ABSTRACT. Let (M, g, f) be a 5-dimensional complete noncompact gradient shrinking Ricci soliton with the equation  $Ric+\nabla^2 f =$  $\lambda g$ , where Ric is the Ricci tensor and  $\nabla^2 f$  is the Hessian of the potential function f. We prove that it is a finite quotient of  $\mathbb{R}^2 \times \mathbb{S}^3$ if M has constant scalar curvature  $R = 3\lambda$ .

**RIGIDITY OF FIVE-DIMENSIONAL SHRINKING** GRADIENT RICCI SOLITONS WITH CONSTANT

SCALAR CURVATURE

#### 1. INTRODUCTION

Let (M, g) be an *n*-dimensional complete gradient Ricci soliton with the potential function f satisfying

(1.1) 
$$\operatorname{Ric} + \nabla^2 f = \lambda g$$

for some constant  $\lambda$ , where Ric is the Ricci tensor of q and  $\nabla^2 f$  denotes the Hessian of the potential function f. The Ricci soliton is said to be shrinking, steady, or expanding accordingly as  $\lambda$  is positive, zero, or negative, respectively.

A gradient Ricci soliton is a self-similar solution to the Ricci flow which flows by diffeomorphism and homothety. The study of solitons has become increasingly important in both the study of the Ricci flow introduced by Hamilton [24] and metric measure theory. Solitons play a direct role as singularity dilations in the Ricci flow proof of uniformization. In [34], Perelman introduced the ancient  $\kappa$ -solutions, which play an important role in the singularity analysis, and he also proved that suitable blow down limit of ancient  $\kappa$ -solutions must be a shrinking gradient Ricci soliton. In [35], Perelman proved that any two dimensional non-flat ancient  $\kappa$ -solution must be the standard  $\mathbb{S}^2$ , and he also classified three dimensional shrinking gradient Ricci soliton under the assumption of nonnegative curvature and  $\kappa$ -noncollapseness. Due to the work of Perelman [35], Ni-Wallach [33], Cao-Chen-Zhu [9], the classification of three dimensional shrinking gradient Ricci soliton is

<sup>2020</sup> Mathematics Subject Classification. 53C21; 53C44.

Key words and phrases. Ricci soliton, Constant scalar curvature, Weighted Laplacian.

complete. For more work on the classification of gradient Ricci soliton under various curvature condition, see [3, 4, 7, 11, 9, 8, 12, 17, 26, 29, 31, 32, 36, 38, 42, 41, 43].

In general, it is hard to understand the geometry and topology of Ricci soliton in high dimensions, even in dimension four. Cao [6] and Koiso [25] independently constructed a gradient Kähler Ricci soliton on  $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$  which has U(2) symmetry and Ric > 0. Wang-Zhu [40] found a gradient Kähler Ricci soliton on  $\mathbb{CP}^2 \# (-2\mathbb{CP}^2)$  which has  $U(1) \times U(1)$  symmetry. Feldman-Ilmanen-Knopf [21] constructed the first noncompact U(2) invariant shrinking Kähler Ricci soliton on the tautological line bundle of  $\mathbb{CP}^1$  (we call it *FIK* solution in the following) whose Ricci curvature changes sign. Recently, Bamler-Cifarelli-Conlon-Deruelle [1] proved the existence of a unique complete shrinking gradient Kähler-Ricci soliton with bounded scalar curvature on the blowup of  $\mathbb{C} \times \mathbb{CP}^1$  at one point, called BCCD soliton. Li-Wang ([28]) proved that any Kähler Ricci shrinker surface has bounded sectional curvature. Combining the work by Conlon-Deruelle-Sun [14] and Cifarelli-Conlon-Deruelle [20], they provided a complete classification of all Kähler Ricci shrinker surfaces.

In this paper, we focus our attention on 5-dimensional gradient shrinking Ricci solitons with constant scalar curvature. Recall that in Petersen and Wylie's paper [36], a gradient Ricci soliton (M, g) is said to be rigid if it is isometric to a quotient  $N \times \mathbb{R}^k$ , the product soliton of an Einstein manifold N of positive scalar curvature with the Gaussian soliton  $\mathbb{R}^k$ . Conversely, for the complete shrinking case, Prof. Huai-Dong Cao raised the following

**Conjecture**: Let  $(M^n, g, f)$ ,  $n \ge 4$ , be a complete *n*-dimensional gradient shrinking Ricci soliton. If (M, g) has constant scalar curvature, then it must be rigid, i.e., a finite quotient of  $\mathbb{N}^k \times \mathbb{R}^{n-k}$  for some Einstein manifold  $\mathbb{N}$  of positive scalar curvature.

Petersen and Wylie [36] proved that a complete gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat, that is, the sectional curvature  $K(\cdot, \nabla f) = 0$ . Particularly, Petersen and Wylie [36] also showed that the scalar curvature R of a gradient Ricci soliton is 0 or  $n\lambda$ , if and only if the underlying Riemannian structure is Einstein. Subsequently, Fernández-López and García-Río [23] proved that a complete gradient Ricci soliton is rigid iff it has constant scalar curvature and the Ricci curvature has constant rank. They also derived the following results for complete *n*-dimensional gradient Ricci solitons (1.1) with constant scalar curvature R: (i) The possible value of R is  $\{0, \lambda, \dots, (n-1)\lambda, n\lambda\}$ . (ii) If R takes the value  $(n-1)\lambda$ , then the soliton must be rigid. (iii) In the shrinking case, there is no any complete gradient shrinking Ricci soliton with  $R = \lambda$ . (iv) Any four-dimensional gradient shrinking Ricci soliton with constant scalar curvature  $R = (n-2)\lambda$  has non-negative Ricci curvature.

Very recently, Cheng and Zhou [19] confirmed Cao's conjecture in dimension n = 4, together with works of Petersen-Wylie [36] and Fernández-López and García-Río [23]. Precisely, for 4-dimensional complete shrinking gradient Ricci solitons with constant scalar curvature. By the results stated above, their scalar curvature  $R \in \{0, 2\lambda, 3\lambda, 4\lambda\}$ . Moreover, if R = 0 or  $4\lambda$ , they are Einstein [36]; If  $R = 3\lambda$ , they are finite quotient of  $\mathbb{S}^3 \times \mathbb{R}$  [23]. If  $R = 2\lambda$ , then they are isometric to a finite quotient of  $\mathbb{S}^2 \times \mathbb{R}^2$  [19].

For 4-dimensional complete shrinking gradient Ricci solitons with constant scalar curvature, Cheng and Zhou [19] handled the most subtle case  $R = (n - 2)\lambda = 2\lambda$ . In this case, the Ricci curvature is non-negative by the work of Fernández-López and García-Río [23]. In addition, the Riemannian curvature is bounded since Munteanu-Wang [30] proved 4-dimensional complete shrinking gradient Ricci solitons with bounded scalar curvature has bounded Riemannian curvature. However, for *n*-dimensional  $(n \ge 5)$ , we cannot even guarantee the non-negativity of Ricci curvature and the boundedness of Riemannian curvature. Moreovre, we point out that even if the *n*-dimensional  $(n \ge 5)$  shrinking gradient Ricci soliton has non-negative Ricci curvature and bounded Riemannian curvature, Cao's **Conjecture** is still open.

Now we focus on 5-dimensional complete shrinking gradient Ricci solitons with constant scalar curvature. By the results in [23] and [37] stated above, their scalar curvature  $R \in \{0, 2\lambda, 3\lambda, 4\lambda, 5\lambda\}$ . Moreover, there is no any complete gradient shrinking Ricci soliton with  $R = \lambda$ ; if R = 0 or  $5\lambda$ , they are Einstein; if  $R = 4\lambda$ , they are finite quotient of  $\mathbb{S}^4 \times \mathbb{R}$ . Therefore,  $R = 2\lambda$  and  $3\lambda$  are unknown cases.

In this paper, motivated by the 4-dimension works ([19, 36, 23]), we study 5-dimensional complete gradient Ricci solitons with constant scalar curvature  $3\lambda$ . Our main theorem is as follows.

**Theorem 1.1.** Suppose  $(M^5, g, f)$  is a five dimensional shrinking gradient Ricci soliton with  $R = 3\lambda$ , then it is isometric to a finite quotient of  $\mathbb{R}^2 \times \mathbb{S}^3$ .

Next, we discuss some methods to address this problem. Note that, as mentioned before, a complete gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat ([36]). Fern-

ández-López and García-Río [23] proved the radial flatness is equivalent to the constant rank of Ricci curvature. Based on this, it is only necessary to prove that the Ricci curvature has constant rank for Cao's Conjecture.

Recently, Cheng-Zhou [19] proved that 4-dimensional gradient shrinking Ricci soliton with constant scalar curvature  $2\lambda$  is rigid. They applied the weighted Laplacian  $\Delta_f$  to the quantity tr(Ric<sup>3</sup>), the trace of the tensor Ric<sup>3</sup>, for 4-dimensional gradient shrinking Ricci soliton with constant scalar curvature  $2\lambda$  and then derived the following nice inequality

$$\Delta_f \left[ f(\operatorname{tr}(\operatorname{Ric}^3) - \frac{1}{4}) \right] \ge 9f \left[ \operatorname{tr}(\operatorname{Ric}^3) - \frac{1}{4} \right].$$

Using integration by parts, they concluded that  $tr(Ric^3) - \frac{1}{4} = 0$  over M, implying that the Ricci curvature has rank 2, and thus they obtained the rigidity result.

We would like to point out that  $\frac{1}{3} \left[ \text{tr}(\text{Ric}^3) - \frac{1}{4} \right] = \sigma_3(Ric)$ , (see equation (3.11) in [19], since 0 is a Ricci-eigenvalue of gradient shrinking Ricci soliton with constant scalar curvature. When the gradient shrinking Ricci soliton has nonnegative sectional curvature, Guan-Lu-Xu [22] used a combination of  $\sigma_k(Ric)$  to prove that Ricci curvature has constant rank. Naber ([32]) showed that any 4-dimensional nonflat complete noncompact gradient shrinking Ricci soliton with the bounded non-negative curvature operator is also rigid.

Our new proof is inspired by the work of Petensen-Wylie [37]. If the sectional is curvature is nonnegative, it is easy to observe that  $Rm * Ric \geq 0$ , and they applied  $\Delta_f$  directly to the sum of the smallest k Ricci-eigenvalues, obtaining:

$$\Delta_f(\lambda_1 + \lambda_2 + \dots + \lambda_k) \le (\lambda_1 + \lambda_2 + \dots + \lambda_k)$$

in the barrier sense. Then they derived the desired result by the standard maximum principle.

In the case of Cheng-Zhou [19], suppose a 4-dimensional gradient shrinking Ricci soliton has constant scalar curvature  $2\lambda$ , inspired by Pertersen-Wylie [36], Ou-Qu-Wu [39] applied  $\Delta_f$  directly to the sum of the smallest 2 Ricci-eigenvalues and showed it is isometric to a finite quotient of  $\mathbb{S}^2 \times \mathbb{R}^2$ .

As noted in Cheng-Zhou's paper [19], in the proof of the nice inequality, the curvature tensor of M is related with the curvature tensors of the level sets of f by the Gauss equations. One crucial fact is that the level sets of the potential function f is three-dimensional, and their curvature tensor can be expressed by its Ricci tensor, as their Weyl curvature is identically zero. Again, this crucial fact also plays an important role in Ou-Qu-Wu's calculations of  $\Delta_f(\lambda_1 + \lambda_2)$ . However, difficulties arise in the higher dimensional case. Indeed, as the dimension increases, it is necessary to handle the terms involving the Weyl curvature of the level set and the situation becomes more subtle.

The paper is organized as follows. In Section 2, we recall the notations and basic formulas on gradient shrinking Ricci solitons with constant scalar curvature. In Section 3, we will apply  $\Delta_f$  directly to the sum of the smallest 2 eigenvalues, denoted  $\lambda_1$  and  $\lambda_2$  and then derive the estimate of  $\Delta_f(\lambda_1 + \lambda_2)$ , which involves the Weyl curvature of the level set as mentioned before, see Proposition 3.7. In Section 4, we prove a key estimate of  $|\nabla Ric|^2$ . Thanks to the four-dimensional Gauss-Bonnet-Chern formula on the level set. the key estimate controls the Weyl curvature effectively, In Section 5, based the point-picking argument, we prove the Riemannian curvature is bounded. By similar argument, we can also prove that  $\lambda_1 + \lambda_2 \rightarrow 0$  and  $\nabla_{\nabla f} Ric$  also tends to zero at infinity. In Section 6, we prove Theorem 1.1.

# 2. Notations and basic formulas on gradient shrinking Ricci solitons

In this section, we recall the notations and basic formulas on gradient shrinking Ricci solitons with constant scalar curvature. For details, we refer to [6, 24, 36, 19].

Let (M, g) be an *n*-dimensional complete gradient shrinking Ricci soliton satisfying (1.1). By scaling the metric g, one can normalize  $\lambda$  so that  $\lambda = \frac{1}{2}$ . In this paper, we always assume  $\lambda = \frac{1}{2}$  and the gradient shrinking Ricci soliton equation is as follows,

(2.2) 
$$\operatorname{Ric} + \nabla^2 f = \frac{1}{2}g$$

At first we recall some basic formulas which will be used throughout the paper:

(2.3) 
$$dR = 2Ric(\nabla f, \cdot),$$

(2.4) 
$$R + \Delta f = \frac{n}{2},$$

$$(2.5) R + |\nabla f|^2 = f,$$

(2.6) 
$$\Delta_f R = R - 2|Ric|^2,$$

(2.7) 
$$\Delta_f R_{ij} = R_{ij} - 2R_{ikjl}R_{kl}.$$

where  $\Delta_f = \Delta - \langle \nabla f, \nabla \rangle$  is the weighted Laplacian, and  $\Delta_f$  acting on the function is self-adjoint on the space of square integrable functions with respect to the weighted measure  $e^{-f}dv$ . In general, the weighted Laplacian  $\Delta_f$  acting on tensors is given by  $\Delta_f = \Delta - \nabla_{\nabla f}$ .

Next we state the key estimate of potential function f in Cao-Zhou [10].

**Theorem 2.1** ([10]). Suppose  $(M^n, g, f)$  is an noncompact shrinking gradient Ricci soliton, then there exist  $C_1$  and  $C_2$  such that

$$\left(\frac{1}{2}d(x,p) - C_1\right)^2 \le f(x) \le \left(\frac{1}{2}d(x,p) + C_2\right)^2$$

where p is the minimal point of f, which always exists.

**Remark**. Chen [2] proved that any shrinking gradient Ricci soliton has  $R \ge 0$ . Hence, from  $R + |\nabla f|^2 = f$ , we derive that  $|\nabla f|(x) \le \frac{1}{2}d(x,p) + C_2$ .

Now we consider complete gradient shrinking Ricci solitons with constant scalar curvature R. In this case, the potential function f is isoparametric and the isoparametric property of plays a very important role. concretely, the potential function f can be renormalized, by replacing f - R with f, so that  $f: M \to [0, +\infty)$ ,

$$(2.8) \qquad |\nabla f|^2 = f$$

which implies that f is transnormal. Recall (2.4)

(2.9) 
$$\Delta f = \frac{n}{2} - R.$$

Therefore the (nonconstant) renormalized f is an isoparametric function on M. From the potential function estimate (2.8), f is proper and unbounded. By the theory of isoparametric functions, Cheng-Zhou ([19]) derived the following results.

**Theorem 2.2** ([19]). Let (M, g, f) be an n-dimensional complete noncompact gradient shrinking Ricci soliton satisfying (2.2) with constant scalar curvature R and let f be normalized as

$$|\nabla f|^2 = f.$$

Then the following results hold.

(i)  $M_{-} = f^{-1}(0)$  is a compact and connected minimal submanifold of M.

(ii) The function f can be expressed as

$$f(x) = \frac{1}{4}dist^2(x, M_-).$$

(iii) For any point  $p \in M_-$ ,  $\nabla^2 f$  has two eigenspaces  $T_p M_-$  and  $\nu_p M_-$  corresponding eigenvalues 0 and  $\frac{1}{2}$ , and dim $(M_-) = 2R$ .

(iv) Let  $D_a := \{x \in M : f(x) \le a\}$ , for t > 0. Then mean curvature H(a) of the smooth hypersurface  $\Sigma(a) = \partial D_a$  satisfies

$$H(a) = \frac{(n-2R-1)}{\sqrt{a}}.$$

(v) The volume of the set  $D_a$  satisfies

$$Vol(D_a) = \frac{2^k}{k} |M_-|\omega_k a^{\frac{k}{2}}, Vol(\Sigma_a) = 2^{k-1} |M_-|\omega_k a^{\frac{k-1}{2}},$$

where k = n - 2R,  $|M_{-}|$  denotes the volume of the submanifold  $M_{-}$ , and  $\omega_{k-1}$  is the area of the unit sphere in  $\mathbb{R}^{k}$ .

Finally, we recall Naber's splitting theorem as follows.

**Theorem 2.3** ([32]). For any n-dimensional shrinking gradient Ricci soliton  $(M^n, g, f)$  with bounded curvature and a sequence of points  $x_i \in M$  going to infinity along an integral curve of  $\nabla f$ , by choosing a subsequence if necessary,  $(M, g, x_i)$  converges smoothly to a product manifold  $\mathbb{R} \times \mathbb{N}^{n-1}$ , where  $\mathbb{N}$  is a shrinking gradient Ricci soliton.

# 3. Estimate on the sum of the smallest two Ricci-eigenvalues of weighted Laplacian operator

In this section, let (M, g, f) be a 5-dimensional complete noncompact gradient shrinking Ricci soliton satisfying (2.2) with constant scalar curvature  $R = \frac{3}{2}$ . We will apply  $\Delta_f$  directly to the sum of the smallest 2 Ricci-eigenvalues, denoted  $\lambda_1$  and  $\lambda_2$ , and derive the estimate of  $\Delta_f(\lambda_1+\lambda_2)$  involving the Weyl curvature of the level set, see Propositon 3.7.

First, we recall the non-negativity of Ricci curvature for 5-dimensional gradient shrinking Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ , also see [23].

**Proposition 3.1** ([23]). Let (M, g, f) be a 5-dimensional complete noncompact gradient shrinking Ricci soliton satisfying (2.2) with constant scalar curvature  $R = \frac{3}{2}$ . Then it has nonnegative Ricci curvature, the smallest Ricci-eigenvalue  $\lambda_1 = 0$  and  $\operatorname{Ric}(\nabla f, \cdot) = 0$ .

*Proof.* Let (M, g, f) be an *n*-dimensional complete noncompact gradient shrinking Ricci soliton satisfying (2.2) with constant scalar curvature  $R = \frac{n-2}{2}$ , the authors proved that the Ricci curvature is non-negative In [23], Thus for n = 5 and  $R = \frac{3}{2}$ , it has nonnegative Ricci curvature.

(2.3) implies  $Ric(\nabla f, \cdot) = 0$  since the scalar curvature is constant. Hence  $\lambda_1 = 0$  is the smallest Ricci-eigenvalue corresponding to the Ricci-eigenvector  $\nabla f$ .

From Proposition 3.1, throughout this paper we alway denote the eigenvalues of Ricci curvature by

$$0 = \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5.$$

In the following, we calculate  $\Delta_f(\lambda_1 + \lambda_2)$  in the barrier sense, and have the following lemma.

**Lemma 3.2.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature. Then

(3.10) 
$$\Delta_f(\lambda_1 + \lambda_2) \le (\lambda_1 + \lambda_2) - 2\sum_{\alpha=2}^5 K_{1\alpha}\lambda_\alpha - 2\sum_{\alpha=3}^5 K_{2\alpha}\lambda_\alpha$$

in the sense of barrier, where  $K_{ij}$  denotes the sectional curvature of the plane spanned by  $e_i$  and  $e_j$ , and  $e_i$  is the orthonormal eigenvectors corresponding to the Ricci-eigenvectors  $\lambda_i$ .

*Proof.* Actually, at x, because  $R = \frac{3}{2}$ ,  $Ric(\nabla f) = 0$ , so we choose  $e_1 = \frac{\nabla f}{|\nabla f|}$ , then extend  $e_1$  to an orthonormal basis  $\{e_1, e_2, e_3, e_4, e_5\}$  such that  $\{e_i\}_{i=1}^5$  are the eigenvectors of Ric(x) corresponding to eigenvalues  $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ . Take parallel transport of  $\{e_i\}_{i=1}^5$  along all the geodesics from x, then in a neighborhood  $B(x, \delta)$  we get a smooth function  $u(y) = Ric(y)(e_1(y), e_1(y)) + Ric(y)(e_2(y), e_2(y))$  satisfying

$$u(y) \ge \lambda_1(y) + \lambda_2(y) \text{ and } u(x) = \lambda_1(x) + \lambda_2(x). \text{ Thus, at } x,$$
  

$$\Delta_f(\lambda_1 + \lambda_2)$$
  

$$\le (\lambda_1 + \lambda_2) - 2(\sum_{i=1}^5 K_{1i}\lambda_i + \sum_{i=1}^5 K_{2i}\lambda_i)$$
  

$$= (\lambda_1 + \lambda_2) - 2\sum_{\alpha=2}^5 K_{1\alpha}\lambda_\alpha - 2\sum_{\alpha=3}^5 K_{2\alpha}\lambda_\alpha;$$

this completes the proof.

Next, we deal with the term  $\sum_{\alpha=2}^{5} K_{1\alpha} \lambda_{\alpha}$ . First we will give the following lemma for preparation.

**Lemma 3.3.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature. Then

(3.11) 
$$K_{1\alpha} = \frac{\nabla_{\nabla f} R_{\alpha\alpha} + \lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha})}{f},$$

for  $\alpha = 2, 3, 4, 5$ .

*Proof.* From the Ricci identity, we have

$$-R(\nabla f, e_{\alpha}, \nabla f, e_{\beta})$$

$$= -(\nabla_{\beta} f_{\alpha k} - \nabla_{k} f_{\alpha \beta}) f_{k}$$

$$= (\nabla_{\beta} R_{\alpha k} - \nabla_{k} R_{\alpha \beta}) f_{k}$$

$$= -\nabla_{\nabla f} R_{\alpha \beta} + \nabla_{\beta} (R_{\alpha k} f_{k}) - R_{\alpha k} f_{k \beta}$$

$$= -\nabla_{\nabla f} R_{\alpha \beta} - R_{\alpha k} \left(\frac{1}{2}g_{k \beta} - R_{k \beta}\right)$$

$$= -\nabla_{\nabla f} R_{\alpha \beta} - \left(\frac{1}{2}R_{\alpha \beta} - \sum_{k=1}^{5}R_{\alpha k} R_{k \beta}\right)$$

where (2.3) was used in the third equality. Therefore, we see

(3.12) 
$$R(e_1, e_\alpha, e_1, e_\beta) = \frac{\nabla_{\nabla f} R_{\alpha\beta} + \left(\frac{1}{2} R_{\alpha\beta} - \sum_{k=1}^5 R_{\alpha k} R_{k\beta}\right)}{f}$$

due to  $|\nabla f|^2 = f$ . (3.11) holds by setting  $\beta = \alpha$  in (3.12). This completes the proof of the lemma.

,

**Lemma 3.4.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $\frac{3}{2}$ . Then we have

$$-\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} = -\frac{1}{f}\sum_{\alpha=2}^{5}\lambda_{\alpha}^{2}(\frac{1}{2}-\lambda_{\alpha}) = \frac{1}{f}\sum_{\alpha=2}^{5}(\lambda_{\alpha}-\frac{1}{2})^{2}\lambda_{\alpha}.$$

*Proof.* First, since the scalar curvature is constant, (2.6) implies

$$|Ric|^2 = \frac{R}{2} = \frac{3}{4}.$$

This means  $\sum_{\alpha=2}^{5} \lambda_{\alpha} = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = \frac{3}{2}$  and  $\sum_{\alpha=2}^{5} \lambda_{\alpha}^2 = \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 = \frac{3}{4}$ . Hence

$$\sum_{\alpha=2}^{5} \lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha}) = 0.$$

Recall (3.11), and we obtain

$$-\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}$$

$$= -\frac{1}{f}\sum_{\alpha=2}^{5} \left[\nabla f \cdot \nabla \lambda_{\alpha} + \lambda_{\alpha}(\frac{1}{2} - \lambda_{\alpha})\right]\lambda_{\alpha}$$

$$= -\frac{1}{f}\left[\frac{1}{2}\nabla f \cdot \sum_{\alpha=2}^{5}\lambda_{\alpha}^{2} + \sum_{\alpha=2}^{5}\lambda_{\alpha}^{2}(\frac{1}{2} - \lambda_{\alpha})\right]$$

$$= -\frac{1}{f}\sum_{\alpha=2}^{5}\lambda_{\alpha}^{2}(\frac{1}{2} - \lambda_{\alpha})$$

$$= -\frac{1}{f}\sum_{\alpha=2}^{5}\left[-\lambda_{\alpha}(\frac{1}{2} - \lambda_{\alpha})^{2} + \frac{1}{2}\lambda_{\alpha}(\frac{1}{2} - \lambda_{\alpha})\right]$$

$$= \frac{1}{f}\sum_{\alpha=2}^{5}\lambda_{\alpha}(\frac{1}{2} - \lambda_{\alpha})^{2}.$$

We have completed the proof of this proposition.

Subsequently, we consider the level set  $\Sigma$  of the potential function f to handle the term  $-2\sum_{\alpha=3}^{5} K_{2\alpha}\lambda_{\alpha}$ . For this purpose, recall that the intrinsic curvature tensor  $R_{\alpha\beta\gamma\eta}^{\Sigma}$  and the extrinsic curvature tensor  $R_{\alpha\beta\gamma\eta}$  of  $\Sigma$  where  $\{\alpha, \beta, \gamma, \eta\} \in \{2, 3, 4, 5\}$ , are related by the Gauss equation:

$$R^{\Sigma}_{\alpha\beta\gamma\eta} = R_{\alpha\beta\gamma\eta} + h_{\alpha\gamma}h_{\beta\eta} - h_{\alpha\eta}h_{\beta\gamma},$$

where  $h_{\alpha\beta}$  denotes the components of the second fundamental form A of  $\Sigma$ . Moreover,

Claim

(3.13) 
$$R^{\Sigma} = R = \frac{3}{2};$$

(3.14) 
$$Ric^{\Sigma} = Ric + \frac{\nabla_{\nabla f}Ric}{f};$$

(3.15) 
$$|Ric^{\Sigma}|^{2} = |Ric|^{2} + \frac{|\nabla_{\nabla f}Ric|^{2}}{f^{2}}$$

(3.16)

$$K_{\alpha\beta} = \frac{1}{2} (\lambda_{\alpha} + \lambda_{\beta}) - \frac{1}{4} - \frac{1}{2} (K_{1\alpha} + K_{1\beta}) + \frac{1}{4f} [1 - (\lambda_{\alpha} + \lambda_{\beta})] - \frac{1}{2f} \left[ (\frac{1}{2} - \lambda_{\alpha})^2 + (\frac{1}{2} - \lambda_{\beta})^2 \right] - \frac{1}{f} (\frac{1}{2} - \lambda_{\alpha}) (\frac{1}{2} - \lambda_{\beta}) + W_{\alpha\beta}^{\Sigma},$$

where  $W_{\alpha\beta}^{\Sigma} = W_{\alpha\beta\alpha\beta}^{\Sigma}$  denotes the components of the Weyl curvature of  $\Sigma$ .

In fact, it follows from the Gauss equation that

$$R_{\alpha\beta}^{\Sigma} = R_{\alpha\beta} - R_{1\alpha1\beta} + Hh_{\alpha\beta} - h_{\alpha\gamma}h_{\gamma\beta}$$

and the scalar curvature  $R^{\Sigma}$  of  $\Sigma$  satisfies

$$R^{\Sigma} = R - 2R_{11} + H^2 - |A|^2.$$

Since  $R = \frac{3}{2}$ ,  $Ric(\nabla f, \cdot) = 0$ ,  $R_{1i} = 0$ , i = 1, ..., 5, then

$$R^{\Sigma} = R + H^2 - |A|^2.$$

Noting

$$h_{\alpha\beta} = \frac{f_{\alpha\beta}}{|\nabla f|} = \frac{\frac{1}{2} - \lambda_{\alpha}}{\sqrt{f}} \delta_{\alpha\beta},$$

then the mean curvature satisfies

$$H = \frac{\frac{4}{2} - \sum \lambda_{\alpha}}{\sqrt{f}} = \frac{1}{2\sqrt{f}}$$

and

$$|A|^{2} = \frac{1}{f} \sum \left(\frac{1}{2} - \lambda_{\alpha}\right)^{2} = \frac{1}{f} (1 - \sum \lambda_{\alpha} + \sum \lambda_{\alpha}^{2})$$
$$= \frac{1}{f} (1 - \frac{3}{2} + \frac{3}{4}) = \frac{1}{4f}.$$

Hence,  $R^{\Sigma} = R = \frac{3}{2}$ . Together with (3.12), we see

$$\begin{split} R_{\alpha\beta}^{\Sigma} = & R_{\alpha\beta} - \frac{\nabla_{\nabla f} R_{\alpha\beta} + \left(\frac{1}{2} R_{\alpha\beta} - R_{\alpha k} R_{k\beta}\right)}{f} + \frac{\frac{1}{2} - \lambda_{\alpha}}{2f} \delta_{\alpha\beta} \\ & - \frac{\left(\frac{1}{2} - \lambda_{\alpha}\right)\left(\frac{1}{2} - \lambda_{\beta}\right)}{f} \delta_{\alpha\gamma} \delta_{\gamma\beta} \\ = & R_{\alpha\beta} - \frac{\nabla_{\nabla f} R_{\alpha\beta}}{f} + \frac{1}{f} [-\lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha}) + \frac{1}{2} (\frac{1}{2} - \lambda_{\alpha}) - (\frac{1}{2} - \lambda_{\alpha})^2] \delta_{\alpha\beta} \\ = & R_{\alpha\beta} - \frac{\nabla_{\nabla f} R_{\alpha\beta}}{f}, \end{split}$$

which implies (3.14) holds.

By  $|Ric|^2 = \frac{3}{4}$  agian, we have  $Ric \cdot \nabla_{\nabla f} Ric = 0$ .

$$|Ric^{\Sigma}|^{2} = |Ric|^{2} + \frac{|\nabla_{\nabla f}Ric|^{2}}{f^{2}} - 2\frac{Ric \cdot \nabla_{\nabla f}Ric}{f}$$
$$= |Ric|^{2} + \frac{|\nabla_{\nabla f}Ric|^{2}}{f^{2}}.$$

Finally, recall that the relationship between curvature and the Weyl curvature

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2} (g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}) - \frac{1}{(n-1)(n-2)} R(g_{ik}g_{jl} - g_{il}g_{jk}),$$

and we get

$$K_{\alpha\beta}^{\Sigma} = W_{\alpha\beta}^{\Sigma} + \frac{1}{2} (R_{\alpha\alpha}^{\Sigma} + R_{\beta\beta}^{\Sigma}) - \frac{1}{6} R^{\Sigma}$$
  
$$= W_{\alpha\beta}^{\Sigma} + \frac{1}{2} (R_{\alpha\alpha} - K_{1\alpha} + Hh_{\alpha\alpha} - h_{\alpha\alpha}^{2} + R_{\beta\beta} - K_{1\beta} + Hh_{\beta\beta} - h_{\beta\beta}^{2}) - \frac{1}{4}$$
  
$$= W_{\alpha\beta}^{\Sigma} + \frac{1}{2} (\lambda_{\alpha} + \lambda_{\beta} - K_{1\alpha} - K_{1\beta} + H(h_{\alpha\alpha} + h_{\beta\beta}) - h_{\alpha\alpha}^{2} - h_{\beta\beta}^{2}) - \frac{1}{4}$$

on the hypersurface  $\Sigma$ . It follows from the Gauss equation that

$$\begin{split} K_{\alpha\beta} &= K_{\alpha\beta}^{\Sigma} - h_{\alpha\alpha}h_{\beta\beta} + h_{\alpha\beta}^{2} \\ &= \frac{1}{2} \left( \lambda_{\alpha} + \lambda_{\beta} - K_{1\alpha} - K_{1\beta} + H(h_{\alpha\alpha} + h_{\beta\beta}) - h_{\alpha\alpha}^{2} - h_{\beta\beta}^{2} \right) \\ &- \frac{1}{4} - h_{\alpha\alpha}h_{\beta\beta} + W_{\alpha\beta}^{\Sigma} \\ &= \frac{1}{2} (\lambda_{\alpha} + \lambda_{\beta}) - \frac{1}{4} - \frac{1}{2} (K_{1\alpha} + K_{1\beta}) + \frac{1}{4f} [(\frac{1}{2} - \lambda_{\alpha}) + (\frac{1}{2} - \lambda_{\beta})] \\ &- \frac{1}{2f} \left[ (\frac{1}{2} - \lambda_{\alpha})^{2} + (\frac{1}{2} - \lambda_{\beta})^{2} \right] - \frac{1}{f} (\frac{1}{2} - \lambda_{\alpha}) (\frac{1}{2} - \lambda_{\beta}) + W_{\alpha\beta}^{\Sigma} \end{split}$$

We have completed the proof of equations (3.13)-(3.16) in Claim.

Using equations (3.13)-(3.16), we have the following result.

**Lemma 3.5.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $\frac{3}{2}$ . Then

$$-2\sum_{\alpha=3}^{5} K_{2\alpha}\lambda_{\alpha}$$
  
=  $-2(\lambda_1 + \lambda_2)[1 - (\lambda_1 + \lambda_2)] + \frac{3}{2f}\nabla f \cdot \nabla(\lambda_1 + \lambda_2)$   
 $-\frac{1}{f}\nabla f \cdot \nabla(\lambda_1 + \lambda_2)^2 + \sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} + \frac{1}{f}\sum_{\alpha=3}^{5}(\frac{1}{2} - \lambda_{\alpha})^2\lambda_{\alpha}$   
 $-2\sum_{\alpha=3}^{5} W_{2\alpha}^{\Sigma}\lambda_{\alpha}.$ 

*Proof.* It follows from (3.16) that

$$-2\sum_{\alpha=3}^{5} K_{2\alpha}\lambda_{\alpha}$$

$$=-\sum_{\alpha=3}^{5} (\lambda_{2} + \lambda_{\alpha})\lambda_{\alpha} + \frac{1}{2}\sum_{\alpha=3}^{5} \lambda_{\alpha} + \sum_{\alpha=3}^{5} (K_{12} + K_{1\alpha})\lambda_{\alpha} - 2\sum_{\alpha=3}^{5} W_{2\alpha}^{\Sigma}\lambda_{\alpha}$$

$$+\frac{1}{2f}\sum_{\alpha=3}^{5} \left[ (\frac{1}{2} - \lambda_{2}) + (\frac{1}{2} - \lambda_{\alpha}) \right]\lambda_{\alpha} - \frac{2}{f}(\frac{1}{2} - \lambda_{2})\sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})\lambda_{\alpha}$$

$$+\frac{1}{f} \left[ (\frac{1}{2} - \lambda_{2})^{2}\sum_{\alpha=3}^{5} \lambda_{\alpha} + \sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^{2}\lambda_{\alpha} \right].$$

Next, we handle these items one by one. First, we see

$$-\sum_{\alpha=3}^{5} (\lambda_{2} + \lambda_{\alpha})\lambda_{\alpha} + \frac{1}{2}\sum_{\alpha=3}^{5} \lambda_{\alpha}$$
  
=  $-\lambda_{2}(\frac{3}{2} - \lambda_{2}) - (\frac{3}{4} - \lambda_{2}^{2}) + \frac{1}{2f}(\frac{3}{2} - \lambda_{2})$   
=  $-2\lambda_{2}(1 - \lambda_{2}),$ 

Notice that  $\sum_{\alpha=3}^{5} \lambda_{\alpha} = \frac{3}{2} - \lambda_2$  and  $\sum_{\alpha=3}^{5} \lambda_{\alpha}^2 = \frac{3}{4} - \lambda_2^2$ , it is easy to check

$$\frac{1}{2f} \sum_{\alpha=3}^{5} \left[ \left(\frac{1}{2} - \lambda_2\right) + \left(\frac{1}{2} - \lambda_\alpha\right) \right] \lambda_\alpha$$
$$- \frac{2}{f} \left(\frac{1}{2} - \lambda_2\right) \sum_{\alpha=3}^{5} \left(\frac{1}{2} - \lambda_\alpha\right) \lambda_\alpha + \frac{1}{f} \left(\frac{1}{2} - \lambda_2\right)^2 \sum_{\alpha=3}^{5} \lambda_\alpha$$
$$= -\frac{1}{f} \lambda_2 \left(\frac{1}{2} - \lambda_2\right) \left(\frac{3}{2} - 2\lambda_2\right).$$

It follows from (3.11) that

$$\sum_{\alpha=3}^{5} (K_{12} + K_{1\alpha})\lambda_{\alpha}$$
  
= $K_{12}(\frac{3}{2} - 2\lambda_2) + \sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}$   
= $(\frac{3}{2} - 2\lambda_2)\frac{1}{f} \left[\nabla f \cdot \nabla \lambda_2 + \lambda_2(\frac{1}{2} - \lambda_2)\right] + \sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}$   
= $\frac{3}{2f}\nabla f \cdot \nabla \lambda_2 - \frac{1}{f}\nabla f \cdot \nabla \lambda_2^2 + \lambda_2(\frac{1}{2} - \lambda_2)(\frac{3}{2} - 2\lambda_2) + \sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}$ 

Therefore, we have

$$-2\sum_{\alpha=3}^{5} K_{2\alpha}\lambda_{\alpha}$$
  
=  $-2\lambda_{2}(1-\lambda_{2}) + \frac{3}{2f}\nabla f \cdot \nabla\lambda_{2} - \frac{1}{f}\nabla f \cdot \nabla\lambda_{2}^{2}$   
+  $\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} + \frac{1}{f}\sum_{\alpha=3}^{5}(\frac{1}{2}-\lambda_{\alpha})^{2}\lambda_{\alpha} - 2\sum_{\alpha=3}^{5} W_{2\alpha}^{\Sigma}\lambda_{\alpha}.$ 

The proof of Lemma 3.5 was thus completed.

We have the following two basic inequalities to obtain our desired estimate.

**Lemma 3.6.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $\frac{3}{2}$ . Then

(3.17) 
$$(\frac{1}{2} - \lambda_{\alpha})^2 \le \sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^2 = \lambda_2 (1 - \lambda_2)$$

for  $\alpha = 3, 4, 5$ .

(3.18) 
$$\sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^2 \lambda_{\alpha} \le \lambda_2 (1 - \lambda_2) (\frac{3}{2} - \lambda_2).$$

(3.19) 
$$\sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^2 \lambda_{\alpha}^2 \le \lambda_2 (1 - \lambda_2) (\frac{3}{4} - \lambda_2^2).$$

*Proof.* It follows from (2.6) that

$$0 = \sum_{\alpha=2}^{5} \lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha})$$
  
=  $-\sum_{\alpha=2}^{5} \left[ (\frac{1}{2} - \lambda_{\alpha})^{2} + \frac{1}{2} (\frac{1}{2} - \lambda_{\alpha}) \right]$   
=  $-\sum_{\alpha=2}^{5} (\frac{1}{2} - \lambda_{\alpha})^{2} + \frac{1}{4},$ 

which yeilds that

$$\sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^{2} = \frac{1}{4} - (\frac{1}{2} - \lambda_{2})^{2}$$
$$= \lambda_{2}(1 - \lambda_{2}).$$

Therefore, we have

$$\begin{split} &\sum_{\alpha=3}^{5} (\frac{1}{2} - \lambda_{\alpha})^2 \lambda_{\alpha} \\ &\leq \lambda_2 (1 - \lambda_2) (\lambda_3 + \lambda_4 + \lambda_5) \\ &= \lambda_2 (1 - \lambda_2) (\frac{3}{2} - \lambda_2). \end{split}$$

Similarly, (3.19) holds.

Finally, we have the following proposition.

**Proposition 3.7.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ . We have the following inequality holds in the barrier sense,

$$\begin{split} &\Delta_f(\lambda_1 + \lambda_2) \\ \leq &- (\lambda_1 + \lambda_2) + 2(\lambda_1 + \lambda_2)^2 + \frac{3}{2f} \nabla f \cdot \nabla(\lambda_1 + \lambda_2) \\ &- \frac{1}{f} \nabla f \cdot \nabla(\lambda_1 + \lambda_2)^2 + \frac{2}{f} (\lambda_1 + \lambda_2) [1 - (\lambda_1 + \lambda_2)] \left[ \frac{3}{2} - (\lambda_1 + \lambda_2) \right] \\ &- 2\sum_{\alpha=3}^5 W_{2\alpha}^{\Sigma} \lambda_{\alpha} \end{split}$$

on  $M \setminus D(a)$  for some a > 0.

*Proof.* By inserting equations from Lemma 3.5 and 3.4 into (3.10), we obtain:

$$\begin{split} &\Delta_f(\lambda_1 + \lambda_2) \\ \leq \lambda_2 - 2\sum_{\alpha=2}^5 K_{1\alpha}\lambda_\alpha - 2\sum_{\alpha=3}^5 K_{2\alpha}\lambda_\alpha \\ &= -\lambda_2 + 2\lambda_2^2 + \frac{3}{2f}\nabla f \cdot \nabla\lambda_2 - \frac{1}{f}\nabla f \cdot \nabla\lambda_2^2 \\ &+ \frac{2}{f}\sum_{\alpha=3}^5 (\frac{1}{2} - \lambda_\alpha)^2\lambda_\alpha - 2\sum_{\alpha=3}^5 W_{2\alpha}^{\Sigma}\lambda_\alpha \\ \leq -\lambda_2 + 2\lambda_2^2 + \frac{3}{2f}\nabla f \cdot \nabla\lambda_2 - \frac{1}{f}\nabla f \cdot \nabla\lambda_2^2 \\ &+ \frac{2}{f}\lambda_2(1 - \lambda_2)(\frac{3}{2} - \lambda_2) - 2\sum_{\alpha=3}^5 W_{2\alpha}^{\Sigma}\lambda_\alpha, \end{split}$$

where (3.18) was used in the last inequality, and the proof of this proposition was completed.

**Remark**. In Proposition 3.7, the estimate of  $\Delta_f(\lambda_1 + \lambda_2)$  involves the term related to the Weyl curvature of the level set, which does not appear in the case of 4-dimensional gradient Ricci solitons, thereby making the higher-dimensional case more difficult.

# 4. A key estimate of $|\nabla Ric|^2$

In this section, we will derive a key estimate of  $|\nabla Ric|^2$ , focusing on effective control of the Weyl curvature.

Next, we proceed to calculate  $|\nabla Ric|^2$ .

**Lemma 4.1.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ . Then we have

$$|\nabla Ric|^2 = 2\sum_{\alpha,\beta=2,\alpha\neq\beta}^5 K_{\alpha\beta}\lambda_\alpha\lambda_\beta - \frac{3}{4}$$

*Proof.* Notice that  $|Ric|^2 = \frac{1}{2}R = \frac{3}{4}$  agian, we start with

$$\frac{1}{2}\Delta_f(|Ric|^2) = R_{ij}\Delta_f R_{ij} + |\nabla Ric|^2,$$

which implies

$$\nabla Ric|^{2} = -\sum_{i,j=1}^{5} R_{ij} \Delta_{f} R_{ij}$$
$$= \sum_{i,j=1}^{5} R_{ij} (2R_{ikjl}R_{kl} - R_{ij})$$
$$= 2\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} R_{\alpha\beta\alpha\beta}\lambda_{\alpha}\lambda_{\beta} - |Ric|^{2}$$
$$= 2\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} K_{\alpha\beta}\lambda_{\alpha}\lambda_{\beta} - \frac{3}{4}.$$

This completes the proof.

Subsequently, we deal with the right hand side of the equation in Lemma 4.1, and have the following key estimate of  $|\nabla Ric|^2$ .

**Proposition 4.2.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ . There exists a constant C > 0, such that

$$|\nabla Ric|^2 \leq C(\lambda_1 + \lambda_2) + K_{12} + C|W^{\Sigma(s)}|^2$$

on  $M \setminus D(a)$  for some a > 0 and  $s \ge a$ .

*Proof.* It follows from (3.16) that

$$\begin{split} |\nabla Ric|^2 &= \sum_{\alpha,\beta=2,\alpha\neq\beta}^5 2K_{\alpha\beta}\lambda_\alpha\lambda_\beta - \frac{3}{4} \\ &= \left(\sum_{\alpha,\beta=2,\alpha\neq\beta}^5 (\lambda_\alpha + \lambda_\beta) - \frac{1}{2}\right)\lambda_\alpha\lambda_\beta - \frac{3}{4} \\ &- \sum_{\alpha,\beta=2,\alpha\neq\beta}^5 (K_{1\alpha} + K_{1\beta})\lambda_\alpha\lambda_\beta \\ &+ \sum_{\alpha,\beta=2,\alpha\neq\beta}^5 \frac{1}{2f}[(\frac{1}{2} - \lambda_\alpha) + (\frac{1}{2} - \lambda_\beta)]\lambda_\alpha\lambda_\beta \\ &- \frac{1}{f}\sum_{\alpha,\beta=2,\alpha\neq\beta}^5 \left[(\frac{1}{2} - \lambda_\alpha)^2 + (\frac{1}{2} - \lambda_\beta)^2\right]\lambda_\alpha\lambda_\beta \\ &- \frac{2}{f}\sum_{\alpha,\beta=2,\alpha\neq\beta}^5 (\frac{1}{2} - \lambda_\alpha)(\frac{1}{2} - \lambda_\beta)\lambda_\alpha\lambda_\beta + 2\sum_{\alpha,\beta=2,\alpha\neq\beta}^5 W_{\alpha\beta}^{\Sigma}\lambda_\alpha\lambda_\beta. \end{split}$$

Next, we handle these terms one by one.

Claim 1.

$$I := \left(\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\lambda_{\alpha}+\lambda_{\beta}) - \frac{1}{2}\right) \lambda_{\alpha}\lambda_{\beta} - \frac{3}{4}$$
  
$$\leq -\frac{3}{2} (\lambda_{1}+\lambda_{2}) + 6(\lambda_{1}+\lambda_{2})^{2} - 6(\lambda_{1}+\lambda_{2})^{3} + 6(\lambda_{1}+\lambda_{2})^{\frac{3}{2}}.$$

In fact, we rewrite I as follows,

(4.20)  

$$I = 2 \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \lambda_{\alpha}^{2} \lambda_{\beta} - \frac{1}{2} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \lambda_{\alpha} \lambda_{\beta} - \frac{3}{4}$$

$$= 2 \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} (\frac{3}{2} - \lambda_{\alpha}) - \frac{1}{2} \sum_{\alpha=2}^{5} \lambda_{\alpha} (\frac{3}{2} - \lambda_{\alpha}) - \frac{3}{4}$$

$$= -2 \sum_{\alpha=2}^{5} \lambda_{\alpha}^{3} + \frac{7}{2} \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} - \frac{3}{4} \sum_{\alpha=2}^{5} \lambda_{\alpha} - \frac{3}{4}$$

$$= -2 \sum_{\alpha=3}^{5} \lambda_{\alpha}^{3} - 2\lambda_{2}^{3} + \frac{3}{4}.$$

Here we used the facts

$$\sum_{\alpha=2}^{5} \lambda_{\alpha} = \frac{3}{2} \text{ and } \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} = \frac{3}{4}$$

in the last equality.

It follows from the appendix of Cheng-Zhou's paper [19] that

$$-2\sigma_3 = \sigma_1^{3} - 3\sigma_1\sigma_2 - 6\lambda_3\lambda_4\lambda_5,$$

where  $\sigma_i =: \sum_{\alpha=1}^{5} \lambda_{\alpha}^i$  for i = 1, 2, 3. Therefore, by using the facts

$$\sum_{\alpha=3}^{5} \lambda_{\alpha} = \frac{3}{2} - \lambda_{2} \text{ and } \sum_{\alpha=3}^{5} \lambda_{\alpha}^{2} = \frac{3}{4} - \lambda_{2}^{2}$$

again, we have

$$-2\sum_{\alpha=3}^{5}\lambda_{\alpha}^{3} = \left(\sum_{\alpha=3}^{5}\lambda_{\alpha}\right)^{3} - 3\left(\sum_{\alpha=3}^{5}\lambda_{\alpha}\right)\left(\sum_{\alpha=3}^{5}\lambda_{\alpha}\right)^{2} - 6\lambda_{3}\lambda_{4}\lambda_{5}$$
$$= \left(\frac{3}{2} - \lambda_{2}\right)^{3} - 3\left(\frac{3}{2} - \lambda_{2}\right)\left(\frac{3}{4} - \lambda_{2}^{2}\right) - 6\lambda_{3}\lambda_{4}\lambda_{5}.$$

It can be calculated directly that

$$\begin{aligned} \lambda_3 \lambda_4 \lambda_5 \\ &= (\lambda_3 - \frac{1}{2})(\lambda_4 - \frac{1}{2})(\lambda_5 - \frac{1}{2}) + \frac{1}{2}(\lambda_3 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5) \\ &- \frac{1}{4}(\lambda_3 + \lambda_4 + \lambda_5) + \frac{1}{8} \\ &= (\lambda_3 - \frac{1}{2})(\lambda_4 - \frac{1}{2})(\lambda_5 - \frac{1}{2}) + \frac{1}{4}\left[(\lambda_3 + \lambda_4 + \lambda_5)^2 - (\lambda_3^2 + \lambda_4^2 + \lambda_5^2)\right] \\ &- \frac{1}{4}(\lambda_3 + \lambda_4 + \lambda_5) + \frac{1}{8} \\ &= (\lambda_3 - \frac{1}{2})(\lambda_4 - \frac{1}{2})(\lambda_5 - \frac{1}{2}) + \frac{1}{4}(\frac{3}{2} - \lambda_2)^2 - (\frac{3}{4} - \lambda_2^2) \\ &- \frac{1}{4}(\frac{3}{2} - \lambda_2) + \frac{1}{8} \\ &= (\lambda_3 - \frac{1}{2})(\lambda_4 - \frac{1}{2})(\lambda_5 - \frac{1}{2}) + \frac{1}{2}\lambda_2(\lambda_2 - 1) + \frac{1}{8}. \end{aligned}$$

Hence, plugging the above two equalities into (4.20), we obtain

$$I = \left(\frac{3}{2} - \lambda_2\right)^3 - 3\left(\frac{3}{2} - \lambda_2\right)\left(\frac{3}{4} - \lambda_2^2\right) - 6\left(\lambda_3 - \frac{1}{2}\right)\left(\lambda_4 - \frac{1}{2}\right)\left(\lambda_5 - \frac{1}{2}\right)$$
$$-3\lambda_2(\lambda_2 - 1) - \frac{3}{4} + \frac{3}{4}$$
$$= -\frac{3}{2}\lambda_2 + 6\lambda_2^2 - 6\lambda_2^3 - 6\left(\lambda_3 - \frac{1}{2}\right)\left(\lambda_4 - \frac{1}{2}\right)\left(\lambda_5 - \frac{1}{2}\right).$$

Moreover,  $|\lambda_{\alpha} - \frac{1}{2}|^2 \leq \lambda_2(1 - \lambda_2) \leq \lambda_2$  holds for  $\alpha = 3, 4, 5$  due to (3.17), and thus

$$I \le -\frac{3}{2}\lambda_2 + 6\lambda_2^2 - 6\lambda_2^3 + 6\lambda_2^{\frac{3}{2}}.$$

We have completed the proof of Claim 1.

Second, we would like to handle the term

$$II := -\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (K_{1\alpha} + K_{1\beta})\lambda_{\alpha}\lambda_{\beta}.$$

Claim 2.

$$II \le 2\frac{|\nabla Ric|^2}{f} + C(\lambda_1 + \lambda_2) + \frac{1}{2}K_{12}.$$

In fact, we rewrite this term as follows

$$II = -\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (K_{1\alpha} + K_{1\beta})\lambda_{\alpha}\lambda_{\beta}$$
  
$$= -2\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} K_{1\alpha}\lambda_{\alpha}\lambda_{\beta}$$
  
$$= -2\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}(R - \lambda_{\alpha})$$
  
$$= -3\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} + 2\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha}^{2}$$
  
$$= -3\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} + 2\left[\sum_{\alpha=2}^{5} K_{1\alpha}(\lambda_{\alpha} - \frac{1}{2})^{2} + \sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} - \frac{1}{4}\sum_{\alpha=2}^{5} K_{1\alpha}\right]$$
  
$$= -\sum_{\alpha=2}^{5} K_{1\alpha}\lambda_{\alpha} + 2\sum_{\alpha=2}^{5} K_{1\alpha}(\lambda_{\alpha} - \frac{1}{2})^{2}.$$

where

$$\sum_{\alpha=2}^{5} K_{1\alpha} = R_{11} = 0$$

was used in the last equality. Due to Lemma 3.4 and (3.18), we obtain

$$-\sum_{\alpha=2}^{5} K_{1\alpha} \lambda_{\alpha} = \frac{1}{f} [(\lambda_2 - \frac{1}{2})^2 \lambda_2 + \sum_{\alpha=3}^{5} (\lambda_\alpha - \frac{1}{2})^2 \lambda_{\alpha}]$$
  
$$\leq \frac{1}{f} \left[ (\lambda_2 - \frac{1}{2})^2 \lambda_2 + \lambda_2 (1 - \lambda_2) (\frac{3}{2} - \lambda_2) \right]$$
  
$$\leq C \frac{\lambda_1 + \lambda_2}{f}.$$

It follows from (3.11) that

$$2\sum_{\alpha=2}^{5} K_{1\alpha} (\lambda_{\alpha} - \frac{1}{2})^{2}$$

$$= 2K_{12} (\lambda_{2} - \frac{1}{2})^{2} + 2\sum_{\alpha=3}^{5} K_{1\alpha} (\lambda_{\alpha} - \frac{1}{2})^{2}$$

$$\leq 2K_{12} (\lambda_{2}^{2} - \lambda_{2} + \frac{1}{4}) + \sum_{\alpha=3}^{5} K_{1\alpha}^{2} + \sum_{\alpha=3}^{5} (\lambda_{\alpha} - \frac{1}{2})^{4}$$

$$\leq 2K_{12} (\lambda_{2} - 1)\lambda_{2} + \frac{1}{2}K_{12} + \sum_{\alpha=3}^{5} K_{1\alpha}^{2} + \sum_{\alpha=3}^{5} (\lambda_{\alpha} - \frac{1}{2})^{4}$$

$$\leq K_{12}^{2} + (\lambda_{2} - 1)^{2}\lambda_{2}^{2} + \frac{1}{2}K_{12} + \sum_{\alpha=3}^{5} K_{1\alpha}^{2} + \left[\sum_{\alpha=3}^{5} (\lambda_{\alpha} - \frac{1}{2})^{2}\right]^{2}$$

$$= \sum_{\alpha=2}^{5} K_{1\alpha}^{2} + \left[(\lambda_{2} - 1)^{2}\lambda_{2}^{2} + (1 - \lambda_{2})^{2}\lambda_{2}^{2}\right] + \frac{1}{2}K_{12}$$

$$\leq 2\frac{|\nabla Ric|^{2}}{f} + C(\lambda_{1} + \lambda_{2}) + \frac{1}{2}K_{12}$$

for some constant C. Here the following inequality was used in the last equality,

$$\begin{split} \sum_{\alpha=2}^{5} K_{1\alpha}^{2} &= \frac{1}{f^{2}} \sum_{\alpha=2}^{5} \left[ \nabla f \cdot \nabla \lambda_{\alpha} + \lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha}) \right]^{2} \\ &\leq \frac{2}{f^{2}} \left[ \sum_{\alpha=2}^{5} (\nabla f \cdot \nabla \lambda_{\alpha})^{2} + \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} (\frac{1}{2} - \lambda_{\alpha})^{2} \right] \\ &\leq 2 \frac{|\nabla Ric|^{2}}{f} + \frac{2}{f^{2}} \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} (\frac{1}{2} - \lambda_{\alpha})^{2} \\ &\leq 2 \frac{|\nabla Ric|^{2}}{f} + \frac{2}{f^{2}} \left[ \lambda_{2}^{2} (\frac{1}{2} - \lambda_{2})^{2} + \sum_{\alpha=3}^{5} \lambda_{\alpha}^{2} \lambda_{2} (1 - \lambda_{2}) \right] \\ &\leq 2 \frac{|\nabla Ric|^{2}}{f} + \frac{2}{f^{2}} \lambda_{2} \left[ \lambda_{2} (\frac{1}{2} - \lambda_{2})^{2} + (\frac{3}{4} - \lambda_{2}) (1 - \lambda_{2}) \right] \\ &\leq 2 \frac{|\nabla Ric|^{2}}{f} + \frac{C}{f^{2}} (\lambda_{1} + \lambda_{2}) \end{split}$$

due to the boundedness of Ricci curvature. In conclusion, we obtain

$$II \le 2\frac{|\nabla Ric|^2}{f} + C(\lambda_1 + \lambda_2) + \frac{1}{2}K_{12}.$$

Thus, we have completed the proof of Claim 2.

Claim 3.

$$III := \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \frac{1}{2f} [(\frac{1}{2} - \lambda_{\alpha}) + (\frac{1}{2} - \lambda_{\beta})] \lambda_{\alpha} \lambda_{\beta}$$
$$-\frac{1}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \left[ (\frac{1}{2} - \lambda_{\alpha})^{2} + (\frac{1}{2} - \lambda_{\beta})^{2} \right] \lambda_{\alpha} \lambda_{\beta}$$
$$-\frac{2}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2} - \lambda_{\alpha}) (\frac{1}{2} - \lambda_{\beta}) \lambda_{\alpha} \lambda_{\beta}$$
$$\leq \frac{C_{2}}{f} (\lambda_{1} + \lambda_{2}).$$

In fact, we rewrite the first two term of III as follows,

$$\begin{aligned} \frac{1}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \left(\frac{1}{2} - \lambda_{\alpha}\right)\lambda_{\alpha}\lambda_{\beta} - \frac{2}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} \left(\frac{1}{2} - \lambda_{\alpha}\right)^{2}\lambda_{\alpha}\lambda_{\beta} \\ &= \frac{1}{f} \sum_{\alpha=2}^{5} \left(\frac{1}{2} - \lambda_{\alpha}\right)\lambda_{\alpha}\left(\frac{3}{2} - \lambda_{\alpha}\right) - \frac{2}{f} \sum_{\alpha=2}^{5} \left(\frac{1}{2} - \lambda_{\alpha}\right)^{2}\lambda_{\alpha}\left(\frac{3}{2} - \lambda_{\alpha}\right) \\ &= \frac{1}{f} \sum_{\alpha=2}^{5} \left(\frac{1}{2} - \lambda_{\alpha}\right)\lambda_{\alpha}\left(\frac{3}{2} - \lambda_{\alpha}\right)\left(1 - 1 + 2\lambda_{\alpha}\right) \\ &= \frac{2}{f} \left[\sum_{\alpha=2}^{5} \lambda_{\alpha}^{2}\left(\frac{1}{2} - \lambda_{\alpha}\right) + \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2}\left(\frac{1}{2} - \lambda_{\alpha}\right)^{2}\right] \\ &= \frac{2}{f} \left[-\sum_{\alpha=2}^{5} \lambda_{\alpha}\left(\frac{1}{2} - \lambda_{\alpha}\right)^{2} + \frac{1}{2}\sum_{\alpha=2}^{5} \lambda_{\alpha}\left(\frac{1}{2} - \lambda_{\alpha}\right) + \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2}\left(\frac{1}{2} - \lambda_{\alpha}\right)^{2}\right] \end{aligned}$$

 $\sum_{\alpha=2}^{5} \lambda_{\alpha}(\frac{1}{2} - \lambda_{\alpha}) = 0$  since

$$\sum_{\alpha=2}^{5} \lambda_{\alpha} = R = 2|Ric|^2 = 2\sum_{\alpha=2}^{5} \lambda_{\alpha}^2.$$

It follows from (3.18) and (3.19) that

$$\begin{aligned} |\sum_{\alpha=2}^{5} \lambda_{\alpha} (\frac{1}{2} - \lambda_{\alpha})^{2}| &\leq \lambda_{2} (\frac{1}{2} - \lambda_{2})^{2} + \lambda_{2} (1 - \lambda_{2}) (\frac{3}{2} - \lambda_{2}) \\ &= \lambda_{2} \left[ (\frac{1}{2} - \lambda_{2})^{2} + (1 - \lambda_{2}) (\frac{3}{2} - \lambda_{2}) \right] \end{aligned}$$

and

$$\sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} (\frac{1}{2} - \lambda_{\alpha})^{2} \leq \lambda_{2}^{2} (\frac{1}{2} - \lambda_{2})^{2} + \lambda_{2} (1 - \lambda_{2}) (\frac{3}{4} - \lambda_{2}^{2})$$
$$= \lambda_{2} \left[ \lambda_{2} (\frac{1}{2} - \lambda_{2})^{2} + (1 - \lambda_{2}) (\frac{3}{4} - \lambda_{2}^{2}) \right]$$

Hence,

$$\frac{1}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2} - \lambda_{\alpha})\lambda_{\alpha}\lambda_{\beta} - \frac{2}{f} \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2} - \lambda_{\alpha})^{2}\lambda_{\alpha}\lambda_{\beta}$$

$$\leq \frac{C}{f} (\lambda_{1} + \lambda_{2})$$

.

for some constant C, since the Ricci curvature is nonnegative and bounded.

We consider the third term of *III* as follows,

$$\begin{aligned} &-\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2}-\lambda_{\alpha})(\frac{1}{2}-\lambda_{\beta})\lambda_{\alpha}\lambda_{\beta} \\ &= -\frac{1}{2}\sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}\lambda_{\beta} + \sum_{\alpha,\beta=2,\alpha\neq\beta}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}\lambda_{\beta}^{2} \\ &= -\frac{1}{2}\sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}(\frac{3}{2}-\lambda_{\alpha}) + \sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}(\frac{3}{4}-\lambda_{\alpha}^{2}) \\ &= -\frac{3}{4}\sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha} + \frac{1}{2}\sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}^{2} \\ &+ \frac{3}{4}\sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha} - \sum_{\alpha=2}^{5} (\frac{1}{2}-\lambda_{\alpha})\lambda_{\alpha}^{3} \\ &= \sum_{\alpha=2}^{5} \lambda_{\alpha}^{2} (\frac{1}{2}-\lambda_{\alpha})^{2} \\ &\leq \lambda_{2} \left[\lambda_{2} (\frac{1}{2}-\lambda_{2})^{2} + (1-\lambda_{2})(\frac{3}{4}-\lambda_{2}^{2})\right], \end{aligned}$$

which implies that

$$-\frac{2}{f}\sum_{\alpha,\beta=2,,\alpha\neq\beta}^{\mathfrak{d}}(\frac{1}{2}-\lambda_{\alpha})(\frac{1}{2}-\lambda_{\beta})\lambda_{\alpha}\lambda_{\beta}\leq\frac{C}{f}(\lambda_{1}+\lambda_{2}).$$

for some constant C, since the Ricci curvature is nonnegative and bounded. Therefore we have

$$III \le \frac{C}{f} (\lambda_1 + \lambda_2).$$

We have completed the proof of Claim 3.

Finally, we consider the term  $IV := 2 \sum_{\alpha,\beta=2}^{5} W_{\alpha\beta}^{\Sigma(s)} \lambda_{\alpha} \lambda_{\beta}$ . Claim 4.

$$IV \le \frac{17}{4} \left[ \epsilon(\lambda_1 + \lambda_2) + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2 \right]$$

for any constant  $\epsilon > 0$  satisfying  $\frac{17}{4}\epsilon \leq \frac{1}{4}$ .

In fact, we rewrite the first two term of IV as follows,

$$IV = 4W_{23}^{\Sigma(s)}\lambda_2\lambda_3 + 4W_{24}^{\Sigma(s)}\lambda_2\lambda_4 + 4W_{25}^{\Sigma(s)}\lambda_2\lambda_5 + 4W_{34}^{\Sigma(s)}\lambda_3\lambda_4 + 4W_{35}^{\Sigma(s)}\lambda_3\lambda_5 + 4W_{45}^{\Sigma(s)}\lambda_4\lambda_5 = 4\lambda_2(W_{23}^{\Sigma(s)}\lambda_3 + W_{24}^{\Sigma(s)}\lambda_4 + W_{25}^{\Sigma(s)}\lambda_5) + 2\lambda_3(W_{34}^{\Sigma(s)}\lambda_4 + W_{35}^{\Sigma(s)}\lambda_5) + 2\lambda_4(W_{34}^{\Sigma(s)}\lambda_3 + W_{45}^{\Sigma(s)}\lambda_5) + 2\lambda_5(W_{35}^{\Sigma(s)}\lambda_3 + W_{45}^{\Sigma(s)}\lambda_4).$$

Using the fact that  $\sum_{\alpha=2}^{5} W_{3\alpha}^{\Sigma(s)} = 0$  again, we have

$$2\lambda_{3}(W_{34}^{\Sigma(s)}\lambda_{4} + W_{35}^{\Sigma(s)}\lambda_{5})$$

$$= 2\lambda_{3}\left[W_{34}^{\Sigma(s)}(\lambda_{4} - \frac{1}{2}) + W_{35}^{\Sigma(s)}(\lambda_{5} - \frac{1}{2}) - \frac{1}{2}W_{32}^{\Sigma(s)}\right]$$

$$\leq 2(|W_{34}^{\Sigma(s)}|^{2} + |W_{35}^{\Sigma(s)}|^{2})^{\frac{1}{2}}\left[(\lambda_{4} - \frac{1}{2})^{2} + (\lambda_{5} - \frac{1}{2})^{2}\right]^{\frac{1}{2}} - W_{23}^{\Sigma(s)}\lambda_{3}$$

$$\leq 2|W^{\Sigma(s)}|\lambda_{2}^{\frac{1}{2}} - W_{23}^{\Sigma(s)}\lambda_{3}$$

$$\leq \left(\epsilon\lambda_{2} + \frac{1}{\epsilon}|W^{\Sigma(s)}|^{2}\right) - W_{23}^{\Sigma(s)}\lambda_{3}$$

for any  $\epsilon > 0$ , where we used the fact

$$(\lambda_4 - \frac{1}{2})^2 + (\lambda_5 - \frac{1}{2})^2 \le \lambda_2(1 - \lambda_2) \le \lambda_2$$

in the second inequality. Similarly,

$$2\lambda_4(W_{34}^{\Sigma(s)}\lambda_3 + W_{45}^{\Sigma(s)}\lambda_5) \leq \left(\epsilon\lambda_2 + \frac{1}{\epsilon}|W^{\Sigma(s)}|^2\right) - W_{24}^{\Sigma(s)}\lambda_4$$

and

$$2\lambda_5(W_{35}^{\Sigma(s)}\lambda_3 + W_{45}^{\Sigma(s)}\lambda_4) \leq \left(\epsilon\lambda_2 + \frac{1}{\epsilon}|W^{\Sigma(s)}|^2\right) - W_{25}^{\Sigma(s)}\lambda_5.$$

Therefore,

$$IV \leq 4\lambda_2 \sum_{\alpha=3}^5 W_{2\alpha}^{\Sigma(s)} \lambda_{\alpha} + 3\left(\epsilon \lambda_2 + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2\right) - \sum_{\alpha=3}^5 W_{2\alpha}^{\Sigma(s)} \lambda_{\alpha}.$$

Next, we would like to deal with the term  $\sum_{\alpha=3}^{5} W_{2\alpha}^{\Sigma(s)} \lambda_{\alpha}$ . Since the Weyl curvature is tracefree, we see

$$\sum_{\alpha=3}^{5} W_{2\alpha}^{\Sigma(s)} = \sum_{\alpha=3}^{5} W_{2\alpha 2\alpha}^{\Sigma(s)} = 0,$$

and then

$$|W_{23}^{\Sigma(s)}\lambda_{3} + W_{24}^{\Sigma(s)}\lambda_{4} + W_{25}^{\Sigma(s)}\lambda_{5}|$$

$$=|W_{23}^{\Sigma(s)}(\lambda_{3} - \frac{1}{2}) + W_{24}^{\Sigma(s)}(\lambda_{4} - \frac{1}{2}) + W_{25}^{\Sigma(s)}(\lambda_{5} - \frac{1}{2})|$$

$$\leq |W^{\Sigma(s)}|[(\lambda_{3} - \frac{1}{2})^{2} + (\lambda_{4} - \frac{1}{2})^{2} + (\lambda_{5} - \frac{1}{2})^{2}]^{\frac{1}{2}}$$

$$=|W^{\Sigma(s)}|[\lambda_{2}(1 - \lambda_{2})]^{\frac{1}{2}}$$

$$\leq |W^{\Sigma(s)}|\lambda_{2}^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\left(\epsilon\lambda_{2} + \frac{1}{\epsilon}|W^{\Sigma(s)}|^{2}\right).$$

Therefore, we see

$$IV \leq 2\lambda_2 \left(\epsilon \lambda_2 + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2\right) + 3 \left(\epsilon \lambda_2 + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2\right) + \frac{1}{2} \left(\epsilon \lambda_2 + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2\right) \leq \frac{17}{4} \left(\epsilon \lambda_2 + \frac{1}{\epsilon} |W^{\Sigma(s)}|^2\right),$$

due to the fact  $4\lambda_2 \leq R = \frac{3}{2}$ . We have completed the proof of **Claim** 4.

Consequently, from Claims 1–4, we obtain that

$$\begin{aligned} |\nabla Ric|^2 &= I + II + III + IV \\ &\leq -\frac{3}{2}(\lambda_1 + \lambda_2) + 6(\lambda_1 + \lambda_2)^2 - 6(\lambda_1 + \lambda_2)^3 + 6(\lambda_1 + \lambda_2)^{\frac{3}{2}} \\ &+ 2\frac{|\nabla Ric|^2}{f} + C(\lambda_1 + \lambda_2) + \frac{1}{2}K_{12} + \frac{C_2}{f}(\lambda_1 + \lambda_2) \\ &+ \frac{17}{4}\left(\epsilon(\lambda_1 + \lambda_2) + \frac{1}{\epsilon}|W^{\Sigma(s)}|^2\right) \\ &\leq 2\frac{|\nabla Ric|^2}{f} + C(\lambda_1 + \lambda_2) + \frac{1}{2}K_{12} + C|W^{\Sigma(s)}|^2, \end{aligned}$$

for some constant C and small  $\epsilon$  satisfying  $\frac{17}{4}\epsilon \leq \frac{1}{4}$ .

Hence

$$\nabla Ric|^2 \le C(\lambda_1 + \lambda_2) + K_{12} + C|W^{\Sigma(s)}|^2$$

on  $M \setminus D(a)$ , where C and a > 0 are constants. We have completed the proof of Proposition 4.2.

# 5. Curvature bound and uniform decay of $\lambda_1 + \lambda_2$

In this section, based the point-picking argument, we will prove the Riemannian curvature is bounded. This is because  $\int_{\Sigma} |W|^2$  tends to zero at infinity, which implies the blowing up limit must be flat. By the similar argument, we can also prove that  $\lambda_1 + \lambda_2 \to 0$  and  $\nabla_{\nabla f} Ric$  also tend to zero.

In order to prove the curvature bound, we continue to handle the term  $K_{12}$  in Proposition 4.2 and have the following lemma.

**Lemma 5.1.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ . There is some constant C > 0, such that

$$|\nabla Ric|^2 \le C(\lambda_1 + \lambda_2) + C|W^{\Sigma(s)}|^2 + C$$

on  $M \setminus D(a)$  for some a > 0 and  $s \ge a$ .

*Proof.* We need to rewrite the equation in Propositon 4.2 in the following way.

$$\begin{aligned} |\nabla Ric|^2 &\leq C(\lambda_1 + \lambda_2) + K_{12} + C|W^{\Sigma(s)}|^2 \\ &\leq C(\lambda_1 + \lambda_2) + K_{12}^2 + \frac{1}{16} + C|W^{\Sigma(s)}|^2 \end{aligned}$$

From equation (4.21), we have

$$K_{12}^2 \le 2\frac{|\nabla Ric|^2}{f} + \frac{C}{f^2}(\lambda_1 + \lambda_2)$$

Thus, we have

$$|\nabla Ric|^{2} \leq C(\lambda_{1} + \lambda_{2}) + 2\frac{|\nabla Ric|^{2}}{f} + \frac{1}{16} + C|W^{\Sigma(s)}|^{2},$$

which yields

$$|\nabla Ric|^2 \le C(\lambda_1 + \lambda_2) + C|W^{\Sigma(s)}|^2 + C$$

for some constant C.

Next, we have the relationship between  $\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma$  and the Ricci curvature by the Gauss-Bonnet-Chern formula.

**Lemma 5.2.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with constant scalar curvature  $R = \frac{3}{2}$ . Then we have

(5.23) 
$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma = 2 \int_{\Sigma(s)} \frac{|\nabla_{\nabla f} Ric|^2}{f^2} d\sigma \le 2 \int_{\Sigma(s)} \frac{|\nabla Ric|^2}{f} d\sigma.$$

*Proof.* Recall the Gauss-Bonnet-Chern formula: for a closed, oriented 4-dimensional Riemannian manifold  $(N^4, g)$ ,

$$\chi(N^4) = \frac{1}{4\pi^2} \int_{N^4} \left[ \frac{1}{8} |W_g|^2 - \frac{1}{12} \left( \Delta_g R_g - R_g^2 + 3 |\text{Ric}_g|^2 \right) \right] dV,$$

where  $\chi(N^4)$  is the Euler characteristic of  $N^4$ ,  $W_g$ ,  $R_g$ ,  $\operatorname{Ric}_g$  are its Weyl curvature, scalar curvature and Ricci curvature, respectively,  $\Delta_g$  is the Laplacian operator with the metric g, and dV is the volume element of  $N^4$ .

On a five dimensional simply connected shrinking Ricci soliton with  $R = \frac{3}{2}$ , it follows from Proposition 2.2 that the zero set  $f^{-1}(0)$  is a three dimensional simply connected closed manifold. Actually,  $f^{-1}(0)$  is the deformation contraction of  $M^5$ , hence has to be diffeomorphic to  $\mathbb{S}^3$ . Because

$$f(x) = \frac{1}{4}d(x, f^{-1}(0))^2, \quad f = |\nabla f|^2$$

and the exponential map is a local diffeomorphism, it is easy to see that  $f^{-1}(s)$  is diffeomorphic to  $\mathbb{S}^1 \times \mathbb{S}^3$  when s is small and positive. Hence all the level set of f are diffeomorphic to  $\mathbb{S}^1 \times \mathbb{S}^3$  since f has no critical point away from  $f^{-1}(0)$ . Therefore,

$$\chi(\Sigma(s)) = \chi(\mathbb{S}^3 \times \mathbb{S}^1) = 0,$$

where  $\Sigma(s) = f^{-1}(s)$  for s > 0.

As for  $\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma$ , putting equations (3.13) and (3.15) into the Gauss-Bonnet formula, we get

$$\begin{split} & \int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma \\ = & \int_{\Sigma(s)} 2\left[ |Ric^{\Sigma(s)}|^2 - \frac{1}{3}(R^{\Sigma(s)})^2 \right] + 32\pi^2 \chi(\Sigma(s)) \\ = & \int_{\Sigma(s)} 2\left( |Ric|^2 + \frac{|\nabla_{\nabla f}Ric|^2}{f^2} - \frac{3}{4} \right) d\sigma \\ = & 2\int_{\Sigma(s)} \frac{|\nabla_{\nabla f}Ric|^2}{f^2} d\sigma \\ \leq & 2\int_{\Sigma(s)} \frac{|\nabla Ric|^2 |\nabla f|^2}{f^2} d\sigma \\ = & 2\int_{\Sigma(s)} \frac{|\nabla Ric|^2}{f} d\sigma, \end{split}$$

where the fact  $|\nabla f|^2 = f$  was used in the last equality.

Combining the above two lemmas, we have the following proposition.

**Proposition 5.3.** Suppose  $(M^5, g, f)$  is a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ , then

$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma \le \int_{\Sigma(s)} \frac{C}{s} d\sigma \le \frac{C}{\sqrt{s}}.$$

*Proof.* It follows from the above two lemmas that

$$\begin{split} \int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma &\leq 2 \int_{\Sigma(s)} \frac{|\nabla Ric|^2}{f} d\sigma \\ &\leq 2 \int_{\Sigma(s)} \frac{1}{s} \left[ C(\lambda_1 + \lambda_2) + C |W^{\Sigma(s)}|^2 + C \right] d\sigma \\ &\leq 2 \int_{\Sigma(s)} \frac{C}{s} d\sigma + \int_{\Sigma(s)} \frac{C}{s} |W^{\Sigma(s)}|^2 d\sigma \end{split}$$

because of bounded Ricci curvature. Hence

$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 d\sigma \le \int_{\Sigma(s)} \frac{C}{s} d\sigma$$

for enough large s.

From item (v) in Poposition 2.2, the volume of  $\Sigma(s)$  satisfies

$$\operatorname{Vol}(\Sigma(s)) = c\sqrt{s}$$

for some constant c. Hence

$$\int_{\Sigma(s)} \frac{C}{s} d\sigma \le \frac{C}{\sqrt{s}},$$

and we have completed the proof of this proposition.

In order to prove the curvature bound, we recall the following results.

**Lemma 5.4** ([18]). Given a complete noncompact Riemannian manifold with unbounded curvature, we can find a sequence of point  $p_j$ divergent to infinity such that for each positive integer j, we have  $|Rm(p_j)| \ge j$  and

$$|Rm(x)| \le 4|Rm(p_j)|$$

for  $x \in B(p_j, \frac{j}{\sqrt{|Rm(p_j)|}}).$ 

**Theorem 5.5.** Suppose  $(M^5, g, f)$  is a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ , then its curvature is bounded.

*Proof.* Suppose not, by Lemma 5.4, then there exists a sequence of point  $p_j$  divergent to infinity such that for each positive integer j, we have  $|Rm(p_j)| \ge j$  and

$$|Rm(x)| \le 4|Rm(p_j)|$$

for  $x \in B(p_j, \frac{j}{\sqrt{|Rm(p_j)|}}).$ 

By the  $\kappa$  noncollapsed theorem in [27] and the scalar curvature is bounded for (M, g), we get that  $\operatorname{Vol}(B(p_j, 1))$  has a uniform positive lower bound [27]. Then we can apply Hamilton's compactness theorem to obtain that the rescaled manifolds  $\left(B(p_j, g, \frac{j}{\sqrt{|Rm(p_j)|}}), |Rm(p_j)|g, p_j\right)$ converge to a smooth complete Riemannian manifold  $(M_{\infty}, g_{\infty}, p_{\infty})$ with  $|Rm(p_{\infty})| = 1$  which is Ricci flat because  $(M^5, g)$  has bounded Ricci curvature and  $|Rm(p_j)| \to \infty$ , moreover  $(M_{\infty}, g_{\infty}, p_{\infty})$  has Euclidean volume growth.

Since the integral curves of f passing through  $p_j$  is a geodesic with respect to (M, g), the geodesic segment of these curves contained in  $B\left(p_j, \frac{j}{\sqrt{|Rm(p_j)|}}\right)$  will converge to a geodesic line in  $(M_{\infty}, g_{\infty})$ , then Cheeger-Gromoll's splitting theorem ([15]) implies that  $M_{\infty} = \mathbb{R} \times N^4$ , where  $N^4$  is Ricci flat and of Euclidean Volume growth.

Because  $(\Sigma(s), g)$  has the second fundamental form

$$h = \frac{\frac{1}{2}g - Ric}{|\nabla f|},$$

which tends to zero as  $f \to \infty$ , so the second fundamental form of  $B\left(p_j, g, \frac{j}{\sqrt{|Rm(p_j)|}}\right) \cap \Sigma(f(p_j))$  with metric  $|Rm(p_j)|g$  converges to zero. This implies the level set  $B\left(p_j, g, \frac{j}{\sqrt{|Rm(p_j)|}}\right) \cap \Sigma(f(p_j))$  with the induced rescaled metrics  $|Rm(p_j)|g$  will converge to  $N^4$ .

On the other hand, we have the key estimate by Proposition 5.3,

$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 \le \frac{C}{\sqrt{s}} \to 0$$

as  $s \to \infty$ .

It is known that integral of the weyl curvature is scaling invariant in dimension 4. All the above implies that

$$\int_{B(p_j,|Rm(p_j)|g,j)\cap\Sigma(f(p_j))} |W^{\Sigma(f(p_j))}|^2 \leq \frac{C}{\sqrt{f(p_j)}} \to 0.$$

So  $N^4$  has vanishing weyl curvature, hence it is flat. This contradicts the fact that  $|Rm(p_{\infty})| = 1$ .

**Theorem 5.6.** Suppose  $(M^5, g, f)$  is a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ , then  $\lambda_1 + \lambda_2 \to 0$  at infinity.

*Proof.* Suppose on the contrary, then there exists a sequence of  $q_j$  divergent to infinity with  $(\lambda_1 + \lambda_2)(q_j) \ge \delta$  for some  $\delta > 0$ .

As [32], define  $f_j(x) = \frac{f(x) - f(q_j)}{|\nabla f(q_j)|}$ . By Theorem 5.5, we have the curvature is bounded, so the pointed manifolds  $(M, g, q_j)$  converge in Cheeger-Gromov sense to  $(M_{\infty}, g_{\infty}, q_{\infty})$ . Since  $|\nabla f_j(q_j)| = 1$ ,

$$\nabla^2 f_j = \frac{\nabla^2 f}{|\nabla f(q_j)|} = \frac{\frac{1}{2}g - Ric}{|\nabla f(q_j)|}$$

tends to zero at infinity,  $f_j$  converges to a smooth function  $f_{\infty}$  with  $|\nabla f_{\infty}|(q_{\infty}) = 1$  and  $\nabla^2 f_{\infty} = 0$ . Hence  $M_{\infty} = \mathbb{R} \times N^4$ , where  $N^4$  is a four dimensional complete Riemannian manifold with  $Ric \geq 0$  and of constant scalar curvature  $\frac{3}{2}$ . Moreover,  $\lambda_1(q_{\infty}) \geq \delta$ .

Again as the above theorem,  $(\Sigma(s), g)$  has second fundamental form

$$h = \frac{\frac{1}{2}g - Ric}{|\nabla f|},$$

which tends to zero as  $s \to \infty$ ; this implies the level set  $\Sigma(f(q_j))$  with the induced induced metric will converge to  $N^4$  with  $\lambda_1(\widetilde{q_{\infty}}) \ge \delta$ , where  $q_{\infty} = (0, \widetilde{q_{\infty}})$ . Proposition 5.3 gives

$$\int_{\Sigma(f(q_j))} |W^{\Sigma(f(q_j))}|^2 \leq \frac{C}{\sqrt{f(q_j)}} \to 0$$

as  $s \to \infty$ . So  $N^4$  has vanishing Weyl curvature. Since  $\operatorname{Vol}(\Sigma(s)) = c\sqrt{s}$ ,  $N^4$  must be noncompact.

Thanks to the classification of complete locally conformally flat manifolds with nonnegative Ricci curvature by Zhu [44] and Carron-Herzlich [13],  $N^4$  is one of the following:

(1) N is non-flat and globally conformally equivalent to  $\mathbb{R}^4$ ;

(2) N is globally conformally equivalent to a space form of positive curvature;

(3) N is locally isometric to the cylinder  $\mathbb{R} \times \mathbb{S}^3$ ;

(4) N is isometric to a complete flat manifold.

If case (1) happens, then there is a positive function u such that  $g_N = u^2 g_E$  has constant scalar curvature  $\frac{3}{2}$ , where  $g_E$  is the Euclidean metric. Equivalently,

(5.24) 
$$\Delta u + \frac{1}{4}u^2 = 0 \quad on \quad \mathbb{R}^4.$$

By Caffarelli-Gidas-Spruck [5] or Chen-Li [16], the solution to (5.24) has been classified, and none give a complete metric. Contradiction. Case (2) is impossible, since N is noncompact. If case (3) happens, it contradicts with  $\lambda_1(\widetilde{q_{\infty}}) \geq \delta$ . Case (4) can not happen, since N is nonflat.

In all, the Weyl curvature of  $N^4$  couldn't be zero, contradiction.  $\Box$ 

Similarly, we have the following corollary.

**Corollary 5.7.** Suppose  $(M^5, g, f)$  is a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ , then

$$\nabla_{\nabla_f} Ric \to 0$$

and

$$Ric - 2Rm * Ric \rightarrow 0$$

at infinity.

*Proof.* The proof is similar to the above theorem, and notice that

$$\Delta Ric = 0$$

and

$$Ric - Rm * Ric = 0$$

on  $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^3$ .

Combining with

$$\Delta_f Ric = \Delta Ric - \nabla_{\nabla f} Ric = Ric - Rm * Ric$$

gives the desired result.

6. The Proof of Theorem 1.1

In this section, we improve the estimate on  $\int_{\Sigma(s)} |W^{\Sigma(s)}|^2$ , then we show  $\lambda_1 + \lambda_2 = 0$  by the integral method, thereby proving Theorem 1.1.

To complete the proof of Theorem 1.1, we address  $|\nabla Ric|^2$  in Proposition 4.2 and present the following results.

**Proposition 6.1.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ . Denote  $u = \lambda_1 + \lambda_2$ ,  $h = f + \frac{3}{2} \log f - \frac{40}{f}$ , then

$$\int_{M \setminus D(a)} \Delta_h u \cdot e^{-h} \le -0.1 \int_{M \setminus D(a)} u \cdot e^{-h} dvol \le 0.$$

where a is sufficiently large.

*Proof.* First, we claim that

(6.25) 
$$\int_{M \setminus D(a)} |W^{\Sigma(s)}|^2 \cdot e^{-h} dvol$$
$$\leq \int_{M \setminus D(a)} \left[ \frac{C(\lambda_1 + \lambda_2)}{f} + \frac{2\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{f^2} \right] \cdot e^{-h} dvol$$

In fact, due to  $\lambda_1 + \lambda_2 \rightarrow 0$  at infinity, it follows from Proposition 4.2 that

$$|\nabla Ric|^2 \le C(\lambda_1 + \lambda_2) + K_{12} + C|W^{\Sigma(s)}|^2,$$

for some constant C. Then by Lemma 5.2

$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 \le \int_{\Sigma(s)} \left[ \frac{C(\lambda_1 + \lambda_2)}{f} + \frac{K_{12}}{f} + \frac{C|W^{\Sigma(s)}|^2}{f} \right].$$

By the absorbing inequality,

$$\int_{\Sigma(s)} (1 - \frac{C}{f}) |W^{\Sigma(s)}|^2 \le \int_{\Sigma(s)} \frac{C(\lambda_1 + \lambda_2)}{f} + \frac{K_{12}}{f}$$

for  $s \ge a$  with a large. Thus, we immediately derive that

$$\int_{\Sigma(s)} |W^{\Sigma(s)}|^2 \le \int_{\Sigma(s)} \frac{C(\lambda_1 + \lambda_2)}{f} + 2\frac{K_{12}}{f}.$$

Combining this with equation (3.11), we see that

$$\begin{split} & \int_{M \setminus D(a)} |W^{\Sigma(s)}|^2 \cdot e^{-h} dvol \\ & \leq \int_{M \setminus D(a)} \left[ \frac{C(\lambda_1 + \lambda_2)}{f} + 2\frac{K_{12}}{f} \right] \cdot e^{-h} dvol \\ & \leq \int_{M \setminus D(a)} \left[ \frac{C(\lambda_1 + \lambda_2)}{f} + \frac{2\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{f^2} \right] \cdot e^{-h} dvol. \end{split}$$

Noticing that  $\lambda_1 + \lambda_2 \to 0$  at infinity and substituting inequality (4.22) into Proposition 3.7, we get

$$\Delta_f(\lambda_1 + \lambda_2) \le -0.9(\lambda_1 + \lambda_2) + \frac{3}{2f} \nabla f \cdot \nabla(\lambda_1 + \lambda_2) + \epsilon |W^{\Sigma}|^2 + \frac{1}{\epsilon} (\lambda_1 + \lambda_2)$$

for any  $\epsilon > 0$ . Together with (6.25), taking  $\epsilon = 20$ , it is easy to see that

$$\begin{split} & \int_{M \setminus D(a)} \Delta_f(\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ \leq & -0.9 \int_{M \setminus D(a)} (\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ & + \int_{M \setminus D(a)} \frac{3\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{2f} \cdot e^{-h} dvol \\ & + \varepsilon \int_{M \setminus D(a)} |W^{\Sigma(s)}|^2 \cdot e^{-h} dvol + \frac{1}{\varepsilon} \int_{M \setminus D(a)} (\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ \leq & -0.8 \int_{M \setminus D(a)} (\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ & + \int_{M \setminus D(a)} \frac{3\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{2f} \cdot e^{-h} dvol \\ & + \epsilon \int_{M \setminus D(a)} \frac{2\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{f^2} \cdot e^{-h} dvol \\ \leq & -0.8 \int_{M \setminus D(a)} (\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ + \frac{3}{2} \int_{M \setminus D(a)} \nabla \log f \cdot \nabla(\lambda_1 + \lambda_2) \cdot e^{-h} dvol \\ & + 40 \int_{M \setminus D(a)} \frac{\nabla f \cdot \nabla(\lambda_1 + \lambda_2)}{f^2} \cdot e^{-h} dvol. \end{split}$$

for sufficiently large a > 0. Then we obtain

$$\int_{M \setminus D(a)} \Delta_h(\lambda_1 + \lambda_2) e^{-h} \le -0.8 \int_{M \setminus D(a)} (\lambda_1 + \lambda_2) \cdot e^{-h} dvol_2$$

where

$$h = f + \frac{3}{2}\log f - \frac{40}{f}.$$

For simplicity, we denote  $u = \lambda_1 + \lambda_2$ , and then the above inequality becomes

$$\int_{M \setminus D(a)} \Delta_h u \cdot e^{-h} \le -0.8 \int_{M \setminus D(a)} u \cdot e^{-h} dvol \le 0,$$

where *a* is sufficiently large. Thus, we have completed the proposition.  $\Box$ 

**Proposition 6.2.** Let  $(M^5, g, f)$  be a five dimensional shrinking gradient Ricci soliton with  $R = \frac{3}{2}$ . Then

$$\int_{M \setminus D(b)} \Delta_h u \cdot e^{-h} \ge 0$$

for some  $b \ge a$ , where  $u = \lambda_1 + \lambda_2$ ,  $h = f + \frac{3}{2} \log f - \frac{40}{f}$  and constant a is the same as in Proposition 6.1.

*Proof.* Notice that

$$\int_{M\setminus D(b)} \Delta_h u \cdot e^{-h} = -\int_{\Sigma(b)} \langle \nabla u, \frac{\nabla h}{|\nabla h|} \rangle \cdot e^{-h},$$

so it sufficies to prove

(6.26) 
$$\int_{\Sigma(b)} \langle \nabla u, \frac{\nabla h}{|\nabla h|} \rangle \le 0$$

for some  $b \geq a$ .

For this purpose, we consider the following one parameter family of diffeomorphisms,

$$\begin{cases} \frac{\partial F}{\partial s} = \frac{\nabla f}{|\nabla f|^2}, \\ F(x, a) = x \in \Sigma(a) \end{cases}$$

Then  $\frac{\partial}{\partial s}f(F(x,s)) = \langle \nabla f, \frac{\nabla f}{|\nabla f|^2} \rangle = 1$ , and the advantage of F is that it maps level set of f to another level set, in particular f(F(x,s)) = s for any  $x \in \Sigma(a)$ .

Suppose  $\{x_1, x_2, x_3, x_4\}$  are local coordinate chart of  $\Sigma(a)$ , on  $\Sigma(s)$ , let  $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) := g(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}), \ d\sigma_{\Sigma(s)} = \sqrt{\det(g_{ij})}dx$ , where  $dx = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ . Next we compute the derivatives of  $dvol_{\Sigma(s)}$ .

$$\begin{aligned} \frac{\partial}{\partial s} dvol_{\Sigma(s)} &= \frac{\partial}{\partial s} \sqrt{\det(g_{ij})} dx \\ &= \frac{1}{2} \cdot 2g^{ij} \langle \nabla_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial s}, \frac{\partial F}{\partial x_j} \rangle d\sigma_{\Sigma(s)} \\ &= g^{ij} \langle \nabla_{\frac{\partial F}{\partial x_i}} \frac{\nabla f}{|\nabla f|^2}, \frac{\partial F}{\partial x_j} \rangle d\sigma_{\Sigma(s)} \\ &= \frac{1}{|\nabla f|^2} g^{ij} \nabla^2 f(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}) d\sigma_{\Sigma(s)} \\ &= \frac{1}{|\nabla f|^2} g^{ij} \left( \frac{1}{2} g(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}) - Ric(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}) \right) d\sigma_{\Sigma(s)} \\ &= \frac{1}{|\nabla f|^2} (2 - \frac{3}{2}) d\sigma_{\Sigma(s)} = \frac{1}{2s} d\sigma_{\Sigma(s)}. \end{aligned}$$

Hence, it is easy to check that

$$\frac{\partial}{\partial s} \left( \frac{1}{\sqrt{s}} d\sigma_{\Sigma(s)} \right) = \left( -\frac{1}{2} s^{-\frac{3}{2}} + \frac{1}{\sqrt{s}} \frac{1}{2s} \right) d\sigma_{\Sigma(s)} = 0.$$

Again, recall the volume of  $\Sigma(s)$  satisfies  $\operatorname{Vol}(\Sigma(s)) = c\sqrt{s}$  for some constant c by Poposition 2.2, and then

$$\frac{\partial}{\partial s}(d\sigma_{\Sigma(s)}) = \frac{c}{2\sqrt{s}}d\sigma_{\Sigma(s)}.$$

Next, define a function

$$I(s) = \int_{\Sigma(s)} u \cdot \frac{1}{\sqrt{s}} d\sigma_{\Sigma(s)}.$$

and then we can compute the derivative of I(s) as follows:

$$\begin{split} I'(s) &= \frac{\partial}{\partial s} \int_{\Sigma(s)} u \cdot \frac{c}{2\sqrt{s}} d\sigma_{\Sigma(s)} \\ &= \int_{\Sigma(s)} \langle \nabla u, \frac{\nabla f}{|\nabla f|^2} \rangle \frac{1}{\sqrt{s}} d\sigma_{\Sigma(s)} + \int_{\Sigma(s)} u \frac{\partial}{\partial s} \left( \frac{1}{\sqrt{s}} d\sigma_{\Sigma(s)} \right) \\ &= \int_{\Sigma(s)} \langle \nabla u, \frac{\nabla f}{|\nabla f|^2} \rangle \frac{1}{\sqrt{s}} d\sigma_{\Sigma(s)} \\ &= \frac{1}{s} \int_{\Sigma(s)} \langle \nabla u, \frac{\nabla h}{|\nabla h|} \rangle d\sigma_{\Sigma(s)}, \end{split}$$

where  $\frac{\nabla f}{\nabla f} = \frac{\nabla h}{|\nabla h|}$  and  $|\nabla f| = \sqrt{s}$  were used in the last equality.

Moreover, since I(s) tends to zero as  $s \to \infty$ , there exists b > a such that  $I'(b) \leq 0$ , i.e.

$$\int_{\Sigma(b)} \langle \nabla u, \frac{\nabla h}{|\nabla h|} \rangle d\sigma_{\Sigma(b)} \le 0;$$

this finish the proof of (6.26).

The Proof of Theorem 1.1. Notice that Prosition 6.1 also holds on  $M \setminus D(b)$ , together with Prosition 6.2 implies that

$$\lambda_1 + \lambda_2 = 0$$

on  $M \setminus D(b)$  for some b. This implies that

$$\lambda_3 = \lambda_4 = \lambda_5 \equiv \frac{1}{2}$$

on  $M \setminus D(b)$ . Hence the function

$$G = tr(Ric^3) - \frac{1}{2}|Ric|^2,$$

is 0 on  $M \setminus D(b)$ .

Because G is an analytic function, has to be zero, we obtain that  $G \equiv 0$  on M. Moreover, the equation  $0 = \Delta_f R = R - 2|Ric|^2$  implies that

$$G = tr(Ric^{3}) - |Ric|^{2} + \frac{1}{4}R$$
  
=  $\sum_{i=1}^{5} (\lambda_{i} - \frac{1}{2})^{2} \lambda_{i} = 0.$ 

Finally we get  $\lambda_1 = \lambda_2 \equiv 0$  and  $\lambda_3 = \lambda_4 = \lambda_5 \equiv \frac{1}{2}$  due to  $Ric \geq 0$  and the continuity of  $\lambda_1 + \lambda_2$ . This implies the Ricci curvature has constant rank 3. Therefore, any 5-dimensional shrinking gradient Ricci soliton with  $R = 3\lambda$  is rigid isometric to a finite quotient of  $\mathbb{R}^2 \times \mathbb{S}^3$  by [23]. We have completed the proof of Theorem 1.1.

Acknowledgements. The authors would like to thank Professor Xi-Nan Ma, Professor Huai-Dong Cao, Professor Yongjia Zhang, Professor Yu Li, Professor Xiaolong Li and Professor Mijia Lai for helpful discussions.

#### References

- R. H. Bamler, C. Cifarelli, R. J. Conlon, A. Deruelle, A new complete two-dimensional shrinking gradient Kähler-Ricci soliton, Geom. Funct. Anal., 34(2024), no.2, 377-392.
- [2] B. L. Chen, Strong uniqueness of the Ricci flow, J. Differential Geom. 82 (2009), no. 2, 363-382.

- [3] S. Brendle, Rotational symmetry of self-similar solutions to the Ricci flow, Invent. Math. 194 (2013), no. 3, 731-764.
- [4] S. Brendle, Rotational symmetry of Ricci solitons in higher dimensions, J. Differential Geom. 97 (2014), no. 2, 191-214.
- [5] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), no. 3, 271-297.
- [6] H. D. Cao, Existence of gradient Kähler-Ricci solitons, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1-16.
- H. D. Cao, Q. Chen, On locally conformally flat gradient steady Ricci solitons, Trans. Amer. Math. Soc. 364 (2012), no. 5, 2377-2391.
- [8] H. D. Cao, Q. Chen, On Bach-flat gradient shrinking Ricci solitons, Duke Math. J. 162 (2013), no. 6, 1149-1169.
- [9] H. D. Cao, B. L. Chen, X. P. Zhu, Recent developments on Hamilton's Ricci flow, Surveys in differential geometry. Vol. XII. Geometric flows, 47-112.
- [10] H. D. Cao, D. T. Zhou, On complete gradient shrinking Ricci solitons, J. Differential Geom. 85 (2010), no. 2, 175-185.
- [11] H. D. Cao, J. M. Xie, Four-dimensional complete gradient shrinking Ricci solitons with half positive isotropic curvature, Math. Z., 305 (2023), no.2, 22 pp.
- [12] X. D. Cao, B. Wang, Z. Zhang, On locally conformally flat gradient shrinking Ricci solitons, Commun. Contemp. Math. 13 (2011), no. 2, 269-282.
- [13] G. Carron, M. Herzlich, Conformally flat manifolds with nonnegative Ricci curvature Compos. Math. 142 (2006), no. 3, 798-810.
- [14] R. J. Conlon, A. Deruelle, S. Sun, Classification results for expanding and shrinking gradient Kähler-Ricci solitons, Geem. Topol. 28(2024), no. 1, 267-351.
- [15] J. Cheeger, D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geometry 6 (1971/72), 119-128.
- [16] W. X. Chen, C. M. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991), no. 3, 615-622.
- [17] X. X. Chen, Y. Q. Wang, On four-dimensional anti-self-dual gradient Ricci solitons, J. Geom. Anal. 25 (2015), no. 2, 1335-1343.
- [18] B.L. Chen, X.P. Zhu, Uniqueness of the Ricci flow on complete noncompact manifolds, J. Differential Geom.,74(2006), no. 1, 119-154.
- [19] X. Cheng, D. T. Zhou, Rigidity of Four-dimensional gradient shrinking Ricci solitons, J. Reine Angew. Math., 802 (2023), no. 2, 255-274.
- [20] C. Cifarelli, R. J. Conlon, A. Deruelle, On finite time Type-I singularities of the Kähler-Ricci flow on compact Kähler surfaces, arXiv:2203.04380v3.
- [21] M. Feldman, T. Ilmanen, D. Knopf, Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons, J. Differential Geom. 65 (2003), no. 2, 169-209.
- [22] P. Guan, P. Lu, Y. Xu, A rigidity theorem for codimension one shrinking gradient Ricci solitons in  $\mathbb{R}^{n+1}$ , Calc. Var. Partial Differential Equations, 54 (2015), no. 4, 4019-4036.
- [23] M. Fernández-López, E. García-Río, On gradient Ricci solitons with constant scalar curvature, Proc. Amer. Math. Soc. 144 (2016), no. 1,369-378.

- [24] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no. 2, 255-306.
- [25] N. Koiso, On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics, Recent topics in differential and analytic geometry, 327-337.
- [26] B. Kotschwar, On rotationally invariant shrinking Ricci solitons, Pacific J. Math. 236 (2008), no. 1, 73-88.
- [27] Y. Li, B. Wang, *Heat kernel on Ricci shrinkers*, Calc. Var. Partial Differential Equations 59 (2020), no. 6, Paper No. 194, 84 pp.
- [28] Y. Li, B. Wang, On Kähler Ricci shrinker surfaces, arXiv:2301.09784.
- [29] M. Eminenti, G. LaNave, C. Mantegazza, Ricci solitons: the equation point of view, Manuscripta Math. 127 (2008), no. 3, 345-367.
- [30] O. Munteanu, J. P. Wang, Geometry of shrinking Ricci solitons, Compos. Math. 151 (2015), no. 12, 2273-2300.
- [31] O. Munteanu, J. P. Wang, Positively curved shrinking Ricci solitons are compact, J. Differential Geom. 106 (2017), no. 3, 499-505.
- [32] A. Naber, Noncompact shrinking four solitons with nonnegative curvature, J. Reine Angew. Math. 645 (2010), 125-153.
- [33] L. Ni, N. Wallach, On a classification of gradient shrinking solitons, Math. Res. Lett. 15 (2008), no. 5, 941-955.
- [34] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159v1.
- [35] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv: math/0303109.
- [36] P. Petersen, W. Wylie, On the classification of gradient Ricci solitons, Geom. Topol. 14 (2010), no. 4, 2277-2300.
- [37] P. Petersen, W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. 241 (2009), no. 2, 329-345.
- [38] S. Pigola, M. Rimoldi, A. Setti, Remarks on non-compact gradient Ricci solitons, Math. Z. 268 (2011), no. 3-4, 777-790.
- [39] J. Y. Ou, Y. Y. Qu, G. Q. Wu, Some rigidity results on shrinking gradient Ricci soliton, arXiv:2411.06395.
- [40] X. J. Wang, X. H. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (2004), no. 1, 87-103.
- [41] G. Q. Wu, S. J. Zhang, Remarks on shrinking gradient Kähler-Ricci solitons with positive bisectional curvature, C. R. Math. Acad. Sci. Paris 354 (2016), no. 7, 713-716.
- [42] P. Wu, J. Y. Wu, W. Wylie, Gradient shrinking Ricci solitons of half harmonic Weyl curvature, Calc. Var. Partial Differential Equations 57 (2018), no. 5, Art. 141, 15 pp.
- [43] Z. H. Zhang, Gradient shrinking solitons with vanishing Weyl tensor, Pacific J. Math. 242 (2009), no. 1, 189-200.
- [44] S.H. Zhu, The classification of complete locally conformally flat manifolds of nonnegative Ricci curvature, Pacific J. Math. 163 (1994), no. 1, 189-199.

(Fengjiang Li) MATHEMATICAL SCIENCE RESEARCH CENTER, CHONGQING UNI-VERSITY OF TECHNOLOGY, CHONGQING 400054, CHINA *Email address*: fengjiangli@cqut.edu.cn

(Jianyu Ou) Department of Mathematics, Xiamen University, Xiamen 361005, China

Email address: oujianyu@xmu.edu.cn

(Yuanyuan Qu) School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, China

 $Email \ address: \ {\tt 52285500012@stu.ecnu.edu.cn}$ 

(Guoqiang Wu) School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China

Email address: gqwu@zstu.edu.cn