COVERING CONDITIONS FOR IDEALS IN SEMIRINGS

PEYMAN NASEHPOUR

ABSTRACT. In this paper, we prove prime avoidance for ringoids. We also generalize McCoy's and Davis' prime avoidance theorems in the context of semiring theory. Next, we proceed to define and characterize compactly packed semirings and show that a commutative semiring is compactly packed if and only if each prime ideal is the radical of a principal ideal. Finally, we calculate the set of zero-divisors of some monoid semimodules over compactly packed semirings in terms of their prime ideals.

0. INTRODUCTION

Prime avoidance is a fundamental result in commutative ring theory, discussed in many commutative algebra textbooks including [11, 16, 24, 25, 27], which has applications in algebraic geometry (cf. [10, p. 200], [17, p. 204], [22, p. 555], [50, p. 44], and [56]), algebraic number theory ([1, p. 65]), group schemes of finite type (cf. [30, p. 597]) and homological algebra (cf. [8, p. 145] and [46, p. 490]).

In his celebrated paper, McCoy [29, Theorem 1] proved that if I and $\{A_i\}_{i=1}^n$ are ideals of a commutative ring R with $I \subseteq \bigcup_{i=1}^n A_i$ such that I is not contained in the union of any $n-1$ of ideals A_i, then there is a positive integer k such that $I^k \subseteq \bigcap_{i=1}^n A_i$. A corollary to this beautiful result is a covering condition for radical ideals in this sense that if an ideal I is contained in a union of finitely many radical ideals $\{A_i\}_{i=1}^n$, then I is contained in at least one of them [28, p. 163-4]. What is known as "prime avoidance" in commutative algebra is, in fact, a corollary to the latter result, and this is why Kaplansky attributes it to McCoy. However, the only book that the author found discussing prime avoidance in the context of noncommutative algebra is Rowen's work (see Proposition 2.12.7 in [47]). The main purpose of this paper is to investigate these covering conditions in ringoid theory, especially within the framework of semiring theory. Since the language of ringoid theory is not standardized yet, we need to establish some terminology.

Let us recall that a set M with a map $M \times M \to M$ defined by $(x, y) \mapsto xy$ is a magma [52, Definition 1.1]. A bimagma $(R, +, \cdot)$ is a ringoid [45, p. 206] if multiplication "·" distributes on addition "+" from both sides, i.e., for all r, s, and t in R,

$$r(s + t) = rs + rt \text{ and } (s + t)r = sr + tr.$$

If $(R, +, \cdot)$ is a ringoid then $(R, +)$ and $(R, \cdot)$ are the additive and multiplicative magmas of R, respectively. Ringoids are the most general among all ring-like algebraic structures although different from the so-called ringoids investigated in [51]. In §1, we discuss ringoids and provide examples, including nonassociative semirings

and Newman algebras (check Theorem 1.4). In §2, we discuss ideals of ringoids. Similar to semiring theory, we say that a nonempty subset I of a ringoid R is a left (right) ideal of R if $(I, +)$ is a submagma of $(R, +)$ and $ra \in I$ ($ar \in I$), for all $r \in R$ and $a \in I$. A nonempty subset I of a ringoid R is an ideal of R if it is both a left and a right ideal of R. We collect all left (right) ideals of a ringoid R in $\mathrm{Id}_l(R)$ ($\mathrm{Id}_r(R)$). Also, we collect all ideals of a ringoid R in $\mathrm{Id}(R)$.

A left (right) ideal I of a ringoid R is subtractive if $x + y \in I$ and $x \in I$ imply that $y \in I$, and $x + y \in I$ and $y \in I$ imply that $x \in I$, for all $x, y \in R$ (see Definition 2.9). A ringoid is subtractive if each of its ideals is subtractive. In Theorem 2.12, we show that if A, B, and C are left (right) ideals of a ringoid $(R, +, \cdot)$ such that $C \subseteq A \cup B$ and A and B are subtractive, then either $C \subseteq A$ or $C \subseteq B$.

A left (right) ideal I of a ringoid R is proper if $I \neq R$. Inspired by the definition of prime ideals in nonassociative ring theory (see Definition 1 in [4]), we say that a proper ideal P of a ringoid R is prime if $(a)(b) \subseteq P$ implies either $a \in P$ or $b \in P$, for all a and b in R (see Definition 2.16). We collect all prime ideals of a ringoid R in $\mathrm{Spec}(R)$.

In Theorem 2.18, we prove that a proper ideal P of a ringoid R is prime if and only if $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$, for all ideals I and J of R. In Theorem 2.19, we verify that if $\{P_i\}_{i=1}^n$ is a family of prime ideals of a ringoid R and I is an ideal of R such that for each $i \in \mathbb{N}_n$ there is an element $a_i \in I$ with $a_i \in P_i$ but $a_i \notin P_k$ for each $k \neq i$, then for each $l \in \mathbb{N}_n$, there is an element b_l belonging to I and each P_i except P_l. We use this, to generalize prime avoidance for ringoids as follows:

Let $P_1, \ldots, P_n$ be subtractive prime ideals of a ringoid $(R, +, \cdot)$. Let I be an ideal of R with $I \nsubseteq P_i$, for each $i \in \mathbb{N}_n$. Then, there is an element $a \in I$ avoiding to be in any of the P_is, i.e., $a \in I \setminus \bigcup_{i=1}^n P_i$ (see Theorem 3.1). Note that this is a generalization of Behren's prime avoidance for nonassociative rings [4, Satz 3].

Semirings, which have recently gained the attention of many algebraists and computer scientists, are interesting generalizations of rings and bounded distributive lattices. They also have essential applications across various fields in science and engineering [19, 20, 21, 23]. In this paper, a ringoid $(S, +, \cdot)$ is a semiring if $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, and 0 ($\neq 1$) is an absorbing element of $(S, \cdot)$, i.e., $s \cdot 0 = 0 = 0 \cdot s$, for all $s \in S$. If all above conditions are satisfied, but the multiplication is not necessarily associative, we say that S is a nonassociative semiring.

In §3, we prove a generalization of prime avoidance in another direction. In Theorem 3.3, as a generalization to prime avoidance in noncommutatve algebra [47, Proposition 2.12.7], we prove that if an ideal of a semiring S is contained in a union of n subtractive ideals of S and at least $n - 2$ of them are prime, then it is contained in some of them. Note that the topological interpretation of the given result is that if S is a subtractive semiring and a finite number of points are contained in an open subset then they are contained in a smaller principal open subset (check Theorem 3.5).

Next in this section, we proceed to prove Davis' version for prime avoidance in the context of semiring theory (check Theorem 3.6). Note that Davis' version of prime avoidance in commutative ring theory has some applications in grades of ideals (see Theorem 124 and Theorem 125 in [24]).

In §4, we generalize McCoy's results on finite unions of ideals in commutative semirings. Note that a semiring S is commutative if $ab = ba$, for all a and b in S. In Theorem 4.2, we show that if $n \geq 3$ is a positive integer, and I and $\{A_i\}_{i=1}^n$ are ideals of a subtractive commutative semiring S such that the covering $I \subseteq \bigcup_{i=1}^n A_i$ is efficient, then there is a positive integer k such that

$$I^k \subseteq \bigcap_{i=1}^n A_i.$$

Two corollaries to this result are two covering conditions which we call them "McAdam's radical ideal avoidance for semirings" (see Corollary 4.3) and "McCoy's semiprime avoidance for semirings" (see Corollary 4.4). Recall that a covering $I \subseteq \bigcup_{i=1}^n A_i$ is efficient if I is not contained in the union of any $n-1$ of A_is [28, §2].

We devote §5 to "arbitrary prime avoidance property" (see [12] and Definition 2.2 in [31]) which is defined in the context of semiring theory as follows:

- For an arbitrary family of prime ideals $\{P_\alpha\}$ and any ideal I of a semiring S, the inclusion $I \subseteq \bigcup_\alpha P_\alpha$ implies $I \subseteq P_\alpha$, for some α.

The ring version of this property has been investigated in [43, 44, 53, 55]. In Definition 5.1, we call semirings with this covering condition "compactly packed", and next, we prove that the following statements are equivalent (see Theorem 5.3) for a commutative semiring S:

(1) The semiring S is compactly packed.
(2) For an arbitrary family of prime ideals $\{P_\alpha\}$ and any prime ideal Q of S, the inclusion $Q \subseteq \bigcup_\alpha P_\alpha$ implies $Q \subseteq P_\alpha$, for some α.
(3) Each prime ideal of S is the radical of a principal ideal in S.
(4) Each radical ideal is the radical of a principal ideal.

A semiring is considered proper if it is not a ring [23, p. 9]. Our generalizations will be more meaningful if we can provide some compactly packed proper semirings. This task is accomplished in Example 5.2, Proposition 5.5, and Proposition 5.6.

Let S be a semiring. A commutative monoid $(M, +, 0)$ is, by definition, an S-semimodule [20, §14] if there exists a scalar multiplication function

$$\lambda : (S, M) \to M \quad \text{defined by} \quad \lambda(s, m) = sm,$$

with the following properties for all $s, t \in S$ and $m, n \in M$:

(1) $(st)m = s(tm)$ and $1m = m$,
(2) $(s+t)m = sm + tm$ and $s(m+n) = sm + sn$,
(3) $0m = 0$ and $s0 = 0$.

In §6, we discuss zero-divisors on semimodules. Recall that an element s in a commutative semiring S is a zero-divisor on the S-semimodule M if there is a nonzero element m in M such that $sm = 0$. All zero-divisors on M are collected in $Z(M)$. In Proposition 1.15 in [2], it is proved that the set of zero-divisors of a commutative ring is a union of radicals of annihilator ideals. As a generalization to this result, we show that the set of zero-divisors of a semimodule over a commutative semiring is a union of radicals of M-annihilator ideals of S (see Proposition 6.3). By definition, an ideal I of S is an M-annihilator ideal if $I = \text{Ann}(X)$, for some nonempty $X \subseteq M$.

Similar to module theory, an S-semimodule M over a commutative semiring S has a.c.c. (d.c.c.) on its M-annihilator ideals if any ascending (descending) chain

of M-annihilator ideals of S terminates at some point. Recall that a semimodule M over a commutative semiring S has "very few zero-divisors" if the set of zero-divisors $Z(M)$ of M is a finite union of prime ideals in $\mathrm{Ass}(M)$ (see Definition 2.7 in [33]). As a generalization of Corollary 2.9 in [33], we show that if S has a.c.c. and d.c.c. on its M-annihilator ideals, then M has very few zero-divisors (see Theorem 6.5).

Recall that a prime ideal P of a commutative semiring S is an associated prime ideal of an S-semimodule M if there is an $m \in M$ such that

$$P = \mathrm{Ann}(m) = \{s \in S : sm = 0\}.$$

All associated prime ideals of a semimodule M are collected in $\mathrm{Ass}(M)$ [33]. An S-semimodule M has Property (A) if each finitely generated ideal I of S with $I \subseteq Z(M)$ has a nonzero annihilator in M (Definition 2.14 in [33]). In Theorem 6.7, we use semiring version of prime avoidance to prove that if I is an ideal of a commutative semiring S and M a Noetherian S-semimodule, then $I \subseteq Z(M)$ implies that $I \subseteq P$, for some $P \in \mathrm{Ass}(M)$. This fact that a Noetherian module over a Noetherian commutative ring has Property (A) is considered "among the most useful" results in commutative algebra by Kaplansky [24, p. 56].

In §6, we proceed to generalize another result of Davis and prove that if a commutative semiring S has few zero-divisors, then its total quotient semiring $Q(S)$ is semi-local (check Theorem 6.11). Recall that a commutative semiring is semi-local if it has finitely many maximal ideals. Moreover, in Definition 6.9, we say a commutative semiring S has few zero-divisors if $Z(S)$ is a finite union of subtractive prime ideals of S. A corollary to Theorem 6.11 is that if a commutative semiring S has a.c.c. on its annihilator ideals, then $Q(S)$ is a semi-local Kasch semiring (see Corollary 6.14). Note that we say a commutative semiring S is a Kasch semiring if each maximal ideal of S is of the form $\mathrm{Ann}(x)$, for some $x \in S$ (see Definition 6.13).

In the final phase of §6, we prove that if S is a compactly packed commutative semiring, M an S-semimodule with Property (A) and $Z(M) = \bigcup_{\alpha \in A} P_\alpha$, where P_αs are prime ideals of S, and G is a cancellative torsion-free commutative monoid, then $Z(M[G]) = \bigcup_{\alpha \in A} P_\alpha[G]$ (see Theorem 6.16). Similarly in Theorem 6.17, we show that if S is a compactly packed Noetherian commutative semiring, M an S-semimodule, and G is a cancellative torsion-free commutative monoid, then $Z(M[G]) = \cup_{P \in \mathrm{Ass}(M)} P[G]$.

Golan's book [20] is a general reference for semiring theory, and our terminology closely follows it.

1. Ringoids and some examples

A bimagma $(R, +, \cdot)$ is a ringoid [45, p. 206] if multiplication "$\cdot$" distributes on addition "$+$" from both sides. We say a ringoid R is with zero if there is an element 0 in R such that 0 is a neutral element for the magma $(R, +)$ and an absorbing element for the magma $(R, \cdot)$.

Now, let $\{R_\alpha\}_\alpha$ be a family of ringoids. Define addition and multiplication on the direct product $\prod_\alpha R_\alpha$ of the given ringoids componentwisely. Additionally, let each ringoid R_α be with zero. Define their direct sum $\bigoplus_\alpha R_\alpha$ consisting those elements in $\prod_\alpha R_\alpha$ whose components are zero except finitely many of them.

Proposition 1.1. *If $\{R_\alpha\}_\alpha$ is a family of ringoids, then their direct product $\prod_\alpha R_\alpha$ equipped with componentwise addition and multiplication is also a ringoid. On the other hand, if each R_α is with zero, then $\prod_\alpha R_\alpha$ is with zero and $\bigoplus_\alpha R_\alpha$ is a subringoid of $\prod_\alpha R_\alpha$.*

Proof. Straightforward. $\square$

Examples 1.2. In the following, we give some examples of ringoids:
   (1) Examples of ringoids with zero include neorings introduced in §2 of Bruck's paper [9].
   (2) We say a ringoid H with zero is an NA-hemiring (i.e., a nonassociative hemiring) if $(H, +, 0)$ is a commutative monoid. Note that if H is an NA-hemiring and X an indeterminate over H, then the set of all polynomials $H[X]$ and formal power series $H[[X]]$ are NA-hemirings.

Definition 1.3. Let $(R, +, \cdot)$ be a ringoid.
   (1) R is multiplicatively unital, if there is an element $1 \in R$ with $1r = r1 = r$, for all $r \in R$.
   (2) R is multiplicatively idempotent if $rr = r$, for all $r \in R$.
   (3) A multiplicatively unital ringoid R with zero is complemented if for any $r \in R$, there is a unique $r' \in R$ with

$$r \cdot r' = r' \cdot r = 0 \text{ and } r + r' = r' + r = 1.$$

   (4) A multiplicatively unital ringoid $(S, +, \cdot)$ with zero is an NA-semiring (i.e., a nonassociative semiring [14]) if $0 \neq 1$ and $(S, +, 0)$ is a commutative monoid. One may consider the family of NA-semirings a generalization of NA-rings discussed extensively in Schafer's classical book [49].
   (5) If the multiplication of an NA-semiring S is associative by change, then S is simply called a semiring [20].

Now, we proceed to give a particular family of NA-semirings called in the literature Newman algebras [6, 13*]. Let us recall that an algebraic system $(N, +, \cdot, 0, 1)$ is a Newman algebra [41, 42] if the following conditions hold:
   (1) $(N, +, 0)$ is a unital magma,
   (2) The element 1 is a right identity element for the magma $(N, \cdot)$,
   (3) Multiplication distributes on addition from both sides,
   (4) For any $a \in N$, there is an element $a' \in N$ with $aa' = 0$ and $a + a' = 1$.

Theorem 1.4. *Any Newman algebra $(S, +, \cdot, 0, 1)$ with $0 \neq 1$ is a complemented multiplicatively idempotent NA-semiring.*

Proof. By T1 on p. 49 in [6], S is multiplicatively idempotent. By T2 and N4' and the discussion between the mentioned results, S is complemented. By T3, the element 0 is an absorbing element of the magma $(S, \cdot)$. By N2', the element 1 is an identity element for $(S, \cdot)$, and so, $(S, \cdot, 1)$ is a unital magma. By Exercise 8 on p. 52 in [6] (for a complete proof, see P17 and P18 in [41]), $(S, +, 0)$ is a commutative monoid and the proof is complete. $\square$

2. Some remarks on ideals of ringoids

Proposition 2.1. *Let $\{I_\alpha\}$ be a family of left (right) ideals of a ringoid R such that $\bigcap_\alpha I_\alpha$ is nonempty. Then, $\bigcap_\alpha I_\alpha$ is a left (right) ideal of R. The smallest left*

(right) ideal of a ringoid R containing a nonempty subset X of R, denoted by $(X)_l$ $((X)_r)$ is

$$(X)_l = \bigcap_{X \subseteq I \in \mathrm{Id}_l(S)} I \quad \left((X)_r = \bigcap_{X \subseteq I \in \mathrm{Id}_r(S)} I \right).$$

Proof. It is straightforward to see that $\bigcap_\alpha I_\alpha$ is a submagma of $(R, +)$. Now, let $r \in R$ and $x \in \bigcap_\alpha I_\alpha$. Since I_α is a left (right) ideal of R for each α, the element rx (xr) is in I_α, for each α. Therefore, rx (xr) $\in \bigcap_\alpha I_\alpha$ showing that $\bigcap_\alpha I_\alpha$ is a left (right) ideal of R. Now, it is evident that $(X)_l = \bigcap_{X \subseteq I \in \mathrm{Id}_l(S)} I$ $((X)_r = \bigcap_{X \subseteq I \in \mathrm{Id}_r(S)} I)$ is the smallest left (right) ideal of R containing X. This completes the proof. $\square$

Corollary 2.2. *If I_α is an ideal of a ringoid R for each α, then their intersection $\bigcap_\alpha I_\alpha$ is also an ideal of R if it is nonempty. In particular, if X is a nonempty subset of R, the smallest ideal of R containing X, denoted by (X), exists and is*

$$(X) = \bigcap_{X \subseteq I \in \mathrm{Id}(S)} I.$$

Remark 2.3. If in a ringoid R, there is an element 0 being an absorbing element of the multiplicative magma of R, i.e., $r0 = 0r = 0$, for all $r \in R$, then each ideal I of R contains the element 0, and so, any intersection of ideals of such ringoids is automatically nonempty.

Definition 2.4. The smallest ideal of a ringoid R containing an element $x \in R$, denoted by (x), is called the principal ideal of R generated by x. A ringoid R is called to be a principal ideal ringoid if each ideal of R is principal.

Similar to ring and semiring theory, the addition of two nonempty subsets I and J of a ringoid R, denoted by $I + J$, is defined as follows:

$$I + J = \{a + b : a \in I, b \in J\}.$$

Definition 2.5. We define a ringoid $(R, +, \cdot)$ to be additively medial if for all a, b, c, and d in R, we have

$$(a + b) + (c + d) = (a + c) + (b + d).$$

Proposition 2.6. *Let I and J be left (right) ideals of an additively medial ringoid R. Then, $I + J$ is also a left (right) ideal of R.*

Proof. Let $a + b$ and $c + d$ be elements of $I + J$. Since R is additively medial, we have

$$(a + b) + (c + d) = (a + c) + (b + d) \in I + J$$

showing that $I + J$ is a submagma of $(R, +)$. Now, let $a + b$ be an element of $I + J$. Since I and J are left (right) ideals of R, we observe that

$$r(a + b) = ra + rb \in I + J \qquad ((a + b)r = ar + br \in I + J).$$

Thus $I + J$ is a left (right) ideal of R and the proof is complete. $\square$

Corollary 2.7. *Let R be a ringoid such that its additive magma is a commutative semigroup. Then, addition of two left (right) ideals of R is a left (right) ideal of R.*

Proof. If the additive magma of R is a commutative semigroup, then the ringoid R is additively medial. $\square$

The multiplication of two nonempty subsets I and J of a ringoid R, denoted by IJ, is defined as follows:
$$IJ = \{ab : a \in I, b \in J\}.$$

Proposition 2.8. *Let I be a right and J be a left ideal of a ringoid R. Then, $IJ \subseteq I \cap J$.*

Proof. Since I is a right ideal of R, we have
$$IJ \subseteq IR \subseteq I.$$
Similarly, since J is a left ideal of R, we have $IJ \subseteq RJ \subseteq J$. This completes the proof. $\qquad\square$

Eilenberg and Schützenberger defined and investigated subtractive submonoids of commutative monoids in [15]. Note that subtractive subsemigroups which have applications in automata theory [48] and subtractive ideals of semirings are defined similarly [20, p. 66]. In the following, we define subtractive ideals of ringoids:

Definition 2.9. Let $(R, +, \cdot)$ be a ringoid. A left (right) ideal I of R is subtractive if $x + y \in I$ and $x \in I$ imply that $y \in I$, and $x + y \in I$ and $y \in I$ imply that $x \in I$, for all $x, y \in R$.

Proposition 2.10. *If I_α is a left (right) subtractive ideal of a ringoid R for each $\alpha \in A$ and $\bigcap_{\alpha \in A} I_\alpha \neq \emptyset$, then $\bigcap_{\alpha \in A} I_\alpha$ is also a subtractive left (right) ideal of R.*

Proof. Straightforward. $\qquad\square$

If the additive operation $+$ on a ringoid $(R, +, \cdot)$ is associative, then the term
$$x_1 + x_2 + \cdots + x_n$$
can be unambiguously computed ([7, Theorem 1]). However, if $+$ is not associative, then the value of the term
$$x_1 + x_2 + \cdots + x_n$$
is ambiguous and in order to clarify the value of the term, we need to specify the order of operations using parentheses. Based on this discussion, let
$$A(x_1, x_2, \ldots, x_n)$$
represent the set of all possible ways to parenthesize the term $x_1 + x_2 + \cdots + x_n$ to be able to compute it unambiguously (cf. Corollary 6.2.3(iii) in [54]).

Proposition 2.11. *Let I be a subtractive left (right) ideal of a ringoid $(R, +, \cdot)$. Let $\{x_i\}_{i=1}^n$ be a finite family of elements in R and x an arbitrary element of the set*
$$A(x_1, x_2, \ldots, x_n).$$
If $x \in I$ and $x_i \in I$ for each $i \neq m$, then $x_m \in I$.

Proof. A straightforward strong induction on n finishes the job. $\qquad\square$

The following is an easy but useful "covering condition" for subtractive ideals of ringoids:

Theorem 2.12. *Let A, B, and C be left (right) ideals of a ringoid $(R, +, \cdot)$ such that $C \subseteq A \cup B$ and A and B are subtractive. Then, either $C \subseteq A$ or $C \subseteq B$.*

Proof. On the contrary, assume that C is not contained in any of A and B. Then, we can find an x in C such that x is not an element of $A \setminus B$, and also, we can find a y in C such that y is not an element of $B \setminus A$. Observe that

$$c = x + y \in C \subseteq A \cup B.$$

Now, if $c \in A$, then from $y \in A$, we obtain that $x \in A$ (because A is subtractive), a contradiction. Also, if $c \in B$, then from $x \in B$, we obtain that $y \in B$ (because B is subtractive), again a contradiction. This completes the proof. □

Theorem 2.13. *Let X be a nonempty subset of a ringoid $(R, +, \cdot)$ with zero. Also, let $(R, \cdot)$ be a semigroup. Then,*

$$\mathrm{Ann}_l(X) = \{r \in R : rX = \{0\}\} \qquad (\mathrm{Ann}_r(X) = \{s \in R : Xs = \{0\}\})$$

is a subtractive left (right) ideal of R.

Proof. We prove the statement for $L = \mathrm{Ann}_l(X)$. The statement for $\mathrm{Ann}_r(X)$ is proved similarly. If $a, b \in L$, then $aX = bX = \{0\}$. Since 0 is a neutral element of the magma $(R, +)$, we have

$$(a + b)X \subseteq aX + bX = \{0\} + \{0\} = \{0\}.$$

This shows that $a + b \in L$. Now, let $r \in R$. Since multiplication of R is associative and 0 annihilates all elements of R, we have

$$(ra)X = r(aX) = r\{0\} = \{0\}.$$

So, $ra \in L$. It follows that $\mathrm{Ann}_l(X)$ is a left ideal of R. Now, suppose that $a + b \in L$. If $a \in L$, then,

$$(a + b)X = \{0\} \text{ and } aX = \{0\}.$$

Using distributivity, we can deduce that $bX = \{0\}$ showing that $b \in L$. Similarly, if $b \in L$, then $a \in L$. Thus $\mathrm{Ann}_l(X)$ is subtractive and the proof is complete. □

Definition 2.14. Let $(R, +, \cdot)$ be a ringoid with zero such that $(R, \cdot)$ is a semigroup. A left (right) ideal I of R is a left annihilator ideal of R if

$$I = \mathrm{Ann}_l(X) \qquad (I = \mathrm{Ann}_r(X)).$$

Proposition 2.15. *Let $(R, +, \cdot)$ be a ringoid with zero such that $(R, \cdot)$ is a semigroup. An arbitrary intersection of left (right) annihilator ideals of R is a left (right) annihilator ideal of R.*

Proof. Let $\{X_\alpha\}$ be a family of nonempty subsets of R. It is easy to verify that

$$\bigcap_\alpha \mathrm{Ann}_l(X_\alpha) = \mathrm{Ann}_l(\cup_\alpha X_\alpha) \text{ and } \bigcap_\alpha \mathrm{Ann}_r(X_\alpha) = \mathrm{Ann}_r(\cup_\alpha X_\alpha).$$

This completes the proof. □

Inspired by the definition of prime ideals for nonassociative rings (see Definition 1 in [4]), we give the following definition:

Definition 2.16. A proper ideal P of a ringoid R is called to be prime if $(a)(b) \subseteq P$ implies either $a \in P$ or $b \in P$, for all a and b in R, where by (x), we mean the principal ideal of R generated by $x \in R$.

Lemma 2.17. *Let P be a prime ideal of a ringoid R. If $a, b \notin P$, then there are a' and b' in the principal ideals (a) and (b), respectively, with $a'b' \notin (a)(b)$.*

Proof. Otherwise $(a)(b) \subseteq P$, and so by primness of P, either $a \in P$ or $b \in P$. $\square$

Theorem 2.18. *Let P be a proper ideal of a ringoid R. Then, the following statements are equivalent:*

(1) *P is a prime ideal of R,*
(2) *$IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$, for all ideals I and J of R.*
(3) *$(IJ) \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$, for all ideals I and J of R, where by (IJ), we mean the smallest ideal of R containing IJ.*

Proof. $(3) \Rightarrow (2)$: Let $IJ \subseteq P$. Note that (IJ) is the smallest ideal containing IJ. Therefore, $(IJ) \subseteq P$. Now, P contains one of the ideals I and J.

$(2) \Rightarrow (3)$: Evident.

$(2) \Rightarrow (1)$: Evident.

$(1) \Rightarrow (2)$: Let I and J be arbitrary ideals of R with $I \nsubseteq P$ and $J \nsubseteq P$. This means that there are $a \in I$ and $b \in J$ with $a, b \notin P$. By Lemma 2.17, there are a' and b' in the principal ideals (a) and (b), respectively, with $a'b' \notin P$. Clearly,

$$a'b' \in (a)(b) \subseteq IJ,$$

and so, $IJ \nsubseteq P$. This completes the proof. $\square$

Theorem 2.19. *Let $\{P_i\}_{i=1}^n$ be a finite family of prime ideals of a ringoid R. Let I be an ideal of R such that for each $i \in \mathbb{N}_n$, there is an element $a_i \in I$ with $a_i \in P_i$ but $a_i \notin P_k$ for each $k \neq i$. Then, for each $l \in \mathbb{N}_n$, there is an element b_l belonging to I and each P_i except P_l.*

Proof. Observe that since P_n is a prime ideal of R, by Lemma 2.17, there are two elements a_1' and a_2' in the principal ideals (a_1) and (a_2), respectively, with $a_1' \cdot a_2' \notin P_n$. Using this, we can find a_2'' and a_3' in the principal ideals $(a_1' \cdot a_2')$ and (a_3), respectively, with $a_2'' \cdot a_3' \notin P_n$. Continuing the process, we can find a_{n-2}'' and a_{n-1}' in the principal ideals $(a_{n-3}' \cdot a_{n-2}')$ and (a_{n-1}), respectively, with $b_n = a_{n-2}'' \cdot a_{n-1}' \notin P_n$. It is evident that the process of constructing the element $b_n = a_{n-2}'' \cdot a_{n-1}'$ guarantees to be an element of I and each P_i with $i < n$. By relabeling, the same argument works to find an element b_l belonging to I and all P_is except P_l, for each $l < n$. This completes the proof. $\square$

Remark 2.20. The proof of Theorem 2.19 is based on Behren's technique used in "Satz 3" in [4].

Let X be a subset of a semiring S. It is straightforward to see that the smallest ideal of S containing X is

$$(X) = \left\{ \sum_{i=1}^n s_i x_i t_i : s_i, t_i \in S, x_i \in X, n \in \mathbb{N} \right\},$$

if X is nonempty and the zero ideal $\{0\}$ if X is empty.

Theorem 2.21. *For a proper ideal P of a semiring S, the following statements are equivalent:*

(1) *P is a prime ideal.*
(2) *$IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$ for any left (right) ideals I and J of S.*
(3) *$xSy \subseteq P$ implies $x \in P$ or $y \in P$ for all $x, y \in S$.*

Proof. (1) $\Rightarrow$ (2): Let I and J be left ideals of S. Since multiplication of S is associative, for the ideals (IS) and (JS) of S, we have

$$(IS)(JS) \subseteq (IJ) \subseteq P.$$

Therefore by Theorem 2.18, either $IS \subseteq P$ or $JS \subseteq P$. Now, since S is a semiring and has a multiplicative identity, we have $I \subseteq IS$ and $J \subseteq JS$. Thus either $I \subseteq P$ or $J \subseteq P$. In a similar way, one may prove the implication for right ideals.

(2) $\Rightarrow$ (3): Let $xSy \subseteq P$. This implies that $(Sx)(Sy) \subseteq SP \subseteq P$. Since Sx and Sy are left ideals of S, we have either $Sx \subseteq P$ or $Sy \subseteq P$. Therefore, either $x \in P$ or $y \in P$.

(3) $\Rightarrow$ (1): If the ideals I and J are not subsets of P, then there are elements x and y in S with $x \in I \setminus P$ and $y \in J \setminus P$. Therefore, by assumption, xSy is not a subset of P. Thus IJ is not a subset of P and the proof is complete. $\qquad\square$

Theorem 2.22 (Krull's Separation Lemma). *Let T be a multiplicatively closed set in a commutative semiring S. Also, let I be an ideal of S with $I \cap T = \emptyset$. Then, the ideal P maximal with respect to exclusion of T exists and is prime.*

Proof. Let T be a multiplicatively closed set and assume that $I \cap T = \emptyset$. Put

$$\Gamma = \{J \in \mathrm{Id}(S) : J \cap T = \emptyset\}.$$

Since Γ is nonempty, by Zorn's lemma Γ possesses an ideal of S maximal with respect to disjointness from T. This proves the existence.

Now, let $ab \in P$ with $a, b \notin P$. So, the ideals $P + Sa$ and $P + Sb$ strictly contain P. Therefore, they need to intersect T which means that there are elements $t_1, t_2 \in T$ of the form $t_1 = s_1 + x_1 a$ and $t_2 = s_2 + x_2 b$ where $s_1, s_2 \in P$ and $x_1, x_2 \in S$. Observe that

$$t_1 t_2 = s_1 s_2 + s_1 x_2 b + x_1 a s_2 + x_1 x_2 ab \in P,$$

contradicting that P does not intersect T. This shows that P is prime [20, Corollary 7.6] and the proof is complete. $\qquad\square$

3. Prime avoidance for ringoids

Theorem 3.1 (Prime avoidance for ringoids). *Let $P_1, \ldots, P_n$ be subtractive prime ideals of a ringoid $(R, +, \cdot)$. Let I be an ideal of R with $I \nsubseteq P_i$, for each $i \in \mathbb{N}_n$. Then, $a \in I \setminus \bigcup_{i=1}^n P_i$, for some $a \in R$.*

Proof. The proof is by strong induction on n. The case $n = 1$ is obvious. The case $n = 2$ is an obvious corollary to Lemma 2.12. Let the statement hold for all positive integer numbers less than $n \geq 3$. By induction's hypothesis, for each $i \in \mathbb{N}_n$, there is an element a_i in I such that a_i is not in P_k for each $k \neq i$. If for at least one i, $a_i \notin P_i$, then we can take $a = a_i$ and we are done. Otherwise, we have $a_i \in P_i$, for each $i \in \mathbb{N}_n$. By Theorem 2.19, for each $l \in \mathbb{N}_n$, there is an element b_l belonging to I and each P_i except P_l. Let a be an element of

$$A(b_1, b_2, \ldots, b_n)$$

obtained from the combination of b_is in the magma $(R, +)$. Since each ideal P_i is subtractive in R, by Proposition 2.11, none of the n ideals $\{P_i\}_{i=1}^n$ contains the element a. However, $a \in I$. This completes the proof. $\qquad\square$

The following is an NA-semiring version of Behren's "Satz 3" in [4]:

Corollary 3.2. *If an ideal I of an NA-semiring S is not a subset of a subtractive prime ideal P_i of S, for each $i \in \mathbb{N}_n$, then I cannot be a subset of the unions of P_is.*

The associativity of multiplication in semirings enables us to extend the prime avoidance in semiring theory in another direction:

Theorem 3.3 (Prime avoidance for semirings). *Let S be a semiring, $\{P_i\}_{i=1}^n$ a finite family of subtractive ideals of S. If for each $i > 2$, P_i is prime and I is an ideal of S with $I \subseteq \bigcup_{i=1}^n P_i$, then $I \subseteq P_i$, for some $1 \le i \le n$.*

Proof. The proof is by strong induction on n. The case $n = 1$ is obvious and the case $n = 2$ is nothing but Lemma 2.12. Now, assume that $n \ge 3$ and $I \subseteq \bigcup_{i=1}^n P_i$ but $x_i \in I \setminus \bigcup_{j \ne i} P_j$, for each i. Since $x_i \in I$, we see that $x_i \in P_i$, for each i. In view of Theorem 2.21, primeness of P_n implies that

$$x_1 S x_2 S \cdots x_{n-2} S x_{n-1} \not\subseteq P_n.$$

So, there is an $x \in x_1 S x_2 S \cdots x_{n-2} S x_{n-1} \setminus P_n$. It follows that $x \in \bigcap_{j=1}^{n-1} P_j$. Since P_j is subtractive, $x + x_n \notin P_j$, for each j which is obviously a contradiction because $x + x_n$ is an element of I. Thus there is an i with

$$I \subseteq \bigcup_{j \ne i} P_j$$

and the result follows by induction and the proof is complete. □

Remark 3.4. Our proof for Theorem 3.3 is inspired by the proof of Proposition 2.12.7 in [47]. We have applied prime avoidance for commutative rings and semirings in some of our previous works including [31, 32, 33, 38, 39, 40]. In Theorem 2.6 in [57], Yeşilot proved a version of Theorem 3.3 for commutative semirings.

For each ideal I of a semiring S, set

$$V(I) = \{P \in \mathrm{Spec}(S) : P \supseteq I\} \text{ and } D(I) = \mathrm{Spec}(S) \setminus V(I).$$

It is easy to verify that $\mathcal{C} = \{V(I) : I \in \mathrm{Id}(S)\}$ is the family of closed sets for a topology on $X = \mathrm{Spec}(S)$, called the Zariski topology [20, p. 89]. The topological interpretation of the prime avoidance is the following:

Theorem 3.5. *Let S be a subtractive semiring. In the Zariski topology on $X = \mathrm{Spec}(S)$, if a finite number of points are contained in an open subset then they are contained in a smaller principal open subset.*

Proof. Let S be a subtractive semiring. In the Zariski topology on $X = \mathrm{Spec}(S)$, consider the finitely many points $\{P_i\}_{i=1}^n$ contained in an open subset $D(I) = X \setminus V(I)$, for some ideal I of S. This means that $I \not\subseteq P_i$, for each $1 \le i \le n$. By Theorem 3.3, there is an element $x \in I$ such that $x \notin \bigcup_{i=1}^n P_i$. It follows that

$$P_i \in D(x) \subseteq D(I), \ 1 \le i \le n,$$

where $D(x) = D((x))$ is the principal open subset smaller than $D(I)$ and the proof is complete. □

Theorem 3.6 (Davis' prime avoidance). *Let x be an element of a semiring S and I an ideal of S. Also, suppose that P_is are subtractive prime ideals of the semiring S. Then, $(x) + I \not\subseteq \bigcup_{i=1}^n P_i$ implies that there is a $y \in I$ with $x + y \notin \bigcup_{i=1}^n P_i$.*

Proof. Without loss of generality, we may assume that $P_j \nsubseteq P_i$, for all $j \neq i$. Set

$$A = \{i \in \mathbb{N}_n : x \in P_i\}.$$

Our claim is that for any index $i \in A$, we have $I \nsubseteq P_i$. Otherwise, $I \subseteq P_i$ and $x \in P_i$ imply that $(x) + I \subseteq P_i$, a contradiction with the assumption $(x) + I \nsubseteq \bigcup_{i=1}^{n} P_i$. Since P_i is a prime ideal of S, from all we said, we deduce that

$$I \cdot \prod_{j \notin A} P_j \nsubseteq P_i, \qquad \forall \, i \in A.$$

Since P_is are subtractive prime ideals of S, using Theorem 3.3, we obtain that

$$I \cdot \prod_{j \notin A} P_j \nsubseteq \bigcup_{i \in A} P_i.$$

Let y be an element of $I \cdot \prod_{j \notin A} P_j \setminus \bigcup_{i \in A} P_i$. By Proposition 2.8, y is in I, and also, in P_j with $j \notin A$, but not in P_i with $i \in A$. Our claim is that $x + y \notin P_i$ for each i. On the contrary, assume that there is an i such that $x + y \in P_i$. If $i \in A$, then by the definition of A, $x \in P_i$, and because P_i is subtractive, $y \in P_i$, a contradiction. If $i \notin A$, then $y \in P_i$, and so, $x \in P_i$ because P_i is subtractive. It follows that $i \in A$, again a contradiction. Hence, $x + y \notin \bigcup_{i=1}^{n} P_i$, as required. $\square$

Remark 3.7. Irving Kaplansky attributes a commutative ring version of Theorem 3.6 to his student Edward Dewey Davis. Davis' prime avoidance has applications in grades of ideals in ring theory (see Theorem 124 and Theorem 125 in [24] and Exercise 16.8 and Exercise 16.9 in [27]).

4. Finite unions of subtractive ideals

McCoy, in his paper [29], investigated finite unions of ideals and proved that if $I \subseteq \bigcup_{i=1}^{n} A_i$ is efficient for some ideals I and A_is of a commutative ring R, then $I^k \subseteq \bigcap_{i=1}^{n} P_i$ for some positive integer k. One of the corollaries of this result is the "prime avoidance", which has been widely applied in various areas of commutative algebra. The main purpose of this section is to generalize some of his results in the context of semiring theory. The following is a semiring version of McCoy's lemma given on p. 634 in [29]:

Lemma 4.1. *Let $n \geq 3$ be a positive integer, and I and $\{A_i\}_{i=1}^{n}$ be subtractive ideals of a commutative semiring S such that $I \subseteq \bigcup_{i=1}^{n} A_i$ is efficient. Then, the intersection of any $n - 1$ of the ideals A_i coincides with $\bigcap_{i=1}^{n} A_i$.*

Proof. Since $I \subseteq \bigcup_{i=1}^{n} A_i$, it is clear that $I = \bigcup_{i=1}^{n} (I \cap A_i)$. Therefore in view of Proposition 2.10, without loss of generality, we may assume that $I = \bigcup_{i=1}^{n} A_i$. Since I is not contained in the union of any $n - 1$ ideals A_i, there is an element a_n in I and in A_n such that

$$a_n \notin \bigcup_{i=1}^{n-1} A_i.$$

Now, let $x \in \bigcap_{i=1}^{n-1} A_i$. The element $x + a_n$ which is in I is not in A_i, for each $i \leq n - 1$ because of subtractivity of A_is. Consequently, $x + a_n$ is an element of A_n. Since A_n is subtractive, $x \in A_n$. Thus $\bigcap_{i=1}^{n-1} A_i \subseteq \bigcap_{i=1}^{n} A_i$ and the proof is complete. $\square$

The following is a semiring version of McCoy's result for commutative rings given on p. 634 in [29]:

Theorem 4.2. *Let $n \geq 3$ be a positive integer, and I and $\{A_i\}_{i=1}^n$ be ideals of a subtractive commutative semiring S such that $I \subseteq \bigcup_{i=1}^n A_i$ is efficient. Then, there is a positive integer k such that*

$$I^k \subseteq \bigcap_{i=1}^n A_i.$$

Proof. Similar to Lemma 4.1, we assume that $I = \bigcup_{i=1}^n A_i$. First we establish the theorem for $n = 3$. Since

$$A_1 \cup A_2 \subseteq A_1 + A_2,$$

from $I = \bigcup_{i=1}^3 A_i$, we obtain that $I \subseteq (A_1 + A_2) \cup A_3$. By Lemma 2.12, if an ideal is contained in the union of two subtractive ideals, it is contained in one of them. Therefore, I is subset of $A_1 + A_2$ because I is not contained in A_3. Similarly, I is contained in $A_2 + A_3$ and $A_3 + A_1$. It follows that

$$I^3 \subseteq (A_1 + A_2)(A_2 + A_3)(A_3 + A_1).$$

If the right side of the above inclusion is multiplied out, then each term consists of a product containing at least two different ideals A_i. On the other hand, a product of ideals is contained in their intersection. Now, in view of Lemma 4.1, we have

$$I^3 \subseteq A_1 \cap A_2 \cap A_3.$$

Now, let the theorem hold for all $3 \leq k < n$. The proof will be complete, if we can deduce that the theorem holds for n. Now, assume that $I \subseteq \bigcup_{i=1}^n A_i$. As above, this implies that

$$(4.1) \qquad I \subseteq (A_1 + A_2) \cup A_3 \cup \cdots \cup A_n.$$

Obviously, either $I \subseteq A_1 + A_2$ or I is contained in the union of some $m < n$ but not in the union of any $m - 1$ of the ideals on the right side of the inclusion (4.1). Since

$$I \nsubseteq A_3 \cup \cdots \cup A_n,$$

one of the remaining m ideals of the right side of (4.1) must be $A_1 + A_2$. Now, by induction's hypothesis, we have $I^{k_1} \subseteq A_1 + A_2$, for some positive integer k_1. A similar argument works for each pair of ideals A_i and A_j with $i < j$. Therefore, there is a positive integer k such that

$$(4.2) \qquad I^k \subseteq \prod_{i<j} (A_i + A_j)$$

Now, if the right side of the inclusion (4.2) is multiplied out, each term consists of a product containing at least $n - 1$ different ideals A_i. Hence, by Lemma 4.1, we have $I^k \subseteq \bigcap_{i=1}^n A_i$, as required. $\qquad \square$

Let us recall that an ideal J of a commutative semiring S is radical if $\sqrt{J} = J$, where by the radical of an ideal J, we mean

$$\sqrt{J} = \{s \in S : \exists n \in \mathbb{N} \ (s^n \in J)\}.$$

Corollary 4.3 (McAdam's radical ideal avoidance for semirings)**.** *Let S be a subtractive commutative semiring and I and $\{J_i\}_{i=1}^n$ be ideals of S. If at least $n-2$ of the ideals in $\{J_i\}_{i=1}^n$ are radical and I is contained in $\bigcup_{i=1}^n J_i$, then I is contained in at least one of the $J_i s$.*

Proof. Straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Let us recall that an ideal P of a semiring S is semiprime if for any ideal I of S, $I^2 \subseteq P$ implies $I \subseteq P$ (see p. 90 in [20]). Obviously, any prime ideal is semiprime. On the other hand, since any semiprime ideal of a commutative semiring is a radical ideal, we have the following:

Corollary 4.4 (McCoy's semiprime avoidance for semirings)**.** *Let S be a subtractive commutative semiring and I and $\{P_i\}_{i=1}^n$ be ideals of S. If at least $n-2$ of the ideals in $\{P_i\}_{i=1}^n$ are semiprime and I is contained in $\bigcup_{i=1}^n P_i$, then I is contained in at least one of the $P_i s$.*

5. COMPACTLY PACKED SEMIRINGS

Definition 5.1. We define a semiring S to be compactly packed if the following covering condition holds for prime ideals of S:

- For an arbitrary family of prime ideals $\{P_\alpha\}$ and any ideal I of S, the inclusion $I \subseteq \bigcup_\alpha P_\alpha$ implies $I \subseteq P_\alpha$, for some α.

Example 5.2. In the following, we give some examples for compactly packed semirings:

(1) Recall that a commutative semiring S is weak Gaussian if and only if each prime ideal of S is subtractive (see Definition 18 and Proposition 19 in [32]). Now, if S is a weak Gaussian semiring with finitely many prime ideals, then by Theorem 3.3, S is compactly packed.

(2) Every principal ideal semiring S (see Definition 2.4) is compactly packed because

$$(x) \subseteq \bigcup_\alpha P_\alpha$$

implies that $x \in P_\alpha$, and consequently, $(x) \subseteq P_\alpha$ for some α. Examples of "proper" principal ideal semirings include subtractive $\mathbb{O}_\infty$-Euclidean semirings (see Corollary 1.5 in [36]) and discrete valuation semirings (see Theorem 3.6 in [37]).

Theorem 5.3. *Let S be a commutative semiring. Then, the following statements are equivalent:*

(1) *The semiring S is compactly packed.*

(2) *For an arbitrary family of prime ideals $\{P_\alpha\}$ and any prime ideal Q of S, the inclusion $Q \subseteq \bigcup_\alpha P_\alpha$ implies $Q \subseteq P_\alpha$, for some α.*

(3) *Each prime ideal of S is the radical of a principal ideal in S.*

(4) *Each semiprime ideal of S is the radical of a principal ideal in S.*

(5) *Each radical ideal is the radical of a principal ideal.*

Proof. $(1) \Rightarrow (2)$: Evident.

$(2) \Rightarrow (3)$: On the contrary, let Q be a prime ideal of S such that $Q \neq \sqrt{\langle s \rangle}$, for all $s \in S$. Observe that

$$\sqrt{\langle s \rangle} = \bigcap_{s \in P \in \mathrm{Spec}(S)} P, \qquad \forall\, s \in S.$$

This means that for each $s \in Q$, there is a prime ideal P_s such that $s \in P_s$ and $Q \nsubseteq P_s$. On the other hand, $Q \subseteq \bigcup_{s \in Q} P_s$.

$(3) \Rightarrow (1)$: First, let $Q = \sqrt{\langle s \rangle}$ be a prime ideal of S included in $\bigcup_\alpha P_\alpha$. This implies that $s \in P_\alpha$, for some α. It is, then, easy to see that

$$Q = \sqrt{\langle s \rangle} \subseteq P_\alpha.$$

Now, let I be an arbitrary ideal of S included in $\bigcup_\alpha P_\alpha$. Note that

$$T = S \setminus \bigcup_\alpha P_\alpha$$

is a multiplicatively closed set in S disjoint from I. By Theorem 2.22, there is a prime ideal P of S with

$$I \subseteq P \subseteq \bigcup_\alpha P_\alpha.$$

Therefore, P, and so, I is a subset of P_α, for some α.

$(1) \Rightarrow (5)$: Let S be a compactly packed semiring and I a radical ideal of S, i.e., $I = \sqrt{I}$. Set

$$D(I) = \mathrm{Spec}(S) \setminus V(I).$$

Clearly, I is not a subset of any prime ideal in $D(I)$. Since S is compactly packed, $I \nsubseteq \bigcup D(I)$. This means that there is an element $x \in I$ which is not in $\bigcup D(I)$. This implies that $V(x) \subseteq V(I)$ because if P is a prime ideal in $V(x)$, then P is not among the prime ideals with $P \nsupseteq I$. This means that P is a prime ideal with $P \supseteq I$. On the other hand, $x \in I$ is equivalent to $(x) \subseteq I$ which implies that $V(I) \subseteq V(x)$. Consequently, $V(x) = V(I)$. In view of Theorem 3.2 in [34], this implies that

$$\sqrt{x} = \bigcap V(x) = \bigcap V(I) = \sqrt{I} = I.$$

Note that (5) implies (4), and (4) implies (3). This completes the proof. $\qquad \square$

Remark 5.4. A ring version of the definition of compactly packed semirings given in Definition 5.1 is due to C.M. Reis and T.M. Viswanathan [44] who proved a version of Theorem 5.3 for Noetherian rings. The ring version of Theorem 5.3 is due to William Walker Smith (see the main theorem of the paper [53]), and Pakala and Shores (see Theorem 1 in [43]).

Proposition 5.5. *Let $(L, +, \cdot)$ be a bounded distributive lattice with finitely many prime ideals. Then, L is a compactly packed semiring.*

Proof. In view of Theorem 3 and Theorem 9 in [32], each ideal of L is subtractive. In view of Example 5.2, L is compactly packed because by assumption, L has finitely many prime ideals. This completes the proof. $\qquad \square$

Proposition 5.6. *Let $T = [0, 1]$ be a finite chain with the smallest element 0 and the largest element 1. Define addition on T as $a + b = \max\{a, b\}$ and multiplication to be null except for the case $a \cdot 1 = 1 \cdot a = a$, for all a and b in T. Then, T is a compactly packaged semiring.*

Proof. By Proposition 21 in [32], each prime ideal of the commutative semiring T is subtractive. Clearly, T is compactly packed because T is a finite semiring and has finitely many prime ideals. This completes the proof. $\square$

The topological interpretation of the covering condition for the compactly packed semirings is the following:

Theorem 5.7. *Let S be a compactly packed semiring. In the Zariski topology on $X = \mathrm{Spec}(S)$, if an arbitrary number of points are contained in an open subset then they are contained in a smaller principal open subset.*

Proof. In view of the Definition 5.1 and the proof given for Theorem 5.3, the proof is essentially the same as the one given in Theorem 3.5, and so, omitted. $\square$

6. Zero-divisors of semimodules over commutative semirings

Let X be a nonempty subset of a semimodule M over a commutative semiring S. Annihilator of X is defined to be

$$\mathrm{Ann}(X) = \{s \in S : sX = \{0\}\}.$$

Proposition 6.1. *Let M be a semimodule over a commutative semiring S. Then, the following statements hold:*

(1) *For any nonempty subset X of M, $\mathrm{Ann}(X)$ is a subtractive ideal of S.*
(2) *For a family of nonempty subsets $\{X_\alpha\}_\alpha$ of M, we have*

$$\bigcap_\alpha \mathrm{Ann}(X_\alpha) = \mathrm{Ann}(\cup_\alpha X_\alpha).$$

Proof. The proofs are similar to the proof of Theorem 2.13 and Proposition 2.15, and so, omitted. $\square$

Remark 6.2. An ideal I of S is an M-annihilator ideal if $I = \mathrm{Ann}(X)$, for some nonempty $X \subseteq M$. By Proposition 6.1, an intersection of M-annihilator ideals is an M-annihilator ideal.

Proposition 6.3. *Let S be a commutative semiring and M an S-semimodule. Then, the set of zero-divisors $Z(M)$ of M is a union of radicals of M-annihilator ideals of S; more precisely,*

$$Z(M) = \bigcup_{0 \neq x \in M} \sqrt{\mathrm{Ann}(x)}.$$

Proof. It is clear that

$$Z(M) = \bigcup_{0 \neq x \in M} \mathrm{Ann}(x) \subseteq \bigcup_{0 \neq x \in M} \sqrt{\mathrm{Ann}(x)}.$$

Now, let $s \in \sqrt{\mathrm{Ann}(x)}$, for some nonzero x in M. This implies that $s^n x = 0$, for some positive integer n. If s is not a zero-divisor on M, then $x = 0$, a contradiction. Thus $s \in Z(M)$ and the proof is complete. $\square$

Remark 6.4. Proposition 6.3, which is a generalization of Proposition 1.15 in [2], states that the set of zero-divisors of a semimodule is a union of radical ideals. This condition that the set of zero-divisors of a module over a commutative ring is a union of prime ideals is of special interests [27, Theorem 6.1]. We investigate this condition in the context of semimodule theory.

The following is a generalization of Corollary 2.9 in [33].

Theorem 6.5. *Let M be a semimodule over a commutative semiring S. If S has a.c.c. and d.c.c. on its M-annihilator ideals, then M has very few zero-divisors.*

Proof. Since S has a.c.c. on its M-annihilator ideals, the maximal elements of the set
$$\mathcal{A} = \{\mathrm{Ann}(x) : x \in M \setminus \{0\}\}$$
exist and are prime ideals of S (Theorem 2.8 in [33]). Let $P_i = \mathrm{Ann}(x_i)$ be the maximal elements of $\mathcal{A}$. Our claim is that the number of such P_is is finite. On the contrary, let $\{P_i\}_{i=1}^{+\infty}$ be among the maximal elements of $\mathcal{A}$. Since S has d.c.c. on its M-annihilator ideals, the descending chain
$$P_1 \supseteq P_1 \cap P_2 \supseteq \cdots$$
of M-annihilator ideals of S terminates at some point, say at r. This implies that
$$P_1 \cap \cdots \cap P_r = P_1 \cap \cdots \cap P_r \cap P_{r+1} \subseteq P_{r+1}.$$
Since P_{r+1} is prime, by Proposition 2.8 and Theorem 2.21, $P_i \subseteq P_{r+1}$ for some $i \leq r$, contradicting the maximality of P_is. Thus $Z(M) = \bigcup_{i=1}^{r} P_i$ and the proof is complete. $\square$

The following is a generalization of Theorem 49 in [32].

Corollary 6.6. *Let S be a commutative semiring. If S has a.c.c. on its annihilator ideals, then S has very few zero-divisors.*

Proof. By Corollary 1.4 in [5], if S has a.c.c. on its annihilator ideals, then S has d.c.c. on its annihilator ideals. Thus by Theorem 6.5, S has very few zero-divisors and the proof is complete. $\square$

Let R be a Noetherian commutative ring with identity and M a Noetherian unital R-module. If an ideal I is a subset of $Z(M)$, then $I \subseteq P$, for some $P \in \mathrm{Ass}(M)$ ([24, Theorem 82]). Kaplansky on p. 56 of his book [24] describes this result "among the most useful in the theory of commutative rings". As an application of prime avoidance for semirings, we generalize this in the context of semimodule theory as follows:

Theorem 6.7. *Let I be an ideal of a commutative semiring S and M an S-semimodule. Also, let S have a.c.c. and d.c.c on its M-annihilator ideals. If $I \subseteq Z(M)$, then $I \subseteq P$, for some $P \in \mathrm{Ass}(M)$.*

Proof. If $I \subseteq Z(M)$, then by Theorem 3.3 (also see §7 in [32]) $I \subseteq P$, for some $P \in \mathrm{Ass}(M)$ and the proof is complete. $\square$

Remark 6.8. Theorem 6.7 provides numerous examples of proper semimodules having Property (A). Recall that an S-semimodule M has Property (A) if each finitely generated ideal $I \subseteq Z(M)$ has a nonzero annihilator in M [33, Definition 2.14]. Also, note that we define a semimodule to be proper if it is not a module.

Davis defines a commutative ring R to have few zero-divisors if $Z(R)$ is a finite union of prime ideals. Next, he proves that a commutative ring R has few zero-divisors if and only if its total quotient ring $Q(R)$ is semi-local (see pages 203 and 204 in [13]).

Definition 6.9. We say a commutative semiring S has few zero-divisors if $Z(S)$ is a finite union of subtractive prime ideals of S.

Remark 6.10. Since any prime ideal in $\mathrm{Ass}(S)$ is of the form of $\mathrm{Ann}(x)$, for some $x \in S$, it is subtractive. Therefore, if a commutative semiring has very few zero-divisors, then it needs to have few zero-divisors! It follows that a commutative semiring S having a.c.c. on its annihilator ideals has (very) few zero-divisors (Corollary 6.6).

Localization of a commutative semiring S at a multiplicative closed set U of S is defined similarly to its counterparts in commutative ring theory [35, §5]. Observe that if S is a commutative semiring, then $S \setminus Z(S)$ is multiplicatively closed in S. The localization of S at $S \setminus Z(S)$ is called to be the total quotient semiring of S, denoted by $Q(S)$. Note that a semiring is semi-local if it has finitely many maximal ideals [35, Definition 3.14].

The following is an extension of Davis' result on rings having few zero-divisors and another example of an application of prime avoidance in semiring theory:

Theorem 6.11. *Let a commutative semiring S have few zero-divisors. Then, its total quotient semiring $Q(S)$ is semi-local.*

Proof. Let S be a commutative semiring. By Theorem 5.4 in [35], the prime ideals of $Q(S)$ is in a one-to-one correspondence with the prime ideals of S disjoint from $S \setminus Z(S)$. Consequently, any prime ideal of $Q(S)$ corresponds to a prime ideal P of S with $P \subseteq Z(S)$. By definition, since S has few zero-divisors,

$$Z(S) = P_1 \cup P_2 \cup \cdots \cup P_n,$$

where P_is are subtractive prime ideals of S. By prime avoidance (Theorem 3.3), we have $P \subseteq P_i$, for some $i \in \mathbb{N}_n$. Therefore, each prime ideal of $Q(S)$ is a subset of a prime ideal of $Q(S)$ obtained by the extension of P_i in $Q(S)$ for some i. This means that the only maximal ideals of $Q(S)$ are the extended ideals of P_is in $Q(S)$. Thus $Q(S)$ is semi-local and the proof is complete. $\qquad\square$

Theorem 6.12. *If $I = \mathrm{Ann}(x)$ for some element x in a commutative semiring S and $U \subseteq S \setminus Z(S)$ is multiplicatively closed in S, then the localization of I at U is $I_U = \mathrm{Ann}(x/1)$ for the element $x/1 \in S_U$.*

Proof. $\subseteq$: Let $s/u \in I_U$, where $s \in I$ and $u \in U$. From $sx = 0$, it follows that

$$s/u \cdot x/1 = sx/u = 0/u = 0.$$

So, $s/u \in \mathrm{Ann}(x/1)$.

$\supseteq$: Now, let $s/u \in \mathrm{Ann}(x/1)$. So, $(s/u)(x/1) = 0/1$. This means that $(sx)/u = 0/1$. Consequently, there is an element $t \in U$ such that $tsx = 0$. Since t is not a zero-divisor on S, we have $sx = 0$ and the proof is complete. $\qquad\square$

A commutative ring R is a Kasch ring if and only if each maximal ideal of R is of the form $\mathrm{Ann}(x)$, for some $x \in R$ (see Definition 8.26 and Corollary 8.28 in [26]). Kasch rings were named after Friedrich Kasch who was a student of Friedrich Karl Schmidt. Inspired by this, we give the following definition:

Definition 6.13. We say a commutative semiring S is a Kasch semiring if each maximal ideal of S is of the form $\mathrm{Ann}(x)$, for some $x \in S$.

Corollary 6.14. *Let a commutative semiring S have a.c.c. on its annihilator ideals. Then, $Q(S)$ is a semi-local Kasch semiring having very few zero-divisors.*

Proof. Since S has a.c.c. on its annihilator ideals, S has very few zero-divisors (Corollary 6.6). This means that $Z(S)$ is a finite union of prime ideals in $\mathrm{Ass}(S)$. Therefore by Theorem 6.11, $Q(S)$ is semi-local. On the other hand, by Theorem 6.12, each maximal ideal of $Q(S)$ is an annihilator of an element in $Q(S)$. Now, since $Z(Q(S)) = \bigcup_{0 \neq x \in Q(S)} \mathrm{Ann}(x)$, it follows that $Q(S)$ has very few zero-divisors and the proof is complete. $\square$

Remark 6.15. In Theorem 8.31 in [26], Lam attributes a ring version of Corollary 6.14 to Carl Faith.

Let S be a commutative semiring and G a commutative monoid. Note that if f is an element of a monoid semiring $S[G]$, its content $c(f)$ is defined to be an ideal of S generated by the coefficients of f. In other words, if

$$f = s_0 + s_1 X^{g_1} + \cdots + s_n X^{g_n}, \qquad s_i \in S, \ g_i \in G$$

is the canonical representation [18, p. 68] of an element f in the monoid semiring $S[G]$, then the content of f is:

$$c(f) = (s_0, s_1, \ldots, s_n).$$

Theorem 6.16. *Let S be a commutative semiring and M an S-semimodule with Property (A) and $Z(M) = \bigcup_{\alpha \in A} P_\alpha$, where P_αs are prime ideals of S. If G is a cancellative torsion-free commutative monoid and S is compactly packed, then $Z(M[G]) = \bigcup_{\alpha \in A} P_\alpha[G]$.*

Proof. ($\subseteq$): If f is a zero-divisor on the monoid semimodule $M[G]$, then by Theorem 2.1 in [33], there is a nonzero element $b \in M$ such that $fb = 0$. This means that

$$c(f) \subseteq Z(M) = \bigcup_{\alpha \in A} P_\alpha.$$

Since S is compactly packed, the ideal $c(f)$ needs to be a subset of one of the prime ideals P_αs. This implies f to be an element of $P_\alpha[G]$.

($\supseteq$): On the other hand, if $f \in P_\alpha[G]$, then $c(f)$ is a subset of P_α. This implies that $c(f) \subseteq Z(M)$. Now, since M has Property (A), $c(f)$ can be annihilated by a nonzero element $b \in M$. Hence, f is a zero-divisor of on $M[G]$, as required. $\square$

Theorem 6.17. *Let S be a Noetherian commutative semiring and M an S-semimodule. If S is compactly packed and G is a cancellative torsion-free commutative monoid, then $Z(M[G]) = \cup_{P \in \mathrm{Ass}(M)} P[G]$.*

Proof. By Corollary 2.9 in [33], $Z(M) = \bigcup_{P \in \mathrm{Ass}(M)} P$.

($\subseteq$): Let $f \in Z(M[G])$. By Theorem 2.1 in [33], $f \cdot b = 0$, for some $b \in M$. This implies that $c(f) \cdot b = 0$. Therefore,

$$c(f) \subseteq Z(M) = \bigcup_{P \in \mathrm{Ass}(M)} P.$$

Since S is compactly packed, we have $c(f) \subseteq P$, for some $P \in \mathrm{Ass}(M)$. Thus $f \in P[G]$, for some $P \in \mathrm{Ass}(M)$.

($\supseteq$): If $f \in P[G]$, for some $P \in \mathrm{Ass}(M)$, $c(f) \subseteq P$, where there is a $b \in M$ with $P = \mathrm{Ann}(b)$. This implies that f is annihilated by b, i.e., $f \in\in Z(M[G])$ and the proof is complete. $\square$

Acknowledgments and Dedicatory

References

[1] Ash, R.B.: A course in algebraic number theory. Mineola, NY: Dover Publications. viii, 112 p. (2010).

[2] Atiyah, M.F., Macdonald, I.G.: Introduction to commutative algebra. Addison-Wesley Series in Mathematics. Boulder: Westview Press. ix, 128 p. (2016).

[3] Behrens, E.-A.: Ring theory. Translated from the German by Clive Reis. Pure and Applied Mathematics (Academic Press) 44. New York, NY: Academic Press. vii, 320 p. (1972).

[4] Behrens, E.-A.: Zur additiven Idealtheorie in nichtassoziativen Ringen. Math. Z. 64, 169–182 (1956).

[5] Behzadipour, H., Nasehpour, P: Some remarks on the comparability of ideals in semirings. Asian-Eur. J. Math. 16, No. 1, Article ID 2350006, 14 p. (2023).

[6] Birkhoff, G.: Lattice theory. Third (new) ed. American Mathematical Society (AMS). Colloquium Publications. Vol. 25. Providence, R.I.: American Mathematical Society. VI, 418 p. (1967).

[7] Bourbaki, N.: Elements of mathematics. Algebra I. Chapters 1–3. Transl. from the French. Berlin: Springer. xxiii, 709 p. (1998).

[8] Brodmann, M.P., Sharp, R.Y.: Local cohomology. An algebraic introduction with geometric applications. 2nd ed. Cambridge Studies in Advanced Mathematics 136. Cambridge: Cambridge University Press. xxii, 491 p. (2012).

[9] Bruck, R.H.: Analogues of the ring of rational integers. Proc. Am. Math. Soc. 6, 50–58 (1955).

[10] Bruns, W., Gubeladze, J.: Polytopes, rings, and K-theory. Springer Monographs in Mathematics. New York, NY: Springer. xiii, 461 p. (2009).

[11] Bruns, W., Herzog, J.: Cohen-Macaulay rings. Rev. ed. Cambridge Studies in Advanced Mathematics. 39. Cambridge: Cambridge University Press. xiv, 453 p. (1998).

[12] Chen, J.: Infinite prime avoidance. Beitr. Algebra Geom. 62, No. 3, 587–593 (2021).

[13] Davis, E.D.: Overrings of commutative rings. II: Integrally closed overrings. Trans. Am. Math. Soc. 110, 196-212 (1964).

[14] Dobbertin, H.: Note on associative Newman algebras. Algebra Univers. 9, 396–397 (1979).

[15] Eilenberg, S., Schützenberger, M.P.: Rational sets in commutative monoids. J. Algebra 13, 173–191 (1969).

[16] Eisenbud, D.: Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics. 150. Berlin: Springer-Verlag. xvi, 785 p. (1995).

[17] Eisenbud, D.: The geometry of syzygies. A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics 229. New York, NY: Springer. xvi, 243 p. (2005).

[18] Gilmer, R.: Commutative semigroup rings. Chicago Lectures in Mathematics. Chicago-London: The University of Chicago Press. XII, 380 p. (1984).

[19] Głazek, K.: A guide to the literature on semirings and their applications in mathematics and information sciences. With complete bibliography.: Dordrecht: Kluwer Academic Publishers. viii, 392 p. (2002).

[20] Golan, J.S.: Semirings and their applications. Dordrecht: Kluwer Academic Publishers. xi, 381 p. (1999).

[21] Gondran, M., Minoux, M.: Graphs and algorithms. Transl. from the French by Steven Vajda. Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication. Chichester: John Wiley & Sons Ltd. XIX, 650 p. (1984).

[22] Görtz, U., Wedhorn, T.: Algebraic geometry I. Schemes. With examples and exercises. 2nd edition. Springer Studium Mathematik – Master. Wiesbaden: Springer Spektrum. vii, 626 p. (2020).

[23] Hebisch, U.; Weinert, H.J.: Semirings. Algebraic theory and applications in computer science. Transl. from the German. Series in Algebra. 5. Singapore: World Scientific Publishing. viii, 361 p. (1998).

[24] Kaplansky, I.: Commutative rings. 2nd revised ed. Chicago-London: The University of Chicago Press. VIII, 182 p. (1974).

[25] Kemper, G.: A course in commutative algebra. Graduate Texts in Mathematics 256. Heidelberg: Springer. xi, 246 p. (2011).

[26] Lam, T.Y.: Lectures on modules and rings. Graduate Texts in Mathematics. 189. New York, NY: Springer. xxiii, 557 p. (1999).

[27] Matsumura, H.: Commutative ring theory. Transl. from the Japanese by M. Reid. Paperback ed. Cambridge Studies in Advanced Mathematics, 8. Cambridge etc.: Cambridge University Press. 336 p. (1989).

[28] McAdam, S.: Finite coverings by ideals, in Ring theory, Proc. Conf. Univ. Oklahoma, 1973, Lecture Notes in Pure and Appl. Math. 7, Dekker, New York (1974), 163–171.

[29] McCoy, N.H.: A note on finite unions of ideals and subgroups. Proc. Am. Math. Soc. 8, 633–637 (1957).

[30] Milne, J.S.: Algebraic groups. The theory of group schemes of finite type over a field. Cambridge Studies in Advanced Mathematics 170. Cambridge: Cambridge University Press. xvi, 644 p. (2017).

[31] Nasehpour, P.: Distinguished elements in semiring extensions. Preprint available at arXiv:1811.02142 [math.AC]. To appear in Journal of Algebra and its Applications.

[32] Nasehpour, P.: On the content of polynomials over semirings and its applications. J. Algebra Appl. 15, No. 5, Article ID 1650088, 32 p. (2016).

[33] Nasehpour, P.: On zero-divisors of semimodules and semialgebras. Georgian Math. J. 28, No. 3, 413–428 (2021).

[34] Nasehpour, P.: Pseudocomplementation and minimal prime ideals in semirings. Algebra Univers. 79, No. 1, Paper No. 11, 15 p. (2018).

[35] Nasehpour, P.: Some remarks on ideals of commutative semirings. Quasigroups Relat. Syst. 26, No. 2, 281–298 (2018).

[36] Nasehpour, P.: Some remarks on semirings and their ideals. Asian-Eur. J. Math. 12, No. 7, Article ID 2050002, 14 p. (2019).

[37] Nasehpour, P.: Valuation semirings. J. Algebra Appl. 17, No. 4, Article ID 1850073, 23 p. (2018).

[38] Nasehpour, P.: Zero-divisors of content algebras. Arch. Math., Brno 46, No. 4, 237–249 (2010).

[39] Nasehpour, P.: Zero-divisors of semigroup modules. Kyungpook Math. J. 51, No. 1, 37–42 (2011).

[40] Nasehpour, P., Payrovi, Sh.: Modules having very few zero-divisors. Commun. Algebra 38, No. 9, 3154–3162 (2010).

[41] Newman, M.H.A.: A characterisation of Boolean lattices and rings. J. Lond. Math. Soc. 16, 256–272 (1941).

[42] Newman, M.H.A.: Relatively complemented algebras. J. London math. Soc. 17, 34–47 (1942).

[43] Pakala, J.V., Shores, T.S.: On compactly packed rings. Pac. J. Math. 97, 197–201 (1981).

[44] Reis, C.M., Viswanathan, T.M.: A compactness property for prime ideals in Noetherian rings. Proc. Am. Math. Soc. 25, 353–356 (1970).

[45] Rosenfeld, A.: An introduction to algebraic structures. San Francisco-Cambridge-London-Amsterdam: Holden-Day. XIV, 285 p. (1968).

[46] Rotman, J.J.: An introduction to homological algebra. 2nd ed. Universitext. Berlin: Springer. xiv, 709 p. (2009).

[47] Rowen, L.H.: Ring theory. Abridged student edition. Boston, MA: Academic Press, Inc. xxviii, 623 p. (1991).

[48] Sakarovitch, J.: Elements of automata theory. Translated from the French by Reuben Thomas. Cambridge: Cambridge University Press. xxiv, 758 p. (2009).

[49] Schafer, R.D.: An introduction to nonassociative algebras. Pure and Applied Mathematics, 22. New York and London: Academic Press. x, 166 p. (1966).

[50] Schenck, H.: Computational algebraic geometry. London Mathematical Society Student Texts 58. Cambridge: Cambridge University Press. xiv, 193 p. (2003).

[51] Sehgal, S.K.: Ringoids with minimum condition. Math. Z. 83, 395–408 (1964).

[52] Serre, J.-P.: Lie algebras and Lie groups. 1964 lectures, given at Harvard University. 2nd ed. Lecture Notes in Mathematics. 1500. Berlin etc.: Springer-Verlag. vii, 168 p. (1992).

[53] Smith, W.W.: A covering condition for prime ideals. Proc. Am. Math. Soc. 30, 451–452 (1971).

[54] Stanley, R.P.: Enumerative combinatorics. Volume 2. Paperback ed. Cambridge Studies in Advanced Mathematics. 62. Cambridge: Cambridge University Press. xii, 585 p. (2001).

[55] Uregen, R.N., Tekir, U., Oral, K.H.: On the union of graded prime ideals. Open Phys. 14, 114–118 (2016).

[56] Vakil, R.: The Rising Sea: Foundations of Algebraic Geometry. Princeton, NJ: Princeton University Press. 632 p. To appear in 2025.

[57] Yeşilot, G: On prime and maximal k-subsemimodules of semimodules. Hacet. J. Math. Stat. 39, No. 3, 305–312 (2010).

Email address: nasehpour@gmail.com

ACADEMIC ADVISOR AND EDUCATION MENTOR, EDUCATION DEPARTMENT, THE NEW YORK ACADEMY OF SCIENCES