} \right) \mathrm{d}w.$$

By further taking the w residues at  $x_1(t), \ldots, x_{\alpha}(t)$ , this equals

$$\frac{8\alpha}{\beta} \left( \alpha \sum_{i=a}^{a+\alpha-1} x_i^2(t) - \left( \sum_{i=a}^{a+\alpha-1} x_i(t) \right)^2 \right) = \frac{8\alpha}{\beta} Z(t),$$

and the conclusion follows.

We next establish the collision time estimate, for poles whose indices are in an interval.

Lemma 5.3. Under the same setup as Lemma 5.2, almost surely

$$\int_{t_0}^{\sigma} \mathbb{1}[x_a(t) = x_{a+1}(t) = \dots = x_{a+\alpha-1}(t)] dt = 0.$$

*Proof.* Again, we write  $Z(t) = Z(t)^{a,\alpha}$  and  $V(t) = V(t)^{a,\alpha}$  in this proof.

We use the local time of Z(t) to analyze its boundary behavior at zero. According to Lemma 5.2, Z(t) for  $t \in [t_0, \sigma)$  is a semi-martingale. We let  $L_t^h$  be the level h local time in  $[t_0, \sigma)$  (with  $L_{t_0}^h = 0$  for each  $h \in \mathbb{R}$ ). Then by [114, Chapter VI, Theorem 1.7], almost surely  $L_t^h$  is continuous in t and cadlag (right continuous) in h, and  $L_t^0 = L_t^0 - L_t^{0-}$  satisfies

(5.9) 
$$L_{\sigma}^{0} = 2 \int_{t_{0}}^{\sigma} \mathbb{1}(Z(t) = 0) V(t) dt = \left(\alpha^{2}(\alpha - 2) + \frac{2\alpha(\alpha - 1)}{\beta}\right) \int_{t_{0}}^{\sigma} \mathbb{1}(Z(t) = 0) dt,$$

where for the second equality, we used that if Z(t) = 0, then  $x_a(t) = \cdots = x_{a+\alpha-1}(t)$  and  $V(t) = \alpha^2(\alpha-2) + \frac{2\alpha(\alpha-1)}{\beta}$ .

Thanks to the occupation time formula [114, Chapter VI, Corollary 1.6], we have

$$\int_0^\infty h^{-1} L^h_\sigma \mathrm{d}h = \int_{t_0}^\sigma Z(t)^{-1} \mathrm{d}\langle Z(t)\rangle \le \frac{8\alpha}{\beta} < \infty,$$

where we used (5.5) which gives  $d\langle Z(t)\rangle = 8\alpha Z(t)dt/\beta$ , and  $\sigma \leq t_0 + 1$ . Since  $L^h_{\sigma}$  is right continuous in h, it follows that  $L^0_{\sigma} = 0$ , and hence (5.9) implies that almost surely  $\int_{t_0}^{\sigma} \mathbb{1}(Z(t) = 0)dt = 0$ . Thus the conclusion follows.

Proof of Proposition 5.1. We take the following two steps.

**Step 1: Non-collision.** We will first show (5.3), i.e., dt almost everywhere poles do not collide. More precisely, we will prove inductively on  $\ell = 1, 2, 3, \dots, k-1$ 

(5.10) 
$$\int_0^\tau \mathbb{1}(x_1(t), x_2(t), \cdots, x_k(t) \text{ take at most } \ell \text{ distinct values}) dt = 0.$$

The claim of Proposition 5.1 follows from the case of  $\ell = k - 1$  in (5.10).

For the base case where  $\ell = 1$ , it follows from Lemma 5.3 with a = 1 and  $\alpha = k$ , and  $t_0 = 0$ . (Note that in this case, we always have  $\sigma \ge \tau$ )

We next give the induction step: if (5.10) holds for some  $2 \le \ell < k - 1$ , then it holds for  $\ell + 1$ .

Under the induction hypothesis, dt almost everywhere, there exist  $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < k$  such that

(5.11) 
$$x_{\alpha_1}(t) > x_{\alpha_1+1}(t), \quad x_{\alpha_2}(t) > x_{\alpha_2+1}(t), \quad \cdots, \quad x_{\alpha_\ell}(t) > x_{\alpha_\ell+1}(t),$$

Fix the indices  $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell < k$ , the set of time  $t \in [0, \tau]$  such that (5.11) holds is a random open set, and we denote it by  $I \subset [0, \tau]$ .

Next we show that almost surely, for almost every  $t \in I$ ,  $x_1(t), x_2(t), \dots, x_k(t)$  take at least  $\ell + 2$  distinct values. This implies that (5.10) holds for  $\ell + 1$ . For the convenience of notations, we denote  $\alpha_0 = 0$  and  $\alpha_{\ell+1} = k$ . Since  $\ell \leq k-2$ , there exists some  $\nu \in [0, \ell]$  such that  $\alpha_{\nu+1} - \alpha_{\nu} \geq 2$ . We then apply Lemma 5.3 with  $a = \alpha_{\nu} + 1$  and  $\alpha = \alpha_{\nu+1} - \alpha_{\nu}$ , and  $t_0$  taking any rational numbers. We note that the union of all such  $[t_0, \sigma)$  would cover I, therefore

$$\int_I \mathbb{1}(x_a(t) = \dots = x_{a+\alpha-1}(t)) \mathrm{d}t = 0.$$

Thus for almost every  $t \in I$ ,  $x_a(t), \dots, x_{a+\alpha-1}(t)$  would take at least two distinct values, so we finish the induction step.

Then by induction principle, we finish the proof of (5.3).

**Step 2:** Dyson Brownian motion. We next prove (5.2). For that we need to construct the Brownian motions  $B_i(t)$  in (5.2). By (5.3) and that each  $x_i(t)$  is continuous, for any  $t \in [0, \tau]$ 

outside a closed measure zero set (i.e., in a countable union of open intervals, whose closure is  $[0,\tau]$ ), we have  $x_i(t) > x_{i+1}(t)$  for each  $i \in [1,k]$ . Then we can take a small contour  $\mathcal{C}_i = \mathcal{C}_{i,t}$ enclosing  $x_i(t)$  but not any other poles. From (5.7) in the proof of Lemma 5.2, we have

(5.12) 
$$dx_i(t) = \sqrt{\frac{2}{\beta}} dB_i(t) + W_t^{i,1}(x_i(t)) dt,$$

where

$$\mathrm{d}B_i(t) = -\sqrt{\frac{\beta}{2}} \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_i} w \mathrm{d}M_t(w) \mathrm{d}w.$$

We can then further extend  $B_i(t)$  to all of  $[0, \tau]$  as a continuous process. The quadratic variations are given by

which equals  $\mathbb{1}(i=j)$ . Then it follows that  $\{B_i(t)\}_{i=1}^k$  are independent Brownian motions. Noting that  $W_t = W_t^{1,k}$ , we get (5.2) from (5.12).

Remark 5.4. Another approach to study DBM developed in [66] is based on applying Ito's formula to the elementary symmetric functions  $\sum_{1 \leq j_1 < j_2 \cdots < j_n} x_{j_1} x_{j_2} \cdots x_{j_n}$ . There a large family of Dyson type interacting particle systems are considered. For  $\beta \geq 1$ , they show that if the initial data has some particles at the same location, they will separate instantly. The method there could potentially be adapted and derive non-collision in the above proof as well.

# 6. COUPLING AND UNIQUENESS

In this section we prove the uniqueness part of Theorem 1.6.

Take any  $\{Y_t\}_{t\in\mathbb{R}}$  satisfying Assumption 1.4 and Assumption 1.5. Let  $\{x(t)\}_{t\in\mathbb{R}} = \{x_i(t)\}_{i\in\mathbb{N},t\in\mathbb{R}}$ be the line ensemble given by its poles (from Corollary 3.2). We also take another line ensemble  $\{\boldsymbol{y}(t)\}_{t\in\mathbb{R}} = \{y_i(t)\}_{i\in\mathbb{N},t\in\mathbb{R}}$  through the same way.

# **Proposition 6.1.** The two line ensembles $\{x(t)\}_{t\in\mathbb{R}}$ and $\{y(t)\}_{t\in\mathbb{R}}$ have the same law.

Our general strategy is to construct a coupling of the dynamics in t, where x(t) and y(t) would get closer as t increases. Then by sending the starting time of the dynamics to  $-\infty$ , one concludes that these two line ensembles must equal in law.

The coupling. There are four parameters  $\delta, C, T, n$  in the definition of this coupling. Here  $\delta, C > 0$ are small and large real numbers; -T is among the sequence  $t_1, t_2, \dots \to -\infty$  in Assumption 1.4, and T is large enough depending on  $\delta, C$ ; and we let n = |T|. We shall mainly consider the dynamics of the first order n many paths, for  $t \in [-T, T]$ .

For each  $t \in [-T, T]$ , let  $\mathcal{E}[t]$  be the event where

$$|x_i(t) - \mathfrak{a}_i|, |y_i(t) - \mathfrak{a}_i| \le \frac{C(\log T)^{40}}{i^{\delta}},$$

for each  $i \in \mathbb{N}$ .

For each  $\ell \in [0, 2Tn^3]$ , denote  $t_{\ell} = -T + \ell n^{-3}$ . Under  $\mathcal{E}[t_{\ell}]$ , by (2.5), we let  $k_{\ell}$  and  $k'_{\ell}$  be the smallest numbers in [n, 2n - 1], such that

$$|x_{k_{\ell}}(t_{\ell}) - x_{k_{\ell}+1}(t_{\ell})| \ge \frac{2}{C'n^{1/3}}, \quad |y_{k'_{\ell}}(t_{\ell}) - y_{k'_{\ell}+1}(t_{\ell})| \ge \frac{2}{C'n^{1/3}}.$$

where C' is a large enough universal constant. Note that when n is large enough depending on C', such  $k_{\ell}$  and  $k'_{\ell}$  exist.

We introduce a stopping time  $\tau$  (with respect to the filtration  $\mathcal{F}_t = \sigma(\{\boldsymbol{x}(u)\}_{u \leq t}, \{\boldsymbol{y}(u)\}_{u \leq t}))$ , as follows. If there exists any  $t \in [-T, T]$  such that  $\mathcal{E}[t]$  does not hold, or if there is any  $\ell \in [0, 2Tn^3 - 1]$  and  $t \in [t_\ell, t_{\ell+1}]$ , such that

$$|x_{k_{\ell}}(t) - x_{k_{\ell}+1}(t)| \wedge |y_{k_{\ell}'}(t) - y_{k_{\ell}'+1}(t)| \le \frac{1}{C' n^{1/3}},$$

we let  $\tau$  be the smallest such t. Otherwise, we let  $\tau = \infty$ .

**Lemma 6.2.** For any  $\varepsilon > 0$ , there exist  $n, \delta, C$ , such that  $\mathbb{P}[\tau = \infty] \ge 1 - \varepsilon$ .

*Proof.* By Proposition 3.1, for small enough  $\delta$ , large enough C and n, we have  $\mathbb{P}\left[\bigcap_{t\in[-T,T]} \mathcal{E}[t]\right] \geq 1 - \varepsilon/2$ . Then by the Hölder continuity estimate Proposition 4.1,

$$\mathbb{P}[\tau = \infty] \ge 1 - \varepsilon/2 - 2Tn^4 e^{-cn^{1/6}} \ge 1 - \varepsilon$$

where c > 0 is small enough depending on C, and the second inequality is by taking n large.  $\Box$ 

By Proposition 5.1, we can find a family of independent Brownian motions  $\{B_i\}_{i=1}^{2n}$ , such that for each  $\ell \in [0, 2Tn^3 - 1]$  and  $t \in [t_\ell \wedge \tau, t_{\ell+1} \wedge \tau]$ ,  $i \in [1, k_\ell]$ ,

(6.1)  
$$dx_{i}(t) = \sqrt{\frac{2}{\beta}} dB_{i}(t) + \left(\sum_{\substack{1 \le j \le k_{\ell} \\ j \ne i}} \frac{1}{x_{i}(t) - x_{j}(t)} + W_{t}(x_{i}(t))\right) dt,$$
$$W_{t}(w) = \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} + \sum_{i=1}^{k_{\ell}} \frac{1}{\mathfrak{a}_{i}} + \sum_{i=k_{\ell}+1}^{\infty} \frac{1}{w - x_{i}(t)} + \frac{1}{\mathfrak{a}_{i}},$$

where we used Corollary 3.2 for the expression of  $W_t$ . We can similarly find a family of independent Brownian motions  $\{\overline{B}_i\}_{i=1}^{2n}$ , such that for each  $\ell \in [0, 2Tn^3 - 1]$  and  $t \in [t_\ell \wedge \tau, t_{\ell+1} \wedge \tau]$ ,  $i \in [1, k_\ell]$ ,

(6.2)  
$$dy_i(t) = \sqrt{\frac{2}{\beta}} d\overline{B}_i(t) + \left(\sum_{\substack{1 \le j \le k'_\ell \\ j \ne i}} \frac{1}{y_i(t) - y_j(t)} + \overline{W}_t(y_i(t))\right) dt,$$
$$\overline{W}_t(w) = \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} + \sum_{i=1}^{k'_\ell} \frac{1}{\mathfrak{a}_i} + \sum_{i=k'_\ell+1}^{\infty} \frac{1}{w - y_i(t)} + \frac{1}{\mathfrak{a}_i}.$$

We now couple  $\{B_i\}_{i=1}^{2n}$  and  $\{\overline{B}_i\}_{i=1}^{2n}$  so that they equal almost surely. Thereby, we get a coupling between  $\{\boldsymbol{x}(t)\}_{t\in\mathbb{R}}$  and  $\{\boldsymbol{y}(t)\}_{t\in\mathbb{R}}$ .

The following proposition states that under this coupling, these two line ensembles are close to each other with high probability.

**Proposition 6.3.** Fix any  $\varepsilon, \theta > 0$  and S > 0. Then there exist  $n, \delta, C$ , such that under the above coupling with probability  $1 - \varepsilon$ ,

$$|x_i(t) - y_i(t)| \le \theta, \quad \forall t \in [-S, S], \ i \in [\![1, n]\!].$$

## $ALE_{\beta}$ VIA POLE EVOLUTION

In the following, we will prove that with probability  $1 - \varepsilon$ ,

(6.3) 
$$x_i(t) \le y_i(t) + \theta, \quad \forall t \in [-S, S], \ i \in [\![1, n]\!].$$

The lower bound that  $x_i(t) \ge y_i(t) - \theta$  can be proven in the same way.

Our strategy is to consider a shifted version of  $\{\boldsymbol{y}(t)\}_{t\in\mathbb{R}}$ , which at t = -T is much larger than  $\boldsymbol{x}(-T)$ ; then we show that it is larger than  $\{\boldsymbol{y}(t)\}_{t\in\mathbb{R}}$  for  $t \in [-S, S]$  (under the coupling), while the amount of shift is  $\leq \theta$  in [-S, S].

We now define the shifted version of  $\{\boldsymbol{y}(t)\}_{t\in\mathbb{R}}$ . For any  $t\in\mathbb{R}$  and  $i\in\mathbb{N}$ , we let

$$\widetilde{y}_i(t) = y_i(t) + M - \kappa(t+T),$$

where M and  $\kappa$  are taken as follows. By Assumption 1.4 and Corollary 2.4, we take M taken large enough (depending only on  $\varepsilon$ ) such that with probability  $1 - \varepsilon/2$ ,

(6.4) 
$$y_i(-T) + M > x_i(-T), \quad \forall i \in \mathbb{N}.$$

We then take  $\kappa$  such that

$$M - \kappa (T - S) = \theta$$

Then for n = |T| large enough (depending on  $M, S, \theta$ ), the above choice of parameters imply

(6.5) 
$$\kappa < \frac{2M}{n}, \quad M - \kappa(S+T) = \theta - 2S\kappa \ge \frac{\theta}{2}$$

We can now rewrite (6.2) in terms of  $\{\widetilde{\boldsymbol{y}}(t)\}_{t\in\mathbb{R}}$ . For each  $\ell \in [0, 2Tn^3 - 1]$  and  $t \in [t_\ell \wedge \tau, t_{\ell+1} \wedge \tau]$ ,  $i \in [1, k'_\ell]$ , we have

$$d\widetilde{y}_{i}(t) = \sqrt{\frac{2}{\beta}} dB_{i}(t) + \left(\sum_{\substack{1 \le j \le k'_{\ell} \\ j \ne i}} \frac{1}{\widetilde{y}_{i}(t) - \widetilde{y}_{j}(t)} + \widetilde{W}_{t}(\widetilde{y}_{i}(t))\right) dt,$$
$$\widetilde{\psi}_{i}(t) = \sqrt{\frac{k'_{\ell}}{\beta}} \frac{1}{1 - \frac{\infty}{\beta}} \frac{1}{\beta} \int_{0}^{\infty} \frac{1}{\beta} dB_{i}(t) dt,$$

(6.6)

$$\widetilde{W}_t(w) = \frac{\operatorname{Ai}'(0)}{\operatorname{Ai}(0)} - \kappa + \sum_{i=1}^{k_\ell} \frac{1}{\mathfrak{a}_i} + \sum_{i=k_\ell'+1}^{\infty} \frac{1}{w - \widetilde{y}_i(t)} + \frac{1}{\mathfrak{a}_i}.$$

**Lemma 6.4.** There exist  $n, \delta, C$ , such under the above coupling the following holds. Take any  $\ell \in [0, (S+T)n^3]$ . Assuming that

$$\widetilde{y}_i(t_\ell \wedge \tau) > x_i(t_\ell \wedge \tau), \quad \forall i \in [\![1, n]\!],$$

then

(6.7) 
$$\widetilde{y}_i(t) > x_i(t), \quad \forall t \in [t_\ell \land \tau, t_{\ell+1} \land \tau], \ i \in [\![1, n]\!].$$

Assuming this lemma, we can now finish proving the uniqueness in law of line ensembles.

Proof of Proposition 6.3. As already alluded to, it suffices to prove (6.3). From our choice of M (see (6.4)), we have that  $\tilde{y}_i(-T) > x_i(-T)$  for all  $i \ge 1$ . Then by repeatedly applying Lemma 6.4 for  $\ell \in [0, (S+T)n^3]$ , and Lemma 6.2, we conclude that with probability  $1 - \varepsilon$ , we have  $\tau = \infty$  and  $\tilde{y}_i(t) > x_i(t)$  for all  $t \in [-T, S]$  and  $i \in [1, n]$ . In particular for  $t \in [-S, S]$ , this gives

$$y_i(t) + \theta = y_i(t) + (M - \kappa(T - S)) \ge y_i(t) + (M - \kappa(t + T)) > x_i(t).$$

This finishes the proof of (6.3).

*Proof of Proposition 6.1.* The conclusion follows from taking  $\theta, \varepsilon$  to zero and S to infinity in Proposition 6.3.

The rest of this section is devoted to proving Lemma 6.4. The idea is straightforward: from the coupling we take the difference between (6.1) and (6.6), to cancel out the Brownian motions; and the rest are deterministic arguments.

Proof of Lemma 6.4. For simplicity of notation, in this proof we fix  $\ell$ , and write  $k = k_{\ell}$  and  $k' = k'_{\ell}$ . Recall that  $k, k' \in [n, 2n-1]$ . We can take the difference between (6.1) and (6.6), so that for any  $i \in [\![1, n]\!],$ 

$$d(\widetilde{y}_i(t) - x_i(t)) = \sum_{j \in \llbracket 1, n \rrbracket, j \neq i} \frac{(x_i(t) - \widetilde{y}_i(t)) - (x_j(t) - \widetilde{y}_j(t))}{(\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t))} dt$$
$$+ \left(\widetilde{W}_t(\widetilde{y}_i(t)) + \sum_{i=1}^{k'} \frac{1}{\widetilde{w}_i(t) - \widetilde{w}_i(t)} - W_t(x_i(t)) - \sum_{i=1}^{k'} \frac{1}{x_i(t) - x_i(t)}\right)$$

(6.8)

$$+\left(\widetilde{W}_{t}(\widetilde{y}_{i}(t))+\sum_{j=n+1}^{k'}\frac{1}{\widetilde{y}_{i}(t)-\widetilde{y}_{j}(t)}-W_{t}(x_{i}(t))-\sum_{j=n+1}^{k}\frac{1}{x_{i}(t)-x_{j}(t)}\right)\mathrm{d}t.$$

Denote the stopping time  $\sigma$  to be the first time after  $t_{\ell} \wedge \tau$ , such that there exists at least one index  $i_* \in [1,n]$  with  $x_{i_*}(\sigma) = \tilde{y}_{i_*}(\sigma)$  (if there were multiple such indices, take  $i_*$  to be the smallest one). We will prove that  $\sigma \geq t_{\ell+1} \wedge \tau$  then (6.7) holds.

We prove by contradiction, and assume that  $\sigma < t_{\ell+1} \wedge \tau$ . By the definition of the stopping time  $\tau$ , for each  $i \in \mathbb{N}$ ,  $i \ge n/3$ , and  $t \in [-T, S \land \tau]$ , we have

(6.9) 
$$x_i(t) \le \mathfrak{a}_i + \frac{C(\log T)^{40}}{i^{\delta}} \le y_i(t) + \frac{2C(\log T)^{40}}{i^{\delta}} < y_i(t) + M - \kappa(t+T) = \widetilde{y}_i(t).$$

We let a (resp. b) be the smallest (resp. largest) index with  $x_a(\sigma) = x_{i_*}(\sigma)$  (resp.  $x_b(\sigma) = x_{i_*}(\sigma)$ ); and we let  $a', b' \in \mathbb{N}$  be the corresponding indices for  $y_{i_*}(\sigma)$ . By (6.9), and that  $\tilde{y}_i(\sigma) \geq x_i(\sigma)$  for each  $i \in [1, n]$ , necessarily  $1 \le a' \le a \le b' \le b < n/2$ . Now for (6.8), by summing over  $i \in [[a, b']]$ , and integrating from  $\sigma - \iota$  to  $\sigma$  for a sufficiently small  $\iota$ , we have

$$(6.10) \qquad 0 > \sum_{i=a}^{b'} (\widetilde{y}_i(t) - x_i(t)) \Big|_{\sigma-\iota}^{\sigma} = \int_{\sigma-\iota}^{\sigma} \sum_{i=a}^{b'} \sum_{j \in [\![1,n]\!] \setminus [\![a,b']\!]} \frac{(x_i(t) - \widetilde{y}_i(t)) - (x_j(t) - \widetilde{y}_j(t))}{(\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t))} dt \\ + \int_{\sigma-\iota}^{\sigma} \sum_{i=a}^{b'} \widetilde{W}_t(\widetilde{y}_i(t)) + \sum_{j=n+1}^{k'} \frac{1}{\widetilde{y}_i(t) - \widetilde{y}_j(t)} - W_t(x_i(t)) - \sum_{j=n+1}^{k} \frac{1}{x_i(t) - x_j(t)} dt.$$

Consider the first term on the RHS of (6.10). Since  $x_i(t)$ ,  $\tilde{y}_i(t)$  are continuous,  $\lim_{t\to\sigma} \tilde{y}_i(t) - x_i(t) =$ 0 for  $i \in [a, b']$ , and  $\lim_{t\to\sigma} \widetilde{y}_i(t) - x_i(t) > 0$  for at least one  $i \notin [a, b']$ . Also, we have that  $\lim_{t\to\sigma} (\widetilde{y}_i(t) - \widetilde{y}_i(t))(x_i(t) - x_i(t)) \ge 0, \text{ for each } i \in [\![a, b']\!] \text{ and } j \in [\![1, n]\!] \setminus [\![a, b']\!].$  Thus

(6.11) 
$$\lim_{\iota \to 0} \inf \frac{1}{\iota} \int_{\sigma-\iota}^{\sigma} \sum_{i=a}^{b'} \sum_{j \in [\![1,n]\!] \setminus [\![a,b']\!]} \frac{(x_i(t) - \widetilde{y}_i(t)) - (x_j(t) - \widetilde{y}_j(t))}{(\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t))} \mathrm{d}t > 0.$$

We next consider the second term in the RHS of (6.10). From the definition of  $\tau$ ,  $\mathcal{E}[t]$  holds for any  $t \in [t_{\ell} \wedge \tau, t_{\ell+1} \wedge \tau]$ . Thus when n, C are large enough, for any  $i \in [a, b']$  and j > n it holds

(6.12) 
$$|x_i(t) - x_j(t)|, |\widetilde{y}_i(t) - \widetilde{y}_j(t)| \le Cj^{2/3}$$

We can rewrite the integrand as

(6.13) 
$$\sum_{i=a}^{b'} \sum_{j=n+1}^{\infty} \frac{x_i(t) - \widetilde{y}_i(t)}{(\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t))} + \sum_{i=a}^{b'} -\kappa + \sum_{j=n+1}^{\infty} \frac{\widetilde{y}_j(t) - x_j(t)}{(\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t))}.$$

Since  $\lim_{t\to\sigma} \widetilde{y}_i(t) - x_i(t) = 0$  for  $i \in [a, b']$ , and  $\lim_{t\to\sigma} (\widetilde{y}_i(t) - \widetilde{y}_j(t))(x_i(t) - x_j(t)) > 0$  for  $i \in [a, b']$ and j > n, the first term in (6.13) converges to zero as  $t \to \sigma$ . For the second term in (6.13), using (6.12), it is at least

$$\sum_{j=n+1}^{\infty} \frac{\widetilde{y}_j(t) - x_j(t)}{C^2 j^{4/3}} - \kappa \ge \sum_{j=n+1}^{\infty} \frac{M - \kappa(t+T) - \frac{2C(\log T)^{40}}{j^{\delta}}}{C^2 j^{4/3}} - \kappa \ge \frac{\theta}{10C^2 n^{1/3}} - \kappa \ge \kappa,$$

where the first inequality is by  $\mathcal{E}[t]$ ; and in the second and last inequalities we used that

$$M - \kappa(t+T) \ge M - \kappa(S+T) \ge \theta/2 \ge (20C^2 n^{1/3} \kappa) \lor (4C(\log T)^{40}/n^{\delta}),$$

which is by (6.5) and taking n large enough (depending on  $\varepsilon, \theta, S$  and  $\delta, C$ ). Then we conclude that

(6.14) 
$$\liminf_{\iota \to 0} \frac{1}{\iota} \int_{\sigma-\iota}^{\sigma} \sum_{i=a}^{b'} \widetilde{W}_t(\widetilde{y}_i(t)) + \sum_{j=n+1}^{k'} \frac{1}{\widetilde{y}_i(t) - \widetilde{y}_j(t)} - W_t(x_i(t)) - \sum_{j=n+1}^k \frac{1}{x_i(t) - x_j(t)} \mathrm{d}t > 0.$$

Combining (6.11) and (6.14), we conclude that for  $\iota > 0$  small enough, the RHS of (6.10) is positive, which leads to a contradiction.

# 7. Convergence to the $Airy_{\beta}$ line ensemble

The dynamical versions of the three classical ensembles as in (1.2) correspond to Dyson Brownian motion (DBM), the Laguerre process, and the Jacobi process, all of which have been intensively studied in the literature, as seen in [9, 27, 44, 92]. In this section we prove that their edge limit is  $ALE_{\beta}$ , i.e., Theorem 1.7. In particular, for DBM our method covers more general potentials (beyond quadratic ones). Our strategy is to prove tightness, and verify that any subsequential limit satisfies Assumption 1.4 and Assumption 1.5, and then apply Theorem 1.6.

We now formally introduce these processes, from a random matrix theory perspective.

**DBM (and with general potential).** Let  $B_t = (B_{ij}(t))$  be an  $n \times n$  real/complex Brownian matrix (with each entry given by Brownian motion B(t) for the real case; and given by  $(B(t) + \hat{B}(t)i)/\sqrt{2}$  with B(t) and  $\hat{B}(t)$  being independent Brownian motions for the complex case), and define  $X_t = (B_t + B_t^*)/\sqrt{2}$  (where  $B_t^*$  is the complex conjugate of  $B_t$ ). Then the eigenvalues of  $X_t$  (denoted by  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$ ) satisfy

$$\mathrm{d}\lambda_i^{(n)}(t) = \sqrt{\frac{2}{\beta}} \mathrm{d}B_i(t) + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{\mathrm{d}t}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)},$$

with  $\beta = 1$  (real case) or 2 (complex case). At time t = 1 the law is given by the Hermite/Gaussian ensemble.

More generally, one can consider DBM with potential V and any  $\beta > 0$ ,

(7.1) 
$$d\lambda_{i}^{(n)}(t) = \sqrt{\frac{2}{\beta}} dB_{i}(t) + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{dt}{\lambda_{i}^{(n)}(t) - \lambda_{j}^{(n)}(t)} - \frac{\sqrt{n}}{2} V'\left(\frac{\lambda_{i}^{(n)}(t)}{\sqrt{n}}\right) dt.$$

Under specific conditions for the potential V (refer to Assumption 7.1), the rescaled particle configurations  $\lambda_i^{(n)}(t)/\sqrt{n}$  (adopting the notations from [26]) have a  $\beta$ -ensemble as the stationary

measure, with the following probability density proportional to:

(7.2) 
$$\frac{1}{Z_{n,\beta,V}} \prod_{1 \le i < j \le n} |x_i - x_j|^{\beta} \prod_{i=1}^n e^{-\frac{\beta n}{2} \sum_{i=1}^n V(x_i)},$$

where  $Z_{n,\beta,V}$  is a renormalization constant. Under mild conditions of V(x) (see e.g., [39, Theorem 1]), there exists a unique equilibrium measure  $\mu_V$  characterized by the following variational principle

(7.3) 
$$\mu_V := \operatorname{argmin}_{\mu} \left\{ -\int_{\mathbb{R}^2} \log |x - y| \mathrm{d}\mu(x) \mathrm{d}\mu(y) + \int_{\mathbb{R}} V(x) \mathrm{d}\mu(x) \right\}$$

where the minimization is taken over all probability measures on  $\mathbb{R}$ .

We shall work on DBM with potentials under the following technical assumptions.

Assumption 7.1. The potential function V(x) satisfies

- It is analytic on  $\mathbb{R}$ .
- There exist constants  $M_0, C, c > 0$  such that  $V'(x) \ge c$  and  $\sup_{y \in [M_0, x]} V'(y)/y \le CV(x)$  for all  $x \ge M_0$ , and similar estimates apply for  $x \le -M_0$ .

Under the previous assumptions, it is known that there exists a unique equilibrium measure  $\mu_V$  on  $\mathbb{R}$  characterized by the variational principle (7.3). We further assume V(x) satisfies

- The measure  $\mu_V$  has a density  $\varrho_V$ , which is positive and supported on a single interval [A, B], with square root singularities at A and B. More precisely, there exists  $R_A > 0$  and  $R = R_B > 0$ , such that  $\lim_{x\to 0_+} x^{-1/2} \varrho_V(A + x) = R_A/\pi$  and  $\lim_{x\to 0_+} x^{-1/2} \varrho_V(B x) = R/\pi$ .
- The function  $x \mapsto V(x)/2 \int \log |x y| d\mu_V(y)$  achieve its minimum value only in the interval [A, B].

In particular, Assumption 7.1 is satisfied by any V that is analytic and strongly convex (see [39]). Next we recall some estimates of the equilibrium density  $\rho_V$  and its Stieltjes transform

(7.4) 
$$m_V(z) = \int_A^B \varrho_V(x)/(x-z) \mathrm{d}x, \quad z \in \mathbb{C} \setminus [A, B],$$

from [26, Section 2.1].

By Assumption 7.1, V(x) is analytic on  $\mathbb{R}$ , so it can be extended analytically to a simply connected open set  $\Omega$  of the complex plane, which contains [A, B]. The equilibrium density  $\rho_V(x)$  is supported on [A, B], given explicitly by

(7.5) 
$$\varrho_V(x) = \frac{r(x)}{\pi} \sqrt{(x-A)(B-x)} = \frac{R}{\pi} \sqrt{B-x} + \mathcal{O}(|B-x|^{3/2}), \quad x \in [A,B],$$

where

$$\mathbf{r}(z) = \frac{1}{2\pi} \int_{A}^{B} \frac{V'(z) - V'(x)}{z - x} \cdot \frac{\mathrm{d}x}{(x - A)(B - x)},$$

is analytic in  $\Omega$ , with  $R_A = r(A)\sqrt{B-A}$  and  $R = R_B = r(B)\sqrt{B-A}$ . And

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(7.6) 
$$m_V(z) = \frac{-V'(z) + 2r(z)\sqrt{(z-A)(z-B)}}{2} = -\frac{V'(B)}{2} + R\sqrt{z-B} + \mathcal{O}(|z-B|),$$

for  $z \in \Omega \setminus [A, B]$ .

**Laguerre process.** Let  $B_t = (B_{ij}(t))$  be an  $n \times m$  real/complex Brownian matrix, and define  $X_t = B_t B_t^*$ . Assume  $n \leq m$ , then the evolution of eigenvalues of  $X_t$  (denoted by  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$ ) is given by the Laguerre process

(7.7) 
$$d\lambda_i^{(n)}(t) = \frac{2}{\sqrt{\beta}} \sqrt{\lambda_i^{(n)}(t)} dB_i(t) + \left( m + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{\lambda_i^{(n)}(t) + \lambda_j^{(n)}(t)}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)} \right) dt,$$

where  $\beta = 1$  (real case) or 2 (complex case); while the Laguerre process, i.e., solution to (7.7), is also considered for any  $\beta > 0$ . At time t = 1 the law is given by the Laguerre ensemble. There is also a stationary version:

(7.8) 
$$d\lambda_{i}^{(n)}(t) = \frac{2}{\sqrt{\beta}}\sqrt{\lambda_{i}^{(n)}(t)}dB_{i}(t) + \left(m + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{\lambda_{i}^{(n)}(t) + \lambda_{j}^{(n)}(t)}{\lambda_{i}^{(n)}(t) - \lambda_{j}^{(n)}(t)}\right)dt - \lambda_{i}^{(n)}dt$$

whose equilibrium measure is given by the Laguerre ensemble in (1.2).

**Jacobi process.** Let  $\Theta(t)$  be the Brownian motion on the  $m \times m$  orthogonal/unitary group. Take p+q=m and  $p \ge n+1, q \ge n+1$  and denote C(t) the left corner of  $\Theta(t)$  with size  $n \times p$ . Then the evolution of the eigenvalues of  $C(t)C(t)^*$  (denoted by  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$ ) is given by the Jacobi process

(7.9)  
$$d\lambda_{i}^{(n)}(t) = \frac{2}{\sqrt{\beta}} \sqrt{\lambda_{i}^{(n)}(t)(1 - \lambda_{i}^{(n)}(t))} dB_{i}(t) + \left( p - m\lambda_{i}^{(n)}(t) + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{\lambda_{i}^{(n)}(t)(1 - \lambda_{i}^{(n)}(t)) + \lambda_{j}^{(n)}(t)(1 - \lambda_{j}^{(n)}(t))}{\lambda_{i}^{(n)}(t) - \lambda_{j}^{(n)}(t)} \right) dt,$$

where  $\beta = 1$  and 2 correspond to the orthogonal and unitary cases, respectively. Again, the Jacobi process, i.e., solution to (7.9), is also considered for any  $\beta > 0$ . We refer to [48, Chapter 9] and [44, Section 1.2] for detailed discussions of this matrix Jacobi process and the derivation of  $(7.9)^5$ . The equilibrium measure of (7.9) is given by the Jacobi ensemble in (1.2).

For the special case with  $\beta = 1, 2, 4$ , the Jacobi ensemble also describes the eigenvalues of MANOVA (multivariate analysis of variance) matrices. Let  $W = (W_1, W_2)$  be an  $n \times m$  real/complex Gaussian matrix, and  $W_1$  consists of its first p columns and  $W_1$  consists of its last q columns. The eigenvalues of the matrix  $(W_1W_1^* + W_2W_2^*)^{-1/2}W_1W_1^*(W_1W_1^* + W_2W_2^*)^{-1/2}$  are given by the Jacobi ensemble.

Given the above setup, we now state the more precise version of Theorem 1.7.

**Theorem 7.2.** For each  $n \in \mathbb{N}$ , let  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  be either the stationary DBM (7.1) with fixed general potential V satisfying Assumption 7.1, Laguerre process (7.8), or Jacobi process (7.9). Take  $n \to \infty$ , with

- $\limsup m/n < \infty$  and  $\liminf m/n > 1$  in the Laguerre case;
- $\limsup p/n, \limsup q/n < \infty$ ,  $\liminf p/n > 1$ ,  $\liminf q/n > 0$  in the Jacobi case.

Then we have that  $\{(\lambda_i^{(n)}(\zeta t) - E)/\chi\}_{i=1}^{\infty}$  converges to  $ALE_{\beta}$ , under the uniform in compact topology. Here we take the convention of  $\lambda_i^{(n)} = -\infty$  for i > n, and  $E, \zeta, \chi$  are as follows:

<sup>&</sup>lt;sup>5</sup>Note that compared to the definition of the Jacobi process in [44], here we rescale time by a factor of  $\beta$ .

$$DBM: E = Bn^{1/2}, \zeta = R^{-4/3}n^{-1/3}, \chi = R^{-2/3}n^{-1/6}, \text{ where } B, R, \text{ and } \varrho_V \text{ from Assumption 7.1}; Laguerre: E = (\sqrt{m} + \sqrt{n})^2, \zeta = 2^{-1}(\sqrt{m} + \sqrt{n})^{2/3}(mn)^{-1/3}, \chi = (\sqrt{m} + \sqrt{n})^{4/3}(mn)^{-1/6}; Jacobi: E = \left(\frac{\sqrt{p(m-n)} + \sqrt{qn}}{m}\right)^2, \zeta = \frac{(E(1-E))^{1/3}}{2(pq(m-n))^{1/3}}n^{-1/3}, \chi = \frac{(E(1-E))^{2/3}}{(pq(m-n))^{1/6}}n^{-1/6}.$$

The rest of this section is devoted to the proof of this result. We note that, effectively, our proof also gives a self-contained contruction of  $ALE_{\beta}$ , which is defined and shown to be the edge limit of quadratic potential DBM in [64].

7.1. Scaling of particles and Stieltjes transform. Our proof of convergence to  $ALE_{\beta}$  consists of the following tasks:

(1) Write out the SDE satisfied by the (rescaled) Stieltjes transform;

(2) Establish tightness of the (rescaled) Stieltjes transform and particles;

(3) Verify Assumption 1.4 and Assumption 1.5 for any subsequential limit.

This subsection is for task (1).

For each one of the three processes, denote the Stieltjes transform

$$m_t^{(n)}(z) = \sum_{i=1}^n \frac{1}{\lambda_i^{(n)}(t) - z}, \quad z \in \mathbb{C}.$$

And we let  $\widetilde{\lambda}_{i}^{(n)}(t) = (\lambda_{i}^{(n)}(\zeta t) - E)/\chi$ . We shall define a certain (time-evolving) particle-generated Nevanlinna function  $Y_{t}^{(n)}$  with poles  $\{\widetilde{\lambda}_{i}^{(n)}(t)\}_{i=1}^{n}$ , through a rescaling from  $m_{t}^{(n)}$ ; and as  $n \to \infty$ , such  $Y_{t}^{(n)}$  should converge to  $Y_{t}$  in Assumption 1.4 and Assumption 1.5.

In light of the Airy-like property, we will instead work with  $\Delta_t^{(n)}(w) = Y_t^{(n)}(w) - \sqrt{w}$ , which we next define for the three cases, respectively.

# 7.1.1. Scaling of particles and Stieltjes transform.

**DBM.** We consider the DBM (7.19) with general potential V starting from the stationary distribution. Then the law of  $\{\lambda_i^{(n)}(t)/\sqrt{n}\}_{i=1}^n$  for every fixed  $t \in \mathbb{R}$  is the  $\beta$  ensemble (7.2). In light of the measure  $\rho_V$ ,  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  for a fixed t should fill in the interval of  $[A\sqrt{n}, B\sqrt{n}]$ , with square root behavior of density near the edges. Take  $m_V$  as in (7.4). Since  $E = Bn^{1/2}$ and  $\chi = R^{-2/3}n^{-1/6}$ ,  $\chi m_t^{(n)}(E + \chi w)$  is approximately by  $\chi\sqrt{n}m_V(B + n^{-1/2}\chi w)$ , which equals  $-\frac{\chi\sqrt{n}V'(B)}{2} + \sqrt{w} + \mathcal{O}(n^{-1/3}|w|)$  according to (7.6). Therefore, we let

(7.10) 
$$\Delta_t^{(n)}(w) = \chi \left( m_{\zeta t}^{(n)}(E + \chi w) + \frac{\sqrt{n}V'(B)}{2} \right) - \sqrt{w} \\ = \chi \left( m_{\zeta t}^{(n)}(E + \chi w) - \sqrt{n}m_V(B + n^{-1/2}\chi w) \right) + \mathcal{O}(n^{-1/3}|w|).$$

Then since  $\widetilde{\lambda}_i^{(n)}(t) = (\lambda_i^{(n)}(\zeta t) - E)/\chi = R^{2/3}n^{1/6}\lambda_i^{(n)}(tR^{-4/3}n^{-1/3}) - R^{2/3}Bn^{2/3}$ , we also have

$$\Delta_t^{(n)}(w) + \sqrt{w} = \sum_{i=1}^n \frac{1}{\widetilde{\lambda}_i^{(n)}(t) - w} + \frac{R^{-2/3}V'(B)}{2}n^{1/3}.$$

**Laguerre.** It is known (see e.g., [50]) that the density of  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  for a fixed t is approximated by the (rescaled) Marchenko-Pastur law

(7.11) 
$$\varrho_{\rm mp}^{(n)}(x) = \frac{\sqrt{(x-E_-)(E_+-x)}}{2\pi x},$$

supported on the interval  $[E_-, E_+]$ , where  $E_- = (\sqrt{m} - \sqrt{n})^2$  and  $E_+ = E = (\sqrt{m} - \sqrt{n})^2$ . Taking its Stieltjes transform we get

$$m_{\rm mp}^{(n)}(z) = \int \frac{\varrho_{\rm mp}^{(n)}(x) dx}{x-z} = \frac{-(z-m+n) + \sqrt{(z-(m+n))^2 - 4mn}}{2z}$$

which should approximate  $m^{(n)}(z)$ . In particular, we have  $m^{(n)}_{mp}(E) = -\frac{\sqrt{n}}{\sqrt{m}+\sqrt{n}}$ . In light of this, we rescale  $m^{(n)}_t(z)$  in time by  $\zeta$  and in space by  $\chi$ , and denote

$$\Delta_t^{(n)}(w) = \chi \left( m_{\zeta t}^{(n)}(E + \chi w) + \frac{\sqrt{n}}{\sqrt{m} + \sqrt{n}} \right) - \sqrt{w}.$$

Then we have

$$\Delta_t^{(n)}(w) + \sqrt{w} = \sum_{i=1}^n \frac{1}{\widetilde{\lambda}_i^{(n)}(t) - w} + \frac{\chi\sqrt{n}}{\sqrt{m} + \sqrt{n}}$$

**Jacobi.** By e.g., [51], the density of  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  for a fixed t is approximated by the following law:

(7.12) 
$$\varrho_{\rm Ja}^{(n)}(x) = \frac{m\sqrt{(x-E_-)(E_+-x)}}{2\pi x(1-x)}$$

supported on the interval  $[E_{-}, E_{+}]$ , where  $E_{-} = \left(\frac{\sqrt{p(m-n)} - \sqrt{qn}}{m}\right)^{2}$  and  $E_{+} = E = \left(\frac{\sqrt{p(m-n)} + \sqrt{qn}}{m}\right)^{2}$ . Taking its Stieltjes transform we get

$$m_{\rm Ja}^{(n)}(z) = \int \frac{\varrho_{\rm Ja}^{(n)}(x) dx}{x-z} = \frac{-(mz - 2nz + n - p) + \sqrt{(mz + n - p)^2 - 4znq}}{2z(1-z)}$$

which should approximate  $m^{(n)}(z)$ . In particular, we have  $m_{\text{Ja}}^{(n)}(E) = -\frac{mE-2nE+n-p}{2E(1-E)}$ . Thus we denote

$$\Delta_t^{(n)}(w) = \chi \left( m_{\zeta t}^{(n)}(E + \chi w) + \frac{mE - 2nE + n - p}{2E(1 - E)} \right) - \sqrt{w}.$$

and we have

$$\Delta_t^{(n)}(w) + \sqrt{w} = \sum_{i=1}^n \frac{1}{\widetilde{\lambda}_i^{(n)}(t) - w} + \frac{\chi(mE - 2nE + n - p)}{2E(1 - E)}$$

Now we have defined the rescaled particles and Stieltjes transform. In the rest of this section, unless otherwise noted, all the notations set up above  $\{\{\widetilde{\lambda}_i^{(n)}(t)\}_{i\in\mathbb{N},t\in\mathbb{R}} \text{ and } \{\Delta_t^{(n)}(w)+\sqrt{w}\}_{w\in\mathbb{H},t\in\mathbb{R}}, etc.\}$  refer to any one of the three above cases.

7.1.2. Rescaled SDE and error terms. We now present the SDE satisfied by  $\Delta_t^{(n)}$ .

**Proposition 7.3.** The following SDE is satisfied:

(7.13) 
$$d\Delta_t^{(n)}(w) = dM_t^{(n)}(w) + \frac{2-\beta}{2\beta} \partial_w^2(\Delta_t^{(n)}(w) + \sqrt{w}) + \frac{1}{2} \partial_w(\Delta_t^{(n)}(w) + \sqrt{w})^2 - \frac{1}{2} + \mathcal{E}_t^{(n)}(w).$$

Here  $\mathcal{E}_t^{(n)}(w)$  is some error term, and  $M_t^{(n)}(w)$  is the Martingale term, with quadratic variation given by

(7.14) 
$$\frac{\mathrm{d}}{\mathrm{d}t} \langle M_t^{(n)}(w), M_t^{(n)}(w') \rangle = \frac{2}{\beta} \sum_{i=1}^n \frac{1}{(\widetilde{\lambda}_i^{(n)}(t) - w)^2 (\widetilde{\lambda}_i^{(n)}(t) - w')^2} + \hat{\mathcal{E}}_t^{(n)}(w, w'),$$

for any  $w, w' \in \mathbb{H}$ , where  $\hat{\mathcal{E}}_t^{(n)}(w, w')$  is some other error term. These error terms satisfy the following estimates. Take any compact  $\mathcal{K} \subset \mathbb{H}$ . For any  $w, w' \in \mathcal{K}$ , there is

(7.15) 
$$|\mathcal{E}_t^{(n)}(w)|, \ |\partial_w \mathcal{E}_t^{(n)}(w)| \lesssim n^{-1/3} (1 + |\Delta_t^{(n)}(w) + \sqrt{w}|)^2,$$

and

(7.16) 
$$|\hat{\mathcal{E}}_t^{(n)}(w,w')|, \ |\partial_w \partial_{w'} \hat{\mathcal{E}}_t^{(n)}(w,w')| \lesssim n^{-2/3} (1+|\Delta_t^{(n)}(w)+\sqrt{w}|),$$

where all the constants behind  $\lesssim$  can depend on  $\mathcal{K}$ .

To draw connection with Assumption 1.5, we note that the quadratic variation of the Martingale term can also be written as

(7.17) 
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle M_t^{(n)}(w)\rangle = \frac{1}{3\beta}\,\partial_w^3(\Delta_t^{(n)}(w) + \sqrt{w}) + \hat{\mathcal{E}}_t^{(n)}(w,w),$$

and

$$(7.18) \quad \frac{\mathrm{d}}{\mathrm{d}t} \langle M_t^{(n)}(w), M_t^{(n)}(w') \rangle = \frac{2}{\beta} \,\partial_w \,\partial_{w'} \left( \frac{(\Delta_t^{(n)}(w) + \sqrt{w}) - (\Delta_t^{(n)}(w') + \sqrt{w'})}{w - w'} \right) + \hat{\mathcal{E}}_t^{(n)}(w, w'),$$

for  $w \neq w'$ .

Proof of Proposition 7.3. We prove this for the cases respectively. Constants in all  $\leq$  in this proof are allowed to depend on  $\mathcal{K}$ .

**DBM.** For simplicity, below, we only consider the stationary DBM in (7.1) with  $V(x) = x^2/4$ . The same arguments apply to any potential satisfying Assumption 7.1, essentially verbatim. More precisely, we now consider  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  satisfying

(7.19) 
$$d\lambda_i^{(n)}(t) = \sqrt{\frac{2}{\beta}} dB_i(t) + \sum_{\substack{1 \le j \le n \\ j \ne i}} \frac{dt}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)} - \frac{1}{2} \lambda_i^{(n)}(t) dt,$$

with the law of  $\{\lambda_i^{(n)}(t)\}_{i=1}^n$  for every fixed  $t \in \mathbb{R}$  being the Hermite/Gaussian  $\beta$  ensemble. In this setting,  $\Delta_t^{(n)}(w)$  from (7.10) simplifies as

$$\Delta_t^{(n)}(w) = n^{-1/6} (m_{tn^{-1/3}}^{(n)} (2\sqrt{n} + wn^{-1/6}) + \sqrt{n}) - \sqrt{w}.$$

By Itô's formula,  $m_t^{(n)}(z)$  satisfies a Burgers type SDE on  $\mathbb{H}$ :

(7.20) 
$$dm_t^{(n)}(z) = -\sum_{i=1}^n \frac{d\lambda_i^{(n)}(t)}{(\lambda_i^{(n)}(t) - z)^2} + \sum_{i=1}^n \frac{d\langle \lambda_i^{(n)}(t) \rangle}{(\lambda_i^{(n)}(t) - z)^3} \\ = -\sqrt{\frac{2}{\beta}} \sum_{i=1}^n \frac{dB_i(t)}{(\lambda_i^{(n)}(t) - z)^2} + \frac{1}{2} \partial_z ((m_t^{(n)})^2(z) + zm_t^{(n)}(z)) dt + \frac{2-\beta}{2\beta} \partial_z^2 m_t^{(n)}(z) dt$$

where the last line follows from the fact that

$$\begin{split} \sum_{i=1}^{n} \frac{1}{(\lambda_{i}^{(n)}(t)-z)^{2}} \left( \sum_{j:j\neq i} \frac{1}{\lambda_{i}^{(n)}(t)-\lambda_{j}^{(n)}(t)} - \frac{\lambda_{i}^{(n)}(t)}{2} \right) \\ &= \frac{1}{2} \sum_{i\neq j}^{n} \left( \frac{1}{(\lambda_{i}^{(n)}(t)-z)^{2}} \frac{1}{\lambda_{i}^{(n)}(t)-\lambda_{j}^{(n)}(t)} + \frac{1}{(\lambda_{j}(t)-z)^{2}} \frac{1}{\lambda_{j}^{(n)}(t)-\lambda_{i}^{(n)}(t)} \right) - \frac{1}{2} z \,\partial_{z} \,m_{t}(z) - \frac{1}{2} m_{t}(z) \\ &= -\frac{1}{2} \sum_{i\neq j} \frac{\lambda_{i}^{(n)}(t)-z+\lambda_{j}(t)-z}{(\lambda_{i}^{(n)}(t)-z)^{2}(\lambda_{j}^{(n)}(t)-z)^{2}} - \frac{1}{2} z \,\partial_{z} \,m_{t}^{(n)}(z) - \frac{1}{2} m_{t}^{(n)}(z) \\ &= -\sum_{i\neq j} \frac{1}{(\lambda_{i}^{(n)}(t)-z)(\lambda_{j}^{(n)}(t)-z)^{2}} = -\frac{1}{2} \,\partial_{z}((m_{t}^{(n)})^{2}(z) + z m_{t}^{(n)}(z)) + \sum_{i=1}^{n} \frac{1}{(\lambda_{i}^{(n)}(t)-z)^{3}}. \end{split}$$

We can rewrite (7.20) as (7.13), with

$$\mathcal{E}_t^{(n)}(w) = \frac{1}{2n^{1/3}} \partial_w \left( w(\Delta_t^{(n)}(w) + \sqrt{w}) \right), \quad \hat{\mathcal{E}}_t^{(n)}(w, w') = 0.$$

Using Lemma 2.2, we can bound the error term  $\mathcal{E}_t^{(n)}(w)$  and its derivative by

$$\begin{aligned} |\mathcal{E}_t^{(n)}(w)| &\lesssim \frac{|w|\mathrm{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}[w]n^{1/3}} + \frac{|\Delta_t^{(n)}(w) + \sqrt{w}|}{n^{1/3}}, \\ |\partial_w \mathcal{E}_t^{(n)}(w)| &\lesssim \frac{|w|\mathrm{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}[w]^2 n^{1/3}} + \frac{\mathrm{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}[w]n^{1/3}}. \end{aligned}$$

**Laguerre.** By Ito's formula, (7.7) implies that on  $\mathbb{H}$ ,

$$(7.21) \quad \mathrm{d}m_t^{(n)}(z) = -\frac{2}{\sqrt{\beta}} \sum_{i=1}^N \frac{\sqrt{\lambda_i^{(n)}(t)} \mathrm{d}B_i(t)}{(\lambda_i^{(n)}(t) - z)^2} + \partial_z (z(m_t^{(n)})^2(z) + (z - m + n)m_t^{(n)}(z))\mathrm{d}t + \frac{4 - 2\beta}{\beta} \sum_{i=1}^N \frac{\lambda_i^{(n)}(t)\mathrm{d}t}{(\lambda_i^{(n)}(t) - z)^3}.$$

With rescaling, we can rewrite (7.21) as

(7.22) 
$$\chi dm_{\zeta t}^{(n)}(E + \chi w) = -\frac{\sqrt{2}}{\sqrt{\beta}(\sqrt{m} + \sqrt{n})} \sum_{i=1}^{n} \frac{(E + \chi \widetilde{\lambda}_{i}^{(n)}(t))^{1/2} dB_{i}(t)}{(\widetilde{\lambda}_{i}^{(n)}(t) - w)^{2}} + \zeta \, \partial_{w} ((E + \chi w)(m_{\zeta t}^{(n)})^{2}(E + \chi w) + (\chi w + 2\sqrt{n}(\sqrt{m} + \sqrt{n}))m_{\zeta t}^{(n)}(E + \chi w)) dt + \frac{2 - \beta}{\beta E} \sum_{i=1}^{N} \frac{(E + \chi \widetilde{\lambda}_{i}^{(n)}(t))) dt}{(\widetilde{\lambda}_{i}^{(n)}(t) - w)^{3}}.$$

Letting  $M_t^{(n)}(w)$  be the first term in the RHS. Its quadratic variation is then given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle M_t^{(n)}(w), M_t^{(n)}(w') \rangle &= \frac{2}{\beta(\sqrt{m} + \sqrt{n})^2} \sum_{i=1}^n \frac{E + \chi \widetilde{\lambda}_i^{(n)}(t)}{(\widetilde{\lambda}_i^{(n)}(t) - w)^2 (\widetilde{\lambda}_i^{(n)}(t) - w')^2} \\ &= \frac{2}{\beta} \sum_{i=1}^n \frac{1}{(\widetilde{\lambda}_i^{(n)}(t) - w)^2 (\widetilde{\lambda}_i^{(n)}(t) - w')^2} + \hat{\mathcal{E}}_t^{(n)}(w, w'), \end{aligned}$$

with

$$\hat{\mathcal{E}}_{t}^{(n)}(w,w') = \frac{2}{\beta(\sqrt{m} + \sqrt{n})^{2}} \sum_{i=1}^{n} \frac{\chi \widetilde{\lambda}_{i}^{(n)}(t)}{(\widetilde{\lambda}_{i}^{(n)}(t) - w)^{2}(\widetilde{\lambda}_{i}^{(n)}(t) - w')^{2}} \lesssim \frac{\mathrm{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]|w'|}{n^{2/3}\mathrm{Im}[w]\mathrm{Im}[w']^{2}}$$

where we used  $|\tilde{\lambda}_{i}^{(n)}(t) - w'| \geq \operatorname{Im}[w']$  and  $\frac{\operatorname{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]}{\operatorname{Im}[w]} = \sum_{i=1}^{n} \frac{1}{|\tilde{\lambda}_{i}^{(n)}(t) - w|^{2}}$ . This gives the desired bound for  $|\hat{\mathcal{E}}_{t}^{(n)}(w, w')|$ . Similarly we can bound  $|\partial_{w}\partial_{w'}\hat{\mathcal{E}}_{t}^{(n)}(w, w')|$ . We next bound  $\mathcal{E}_{t}^{(n)}(w)$  and its derivative. We can write the second term in the RHS of (7.22) as I + II:

$$\begin{split} \mathbf{I} &= \zeta \,\partial_w \left( \frac{E + \chi w}{\chi^2} (\Delta_t^{(n)}(w) + \sqrt{w})^2 + \left( \frac{\chi \sqrt{n}}{\sqrt{m} + \sqrt{n}} \right)^2 \frac{w}{\chi} - \frac{\chi \sqrt{n}}{\sqrt{m} + \sqrt{n}} w \right) \\ &= \zeta \,\partial_w \left( \frac{E + \chi w}{\chi^2} (\Delta_t^{(n)}(w) + \sqrt{w})^2 \right) - \frac{1}{2} = \frac{1}{2} \,\partial_w (\Delta_t^{(n)}(w) + \sqrt{w})^2 - \frac{1}{2} + \mathcal{E}_t^{(n),\mathrm{I}}(w), \\ \mathrm{II} &= \zeta \,\partial_w \left( \left( -\frac{2\sqrt{n}(E + \chi w)}{\chi(\sqrt{m} + \sqrt{n})} + \frac{2\sqrt{n}(\sqrt{m} + \sqrt{n})}{\chi} + w \right) (\Delta_t^{(n)}(w) + \sqrt{w}) \right) = \mathcal{E}_t^{(n),\mathrm{II}}(w). \end{split}$$

And the third term in the RHS of (7.22) can be written as

$$III = \frac{2-\beta}{\beta E} \sum_{i=1}^{n} \frac{(E+\chi \widetilde{\lambda}_{i}^{(n)}(t))dt}{(\widetilde{\lambda}_{i}^{(n)}(t)-w)^{3}} = \frac{2-\beta}{\beta} \sum_{i=1}^{n} \frac{dt}{(\widetilde{\lambda}_{i}^{(n)}(t)-w)^{3}} + \mathcal{E}_{t}^{(n),III}(w).$$

And  $\mathcal{E}_t^{(n)}(w) = \mathcal{E}_t^{(n),\mathrm{II}}(w) + \mathcal{E}_t^{(n),\mathrm{III}}(w) + \mathcal{E}_t^{(n),\mathrm{III}}(w)$ . We have that

$$\begin{split} \mathcal{E}_{t}^{(n),\mathrm{I}}(w) &= \partial_{w} \left( \frac{\zeta w}{\chi} (\Delta_{t}^{(n)}(w) + \sqrt{w})^{2} \right) \lesssim \frac{|\Delta_{t}^{(n)}(w) + \sqrt{w}|^{2} + |w|| \partial_{w} (\Delta_{t}^{(n)}(w) + \sqrt{w})^{2}|}{n^{1/3}} \\ &\lesssim \frac{1}{n^{1/3}} \left( |\Delta_{t}^{(n)}(w) + \sqrt{w}|^{2} + |w|| \Delta_{t}^{(n)}(w) + \sqrt{w}| \frac{\mathrm{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}[w]} \right), \end{split}$$

$$\begin{split} \mathcal{E}_{t}^{(n),\mathrm{II}}(w) &= \zeta \,\partial_{w} \left( \frac{(\sqrt{m} - \sqrt{n})w}{\sqrt{m} + \sqrt{n}} (\Delta_{t}^{(n)}(w) + \sqrt{w}) \right) \\ &\lesssim \frac{|\Delta_{t}^{(n)}(w) + \sqrt{w}|}{n^{1/3}} + \frac{|w||\,\partial_{w}(\Delta_{t}^{(n)}(w) + \sqrt{w})|}{n^{1/3}} \lesssim \frac{|\Delta_{t}^{(n)}(w) + \sqrt{w}|}{n^{1/3}} + \frac{|w|\mathrm{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]}{n^{1/3}\mathrm{Im}[w]}, \end{split}$$

$$\mathcal{E}_{t}^{(n),\mathrm{III}}(w) = \frac{2-\beta}{\beta E} \sum_{i=1}^{n} \frac{\chi \lambda_{i}^{(n)}(t) \mathrm{d}t}{(\tilde{\lambda}_{i}^{(n)}(t) - w)^{3}} \lesssim \frac{1}{n^{2/3}} \sum_{i=1}^{n} \frac{|\lambda_{i}^{(n)}(t) - w| + |w|}{|\tilde{\lambda}_{i}^{(n)}(t) - w|^{3}} \lesssim \frac{\mathrm{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]|w|}{n^{2/3}\mathrm{Im}[w]^{2}}.$$

These together give the bound for  $|\mathcal{E}_t^{(n)}(w)|$ . Similarly we can get the same bound for  $|\partial_w \mathcal{E}_t^{(n)}(w)|$ . **Jacobi.** By Ito's formula, from (7.9) we get

$$(7.23) \quad \mathrm{d}m_t^{(n)}(z) = -\frac{2}{\sqrt{\beta}} \sum_{i=1}^N \frac{\sqrt{\lambda_i^{(n)}(t)(1-\lambda_i^{(n)}(t))} \mathrm{d}B_i(t)}{(\lambda_i^{(n)}(t)-z)^2} \\ + \partial_z (z(1-z)(m_t^{(n)})^2(z) + (mz-2nz+n-p)m_t^{(n)}(z)) \mathrm{d}t + \frac{4-2\beta}{\beta} \sum_{i=1}^N \frac{\lambda_i^{(n)}(t)(1-\lambda_i^{(n)}(t)) \mathrm{d}t}{(\lambda_i^{(n)}(t)-z)^3}$$

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The remaining arguments follow from computations analogous to the Laguerre case, and we omit the details. 

7.2. **Tightness.** This subsection is for the remaining tasks (2) and (3), i.e., tightness and verifying assumptions. They can be summarized as the following statement

**Proposition 7.4.** For any sequence of integers  $\rightarrow \infty$ , we can take a subsequence  $n_1 < n_2 < \cdots$ , such that as  $k \to \infty$ ,  $\{\Delta_t^{(n_k)}(w) + \sqrt{w}\}_{w \in \mathbb{H}, t \in \mathbb{R}}$  and  $\{\widetilde{\lambda}_i^{(n_k)}(t)\}_{i \in \mathbb{N}, t \in \mathbb{N}}$  converge jointly under the uniform in compact topology. Moreover, the limit of  $\{\Delta_t^{(n_k)}(w) + \sqrt{w}\}_{w \in \mathbb{H}, t \in \mathbb{R}}$  is particle-generated, satisfying Assumption 1.4 and Assumption 1.5, with the limit of  $\{\widetilde{\lambda}_i^{(n_k)}(t)\}_{i\in\mathbb{N}}$  being the poles.

This proposition, together with Theorem 1.6, implies Theorem 7.2.

The proof of this tightness result has two components. First, since all the three processes in Theorem 7.2 are stationary, a single time slice is described by the corresponding  $\beta$ -ensemble. The tightness at one time is established, thanks to the results from [26], which also verify Assumption 1.4. Second, we check the tightness of the Stieltjes transform over a time interval, and verify Assumption 1.5. These are achieved using the SDE (7.13).

7.2.1. Tightness at one time. We now show that for fixed time t,  $\Delta_t^{(n)}(w) + \sqrt{w}$  converges as  $n \to \infty$  to a particle-generated Nevanlinna function along subsequences, and verify Assumption 1.4. We note that along this procedure we also get the tightness of  $\{\widetilde{\lambda}_i^{(n)}(t)\}_{i\in\mathbb{N}}$  at fixed t, which has already been proven to converge to the eigenvalues of the  $\beta$  stochastic Airy operator, in e.g., [73, 96, 112].

**Proposition 7.5.** For any fixed  $t \in \mathbb{R}$ , and any sequence of integers  $\to \infty$ , we can take a subsequence  $n_1 < n_2 < \cdots$ , such that  $\{\Delta_t^{(n_k)}(w) + \sqrt{w}\}_{w \in \mathbb{H}}$  (under the uniform in compact topology) and  $\{\widetilde{\lambda}_{i}^{(n_{k})}(t)\}_{i\in\mathbb{N}}$  converge jointly as  $k\to\infty$ . Besides, the limit of  $\{\Delta_{t}^{(n_{k})}(w)+\sqrt{w}\}_{w\in\mathbb{H}}$  is  $(\mathfrak{d},C_{*})$ -Airy-like, with the limit of  $\{\widetilde{\lambda}_i^{(n_k)}(t)\}_{i\in\mathbb{N}}$  being its poles. Here  $\mathfrak{d}$  is a universal constant, and  $C_*$  is random with law independent of the subsequence or t.

The following lemma states that  $\Delta_t^{(n)}$  satisfies similar statements as being Airy-like.

**Lemma 7.6.** For any  $\varepsilon > 0$ , there exists a large constant C > 0, such that for any  $t \in \mathbb{R}$ , the following holds with probability  $1 - \varepsilon$ :

- the particles are bounded above, i.e., λ<sub>1</sub><sup>(n)</sup>(t) ≤ C;
  for any w = a + ib with |w| ≤ n<sup>1/6</sup> and b ≥ C√a ∨ 0 + 1, it holds

(7.24) 
$$|\Delta_t^{(n)}(w)| \le \frac{C \mathrm{Im}[\sqrt{w}]^{1/2}}{\mathrm{Im}[w]}.$$

Before providing its proof, we derive Proposition 7.5 from it.

Proof of Proposition 7.5. First we show tightness of  $\Delta_t^{(n)}(w) + \sqrt{w}$ . For this, note that  $\Delta_t^{(n)}(w) + \sqrt{w}$ is holomorphic in  $\mathbb{H}$ , so it suffices to show that for any fixed compact subset  $\mathcal{K} \subset \mathbb{H}$ ,  $|\Delta_t^{(n)}(w) + \sqrt{w}|$ is uniformly bounded in  $\mathcal{K}$ .

Take any  $\epsilon > 0$ . Let  $\delta > 0$  small enough depending on  $\epsilon$ , and that  $\mathcal{K} \subset \{w = a + bi : |a| \leq 1$  $1/\delta, \delta \leq b \leq 1/\delta$ . Then (7.24) implies that (with probability  $1-\varepsilon$ ),

(7.25) 
$$|\Delta_t^{(n)}(a+i/\delta) + \sqrt{a+i/\delta}| \le \frac{|a+i/\delta|^{1/4}}{\delta} + |a+i/\delta|^{1/2} < \frac{2}{\delta^{5/4}},$$

for  $|a| \leq 1/\delta$ . Using Lemma 2.2, we have

$$-\partial_w \log |\Delta_t^{(n)}(w) + \sqrt{w}| \le \frac{1}{\operatorname{Im}[w]}.$$

By integrating this expression from  $w = a + i/\delta$  to w = a + bi, we conclude that

(7.26) 
$$|\Delta_t^{(n)}(a+bi) + \sqrt{a+bi}| \le \frac{1}{b\delta} |\Delta_t^{(n)}(a+i/\delta) + \sqrt{a+i/\delta}| \le \frac{2}{b\delta^{9/4}}.$$

This implies that with probability  $1 - \varepsilon$ ,  $|\Delta_t^{(n)}(w) + \sqrt{w}|$  is uniformly bounded in  $\mathcal{K}$ .

Now by taking a subsequence, we have the convergence of  $\Delta_t^{(n_k)}(w) + \sqrt{w}$ . By Lemma 2.7 and Lemma 2.8, the limit is particle-generated. By Lemma 7.6, and taking a further subsequence, the limit is  $(\mathfrak{d}, C_*)$ -Airy-like as asserted, and in particular, has infinitely many poles. Then by Lemma 2.7 and Lemma 2.8 again, we have that for each  $i \in \mathbb{N}$ ,  $\widetilde{\lambda}_i^{(n_k)}(t)$  converges as  $k \to \infty$ , and these give all the poles.

In the rest of this subsection, we derive Lemma 7.6 from an optimal local law for  $\beta$ -ensembles, proved in [26].

**Theorem 7.7.** [26, Corollary 1.6, Proposition 2.5 and Proposition 3.5] We consider  $\beta$ -ensemble  $x_1 \geq \cdots \geq x_n$  whose distribution density is given by (7.2), with potential V(x) satisfying Assumption 7.1. Denote

$$s_n(z) = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i - z}, \quad m_V(z) = \int \frac{\mathrm{d}\mu_V(x)}{x - z}.$$

There exist  $\eta > 0$ , and C > 0 such that for any  $q \ge 1$  and  $n \ge 1$ , the following holds

(1) For any  $0 \le u \le n^{2/3}$ , it holds that

$$\mathbb{P}(\exists k \in [\![1,n]\!], x_k \notin [A - un^{-2/3}, B + un^{-2/3}]) \le Ce^{-u^{3/4}/C}.$$

(2) For any  $z \in \mathbb{H}$ , with  $A - \eta \leq \operatorname{Re}[z] \leq B + \eta$  and  $0 < \operatorname{Im}[z] \leq \eta$ ,

$$\mathbb{E}[|s_n(z) - m_V(z)|^q] \le \frac{(Cq)^{2q}}{(n \operatorname{Im}[z])^q}.$$

For 
$$z \in \mathbb{H}$$
 with  $\kappa = |\operatorname{Re}[z] - A| \wedge |\operatorname{Re}[z] - B|, \ 0 < \operatorname{Im}[z] \le \eta, \ and \ Cq^{1/2}/(n\sqrt{\kappa}) \le \operatorname{Im}[z] \le \kappa,$   
$$\mathbb{E}[|s_n(z) - m_V(z)|^{2q}] \le \frac{(Cq)^{2q}}{(n\operatorname{Im}[z])^{4q}\kappa^q} + \frac{(Cq)^q}{n^{2q}(\kappa\operatorname{Im}[z])^q} + \frac{(Cq)^{2q}}{n^{2q}\kappa^q}$$

*Remark* 7.8. According to the proofs in [26] (see [26, Section 2.1 and Remark 2.4]), for Theorem 7.7 to hold, Assumption 7.1 on the potential V(x) can be relaxed to the following statements:

- V(x) is analytic in  $\Omega$ , which is a simply connected open subset of  $\mathbb{C}$ , containing an interval [A, B].
- There is a unique  $\mu_V$  given by (7.3), where the minimization is taken over all probability measures on  $\mathbb{R} \cap \Omega$ . Moreover,  $\mu_V$  has density  $\rho_V$  whose support is [A, B], and has square root singularities at A and B (with coefficients  $R_A/\pi > 0$  and  $R/\pi = R_B/\pi > 0$ ).
- (7.5) and (7.6) hold, and the function r(z) there is analytic and nonzero in  $\Omega$ .

Besides, Theorem 7.7 remains valid if  $V = V_n$  depends on n, provided that the above conditions are satisfied quantitatively and uniformly for each  $V_n$ . Specifically, there exists a constant C > 0(independent of n) such that

•  $|A|, |B| < C, |B - A| > C^{-1};$ 

- for any z ∈ Ω, C<sup>-1</sup> < |r(z)| < C (note that this implies uniform bounds of R<sub>A</sub> and R<sub>B</sub>);
  Ω (which may depend on n) contains the C<sup>-1</sup> neighborhood of [A, B];
- the constants in  $\mathcal{O}$  in (7.5) and (7.6) are < C.

Under these assumptions, the original proof in [26] applies without modification, ensuring that Theorem 7.7 remains valid.

With these extensions, we can now address both the Laguerre and Jacobi cases. More precisely, in the Laguerre case, (upon a rescaling of particles to constant order) we have

$$V(x) = V_n(x) = -\frac{m-n+1-2/\beta}{n}\log(x) + x,$$

and

$$A = (\sqrt{(m+1-2/\beta)/n} - 1)^2, \quad B = (\sqrt{(m+1-2/\beta)/n} + 1)^2,$$
$$\varrho_V(x) = \varrho_{V_n}(x) = \frac{\sqrt{4(m+1-2/\beta)n - (xn - (m+n+1-2/\beta))^2}}{2\pi xn}$$

See e.g., [50]. Note that this  $\rho_V$  is approximately a rescaling of  $\rho_{\rm mp}^{(n)}$  from (7.11). Moreover, we have

$$r(z) = \frac{2m_V(z) + V'(z)}{2\sqrt{(z-A)(z-B)}} = \frac{1}{2z}.$$

In the Jacobi case, we have

$$V(x) = V_n(x) = -\frac{p - n + 1 - 2/\beta}{n}\log(x) - \frac{q - n + 1 - 2/\beta}{n}\log(1 - x),$$

and

$$A = \left(\frac{\sqrt{(p+1-2/\beta)(m-n+2-4/\beta)} - \sqrt{(q+1-2/\beta)n}}{m+2-4/\beta}\right)^2$$
$$B = \left(\frac{\sqrt{(p+1-2/\beta)(m-n+2-4/\beta)} - \sqrt{(q+1-2/\beta)n}}{m+2-4/\beta}\right)^2$$
$$\varrho_V(x) = \varrho_{V_n}(x) = \frac{(m+2-4/\beta)\sqrt{(x-A)(B-x)}}{2\pi x(1-x)n}.$$

See e.g., [51]. Note that this  $\rho_V$  is approximately a rescaling of  $\rho_{Ja}^{(n)}$  from 7.12. Moreover, we have

$$r(z) = \frac{2m_V(z) + V'(z)}{2\sqrt{(z-A)(z-B)}} = \frac{(m+2-4/\beta)}{2z(1-z)n}.$$

In both cases, it is evident that the above conditions are satisfied, under the limiting scheme specified in Theorem 7.2.

Theorem 7.7 can be translated into the following estimates of  $\widetilde{\lambda}_1^{(n)}(t)$  and  $\Delta_t^{(n)}(w)$ .

**Lemma 7.9.** There exists C > 0 such that (for any  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ ) the following holds:

(1) For any  $0 < u = \mathcal{O}(n^{2/3})$ , it holds that

(7.27) 
$$\mathbb{P}(\widetilde{\lambda}_1^{(n)}(t) > u) \le Ce^{-u^{3/4}/C}.$$

(2) For any  $q \ge 1$  and w = a + ib with  $|w| = \mathcal{O}(n^{2/3})$ ,

(7.28) 
$$\mathbb{E}[|\Delta_t^{(n)}(w) + \mathcal{O}(n^{-1/3}|w|)|^q] \le \frac{(Cq)^{2q}}{b^q},$$

and when  $Cq^{1/2}/\sqrt{a} \leq b \leq a$ ,

(7.29) 
$$\mathbb{E}[|\Delta_t^{(n)}(w) + \mathcal{O}(n^{-1/3}|w|)|^{2q}] \le \frac{(Cq)^{2q}}{b^{4q}a^q} + \frac{(Cq)^q}{(ab)^q} + \frac{(Cq)^{2q}}{n^{2q/3}a^q}$$

*Proof.* We prove the statement for the DBM with general potential, and the other two cases can be proven in the same way (since Theorem 7.7 also applies, as discussed in Remark 7.8).

For the stationary DBM of (7.1), the rescaled particles  $(\lambda_1^{(n)}(t)/\sqrt{n}, \lambda_2^{(n)}(t)/\sqrt{n}, \cdots, \lambda_n^{(n)}(t)/\sqrt{n})$ follow the  $\beta$ -ensemble (7.2) with potential V(x). The first statement (7.27) follows the first statement of Theorem 7.7. We recall from (7.10) that

(7.30) 
$$\Delta_t^{(n)}(w) = \chi \left( m_{\zeta t}^{(n)}(E + \chi w) - \sqrt{n} m_V(B + n^{-1/2} \chi w) \right) + \mathcal{O}(n^{-1/3} |w|).$$

Note that  $m_{\zeta t}^{(n)}(E+\chi w)$  has the same law as  $\sqrt{n}s_n((E+\chi w)/\sqrt{n})$  in Theorem 7.7. Then (7.28) and (7.29) follow from the second statement of Theorem 7.7 and (7.30). 

Proof of Lemma 7.6. The estimate on  $\widetilde{\lambda}_1^{(n)}(t)$  follows from (7.27). In the following we prove (7.24). We denote  $w = z^2$ ,  $z = \kappa + i\eta$  with  $\kappa \ge 0$ . Then  $a = \kappa^2 - \eta^2$ and  $b = 2\kappa\eta$ . Take K > 0 large enough depending on  $\varepsilon$ . Then (7.24) follows from the following two statements:

- (1) Denote  $\mathcal{D}_1 = \{\kappa + i\eta : \eta \geq K, K/\eta \leq \kappa \leq 10\eta\}$ . It holds with probability  $1 \varepsilon/2$  that  $|\Delta_t^{(n)}(w)| \le C/(\kappa\sqrt{\eta})$  uniformly for  $\sqrt{w} \in \mathcal{D}_1, |w| \le n^{1/6}$ .
- (2) Denote  $\mathcal{D}_2 = \{\kappa + i\eta : \eta \ge K, \kappa \ge 10\eta\}$ . It holds with probability  $1 \varepsilon/2$  that  $|\Delta_t^{(n)}(w)| \le \varepsilon$  $C/(\kappa_{\sqrt{\eta}})$  uniformly for  $\sqrt{w} \in \mathcal{D}_2, |w| \leq n^{1/6}$ .

We first prove (1). For  $\kappa < 10\eta$ , by taking  $q = \sqrt{u}/(Ce)$  in (7.28), Markov's inequality implies

(7.31) 
$$\mathbb{P}\left(|\Delta_t^{(n)}(w)| \ge \frac{u}{\kappa\eta} + \frac{C|w|}{n^{1/3}}\right) \le e^{-2\sqrt{u}/(Ce)}, \quad u \ge 1.$$

By taking  $u = \sqrt{\eta}/4$  in (7.31), and noticing that for  $|w| \le n^{1/6}$ ,  $C|w|/n^{1/3} \le 1/(2\kappa\sqrt{\eta})$ ,

(7.32) 
$$\mathbb{P}\left(|\Delta_t^{(n)}(w)| \ge \frac{1}{\kappa\sqrt{\eta}}\right) \le e^{-\eta^{1/4}/(Ce)}$$

We take a lattice  $\mathcal{L}_1 = \{(\kappa + i\eta \in \mathcal{D}_1 : \eta = j^{1/3}, j \in \mathbb{N}, \kappa \in \mathbb{N}/\eta^2\}$ , which is a discretization of  $\mathcal{D}_1$ . By an union bound using (7.32), we have that  $|\Delta_t^{(n)}(w)| \leq 1/(\kappa\sqrt{\eta})$  for each  $w \in \mathcal{L}_1, |w| \leq n^{1/6}$ , except for an event with probability

$$\sum_{i=\lceil K^3\rceil}^{\infty} 10je^{-j^{1/12}/(Ce)} \le \varepsilon/2.$$

By our construction of the lattice  $\mathcal{L}_1$ , for any  $\kappa + i\eta \in \mathcal{D}_1$ , there exists some  $\kappa' + i\eta' \in \mathcal{L}_1$  such that  $|(\kappa + i\eta) - (\kappa' + i\eta')| \lesssim \eta^{-2}$ . By Lemma 2.2 we have

(7.33) 
$$\left|\partial_w(\Delta_t^{(n)}(w) + \sqrt{w})\right| \le \frac{\operatorname{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\operatorname{Im}[w]}$$

Using (7.33), via an argument similar to the proof of Proposition 3.3, the estimates for points in the lattice  $\mathcal{L}_1$  extend to all points in the domain  $\mathcal{D}_1$ , implying (1).

Now consider (2). For  $\kappa/10 \ge \eta \ge K, \kappa^2 \eta \ge Cq^{1/2}$ , and  $|w| \le n^{1/6}$ , (7.29) simplifies to

(7.34) 
$$\mathbb{E}[|\Delta_t^{(n)}(w) + \mathcal{O}(|w|n^{-1/3})|^{2q}] \le \frac{(Cq)^{2q}}{(\kappa\eta)^{2q}\kappa^{2q}}.$$

By taking  $q = \kappa \sqrt{\eta}/(2Ce)$ , and using that  $C|w|/n^{1/3} \le 1/(2\kappa\sqrt{\eta})$  for  $|w| \le n^{1/6}$ , Markov's inequality implies

(7.35) 
$$\mathbb{P}\left(|\Delta_t^{(n)}(w)| \ge \frac{1}{\kappa\sqrt{\eta}}\right) \le e^{-2q} = e^{-\kappa\sqrt{\eta}/(Ce)}$$

Then (2) follows by first taking an union bound of (7.35) over the lattice  $\{(\kappa + i\eta \in \mathcal{D}_2 : \kappa = j^{1/3}, j \in \mathbb{N}, \eta \in \mathbb{N}/\kappa^2\}$ , and (as before) using (7.33) to extend the estimates for all points in  $\mathcal{D}_2$ .  $\Box$ 

7.2.2. *Tightness as time dependent processes.* The claim of Proposition 7.4 follows from the following tightness statements.

**Lemma 7.10.** For any T > 0 and compact  $\mathcal{K} \subset \mathbb{H}$ , both  $\{M_t^{(n)}(w) - M_{-T}^{(n)}(w)\}_{t \in [-T,T], w \in \mathcal{K}}$  and  $\{\Delta_t^{(n)}(w)\}_{t \in [-T,T], w \in \mathcal{K}}$  are tight as  $n \to \infty$ .

Assuming this, we now prove the (subsequential) convergence announced at the beginning of this subsection.

Proof of Proposition 7.4. Using Lemma 7.10, and the Skorokhod representation theorem, we can take a subsequence,  $n_1 < n_2 < \cdots$ , such that almost surely, as  $k \to \infty$ ,  $\{\Delta_t^{(n_k)}(w) + \sqrt{w}\}_{t \in \mathbb{R}, w \in \mathbb{H}}$  converges (uniformly in compact sets) to a random process  $\{Y_t(w)\}_{t \in \mathbb{R}, w \in \mathbb{H}}$ , and  $\{M_t^{(n_k)}(w)\}_{t \in \mathbb{R}, w \in \mathbb{H}}$  also converges to a random process  $\{M_t(w)\}_{t \in \mathbb{R}, w \in \mathbb{H}}$ . Both convergences are under the uniform in compact topology.

Moreover, using Proposition 7.5, and by passing to a further subsequence, we can assume that for each rational  $t, Y_t$  is  $(\mathfrak{d}, C_{*,t})$ -Airy-like, where  $\mathfrak{d}$  is a universal constant, and  $\{C_{*,t}\}_{t\in\mathbb{Q}}$  is a tight family of random variables; and  $\{\widetilde{\lambda}_i^{(n_k)}(t)\}_{i\in\mathbb{N}}$  converges almost surely to  $\{\lambda_i(t)\}_{i\in\mathbb{N}}$ , which are the poles of  $Y_t$ . (Note that here we cannot derive the same for all  $t \in \mathbb{R}$ , as Lemma 7.6 is not uniform in t.)

From the continuity of  $Y_t$  in t, we can deduce that  $Y_t$  is particle-generated from each  $t \in \mathbb{R}$ , by Lemma 2.7 and Lemma 2.8.

Verify limiting SDE. From its construction we see that  $Y_t$  satisfies Assumption 1.4. Next, we check that Assumption 1.5 is also satisfied, i.e., we verify (1.6), (1.7), and (1.8).

We can deduce the following uniform in compact convergence:

 $\begin{array}{l} \bullet \ \partial_w^3(\Delta_t^{(n_k)}(w) + \sqrt{w}) \to \partial_w^3 \, Y_t(w). \\ \bullet \ \partial_w \, \partial_{w'} \left( \frac{(\Delta_t^{(n_k)}(w) + \sqrt{w}) - (\Delta_t^{(n_k)}(w') + \sqrt{w'})}{w - w'} \right) \to \partial_w \, \partial_{w'} \left( \frac{Y_t(w) - Y_t(w')}{w - w'} \right), \text{ with the space of } (w, w') \\ \text{ being } \{(w, w') \in \mathbb{H}^2, w \neq w'\}. \end{array}$ 

Both of these convergences follow from the uniform in compact convergence of  $\Delta_t^{(n_k)}(w) + \sqrt{w}$ , and using Cauchy's integral formula (and taking contour integrals around w and w') to compute the derivatives via taking a contour integral around w. Then from (7.17) and (7.18), and that  $|\hat{\mathcal{E}}_t^{(n_k)}(w,w')| \to 0$  by Proposition 7.3, we get (1.7) and (1.8). Now in (7.13), from the uniform in compact convergence of  $\Delta_t^{(n_k)}(w) + \sqrt{w}$ , we can deduce the following uniform in compact convergence:

•  $\partial_w^2(\Delta_t^{(n)}(w) + \sqrt{w})$  and  $\partial_w(\Delta_t^{(n)}(w) + \sqrt{w})^2$  converge to  $\partial_w^2 Y_t(w)$  and  $\partial_w Y_t(w)^2$ , respectively. This again follows by using Cauchy's integral formula, to compute the derivatives via taking a contour integral around w.

Then since  $|\mathcal{E}_t^{(n_k)}(w)| \to 0$  by Proposition 7.3, we get (1.6).

**Uniform convergence.** Now by Corollary 3.2, the poles of  $\{Y_t\}_{t\in\mathbb{R}}$  give a line ensemble, which we denote by  $\{\lambda_i(t)\}_{i\in\mathbb{N},t\in\mathbb{R}}$ . We note that this is consistent with the previously defined  $\{\lambda_i(t)\}_{i\in\mathbb{N}}$  for rational t.

Take T > 0. We next show that almost surely, for each  $i \in \mathbb{N}$ ,  $\lambda_i^{(n_k)}(t) \to \lambda_i(t)$  as  $k \to \infty$ , uniformly in [-T, T].

First, from Corollary 3.2, there exists a random and large enough number  $X > \max_{t \in [-T,T]} \lambda_1(t) + 1$ , and  $\max_{t \in [-T,T]} \operatorname{Im}[Y_t(X+i)] < 0.01$ . Then by the uniform in compact convergence of  $\{\Delta_t^{(n_k)}(w) + \sqrt{w}\}_{t \in \mathbb{R}, w \in \mathbb{H}}$ , for k large enough, we have  $\max_{t \in [-T,T]} \operatorname{Im}[\Delta_t^{(n_k)}(X+i) + \sqrt{X+i}] < 0.02$ , therefore  $X \notin \{\lambda_i^{(n_k)}(t)\}_{i \in \mathbb{N}}$  for any  $t \in [-T,T]$ . On the other hand, by the convergence of  $\lambda_1^{(n_k)}(0) \to \lambda_i(0)$ , for k large enough, we have  $\lambda_1^{(n_k)}(0) < X$ . Thus,  $\max_{t \in [-T,T]} \lambda_1^{(n_k)}(t) < X$  whenever k is large enough.

Suppose that for some  $i \in \mathbb{N}$ ,  $\lambda_i^{(n_k)}(t)$  does not uniformly converge to  $\lambda_i(t)$  in [-T, T] (as  $k \to \infty$ ). Then there is a sequence  $t_1, t_2, \cdots$ ,  $\lim_{k\to\infty} t_k = t_0 \in [-T, T]$ , such that as  $k \to \infty$ ,  $\lambda_i^{(n_k)}(t_k)$  does not converge to  $\lambda_i(t_0)$ .

However, by the uniform (in t and w) convergence of  $\Delta_t^{(n)}(w) + \sqrt{w}$  to  $Y_t(w)$ , we have that  $\lim_{k\to\infty} \Delta_t^{(n_k)}(w) + \sqrt{w} = Y_{t_0}(w)$ , uniformly for w in any compact subset of  $\mathbb{H}$ . With the upper bound of  $\lambda_1^{(n_k)}(t_k) < X$ , by Lemma 2.7 and Lemma 2.8 we have that  $\lim_{k\to\infty} \lambda_i^{(n_k)}(t_k) = \lambda_i(t_0)$  for each  $i \in \mathbb{N}$ , thereby arriving at a contradiction.

*Proof of Lemma 7.10.* Take p to be a large constant. In this proof, all the constants in  $\leq$  and  $\mathcal{O}(\cdot)$  can depend on  $\mathcal{K}$ , T, and p.

We start with the martingale terms. From (7.14), we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle M_t^{(n)}(w)\rangle\right| \le \frac{2}{\beta} \sum_{i=1}^n \frac{1}{|\tilde{\lambda}_i^{(n)}(t) - w|^4} + |\hat{\mathcal{E}}_t^{(n)}(w, w)| \le \frac{2}{\beta} \cdot \frac{\mathrm{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}^3[w]} + |\hat{\mathcal{E}}_t^{(n)}(w, w)|,$$

and

$$\begin{split} & \left| \frac{\mathrm{d}}{\mathrm{d}t} \langle M_t^{(n)}(w) - M_t^{(n)}(w') \rangle \right| \\ & \leq \frac{2}{\beta} \sum_{i=1}^n \frac{|w - w'|^2 |\tilde{\lambda}_i^{(n)}(t) - w + \tilde{\lambda}_i^{(n)}(t) - w'|^2}{|\tilde{\lambda}_i^{(n)}(t) - w|^4 |\tilde{\lambda}_i^{(n)}(t) - w'|^4} + |\hat{\mathcal{E}}_t^{(n)}(w, w) - 2\hat{\mathcal{E}}_t^{(n)}(w, w') + \hat{\mathcal{E}}_t^{(n)}(w', w')| \\ & \leq \frac{2}{\beta} |w - w'|^2 \left( \frac{\mathrm{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\mathrm{Im}^3[w]\mathrm{Im}^2[w']} + \frac{\mathrm{Im}[\Delta_t^{(n)}(w') + \sqrt{w'}]}{\mathrm{Im}^2[w]\mathrm{Im}^3[w']} \right) \\ & + |\hat{\mathcal{E}}_t^{(n)}(w, w) - 2\hat{\mathcal{E}}_t^{(n)}(w, w') + \hat{\mathcal{E}}_t^{(n)}(w', w')|, \end{split}$$

for  $w \neq w'$ . Using (7.15) to bound the error terms in the Laguerre and Jacobi cases, we have that the above two are  $\mathcal{O}(1 + |\Delta_t^{(n)}(w) + \sqrt{w}|)$  and  $\mathcal{O}(|w - w'|^2(1 + |\Delta_t^{(n)}(w) + \sqrt{w}|))$ , respectively. Then

thanks to Burkholder-Davis-Gundy inequality, for any  $w, w' \in \mathcal{K}$ , and  $-T \leq t \leq t' \leq T$ ,

(7.36) 
$$\mathbb{E}\left[\left|M_{t'}^{(n)}(w) - M_{t}^{(n)}(w)\right|^{p}\right] \lesssim \mathbb{E}\left[\left|\int_{t}^{t'} \mathrm{d}\langle M_{s}^{(n)}(w)\rangle\right|^{p/2}\right] \lesssim |t'-t|^{p/2},$$

and

$$(7.37) \quad \mathbb{E}\left[\left| (M_t^{(n)}(w) - M_{-T}^{(n)}(w)) - (M_t^{(n)}(w') - M_{-T}^{(n)}(w')) \right|^p \right] \\ \lesssim \mathbb{E}\left[ \left| \int_{-T}^t \mathrm{d}\langle M_s^{(n)}(w) \rangle - \mathrm{d}\langle M_s^{(n)}(w') \rangle \right|^{p/2} \right] \lesssim |w - w'|^p,$$

where in the last inequalities we used the above bounds and (7.28).

The tightness of  $\{M_t^{(n)}(w) - M_{-T}^{(n)}(w)\}_{t \in [-T,T], w \in \mathcal{K}}$  then follows from these estimates with p > 4 (see e.g., [117, Lemma 5.9], which can be proved by bounding the modulus of continuity via a union bound across scale, and using e.g., [19, Theorem 7.3]).

We now turn to  $\Delta_t^{(n)}(w)$ . From the one time tightness in Proposition 7.5, it suffices to derive the tightness of  $\{\Delta_t^{(n)}(w) - \Delta_{-T}^{(n)}(w)\}_{t \in [-T,T], w \in \mathcal{K}}$ . From (7.13), and using Lemma 2.2, for any  $w \in \mathcal{K}$  and  $-T \leq t \leq t' \leq T$  we have

$$\Delta_{t'}^{(n)}(w) - \Delta_{t}^{(n)}(w) = M_{t'}^{(n)}(w) - M_{t}^{(n)}(w) + \int_{t}^{t'} \mathcal{O}\left(1 + |\mathcal{E}_{s}^{(n)}(w)| + \frac{\operatorname{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]}{\operatorname{Im}[w]^{2}} + \frac{\operatorname{Im}[\Delta_{t}^{(n)}(w) + \sqrt{w}]|\Delta_{t}^{(n)}(w) + \sqrt{w}|}{\operatorname{Im}[w]}\right) \mathrm{d}s.$$

Then by (7.16), (7.28), and (7.36), we have

$$\mathbb{E}\left[\left|\Delta_{t'}^{(n)}(w) - \Delta_{t}^{(n)}(w)\right|^{p}\right] \lesssim |t' - t|^{p/2}$$

Besides, for any  $w, w' \in \mathcal{K}$ , for each r = 0, 1, 2 we have

$$\begin{aligned} |\partial_w^r(\Delta_t^{(n)}(w) + \sqrt{w}) - \partial_w^r(\Delta_t^{(n)}(w') + \sqrt{w'})| &\leq (r+1)! |w - w'| \sum_{i=1}^n \frac{|\widetilde{\lambda}_t^{(n)} - w|^r + |\widetilde{\lambda}_t^{(n)} - w'|^r}{|\widetilde{\lambda}_t^{(n)} - w|^{r+1} |\widetilde{\lambda}_t^{(n)} - w'|^{r+1}} \\ &\leq (r+1)! |w - w'| \left( \frac{\operatorname{Im}[\Delta_t^{(n)}(w) + \sqrt{w}]}{\operatorname{Im}[w]^{r+1}} + \frac{\operatorname{Im}[\Delta_t^{(n)}(w') + \sqrt{w'}]}{\operatorname{Im}[w']^{r+1}} \right). \end{aligned}$$

Then from (7.13), using this and (7.16), (7.28), (7.37), we get

$$\mathbb{E}\left[\left|\left(\Delta_t^{(n)}(w) - \Delta_{-T}^{(n)}(w)\right) - \left(\Delta_t^{(n)}(w') - \Delta_{-T}^{(n)}(w')\right)\right|^p\right] \lesssim |w - w'|^p.$$

Thereby the tightness of  $\{\Delta_t^{(n)}(w) - \Delta_{-T}^{(n)}(w)\}_{t \in [-T,T], w \in \mathcal{K}}$  follows, from these moments bounds with p > 4.

# APPENDIX A. PARTICLE LOCATION AND STIELTJES TRANSFORM

In this appendix we analyze particle-generated Nevanlinna functions. The main objective is to quantitatively establish the relation between the Airy-like property (from Definition 1.2) and the locations of the poles. In particular, we will prove Lemma 2.3 in this appendix.

We note that most of the arguments presented here are classical in random matrix theory. Our starting point is the following relation between any Nevanlinna function and its corresponding measure. This relation can be viewed as a version of the Helffer-Sjöstrand formula (see e.g., [55, Section 11.2]).

A.1. Helffer-Sjöstrand formula. Take any compactly supported smooth test functions f and  $\chi$  on  $\mathbb{R}$ , such that  $\chi = 1$  in a neighborhood of 0. Define  $\tilde{f}(x + iy) := (f(x) + iyf'(x))\chi(y)$ . Then for any  $\lambda \in \mathbb{R}$ , we have (for  $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ )

$$f(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial_{\bar{z}} \, \tilde{f}(x + \mathrm{i}y)}{\lambda - x - \mathrm{i}y} \mathrm{d}x \mathrm{d}y,$$

which further leads to the following statement.

**Lemma A.1.** Take any Nevanlinna function  $Y : \mathbb{H} \to \mathbb{H} \cup \mathbb{R}$ , with Nevanlinna representation

$$Y(w) = b + cw + \int \left(\frac{1}{\lambda - w} - \frac{\lambda}{1 + \lambda^2}\right) d\mu(\lambda),$$

for some  $b, c \in \mathbb{R}$ ,  $c \ge 0$ , and  $\mu$  being a Borel measure on  $\mathbb{R}$ . Then there is

$$\int f(\lambda) \mathrm{d}\mu(\lambda) = \frac{2}{\pi} \int_{x+\mathrm{i}y \in \mathbb{H}} \operatorname{Re}\left[\partial_{\bar{z}} \tilde{f}(x+\mathrm{i}y)Y(x+\mathrm{i}y)\right] \mathrm{d}x \mathrm{d}y.$$

These identities are classical and follow directly from standard complex analysis arguments; see e.g., [55, Section 11.2]. We omit the proofs here.

A.2. **Proof of Lemma 2.3.** Our strategy is to extract the particle locations from Y, using Lemma A.1.

We denote  $\sigma = \delta/6$ , and take any s > 0, and a smooth test function  $f = f^{(s)} : \mathbb{R} \to \mathbb{R}_{\geq 0}$ , satisfying

• f = 1 on  $[-s^{2/3}, K]$ ; • f = 0 on  $\mathbb{R} \setminus [-s^{-\sigma} - s^{2/3}, 2K]$ ; •  $|f'| \lesssim s^{\sigma}$  and  $|f''| \lesssim s^{2\sigma}$  on  $[-s^{-\sigma} - s^{2/3}, s^{2/3}]$ ; •  $|f'| \lesssim 1$  and  $|f''| \lesssim 1$  on [K, 2K].

We next prove

(A.1) 
$$\left|\sum_{i=1}^{\infty} f(x_i) - \frac{1}{\pi} \int_{\mathbb{R}_+} f(-x) \sqrt{x} \mathrm{d}x\right| \lesssim K^4 s^{1/3-\sigma}.$$

for any  $s > 10K^3$ . Then by the assumption that  $x_1 \leq K$ , and (2.5), the estimate Lemma 2.3 follows.

Take another smooth test function  $\chi = \chi^{(s)} : \mathbb{R} \to \mathbb{R}_{\geq 0}$ , with  $\chi = 1$  on  $[-s^{2/3}, s^{2/3}]$ ,  $\chi = 0$  on  $\mathbb{R} \setminus [-s^{2/3} - 1, s^{2/3} + 1]$ , and  $|\chi'| \leq 1$  on  $\mathbb{R}$ . Let  $\tilde{f}(x + iy) = (f(x) + iyf'(x))\chi(y)$  for any  $x, y \in \mathbb{R}$ . Then by Lemma A.1, we have

(A.2)  

$$\sum_{i=1}^{\infty} f(x_i(t)) - \frac{1}{\pi} \int_{\mathbb{R}_+} f(-x)\sqrt{x} dx = \frac{2}{\pi} \int_{x+iy \in \mathbb{H}} \operatorname{Re}\left[\partial_{\bar{z}} \tilde{f}(x+iy)(Y_t(x+iy) - \sqrt{x+iy})\right] dxdy.$$

Denote  $\mathcal{E}_t(x + iy) := Y_t(x + iy) - \sqrt{x + iy}$ . The RHS of (A.2) decomposes into

(A.3) 
$$-\frac{1}{\pi} \int_{x+\mathrm{i}y \in \mathbb{H}} (f''(x)y\chi(y)\mathrm{Im}[\mathcal{E}_t(x+\mathrm{i}y)] + \mathrm{Im}\left[(f(x)+\mathrm{i}yf'(x))\chi'(y)\mathcal{E}_t(x+\mathrm{i}y)\right]) \,\mathrm{d}x\mathrm{d}y.$$

The first term. For the term  $f''(x)y\chi(y)\text{Im}[\mathcal{E}_t(x+iy)]$ , we note that f'' is non-zero only on the two intervals  $[-s^{-\sigma}-s^{2/3},-s^{2/3}]$  and [K,2K].

For the first interval, we break the integral in y into two parts:  $0 < y < 4K^2s^{-\sigma}$ , and  $y \ge 4K^2s^{-\sigma}$ . Using that  $y \operatorname{Im}[Y_t(x+iy)]$  is increasing in y (from Lemma 2.2), we have that for any  $x \in [-s^{-\sigma} - s^{2/3}, -s^{2/3}]$  and  $0 < y < 4K^2s^{-\sigma}$ ,

$$y \operatorname{Im}[\mathcal{E}(x + \mathrm{i}y)] < y \operatorname{Im}[Y(x + \mathrm{i}y)] \le 4K^2 s^{-\sigma} \operatorname{Im}[Y(x + 4\mathrm{i}K^2 s^{-\sigma})],$$

which, by the second condition of Y, is further bounded by (using that  $\sigma \leq \delta/3$  and  $s > K^3$ )

$$4K^2s^{-\sigma}\sqrt{|x+4iK^2s^{-\sigma}|} + |x+4iK^2s^{-\sigma}|^{(1-\delta)/2} \lesssim K^2s^{1/3-\sigma}$$

It follows that (using  $|f''| \lesssim s^{2\sigma}$ )

$$\frac{1}{\pi} \int_0^{4K^2 s^{-\sigma}} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} |f''(x)| y\chi(y) \operatorname{Im}[\mathcal{E}_t(x + \mathrm{i}y)] \mathrm{d}x \mathrm{d}y \lesssim K^4 s^{1/3 - \sigma}.$$

For integrating over  $y \ge 4K^2 s^{-\sigma}$ , we perform an integration by parts in x, and get

$$\begin{aligned} &\frac{1}{\pi} \int_{4K^2 s^{-\sigma}}^{\infty} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} f''(x) y \chi(y) \operatorname{Im}[\mathcal{E}(x + \mathrm{i}y)] \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{\pi} \int_{4K^2 s^{-\sigma}}^{\infty} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} f'(x) y \chi(y) \operatorname{Im}[\partial_x \mathcal{E}(x + \mathrm{i}y)] \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{\pi} \int_{4K^2 s^{-\sigma}}^{\infty} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} f'(x) y \chi(y) \operatorname{Re}[\partial_y \mathcal{E}(x + \mathrm{i}y)] \mathrm{d}x \mathrm{d}y. \end{aligned}$$

Via another integration by parts in y, this equals

$$\frac{1}{\pi} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} f'(x) y\chi(y) \operatorname{Re}[\mathcal{E}(x + \mathrm{i}y)] \mathrm{d}x \bigg|_{y=4K^2 s^{-\sigma}} \\ -\frac{1}{\pi} \int_{4K^2 s^{-\sigma}}^{\infty} \int_{-s^{-\sigma} - s^{2/3}}^{-s^{2/3}} f'(x) \,\partial_y(y\chi(y)) \operatorname{Re}[\mathcal{E}(x + \mathrm{i}y)] \mathrm{d}x \mathrm{d}y$$

The first line above is bounded (using the second condition of Y) by

$$\frac{1}{\pi} |s^{-\sigma} + s^{2/3} + 4iK^2 s^{-\sigma}|^{(1-\delta)/2} \lesssim s^{(1-\delta)/3};$$

and as for the second line, its absolute value is bounded by

$$\frac{1}{\pi} \int_{4K^2 s^{-\sigma}}^{\infty} |\partial_y(y\chi(y))| \frac{|s^{-\sigma} + s^{2/3} + \mathrm{i}y|^{(1-\delta)/2}}{y} \mathrm{d}y \lesssim s^{(1-\delta)/3} \int_{4K^2 s^{-\sigma}}^{\infty} \frac{|\partial_y(y\chi(y))|}{y} \mathrm{d}y \lesssim s^{1/3-\sigma},$$

where the last inequality holds whenever  $\sigma \leq \delta/6$ .

For the integral of  $f''(x)y\chi(y)\operatorname{Im}[\mathcal{E}_t(x+\mathrm{i}y)]$  for  $x \in [K, 2K]$ , we similarly break the integral in y into two parts:  $0 < y < 4K^2$  and  $y \ge 4K^2$ . Using that  $y\operatorname{Im}[Y_t(x+\mathrm{i}y)]$  is increasing in y (from Lemma 2.2), we have that for any  $x \in [K, 2K]$  and  $0 < y < 4K^2$ ,

$$y \operatorname{Im}[\mathcal{E}(x + \mathrm{i}y)] < y \operatorname{Im}[Y(x + \mathrm{i}y)] \le 4K^2 \operatorname{Im}[Y(x + 4\mathrm{i}K^2)] \lesssim K^3$$

where we used the second condition of Y for the last inequality. Therefore (using  $|f''| \leq 1$  on [K, 2K]) we have

$$\frac{1}{\pi} \int_0^{4K^2} \int_K^{2K} |f''(x)| y\chi(y) \operatorname{Im}[\mathcal{E}_t(x+\mathrm{i}y)] \mathrm{d}x \mathrm{d}y \lesssim K^6.$$

For integrating over  $y \ge 4K^2$ , using a similar integration by parts procedure, we get

$$\frac{1}{\pi} \int_{K}^{2K} f'(x) y \chi(y) \operatorname{Re}[\mathcal{E}(x+\mathrm{i}y)] \mathrm{d}x \bigg|_{y=4K^2}$$
$$-\frac{1}{\pi} \int_{4K^2}^{\infty} \int_{K}^{2K} f'(x) \,\partial_y(y\chi(y)) \operatorname{Re}[\mathcal{E}(x+\mathrm{i}y)] \mathrm{d}x \mathrm{d}y.$$

Using the second condition of Y, the first line above is  $\leq K$ , and the absolute value of the second line is bounded by (using again  $\sigma \leq \delta/3$ )

$$\frac{1}{\pi} \int_{4K^2}^{\infty} |\partial_y(y\chi(y))| \frac{|2K + iy|^{(1-\delta)/2}}{y} dy \lesssim s^{(1-\delta)/3} < s^{1/3-\sigma}.$$

The second term. We now consider the term  $\text{Im}\left[(f(x) + iyf'(x))\chi'(y)\mathcal{E}_t(x + iy)\right]$  in (A.3). Note that  $\chi'(y) \neq 0$  only for  $y \in [s^{2/3}, s^{2/3} + 1]$ . For such y and  $x \in [-s^{-\sigma} - s^{2/3}, -s^{2/3}] \cup [K, 2K]$ , we have  $|\mathcal{E}(x + iy)| \lesssim s^{-1/3 - \delta/3}$ , by the second condition of Y. Therefore,

(A.4) 
$$\frac{1}{\pi} \int_{x+\mathrm{i}y \in \mathbb{H}} |\mathrm{Im}\left[ (f(x) + \mathrm{i}y f'(x))\chi'(y)\mathcal{E}_t(x+\mathrm{i}y) \right] |\,\mathrm{d}x\mathrm{d}y$$
$$e^{s^{2/3}+1} e^{-s^{2/3}+1} e^{-s^{2/3$$

(A.5) 
$$\lesssim s^{-1/3-\delta/3} \int_{s^{2/3}}^{s^{2/3}+1} \int (1+s^{2/3}|f'(x)|)|\chi'(y)| \mathrm{d}x\mathrm{d}y \lesssim s^{1/3-\delta/3} \le s^{1/3-\sigma},$$

using  $\sigma \leq \delta/3$ .

In summary, we have proved (A.1) by putting together the above estimates. Thus the conclusion follows.  $\hfill \square$ 

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