# From the self-dual Yang-Mills equation to the Fokas-Lenells equation

Shangshuai Li<sup>1,2,3</sup> Shuzhi Liu<sup>4</sup> Da-jun Zhang<sup>1,2\*</sup>

<sup>1</sup>Department of Mathematics, Shanghai University, Shanghai 200444, China

<sup>2</sup>Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, China

<sup>3</sup>Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University,

Tokyo 169-8555, Japan

<sup>4</sup>School of Statistics and Data Science, Ningbo University of Technology, Ningbo, 315211, China

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#### Abstract

A reduction from the self-dual Yang-Mills (SDYM) equation to the unreduced Fokas-Lenells (FL) system is described in this paper. It has been known that the SDYM equation can be formulated from the Cauchy matrix schemes of the matrix Kadomtsev-Petviashvili (KP) hierarchy and the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy. We show that the reduction can be realized in these two Cauchy matrix schemes, respectively. Each scheme allows us to construct solutions for the unreduced FL system. We prove that these solutions obtained from different schemes are equivalent under certain reflection transformation of coordinates. Using conjugate reduction we obtain solutions of the FL equation. The paper adds an important example to Ward's conjecture on the reductions of the SDYM equation. It also indicates the Cauchy matrix structures of the Kaup-Newell hierarchy.

**Keywords:** self-dual Yang-Mills equation, Fokas-Lenells equation, Cauchy matrix approach, Miura transformation, explicit solution

# 1 Introduction

The self-dual Yang-Mills (SDYM) equation [1], also referred to as the Yang equation in 4dimensional Wess-Zumino-Witten model [2], is an important motion equation in integrable system and twistor theory [3]. A classical form of the SDYM equation is expressed as

$$\partial_{\tilde{z}}((\partial_z J)J^{-1}) - \partial_{\tilde{w}}((\partial_w J)J^{-1}) = 0, \qquad (1.1)$$

where J is a matrix function of  $(z, \tilde{z}, w, \tilde{w}) \in \mathbb{C}^4$ . The SDYM equation allows many reductions towards lower-dimensional classical integrable equations. For example, it can be reduced to the Ablowitz-Kaup-Newell-Segur (AKNS) system (see section 2 in [4] for the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS) and sine-Gordon equations), the Painlevé equations [5], the chiral field equations [6] and so on. More than that, there is a well-known conjecture proposed by Ward [7]:

<sup>\*</sup>Corresponding author. Email: djzhang@staff.shu.edu.cn

... many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalizations) by reduction.

The purpose of this paper is to build a connection between the SDYM equation and the Fokas-Lenells (FL) equation, which reads

$$u_{xt} - u - 2\mathbf{i}|u|^2 u_x = 0, (1.2)$$

where u = u(x, t) is a complex-valued function,  $|u|^2 = uu^*$ ,  $u^*$  is the conjugate of u, variables x, t are real coordinates, and i is the pure imaginary unit. It bears the names of Fokas and Lenells because Fokas derived it in 1995 from the Hamiltonian triplet of the NLS equation [8] and later Lenells and Fokas initiated the research of it in a sequence of papers [9–11]. They have shown that the equation belongs to the Kaup-Newell hierarchy [9] and it is exactly a reduction of the first member of negative potential Kaup-Newell hierarchy [10]:

$$u_{xt} - u - 2\mathbf{i}uvu_x = 0, \tag{1.3a}$$

$$v_{xt} - v + 2\mathbf{i}vuv_x = 0, \tag{1.3b}$$

which we call the pKN(-1) system for short. It is worth noting that the FL equation (1.2) is equivalent to the two-dimensional massive Thirring model and the pKN(-1) system is also known as Mikhailov model. In 1976 [12], Mikhailov found a Lax pair of the massive Thirring model which is written in light-cone coordinates as

$$\mu_x + \mathrm{i}\nu + \mathrm{i}|\nu|^2 \mu = 0,$$
  
$$\nu_t + \mathrm{i}\mu + \mathrm{i}|\mu|^2 \nu = 0,$$

and which describes interaction of two states of a fermion. For more details of the connection between the FL equation and the massive Thirring model, one can refer to [13–15] or the Appendix A in [16]. As an integrable system, the FL equation has attracted attentions from various aspects, for example, the inverse scattering transform and Riemann-Hilbert method [9,17–19], bilinear approach [16,20–22], Darboux transformation [23–25] and so on.

In this paper, we aim to reduce the SDYM equation to the FL equation. We will introduce a constraint such that we can get the pKN(-1) system (1.3) from a general SDYM equation. Then, we will show that the constraint can be realized based on our recent research of the SDYM equation using the Cauchy matrix approach. Recently, we have successfully formulated the SDYM equation (1.1) with two types of Cauchy matrix structures [26,27]. The Cauchy matrix approach is a direct method that allows us to construct integrable equations together with their Lax pairs and solutions with Cauchy matrix structure through investigating the Sylvester-type equations. This approach was first established by Nijhoff et al. to study integrable quadrilateral (discrete) equations [28] in the Adler-Bobenko-Suris list [29]. In their method (see section 2.1 in [28]), they introduced a dressed Cauchy matrix:

$$\boldsymbol{M} \doteq (M_j)_{i,j=1,\dots,N}, \qquad M_{ij} \doteq \frac{\rho_i c_j}{k_i + k_j}, \tag{1.4}$$

where  $k_i, c_i$  are constants,  $\rho_i = \rho_i(n, m)$  is a discrete plane wave factor defined as

$$\rho_i \doteq \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m \rho_i^0, \quad (n,m) \in \mathbb{Z}^2,$$
(1.5)

and  $\rho_i^0$  is a phase parameter independent of (n, m). It turns out that such a matrix M obey the following Sylvester equation:

$$\boldsymbol{K}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{K} = \boldsymbol{r}\boldsymbol{c}^{T}, \tag{1.6}$$

where

$$\boldsymbol{K} \doteq \operatorname{diag}(k_1, \cdots, k_N), \quad \boldsymbol{r} \doteq (\rho_1, \cdots, \rho_N)^T, \quad \boldsymbol{c} \doteq (c_1, \cdots, c_N)^T.$$
 (1.7)

Then a set of scalar functions  $S^{(i,j)} \doteq c^T K^j (I_N + M)^{-1} K^i r$  are defined for  $i, j \in \mathbb{Z}$ , which turns out to obey some shift and recursive relations. Here  $I_N$  is the N-th order identity matrix. One can also express  $S^{(i,j)}$  as the ratio of determinants by

$$S^{(i,j)} = \frac{g}{f}, \quad f = |\mathbf{I}_N + \mathbf{M}|, \quad g = -\begin{vmatrix} \mathbf{I}_N + \mathbf{M} & \mathbf{K}^i \mathbf{r} \\ \mathbf{c}^T \mathbf{K}^j & 0 \end{vmatrix}.$$
 (1.8)

Equations can be derived as closed form of certain  $S^{(i,j)}$ , which give rise to discrete integrable equations together with their solutions, e.g. see [28]. Later, a generalized Cauchy matrix approach was proposed, which allows K to be arbitrary invertible matrix [30], providing more flexibility in choosing parameters. The obtained explicit solutions can be classified in terms of the canonical form of K (or eigenvalue structure of K), which describe interactions of N solitons, resonance of multiple-pole solutions and interaction between solitons and multiple-pole solutions. Subsequently, this method was extended to continuous integrable systems, enabling the formulation of KdV equation, modified KdV equation, sine-Gordon equation [31], Kadomtsev-Petviashvili (KP) equation [32] and  $2 \times 2$  Ablowitz-Kaup-Newell-Segur (AKNS) system [33]. In the recent work [26] and [27], we have shown that the SDYM equation and its solutions can be formulated from the Cauchy matrix structures of the matrix AKNS system and the matrix KP system. These progresses will help us realize the reduction constraint (see, e.g. (2.10)) using the Cauchy matrix structures. As a result, we will also give a Cauchy-matrix formulation of the FL equation, which has also emerged in [20].

The paper is organized as follows. In section 2, we recall the theory of the SDYM equation and present dimensional reduction from a general SDYM equation to the pKN(-1) system. Then, in section 3 we show how the reduction is realized in the Cauchy matrix schemes of the KP-type and AKNS-type. Equivalence of these solutions obtained from the two Cauchy matrix schemes are also discussed in this section. Section 4 devotes to conjugate reductions such that N-soliton solutions of the FL equation are obtained. Concluding remarks are given in section 5. There are two appendixes. In appendix A multiple-pole solutions in the KP-type Cauchy matrix scheme are constructed. Appendix B compares our solutions with Matsuno's solutions obtained from bilinear approach and demonstrates their uniformity.

# **2** From SDYM to pKN(-1)

In this section, first, we briefly review the construction towards SDYM equation. One can refer to [3, 34-36] for more details and descriptions about the theory of SDYM equation. Then we apply suitable dimensional reduction and coordinate transformation to derive the pKN(-1) system.

#### 2.1 Theory of the SDYM equation

We start from introducing a metric matrix in  $\mathbb{C}^4 = (z_1, z_2, z_3, z_4)$ , which is determined by

$$(\eta^{mn})_{4\times4} \doteq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad m, n = 1, 2, 3, 4.$$

$$(2.1)$$

Let G be a certain Lie group and g be the Lie algebra of G. The Yang-Mills field strengths are determined as follows:

$$F_{ij} \doteq [\mathcal{D}_i, \mathcal{D}_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j], \quad \mathcal{D}_i \doteq \partial_i + A_i, \quad i, j \in \{1, 2, 3, 4\}, \quad i \neq j, \quad (2.2)$$

where  $[\cdot, \cdot]$  denotes the Lie bracket defined as [A, B] = AB - BA, matrix functions  $A_i \in g$  are gauge potentials, operators  $\mathcal{D}_i$  are the covariant derivatives and  $\partial_i = \partial_{z_i}$ .

The anti-self-dual condition of field strength is given by<sup>1</sup>

$$F_{ij} = -\frac{1}{2} \epsilon_{ijkl} \eta^{ka} \eta^{lb} F_{ab}, \qquad (2.3)$$

where  $\epsilon_{ijkl}$  is the Levi-Civita tensor,  $\eta^{mn}$  follows the definition in (2.1), and i, j, k, l, a, b are indices running over  $\{1, 2, 3, 4\}$ , the Einstein summation convention is used. Denoting  $(z, \tilde{z}, w, \tilde{w}) \doteq (z_1, z_2, z_3, z_4)$ , one can rewrite (2.3) as the follows:

$$F_{zw} = 0, \quad F_{\tilde{z}\tilde{w}} = 0, \quad F_{z\tilde{z}} - F_{w\tilde{w}} = 0,$$
 (2.4)

which indicates the existence of h and  $\tilde{h}$  satisfying

$$\mathcal{D}_z h = 0, \quad \mathcal{D}_w h = 0, \quad \mathcal{D}_{\tilde{z}} \dot{h} = 0, \quad \mathcal{D}_{\tilde{w}} \dot{h} = 0.$$
(2.5)

Defining  $J = \tilde{h}^{-1}h$ , one can derive the *J*-matrix formulation of the SDYM equation:

$$\partial_{\tilde{z}}((\partial_z J)J^{-1}) - \partial_{\tilde{w}}((\partial_w J)J^{-1}) = 0.$$
(2.6)

The SDYM equation is an integrable system, whose Lax representation is given by (e.g. [37])

$$L(\phi) \doteq (\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta = 0, \qquad (2.7a)$$

$$M(\phi) \doteq (\partial_z - (\partial_z J)J^{-1})\phi - (\partial_{\tilde{w}}\phi)\zeta = 0.$$
(2.7b)

By introducing a Miura transformation

$$\partial_{\tilde{z}}K = -(\partial_w J)J^{-1}, \quad \partial_{\tilde{w}}K = -(\partial_z J)J^{-1}, \tag{2.8}$$

the compatible condition of (2.7) also gives rise to the K-matrix formulation:

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K - [\partial_{\tilde{z}} K, \partial_{\tilde{w}} K] = 0.$$
(2.9)

Obviously, the SDYM equation (2.6) is also a result of the compatibility of (2.8). In principle, the SDYM equation can be studied either in the form (2.6) or (2.9), or in a more general form (2.8). For convenience in the following, we call (2.8) the general SDYM equation.

<sup>&</sup>lt;sup>1</sup>There is no intrinsic difference between self-dual condition with anti-self-dual condition. They can be transformed to each other under the coordinate transformation  $(z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, z_4, z_3)$ .

### 2.2 Dimensional reduction to the pKN(-1) system

To reduce the SDYM equation to the pKN(-1) system, let us introduce the following constraints:

• choose

$$G = \mathrm{SL}(2), \tag{2.10a}$$

$$(z, \tilde{z}, w, \tilde{w}) \in \mathbb{R}^4$$
, and  $\tilde{w} = w$ ; (2.10b)

• assume that J and K have the following variable separation form

$$J(z,\tilde{z},w) = \mathbf{e}^{-\sigma_3 w} J'(z,\tilde{z}) \mathbf{e}^{\sigma_3 w}, \qquad (2.10c)$$

$$K(z, \tilde{z}, w) = \mathbf{e}^{-\sigma_3 w} K'(z, \tilde{z}) \mathbf{e}^{\sigma_3 w}, \qquad (2.10d)$$

where  $\sigma_3 = \text{diag}(1, -1)$  is the third Pauli matrix.

Note that the condition (2.10a) indicates that |J| is a constant (see, e.g. [1, 38] or [36]), therefore we can always normalize it such that

$$|J| = 1. (2.11)$$

The above decomposition for J and K immediately yields

$$\partial_w J = [J, \sigma_3], \quad \partial_w K = [K, \sigma_3].$$
 (2.12)

Substituting them into the *J*-formulation (2.6), the *K*-formulation (2.9) and the general form (2.8), we get the following 2-dimensional equations:

$$\partial_{\tilde{z}}((\partial_z J)J^{-1}) - [[J,\sigma_3]J^{-1},\sigma_3] = 0, \qquad (2.13)$$

$$\partial_z \partial_{\tilde{z}} K - [[K, \sigma_3], \sigma_3] - [\partial_{\tilde{z}} K, [K, \sigma_3]] = 0, \qquad (2.14)$$

$$\partial_z J = -[K, \sigma_3]J, \quad \partial_{\tilde{z}} K = -[J, \sigma_3]J^{-1}.$$
(2.15)

Since J and K are  $2 \times 2$  matrix functions, we can denote them as

$$J \doteq \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \quad K \doteq \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$
 (2.16)

Note that the setting |J| = 1 in (2.11) indicates

$$J^{-1} = \begin{pmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{pmatrix}$$

In the following, we only focus on (2.15), which yields the explicit relations

$$\begin{pmatrix} J_{11,z} & J_{12,z} \\ J_{21,z} & J_{22,z} \end{pmatrix} = \begin{pmatrix} 2K_{12}J_{21} & 2K_{12}J_{22} \\ -2K_{21}J_{11} & -2K_{21}J_{12} \end{pmatrix},$$
 (2.17a)

$$\begin{pmatrix} K_{11,\tilde{z}} & K_{12,\tilde{z}} \\ K_{21,\tilde{z}} & K_{22,\tilde{z}} \end{pmatrix} = \begin{pmatrix} -2J_{12}J_{21} & 2J_{12}J_{11} \\ -2J_{21}J_{22} & 2J_{21}J_{12} \end{pmatrix}.$$
 (2.17b)

Then we give the following theorem.

**Theorem 1.** Introduce new coordinates

$$(x,t) \doteq (2\tilde{z},2z) \in \mathbb{R}^2$$

and functions

$$(u,v) \doteq \left(K_{21}, \mathrm{i}\frac{J_{12}}{J_{22}}\right).$$
 (2.18)

Then (u, v) solves the pKN(-1) system (1.3).

*Proof.* Using the relations in (2.17), a direct calculation shows that

$$u_{xt} - u = K_{21,xt} - K_{21} = (J_{11}J_{22} + J_{12}J_{21} - 1)K_{21} = 2J_{12}J_{21}K_{21} = 2ivuu_x,$$
(2.19)

where we have made use of the result |J| = 1. This is nothing but the first equation in the pKN(-1) system (1.3). For the second equation (1.3b), taking t-derivative of v yields

$$v_t = i \left(\frac{J_{12}}{J_{22}}\right)_t = i \frac{J_{12,t} J_{22} - J_{12} J_{22,t}}{J_{22}^2} = i K_{12} - i v^2 u.$$
(2.20)

Further, using (2.17) we have

$$v_{xt} = iJ_{12}J_{11} - i\frac{J_{12}^2}{J_{22}}J_{21} - 2iuvv_x = v - 2iuvv_x, \qquad (2.21)$$

which is (1.3b). Thus we complete the proof.

We conclude this subsection with the following remarks.

**Remark 1.** The pKN(-1) system (1.3) can be reduced from the general SDYM equation (2.8) under the reduction constraint (2.10).

**Remark 2.**  $K_{12}$  and  $K_{21}$  in (2.14) enjoy a coupled closed from:

$$\partial_z \partial_{\bar{z}} K_{12} - 4K_{12} + 8K_{12} \partial_z^{-1} \partial_{\bar{z}} (K_{12} K_{21}) = 0, \qquad (2.22a)$$

$$\partial_z \partial_{\bar{z}} K_{21} - 4K_{21} + 8K_{21} \partial_z^{-1} \partial_{\bar{z}} (K_{12} K_{21}) = 0, \qquad (2.22b)$$

which is known as the first member in the negative AKNS hierarchy (AKNS(-1) system for short) or the non-potential sine-Gordon system, (see equation (4.11) in [33] or (2.9) in [39]).

**Remark 3.** The reduction condition (2.10) is not unique. One can replace  $\sigma_3$  in (2.10) with either of the following,

$$P_1 \doteq \text{diag}(1,0), \text{ or } P_2 \doteq \text{diag}(0,-1), \text{ or } P_3 \doteq \text{diag}\left(\frac{1}{2}, -\frac{1}{2}\right),$$
 (2.23)

and introduce  $(x,t) \doteq (\tilde{z},z) \in \mathbb{R}^2$ . Then (u,v) defined as in (2.18) still solves the pKN(-1) system (1.3).

**Remark 4.** In addition to (2.18), the following setting

$$(u,v) \doteq \left(i\frac{J_{21}}{J_{11}}, K_{12}\right)$$
 (2.24)

also satisfies the pKN(-1) system (1.3) under the reduction (2.10).

**Remark 5.** In practice, if J and K satisfy (2.12) together with the setting |J| = 1, one can always reduce the general SDYM equation (2.8) to the pKN(-1) system (1.3). In the next section, we will show how these conditions are fulfilled in the Cauchy matrix approach.

 $\Box$ 

# **3** Realization of reductions in Cauchy matrix approach

Recalling the Cauchy matrix approach presented in [27], the general SDYM equation (2.8) can be well defined; in addition, relations in (2.12) and the setting |J| = 1 can also arise from the Cauchy matrix approach of the SDYM equation. In this section, first, we will recall the two Cauchy matrix schemes, namely, the KP-type and the AKNS-type, in which the general SDYM equation (2.8) and relation (2.12) have been established. This then allows us to realize reductions and present explicit solutions to the pKN(-1) system (1.3).

#### 3.1 Realization of reductions

Recently in [27] we have studied the SDYM equation from two types of the Sylvester equations:

• The KP-type Sylvester equation (asymmetric Sylvester equation):

$$KM - ML = rs^T, (3.1)$$

where  $K, L, M \in \mathbb{C}_{N \times N}$ ,  $r = (r_1, r_2) \in \mathbb{C}_{N \times 2}$ ,  $s = (s_1, s_2) \in \mathbb{C}_{N \times 2}$ . By introducing  $M_1, M_2$  that satisfy

$$KM_1 - M_1L = r_1s_1^T, \quad KM_2 - M_2L = r_2s_2^T,$$
 (3.2)

we have  $M = M_1 + M_2$ . The master functions are defined as

$$S_{[KP]}^{(i,j)} = s^{T} L^{j} M_{1}^{-1} K^{i} r = s^{T} L^{j} (M_{1} + M_{2})^{-1} K^{i} r = \begin{pmatrix} s_{11}^{(i,j)} & s_{12}^{(i,j)} \\ s_{21}^{(i,j)} & s_{22}^{(i,j)} \end{pmatrix}$$
$$= \begin{pmatrix} s_{1}^{T} L^{j} (M_{1} + M_{2})^{-1} K^{i} r_{1} & s_{1}^{T} L^{j} (M_{1} + M_{2})^{-1} K^{i} r_{2} \\ s_{2}^{T} L^{j} (M_{1} + M_{2})^{-1} K^{i} r_{1} & s_{2}^{T} L^{j} (M_{1} + M_{2})^{-1} K^{i} r_{2} \end{pmatrix}.$$
(3.3)

• The AKNS-type Sylvester equation (symmetric Sylvester equation):

$$KM - MK = rs^T, (3.4)$$

where K, M, r, s are block matrices in the forms of

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \end{pmatrix}, \quad (3.5)$$

with  $K_i, M_i \in \mathbb{C}_{N \times N}, r_i, s_i \in \mathbb{C}_{N \times 1}, i = 1, 2$ . Expansion of (3.4) yields

$$K_1 M_1 - M_1 K_2 = r_1 s_2^T, \quad K_2 M_2 - M_2 K_2 = r_2 s_1^T.$$
 (3.6)

The master functions are defined as

$$\begin{aligned} \boldsymbol{S}_{[AKNS]}^{(i,j)} &= \boldsymbol{s}^{T} \boldsymbol{K}^{j} (\boldsymbol{I}_{2N} + \boldsymbol{M})^{-1} \boldsymbol{K}^{i} \boldsymbol{r} = \begin{pmatrix} s_{1}^{(i,j)} & s_{2}^{(i,j)} \\ s_{3}^{(i,j)} & s_{4}^{(i,j)} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{s}_{2}^{T} \boldsymbol{K}_{2}^{j} (\boldsymbol{M}_{1} - \boldsymbol{M}_{2}^{-1})^{-1} \boldsymbol{K}_{1}^{i} \boldsymbol{r}_{1} & \boldsymbol{s}_{2}^{T} \boldsymbol{K}_{2}^{j} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2} \boldsymbol{M}_{1})^{-1} \boldsymbol{K}_{2}^{i} \boldsymbol{r}_{2} \\ \boldsymbol{s}_{1}^{T} \boldsymbol{K}_{1}^{j} (\boldsymbol{I}_{N} - \boldsymbol{M}_{1} \boldsymbol{M}_{2})^{-1} \boldsymbol{K}_{1}^{i} \boldsymbol{r}_{1} & \boldsymbol{s}_{1}^{T} \boldsymbol{K}_{1}^{j} (\boldsymbol{M}_{2} - \boldsymbol{M}_{1}^{-1})^{-1} \boldsymbol{K}_{2}^{i} \boldsymbol{r}_{2} \end{pmatrix}. \end{aligned}$$
(3.7)

They give rise to two different formulations of solutions of the SDYM equation (1.1). For convenience, we call the solutions derived from (3.1)/(3.4) the KP/AKNS-type solution, respectively.

For the case of the KP-type, we introduce the following dispersion relations:

$$\boldsymbol{r}_{x_n} = \boldsymbol{K}^n \boldsymbol{r} \boldsymbol{a}, \quad \boldsymbol{s}_{x_n} = -(\boldsymbol{L}^T)^n \boldsymbol{s} \boldsymbol{a}, \quad n \in \mathbb{Z},$$
(3.8)

where  $\boldsymbol{a} = \text{diag}(a_1, a_2)$ . Then the setting

$$(\boldsymbol{V}, \boldsymbol{U}) \doteq (\boldsymbol{I}_2 - \boldsymbol{S}_{[\text{KP}]}^{(-1,0)}, \boldsymbol{S}_{[\text{KP}]}^{(0,0)})$$
 (3.9)

yields a differential recurrence relation (see Appendix A in [27]):

$$V_{x_{n+1}}V^{-1} = -U_{x_n}. (3.10)$$

This actually provides the general SDYM equation (2.8) through taking different n. In addition, for both U and V, there hold the relations (see equation (2.5b) in [27])

$$\boldsymbol{U}_{x_0} = [\boldsymbol{U}, \boldsymbol{a}], \quad \boldsymbol{V}_{x_0} = [\boldsymbol{V}, \boldsymbol{a}]. \tag{3.11}$$

Thus, in summary, if we take  $\tilde{w} = w = x_0$  in the above relations and n = 0, -1 in (3.10), namely,

$$V_{x_1}V^{-1} = -U_{x_0}, \quad V_{x_0}V^{-1} = -U_{x_{-1}},$$
 (3.12)

we can recover (2.12) and (2.8) by choosing

$$(\boldsymbol{U}, \boldsymbol{V}, x_{-1}, x_1, x_0) = (K, J, \tilde{z}, z, \tilde{w} = w).$$
(3.13)

Note also that in this case  $|\mathbf{V}| = |\mathbf{L}|/|\mathbf{K}|$  (see Theorem 2 in [27]), which can be normalized to be 1. Thus, the reduction from the general SDYM equation (2.8) to the pKN(-1) system (1.3) can be realized in the Cauchy matrix scheme of the KP-type.

For the case of the AKNS-type, the dispersions are introduced by replacing L in (3.8) with K, i.e.

$$\boldsymbol{r}_{x_n} = \boldsymbol{K}^n \boldsymbol{r} \boldsymbol{a}, \quad \boldsymbol{s}_{x_n} = -(\boldsymbol{K}^T)^n \boldsymbol{s} \boldsymbol{a}, \quad n \in \mathbb{Z},$$
 (3.14)

and we introduce two functions as

$$(\boldsymbol{v}, \boldsymbol{u}) \doteq \left(\boldsymbol{I}_2 - \boldsymbol{S}_{[AKNS]}^{(-1,0)}, \boldsymbol{S}_{[AKNS]}^{(0,0)}\right).$$
(3.15)

There are similar relations (see Theorem 1 in [26]):

$$\boldsymbol{v}_{x_{n+1}}\boldsymbol{v}^{-1} = -\boldsymbol{u}_{x_n}, \quad n \in \mathbb{Z},$$
(3.16)

and (see equation (2.5b) in [27])

$$\boldsymbol{u}_{x_0} = [\boldsymbol{u}, \boldsymbol{a}], \quad \boldsymbol{v}_{x_0} = [\boldsymbol{v}, \boldsymbol{a}].$$
 (3.17)

Then, similarly, one can recover (2.12) and (2.8) by choosing

$$(\boldsymbol{u}, \boldsymbol{v}, x_{-1}, x_1, x_0) = (K, J, \tilde{z}, z, \tilde{w} = w).$$
(3.18)

We can also have |v| = 1 in this case (see Theorem 2 in [27]). Thus, the reductions are realized from the Cauchy matrix scheme of the AKNS-type as well.

Now that these reductions can be realized from the Cauchy matrix approach, we can derive explicit solutions for the pKN(-1) system by using the known results of the (matrix) KP hierarchy and AKNS hierarchy.

### **3.2** Solutions of pKN(-1) system from Cauchy matrix scheme

#### 3.2.1 From KP-type Cauchy matrix scheme

We have shown that the reduction conditions can be fulfilled in the two Cauchy matrix schemes. That means we can then get explicit solution (u, v) for the pKN(-1) system through the formulation (2.18). In practice, we need to in the first step present explicit expression for U and V defined in (3.9) and then recover (u, v) from (2.18). To achieve that, we consider the following solutions of the Sylvester equation (3.1) together with the dispersion relation (3.8) (see [27]):

$$\boldsymbol{K} = \operatorname{diag}(k_1, \cdots, k_N), \quad \boldsymbol{L} = \operatorname{diag}(l_1, \cdots, l_N), \quad k_i, l_i \in \mathbb{C},$$
(3.19a)

$$\boldsymbol{r} = (\boldsymbol{r}_1, \boldsymbol{r}_2), \quad \boldsymbol{r}_j = (\rho_j(k_1), \cdots, \rho_j(k_N))^T,$$
(3.19b)

$$s = (s_1, s_2), \quad s_j = (\sigma_j(l_1), \cdots, \sigma_j(l_N))^T, \quad j = 1, 2,$$
  
(3.19c)

$$M = M_1 + M_2, \quad M_j = (M_{is}^{(j)})_{N \times N}, \quad M_{is}^{(j)} = \frac{\rho_j(k_i)\sigma_j(l_s)}{k_i - l_s}, \quad j = 1, 2,$$
 (3.19d)

with the plane wave factors

$$\rho_j(k_i) = \exp\left(a_j \mathcal{L}(k_i) + \lambda_j(k_i)\right), \quad \sigma_j(l_i) = \exp\left(-a_j \mathcal{L}(l_i) + \mu_1(l_i)\right), \quad j = 1, 2,$$
(3.20)

where

$$\mathcal{L}(k) \doteq k^n x_n + k^{n+1} x_{n+1} + k^m x_m + k^{m+1} x_{m+1}, \qquad (3.21)$$

and the phase factors  $\lambda_i(k), \mu_i(k)$  are functions of k.

In light of (3.3), (3.9) and (3.13), we have

$$\boldsymbol{V} \doteq \boldsymbol{I}_2 - \boldsymbol{s}^T \boldsymbol{M}^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}, \quad \boldsymbol{U} \doteq \boldsymbol{s}^T \boldsymbol{M}^{-1} \boldsymbol{r}, \tag{3.22}$$

$$J = \mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad K = \mathbf{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad (3.23)$$

and further, from (2.18) we have

$$(u,v) = \left(U_{21}, i\frac{V_{12}}{V_{22}}\right) = \left(s_2^T M^{-1} r_1, -\frac{is_1^T M^{-1} K^{-1} r_2}{1 - s_2^T M^{-1} K^{-1} r_2}\right),$$
(3.24)

which will be solutions of the pKN(-1) system (1.3) if taking (n, m) = (-1, 0) in (3.21) and

$$a = \sigma_3, \quad (x,t) \doteq (2x_{-1}, 2x_1).$$
 (3.25)

Note that |V| is a constant given by (see Theorem 2 in [27]) |V| = |L|/|K| and gives rise to same (u, v) through (3.24) no matter |V| is normalized to be 1 or not. So, in the following, no need to normalize V. In addition, corresponding to Remark 3 we have the following.

**Remark 6.** In the case *a* takes either of  $P_j$  for j = 1, 2, 3 as given in (2.23), we shall take  $(x,t) \doteq (x_{-1}, x_1)$  and take the plane wave factors as the following,

$$\rho_j(k_i) = \exp\left(a_j\left(\frac{1}{k_i}x + k_it\right) + \lambda'_j(k_i)\right), \quad \sigma_j(l_i) = \exp\left(-a_j\left(\frac{1}{l_i}x + l_it\right) + \mu'_j(l_i)\right), \quad (3.26)$$

for j = 1, 2, where the phase factors have been taken as

$$\lambda'_{j}(k_{i}) = \lambda_{j}(k_{i}) + 2a_{j}x_{0}, \quad \mu'_{j}(k_{i}) = \mu_{j}(l_{j}) - 2a_{j}x_{0}, \quad j = 1, 2.$$
(3.27)

Next, we present the (u, v) formulation in a more explicit form. Note that the dressed Cauchy matrix M can be expressed in the following decomposition form:

$$M = M_1 + M_2, \quad M_1 = R_1 G S_1^T, \quad M_2 = R_2 G S_2^T,$$
 (3.28)

where

$$\boldsymbol{R}_{j} = \operatorname{diag}(\rho_{j}(k_{1}), \cdots, \rho_{j}(k_{N})), \quad \boldsymbol{S}_{j} = \operatorname{diag}(\sigma_{j}(l_{1}), \cdots, \sigma_{j}(l_{N})), \quad j = 1, 2, \quad (3.29a)$$

$$G = (G_{ij})_{N \times N} = \frac{1}{k_i - l_j}, \quad i, j = 1, \cdots, N.$$
 (3.29b)

This fact indicates that we can rewrite  $\boldsymbol{r}_j$  and  $\boldsymbol{s}_j$  as

$$\boldsymbol{r}_j = \boldsymbol{R}_j \boldsymbol{e}_N, \quad \boldsymbol{s}_j = \boldsymbol{S}_j \boldsymbol{e}_N, \quad \boldsymbol{e}_N = (\underbrace{1, 1, \cdots, 1}_{N-\text{dimensional}})^T, \quad j = 1, 2.$$
 (3.30)

Thus for each  $s_{ab}^{(i,j)}$  in  $\boldsymbol{S}_{[\mathrm{KP}]}^{(i,j)}, a, b = 1, 2$ , we have

$$s_{11}^{(i,j)} = \boldsymbol{e}_N^T \boldsymbol{L}^j (\boldsymbol{G} + \boldsymbol{R}_1^{-1} \boldsymbol{R}_2 \boldsymbol{G} \boldsymbol{S}_2^T (\boldsymbol{S}_1^T)^{-1})^{-1} \boldsymbol{K}^i \boldsymbol{e}_N, \qquad (3.31a)$$

$$s_{12}^{(i,j)} = e_N^T L^j (R_2^{-1} R_1 G + G S_2^T (S_1^T)^{-1})^{-1} K^i e_N,$$
(3.31b)

$$s_{21}^{(i,j)} = \boldsymbol{e}_N^T \boldsymbol{L}^j (\boldsymbol{G} \boldsymbol{S}_1^T (\boldsymbol{S}_2^T)^{-1} + \boldsymbol{R}_1^{-1} \boldsymbol{R}_2 \boldsymbol{G})^{-1} \boldsymbol{K}^i \boldsymbol{e}_N, \qquad (3.31c)$$

$$s_{22}^{(i,j)} = \boldsymbol{e}_N^T \boldsymbol{L}^j (\boldsymbol{R}_2^{-1} \boldsymbol{R}_1 \boldsymbol{G} \boldsymbol{S}_1^T (\boldsymbol{S}_2^T)^{-1} + \boldsymbol{G})^{-1} \boldsymbol{K}^i \boldsymbol{e}_N.$$
(3.31d)

Thus we can rewrite the formula (3.24) as

$$u = s_2^T M^{-1} r_1 = e_N^T (G S_1^T (S_2^T)^{-1} + R_1^{-1} R_2 G)^{-1} e_N,$$
(3.32a)

$$v = -\frac{1s_1^T M^{-1} K^{-1} r_2}{1 - s_2^T M^{-1} K^{-1} r_2} = -\frac{1e_N^T (R_2^{-1} R_1 G + GS_2^{-1} (S_1^{-1})^{-1} K^{-1} e_N)}{1 - e_N^T (R_2^{-1} R_1 GS_1^T (S_2^{-1})^{-1} + G)^{-1} K^{-1} e_N}.$$
 (3.32b)

Notice that in this explicit formulation, we can introduce

$$\boldsymbol{R} \doteq \boldsymbol{R}_1^{-1} \boldsymbol{R}_2 = \operatorname{diag} \left( \frac{\rho_2(k_1)}{\rho_1(k_1)}, \cdots, \frac{\rho_2(k_N)}{\rho_1(k_N)} \right),$$
(3.33a)

$$\boldsymbol{S} \doteq \boldsymbol{S}_1^{-1} \boldsymbol{S}_2 = \operatorname{diag} \left( \frac{\sigma_2(l_1)}{\sigma_1(l_1)}, \cdots, \frac{\sigma_2(l_N)}{\sigma_1(l_N)} \right),$$
(3.33b)

where

$$\frac{\rho_2(k_i)}{\rho_1(k_i)} = \exp\left((a_2 - a_1)\left(\frac{1}{k_i}x + k_it\right) + \lambda_2'(k_i) - \lambda_1'(k_i)\right),$$
(3.34a)

$$\frac{\sigma_2(l_i)}{\sigma_1(l_i)} = \exp\left((a_1 - a_2)\left(\frac{1}{l_i}x + l_it\right) + \mu'_2(l_i) - \mu'_1(l_i)\right).$$
(3.34b)

This fact also indicates that different a may lead to the same results. Finally, we have explicit formulas for (u, v) as

$$u = \boldsymbol{e}_N^T (\boldsymbol{G}\boldsymbol{S}^{-1} + \boldsymbol{R}\boldsymbol{G})^{-1} \boldsymbol{e}_N, \qquad (3.35a)$$

$$v = -\frac{\mathrm{i} e_N^T (\mathbf{R}^{-1} \mathbf{G} + \mathbf{G} \mathbf{S})^{-1} \mathbf{K}^{-1} \mathbf{e}_N}{1 - e_N^T (\mathbf{R}^{-1} \mathbf{G} \mathbf{S}^{-1} + \mathbf{G})^{-1} \mathbf{K}^{-1} \mathbf{e}_N},$$
(3.35b)

where we have taken  $\boldsymbol{a}$  to be either of  $P_j$  as given in (2.23), and

$$\boldsymbol{R} = \operatorname{diag}(r(k_1), \cdots, r(k_N)), \quad r(k_i) = \exp\left(-\frac{1}{k_i}x - k_it + \zeta(k_i)\right), \quad (3.36a)$$

$$\boldsymbol{S} = \operatorname{diag}(s(l_1), \cdots, s(l_N)), \quad s(l_i) = \exp\left(\frac{1}{l_i}x + l_it + \eta(l_i)\right)$$
(3.36b)

with  $\zeta(k_i)$  and  $\eta(l_i)$  being phase factors.

### 3.2.2 From AKNS-type Cauchy matrix scheme

In the case of the AKNS-type, the Sylvester equation (3.4) with (3.5) and the dispersion relation (3.14) have solutions (see [26, 27, 33]):

$$\boldsymbol{K}_1 = \operatorname{diag}(k_1, \cdots, k_N), \quad \boldsymbol{K}_2 = \operatorname{diag}(l_1, \cdots, l_N), \quad (3.37a)$$

$$\mathbf{r}_1 = (\rho_1(k_1), \cdots, \rho_1(k_N))^T, \quad \mathbf{s}_1 = (\sigma_1(k_1), \cdots, \sigma_1(k_N))^T, \quad (3.37b)$$

$$\mathbf{r}_2 = (\rho_2(l_1), \cdots, \rho_2(l_N))^T, \quad \mathbf{s}_2 = (\sigma_2(l_1), \cdots, \sigma_2(l_N))^T,$$
 (3.37c)

$$M_1 = (M_{1,ij})_{N \times N}, \qquad M_{1,ij} = \frac{\rho_1(k_i)\sigma_2(l_j)}{k_i - l_j},$$
(3.37d)

$$M_2 = (M_{2,ij})_{N \times N}, \quad M_{2,ij} = \frac{\rho_2(l_i)\sigma_1(k_j)}{l_i - k_j},$$
(3.37e)

where the plane wave factors  $\rho_j, \sigma_j$  are defined as in (3.20).

In light of (3.7), (3.15) and (3.18), we have

$$\boldsymbol{v} \doteq \boldsymbol{I}_2 - \boldsymbol{s}^T (\boldsymbol{I} + \boldsymbol{M})^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}, \quad \boldsymbol{u} \doteq \boldsymbol{s}^T (\boldsymbol{I} + \boldsymbol{M})^{-1} \boldsymbol{r}.$$
 (3.38)

Suppose

$$J = \boldsymbol{v} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, \quad K = \boldsymbol{u} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$
(3.39)

Then, the following functions

$$(u,v) \doteq \left(u_3, i\frac{v_2}{v_4}\right) = \left(\boldsymbol{s}_1^T (\boldsymbol{I}_N - \boldsymbol{M}_1 \boldsymbol{M}_2)^{-1} \boldsymbol{r}_1, \frac{-i\boldsymbol{s}_2^T (\boldsymbol{I}_N - \boldsymbol{M}_2 \boldsymbol{M}_1)^{-1} \boldsymbol{K}_2^{-1} \boldsymbol{r}_2}{1 + \boldsymbol{s}_1^T (\boldsymbol{I}_N - \boldsymbol{M}_1 \boldsymbol{M}_2)^{-1} \boldsymbol{M}_1 \boldsymbol{K}_2^{-1} \boldsymbol{r}_2}\right) \quad (3.40)$$

provide solutions to the pKN(-1) system (1.3) where we take (n,m) = (-1,0) in (3.21) and take (3.25) as well. Note that in the case of the AKNS-type, |J| = 1 (see Theorem 2 in [27] and the Remark 6 holds too and thus in the following we use the plane wave factors defined in the form (3.26).

To give explicit expressions of (u, v), we factorize  $M_1$  and  $M_2$  as the following:

$$\boldsymbol{M}_1 = \boldsymbol{R}_1 \boldsymbol{G} \boldsymbol{S}_2^T, \quad \boldsymbol{M}_2 = -\boldsymbol{R}_2 \boldsymbol{G}^T \boldsymbol{S}_1^T, \quad (3.41)$$

where

$$\mathbf{R}_{1} = \text{diag}(\rho_{1}(k_{1}), \cdots, \rho_{1}(k_{N})), \quad \mathbf{S}_{2} = \text{diag}(\sigma_{2}(l_{1}), \cdots, \sigma_{2}(l_{N})), \quad (3.42a)$$

$$\boldsymbol{R}_2 = \operatorname{diag}(\rho_2(l_1), \cdots, \rho_2(l_N)), \quad \boldsymbol{S}_1 = \operatorname{diag}(\sigma_1(k_1), \cdots, \sigma_1(k_N)), \quad (3.42b)$$

$$\boldsymbol{G} = \left(\frac{1}{k_i - l_j}\right)_{1 \le i, j \le N}, \quad \boldsymbol{G}^T = \left(-\frac{1}{l_i - k_j}\right)_{1 \le i, j \le N}, \quad \boldsymbol{e}_N = \left(\underbrace{1, 1, \cdots, 1}_{N-\text{dimensional}}\right), \quad (3.42c)$$

and the plane wave factors are defined as in (3.26). For each entry in  $S_{[AKNS]}^{(i,j)}$ , we can rewrite them as

$$s_{1}^{(i,j)} = \boldsymbol{e}_{N}^{T} \boldsymbol{K}_{2}^{j} (\boldsymbol{G} + \boldsymbol{R}_{1}^{-1} (\boldsymbol{S}_{1}^{T})^{-1} (\boldsymbol{G}^{T})^{-1} \boldsymbol{R}_{2}^{-1} (\boldsymbol{S}_{2}^{T})^{-1})^{-1} \boldsymbol{K}_{1}^{i} \boldsymbol{e}_{N}, \qquad (3.43a)$$

$$s_{2}^{(i,j)} = \boldsymbol{e}_{N}^{T} \boldsymbol{K}_{2}^{j} (\boldsymbol{R}_{2}^{-1} (\boldsymbol{S}_{2}^{T})^{-1} + \boldsymbol{G}^{T} \boldsymbol{S}_{1}^{T} \boldsymbol{R}_{1} \boldsymbol{G})^{-1} \boldsymbol{K}_{2}^{i} \boldsymbol{e}_{N},$$
(3.43b)

$$s_{3}^{(i,j)} = \boldsymbol{e}_{N}^{T} \boldsymbol{K}_{1}^{j} (\boldsymbol{R}_{1}^{-1} (\boldsymbol{S}_{1}^{T})^{-1} + \boldsymbol{G} \boldsymbol{S}_{2}^{T} \boldsymbol{R}_{2} \boldsymbol{G}^{T}) \boldsymbol{K}_{1}^{i} \boldsymbol{e}_{N}, \qquad (3.43c)$$

$$s_4^{(i,j)} = -\boldsymbol{e}_N^T \boldsymbol{K}_1^j (\boldsymbol{G}^T + \boldsymbol{R}_2^{-1} (\boldsymbol{S}_2^T)^{-1} \boldsymbol{G}^{-1} \boldsymbol{R}_1^{-1} (\boldsymbol{S}_1^T)^{-1})^{-1} \boldsymbol{K}_2^i \boldsymbol{e}_N.$$
(3.43d)

Then we introduce

$$\boldsymbol{P} \doteq \boldsymbol{R}_1 \boldsymbol{S}_1^T = \operatorname{diag}(\rho_1(k_1)\sigma_1(k_1), \cdots, \rho_1(k_N)\sigma_1(k_N)), \quad (3.44a)$$

$$\boldsymbol{Q} \doteq \boldsymbol{R}_2 \boldsymbol{S}_2^T = \operatorname{diag}(\rho_2(l_1)\sigma_2(l_1), \cdots, \rho_2(l_N)\sigma_2(l_N)).$$
(3.44b)

Finally, u and v can be expressed as

$$u = \boldsymbol{e}_N^T (\boldsymbol{P}^{-1} + \boldsymbol{G} \boldsymbol{Q} \boldsymbol{G}^T)^{-1} \boldsymbol{e}_N, \qquad (3.45a)$$

$$v = \frac{-\mathrm{i}\boldsymbol{e}_N^T (\boldsymbol{Q}^{-1} + \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G})^{-1} \boldsymbol{K}_2^{-1} \boldsymbol{e}_N}{1 + \boldsymbol{e}_N^T (\boldsymbol{G}^T + \boldsymbol{Q}^{-1} \boldsymbol{G}^{-1} \boldsymbol{P}^{-1})^{-1} \boldsymbol{K}_2^{-1} \boldsymbol{e}_N},$$
(3.45b)

where we have taken  $\boldsymbol{a}$  to be either of  $P_j$  as given in (2.23), and

$$\boldsymbol{P} = \operatorname{diag}(p(k_1), \cdots, p(k_N)), \quad p(k_i) = \exp\left(\frac{1}{k_i}x + k_it + \omega(k_i)\right), \quad (3.46a)$$

$$\boldsymbol{Q} = \operatorname{diag}(q(l_1), \cdots, q(l_N)), \quad q(l_i) = \exp\left(-\frac{1}{l_i}x - l_it + \theta(l_i)\right).$$
(3.46b)

with  $\omega(k_i)$  and  $\theta(l_i)$  being phase factors.

The following result follows from Remark 4.

**Remark 7.** The pKN(-1) system (1.3) also has solutions:

$$u = i \frac{-s_3^{(-1,0)}}{1 - s_1^{(-1,0)}} = \frac{-i \boldsymbol{e}_N^T (\boldsymbol{P}^{-1} + \boldsymbol{G} \boldsymbol{Q} \boldsymbol{G}^T)^{-1} \boldsymbol{K}_1^{-1} \boldsymbol{e}_N}{1 - \boldsymbol{e}_N^T (\boldsymbol{G} + \boldsymbol{P}^{-1} (\boldsymbol{G}^T)^{-1} \boldsymbol{Q}^{-1})^{-1} \boldsymbol{K}^{-1} \boldsymbol{e}_N},$$
(3.47a)

$$v = s_2^{(0,0)} = \boldsymbol{e}_N^T (\boldsymbol{Q}^{-1} + \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G})^{-1} \boldsymbol{e}_N,$$
 (3.47b)

where P, Q are defined in (3.46).

#### 3.2.3 Equivalence of the two types of solutions

We can prove that the above two types of solutions are equivalent in some sense.

For Cauchy matrix G defined in (3.29b), its inverse can be represented in terms of G as

$$\boldsymbol{G}^{-1} = \boldsymbol{X}\boldsymbol{G}^{T}\boldsymbol{Y},\tag{3.48}$$

where

$$\boldsymbol{X} = \text{diag}(X_1, \cdots, X_N), \quad X_i = \frac{\prod_{s=1}^N (k_s - l_i)}{\prod_{1 \le s \le N}^{s \ne i} (l_s - l_i)},$$
(3.49a)

$$\mathbf{Y} = \text{diag}(Y_1, \cdots, Y_N), \quad Y_i = \frac{\prod_{s=1}^N (k_i - l_s)}{\prod_{1 \le s \le N}^{s \ne i} (k_i - k_s)}.$$
 (3.49b)

In the following, we recover the KP-type solution  $(u_{[KP]}, v_{[KP]})$  given in (3.35) from the AKNS-type solution  $(u_{[AKNS]}, v_{[AKNS]})$  given in (3.47).

Starting from (3.47b), we can rewrite  $v_{[AKNS]}$  as:

$$egin{aligned} v_{ ext{[AKNS]}} &= m{e}_N^T (m{Q}^{-1} + m{G}^T m{P} m{G})^{-1} m{e}_N \ &= m{e}_N^T (m{Q}^{-1} + m{X}^{-1} m{G}^{-1} m{Y}^{-1} m{P} m{G})^{-1} m{e}_N \ &= m{e}_N^T (m{G} m{X} m{Q}^{-1} + m{Y}^{-1} m{P} m{G})^{-1} m{e}_N, \end{aligned}$$

where P, Q are defined in (3.46) and for  $GXe_N$ , we have used a property in Lagrange polynomial (see Lemma 2.4 in [40]):

$$\boldsymbol{GXe}_{N} = \left(\sum_{i=1}^{N} \frac{\Pi_{s\neq 1}(k_{s}-l_{i})}{\Pi_{s\neq i}(l_{s}-l_{i})}, \cdots, \sum_{i=1}^{N} \frac{\Pi_{s\neq N}(k_{s}-l_{i})}{\Pi_{s\neq i}(l_{s}-l_{i})}\right)^{T} = (1, \cdots, 1)^{T} = \boldsymbol{e}_{N}.$$
 (3.50)

Then, letting  $\theta(l_i) = \eta(l_i) + \ln(X_i)$  and  $\omega(k_i) = \zeta(k_i) + \ln(Y_i)$ , we have

$$XQ^{-1}(-x,-t) = S^{-1}(x,t), \quad Y^{-1}P(-x,-t) = R(x,t).$$
 (3.51)

Thus

$$v_{[AKNS]}(-x,-t) = \boldsymbol{e}_{N}^{T}(\boldsymbol{G}\boldsymbol{X}\boldsymbol{Q}^{-1}(-x,-t) + \boldsymbol{Y}^{-1}\boldsymbol{P}(-x,-t)\boldsymbol{G})^{-1}\boldsymbol{e}_{N}$$
  
=  $\boldsymbol{e}_{N}^{T}(\boldsymbol{G}\boldsymbol{S}^{-1}(x,t) + \boldsymbol{R}(x,t)\boldsymbol{G})^{-1}\boldsymbol{e}_{N} = u_{[KP]}(x,t).$  (3.52)

Applying the same trick leads to a similar result for  $u_{[AKNS]}$ :

$$\begin{split} u_{[\text{AKNS}]} &= \frac{-\mathrm{i} \boldsymbol{e}_N^T (\boldsymbol{P}^{-1} + \boldsymbol{G} \boldsymbol{Q} \boldsymbol{G}^T)^{-1} \boldsymbol{K}_1^{-1} \boldsymbol{e}_N}{1 - \boldsymbol{e}_N^T (\boldsymbol{G} + \boldsymbol{P}^{-1} (\boldsymbol{G}^T)^{-1} \boldsymbol{Q}^{-1})^{-1} \boldsymbol{K}_1^{-1} \boldsymbol{e}_N} \\ &= \frac{-\mathrm{i} \boldsymbol{e}_N^T (\boldsymbol{P}^{-1} \boldsymbol{Y} \boldsymbol{G} + \boldsymbol{G} \boldsymbol{Q} \boldsymbol{X}^{-1})^{-1} \boldsymbol{K}_1^{-1} \boldsymbol{e}_N}{1 - \boldsymbol{e}_N^T (\boldsymbol{P}^{-1} \boldsymbol{Y} \boldsymbol{G} \boldsymbol{X} \boldsymbol{Q}^{-1} + \boldsymbol{G})^{-1} \boldsymbol{K}_1^{-1} \boldsymbol{e}_N}. \end{split}$$

Thus we find

$$u_{[\text{AKNS}]}(-x,-t) = \frac{-\mathrm{i}\boldsymbol{e}_N^T (\boldsymbol{R}^{-1}(x,t)\boldsymbol{G} + \boldsymbol{G}\boldsymbol{S}(x,t))^{-1} \boldsymbol{K}^{-1} \boldsymbol{e}_N}{1 - \boldsymbol{e}_N^T (\boldsymbol{R}^{-1}(x,t)\boldsymbol{G}\boldsymbol{S}^{-1}(x,t) + \boldsymbol{G})^{-1} \boldsymbol{K}^{-1} \boldsymbol{e}_N} = v_{[\text{KP}]}(x,t).$$
(3.53)

**Proposition 1.** The solutions derived from the Cauchy matrix scheme of the KP-type in Sec.3.2.1 and derived from the scheme of the AKNS-type in Sec.3.2.2 are connected as

$$(u_{[KP]}(x,t), v_{[KP]}(x,t)) = (v_{[AKNS]}(-x,-t), u_{[AKNS]}(-x,-t)),$$
(3.54)

which coincides with the fact that if (u(x,t), v(x,t)) solves the pKN(-1) system (1.3), so does (v(-x,-t), u(-x,-t)).

# 4 Conjugate reduction to the FL equation

The FL equation (1.2) is a result of the conjugate reduction of the pKN(-1) system (1.3) by taking  $v = u^*$ . Such a reduction can be realized by imposing constraints on K and L (or  $K_1$  or  $K_2$  for the AKNS-type) and generates solutions for the FL equation from those solutions of the pKN(-1) system that we got in Sec.3.2. In this section, we implement such reductions so that we can obtain solution for the FL equation (1.2).

### 4.1 The KP-type solution

For those solutions obtained in Sec.3.2.1 from the KP-type formulation, we describe the reduction and constraint in the following theorem.

**Theorem 2.** For (u, v) given in (3.24), the conjugate reduction  $v = u^*$  holds under the following constraints:

$$\boldsymbol{L} = -\boldsymbol{K}^{\dagger}, \quad \boldsymbol{s}_{1}^{T} = -\mathrm{i}\boldsymbol{r}_{1}^{\dagger}\boldsymbol{K}^{\dagger}, \quad \boldsymbol{s}_{2}^{T} = \boldsymbol{r}_{2}^{\dagger}, \tag{4.1}$$

*i.e.*,

$$l_i = -k_i^*, \quad \sigma_1(l_i) = -i(\rho_1(k_i))^* k_i^*, \quad \sigma_2(l_i) = (\rho_2(k_i))^*, \quad i = 1, \cdots, N.$$
(4.2)

Here  $\mathbf{K}^{\dagger} = (\mathbf{K}^*)^T$ .

*Proof.* First, the dispersion relations in (3.8) are consistent with (4.1). Then, by applying the conjugate reduction (4.2) to (3.19d), one has  $M = M^{\dagger}$  and obtains

$$\boldsymbol{K}\boldsymbol{M} + \boldsymbol{M}^{\dagger}\boldsymbol{K}^{\dagger} = \boldsymbol{r}_{2}\boldsymbol{r}_{2}^{\dagger}. \tag{4.3}$$

Through a direct calculation we have

$$\begin{aligned} U_{21}^{\dagger} V_{22} &= (\boldsymbol{s}_{2}^{T} \boldsymbol{M}^{-1} \boldsymbol{r}_{1})^{\dagger} (1 - \boldsymbol{s}_{2}^{T} \boldsymbol{M}^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}_{2}) \\ &= \boldsymbol{r}_{1}^{\dagger} \boldsymbol{M}^{-\dagger} \boldsymbol{r}_{2} - \boldsymbol{r}_{1}^{\dagger} \boldsymbol{M}^{-\dagger} \boldsymbol{r}_{2} \boldsymbol{r}_{2}^{\dagger} \boldsymbol{M}^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}_{2} \\ &= -\boldsymbol{r}_{1}^{\dagger} \boldsymbol{K}^{\dagger} \boldsymbol{M}^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}_{2} = -\mathrm{i} \boldsymbol{s}_{1}^{T} \boldsymbol{M}^{-1} \boldsymbol{K}^{-1} \boldsymbol{r}_{2} = \mathrm{i} V_{12}, \end{aligned}$$

which gives rise to  $v = u^*$  in light of the formula (3.24). The proof is completed.

Based on the results in Sec.3.2.1, explicit N-soliton solution of the FL equation can be presented as the following.

Theorem 3. Suppose

$$\boldsymbol{K} = \text{diag}(k_1, \cdots, k_N), \quad \boldsymbol{r}_j = (\rho_j(k_1), \cdots, \rho_j(k_N))^T, \quad j = 1, 2, \quad (4.4a)$$

$$\boldsymbol{M} = (M_{ij})_{N \times N}, \qquad M_{ij} = \frac{\rho_2(\kappa_i)(\rho_2(\kappa_j))^* - 1\rho_1(\kappa_i)(\rho_1(\kappa_j))^* \kappa_j}{k_i + k_j^*}, \tag{4.4b}$$

where

$$\rho_1(k_i) = \exp\left(\frac{1}{2k_i}x + \frac{k_1}{2}t + \lambda_1(k_i)\right), \quad \rho_2(k_i) = \exp\left(-\frac{1}{2k_i}x - \frac{k_i}{2}t + \lambda_2(k_i)\right), \quad (4.5a)$$

or alternatively

$$\rho_1(k_i) = \exp\left(\frac{1}{k_i}x + k_it + \lambda_1(k_i)\right), \quad \rho_2(k_i) = \exp\left(\lambda_2(k_i)\right), \quad (4.5b)$$

$$\rho_1(k_i) = \exp(\lambda_1(k_i)), \quad \rho_2(k_i) = \exp\left(-\frac{1}{k_i}x - k_it + \lambda_2(k_i)\right).$$
(4.5c)

Then the following function

$$u_{[\mathrm{KP}]} = \boldsymbol{r}_2^{\dagger} \boldsymbol{M}^{-1} \boldsymbol{r}_1 \tag{4.6}$$

provides a N-soliton solution of the FL equation (1.2).

### 4.2 The AKNS-type solution

For the solutions of the pKN(-1) derived in Sec.3.2.2 from the Cauchy matrix scheme of the AKNS-type, their reductions can be describe below.

**Theorem 4.** For (u, v) defined in (3.40), the conjugate reduction  $v = u^*$  holds under the following constraints:

$$K_2 = -K_1^{\dagger}, \quad s_2 = r_1^*, \quad r_2 = iK_1^*s_1^*, \quad (4.7)$$

*i.e.*,

$$l_i = -k_i^*, \quad \sigma_2(l_i) = (\rho_1(k_i))^*, \quad \rho_2(l_i) = ik_i^*(\sigma_1(k_i))^*.$$
(4.8)

*Proof.* It is not difficult to verify the following Sylvester equations holds for the settings in (3.37):

$$K_2 M_2 - M_2 K_1 = r_2 s_1^T. (4.9)$$

Then, by applying (4.8) to (3.37d) and (3.37e), one obtains

$$M_1^{\dagger} = M_1, \quad M_2^{\dagger} = -K_2^{-1}M_2K_2^{\dagger} = K_2^{-1}M_2K_1.$$
 (4.10)

A direct calculation yields

$$\begin{split} u_{3}^{\dagger} v_{4} &= (\boldsymbol{s}_{1}^{T} (\boldsymbol{I}_{N} - \boldsymbol{M}_{1} \boldsymbol{M}_{2})^{-1} \boldsymbol{r}_{1})^{\dagger} (1 + \boldsymbol{s}_{1}^{T} (\boldsymbol{I}_{N} - \boldsymbol{M}_{1} \boldsymbol{M}_{2})^{-1} \boldsymbol{M}_{1} \boldsymbol{K}_{2}^{-1} \boldsymbol{r}_{2}) \\ &= \boldsymbol{r}_{1}^{\dagger} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2}^{\dagger} \boldsymbol{M}_{1}^{\dagger})^{-1} \boldsymbol{s}_{1}^{*} - \mathrm{i} \boldsymbol{r}_{1}^{\dagger} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2}^{\dagger} \boldsymbol{M}_{1}^{\dagger})^{-1} \boldsymbol{K}_{2}^{-1} \boldsymbol{r}_{2} \boldsymbol{s}_{1}^{T} \boldsymbol{M}_{1} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2} \boldsymbol{M}_{1})^{-1} \boldsymbol{K}_{2}^{-1} \boldsymbol{r}_{2} \\ &= -\mathrm{i} \boldsymbol{r}_{1}^{\dagger} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2}^{\dagger} \boldsymbol{M}_{1}^{\dagger})^{-1} (\boldsymbol{I}_{N} - \boldsymbol{K}_{2}^{-1} \boldsymbol{M}_{2} \boldsymbol{K}_{1} \boldsymbol{M}_{1}) (\boldsymbol{I}_{N} - \boldsymbol{M}_{2} \boldsymbol{M}_{1})^{-1} \boldsymbol{K}_{2}^{-1} \boldsymbol{r}_{2} \\ &= -\mathrm{i} \boldsymbol{s}_{2}^{T} (\boldsymbol{I}_{N} - \boldsymbol{M}_{2} \boldsymbol{M}_{1})^{-1} \boldsymbol{K}_{2}^{-1} \boldsymbol{r}_{2} = \mathrm{i} \boldsymbol{v}_{2}, \end{split}$$

which completes the proof in the light of (3.40).

Explicit N-soliton solution formula of the FL equation can be presented below.

**Theorem 5.** The following function

$$u_{[\text{AKNS}]} = \boldsymbol{s}_1^T (\boldsymbol{I}_N - \boldsymbol{M}_1 \boldsymbol{M}_2)^{-1} \boldsymbol{r}_1$$
(4.11)

gives a N-soliton solution of the FL equation (1.2), where

$$\boldsymbol{K} = \operatorname{diag}(k_1, \cdots, k_N), \tag{4.12a}$$

$$\mathbf{r}_1 = (\rho_1(k_1), \cdots, \rho_1(k_N))^T, \quad \mathbf{s}_1 = (\sigma_1(k_1), \cdots, \sigma_1(k_N))^T, \quad (4.12b)$$

$$\boldsymbol{M}_{1} = (M_{1,ij})_{N \times N}, \qquad M_{1,ij} = \frac{\rho_{1}(k_{i})(\rho_{1}(k_{j}))^{*}}{k_{i} + k_{j}^{*}},$$
(4.12c)

$$\boldsymbol{M}_{2} = (M_{2,ij})_{N \times N}, \qquad M_{2,ij} = -\frac{\mathrm{i}k_{i}^{*}(\sigma_{1}(k_{i}))^{*}\sigma_{1}(k_{j})}{k_{i}^{*} + k_{j}}, \qquad (4.12\mathrm{d})$$

the plane wave factors are give by

$$\rho_1(k_i) = \exp\left(\frac{1}{2k_i}x + \frac{k_i}{2}t + \lambda_1(k_i)\right), \quad \sigma_1(k_i) = \exp\left(\frac{1}{2k_i}x + \frac{k_i}{2}t + \mu_1(k_i)\right), \quad (4.13a)$$

or alternatively,

$$\rho_1(k_i) = \exp\left(\frac{1}{k_i}x + k_it + \lambda_1(k_i)\right), \quad \sigma_1(k_i) = \exp\left(\mu_1(k_i)\right), \quad (4.13b)$$

$$\rho_1(k_i) = \exp(\lambda_1(k_i)), \quad \sigma_1(k_i) = \exp\left(\frac{1}{k_i}x + k_it + \mu_1(k_i)\right).$$
(4.13c)

#### 4.3 Nonlocal reduction and multiple-pole solution

Both the KP-type and AKNS-type solutions for the pKN(-1) system admit a nonlocal reduction

$$v(x,t) = -iu^*(-x,-t),$$
 (4.14)

which gives rise to a nonlocal FL equation from the pKN(-1) system:

$$u_{xt} - u + 2u^*(-x, -t)uu_x = 0. (4.15)$$

The reductions to get solutions of this equation are given by

• Nonlocal reduction for the KP-type:

$$\boldsymbol{L} = \boldsymbol{K}^{\dagger}, \quad \boldsymbol{s}_{1}^{T} = \boldsymbol{r}_{1}^{\dagger}(-x, -t)\boldsymbol{K}^{\dagger}, \quad \boldsymbol{s}_{2}^{T} = \boldsymbol{r}_{2}^{\dagger}(-x, -t), \quad (4.16)$$

• Nonlocal reduction for the AKNS-type:

$$\mathbf{K}_{2} = \mathbf{K}_{1}^{\dagger}, \quad \mathbf{s}_{2} = \mathbf{r}_{1}^{*}(-x, -t), \quad \mathbf{r}_{2} = -\mathbf{K}_{1}^{*}\mathbf{s}_{1}^{*}(-x, -t).$$
 (4.17)

The proof is similar to Theorem 2 and 4 and we skip it.

The Cauchy matrix approach can be used to derive not only N-soliton solutions by considering K as a diagonal matrix, but also multiple-pole solution, in which K and L ( $K_1$  and  $K_2$ in the AKNS-type) are assumed to be Jordan block matrices. The construction of multiple-pole solution had been fully discussed for the case of the SDYM equation in our previous work [27], one can refer to it. As an additional contribution to the completeness of this paper, we will provide the multiple-pole solution of FL equation in appendix A, where we will only consider the KP-type case, while the construction for the AKNS-type multiple-pole solutions is similar.

### 5 Concluding remarks

In this paper we have shown how the unreduced FL system, i.e. the pKN(-1) system (1.3), arose from the reduction of the general SDYM equation (2.8). The reduction was shown to be realized in the two Cauchy matrix schemes, namely, the KP-type and the AKNS-type. Consequently, two types of solutions of the pKN(-1) system were constructed, which turn out to equivalent under certain reflection transformation of coordinates. These solutions allow further (conjugate) reductions and at last yield solutions for the FL equation (1.2).

It can be identified that our solutions for the FL equation are the same as those obtained by Matsuno from bilinear approach [20]. One can refer to Appendix B of this paper for more details. In addition, in [25], vectorial Darboux transformation via bidifferential graded algebra techniques was employed to construct solutions for the unreduced FL system. Their solution (see Theorem 4.2 in [25]) also coincides with our KP-type Cauchy matrix solution with  $\boldsymbol{a} = \sigma_3$ . However, it seems not easy to identify the relation between the solutions in Cauchy matrix form of this paper and in double Wronskian form obtained in [16]. Besides solitons, based on the Cauchy matrix approach, one can construct more solutions other than solitons, for example, multiple-pole solutions. In Appendix A, some formulae of multiple-pole solutions for the pKN(-1) system and the FL equation are presented.

In this paper, apart from the pKN(-1) system, we also derived the AKNS(-1) system (2.22), which naturally appears as the K-formulation of the SDYM equation under a 2-dimensional

reduction. Note that there is a Riccati-type Miura transformation to connect the AKNS(-1) system and the pKN(-1) system:

$$K_{12} = uv^2 - iv_t, (5.1)$$

which was first found in [25] and played a crucial role in [25] in constructing solutions for the FL equation. Actually, (5.1) is nothing but (2.20) in the proof of Theorem 1, which naturally appears from the Miura transformation (2.15).

The paper added an important example to Ward's conjecture on the reduction of the SDYM equation. It may indicate the possibility of other types of solutions for the FL equation in light of such reductions. For example, the SDYM equation is famous for instantons [41] and the Atiyah-Hitchin-Drinfield-Manin (AHDM) ansatz [42]; it also admits quasi-Wronskian type solutions represented using quasideterminant [34, 35]. Whether these solutions could be reformulated in the Cauchy matrix approach and be reduced to the lower dimensional cases would be an interesting topic. In addition, our research conducted in this paper also indicates the Cauchy matrix structure of the Kaup-Newell hierarchy, which will be investigated separately. Finally, considering the Cauchy matrix approach is also a powerful tool to implement integrable discretization, e.g. [43], one can consider discretization of some equations in the Kaup-Newell hierarchy (cf. [44]). Compared with the discretization of the AKNS hierarchy, this is not well understood in literature.

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# A Construction of the KP-type multiple-pole solutions

The construction for multiple-pole solutions is much more complicated. We start our construction by introducing the following Nth-order lower triangle Toeplitz matrix generated by a function a(k):

$$\boldsymbol{F}_{k}^{[N]}[a(k)] = \begin{pmatrix} a & 0 & 0 & \cdots & 0\\ \frac{\partial_{k}a}{1!} & a & 0 & \cdots & 0\\ \frac{\partial_{k}^{2}a}{2!} & \frac{\partial_{k}a}{1!} & a & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{\partial_{k}^{N-1}a}{(N-1)!} & \frac{\partial_{k}^{N-2}a}{(N-2)!} & \frac{\partial_{k}^{N-3}a}{(N-3)!} & \cdots & a \end{pmatrix}_{N \times N}$$
(A.1)

Note that the set of all nonsingular N-order lower triangle Toeplitz matrices compose an Abelian group. Thus we have the following commutative relation

$$F_k^{[N]}[a(k)]F_l^{[N]}[b(l)] = F_l^{[N]}[b(l)]F_k^{[N]}[a(k)].$$

It is notable that by setting a(k) = k in (A.1), matrix  $\mathbf{F}_{k}^{[N]}[a(k)]$  yields a Nth-order Jordan block matrix of k, which is presented as

$$\boldsymbol{F}_{k}^{[N]}[k] = \boldsymbol{J}^{[N]}[k] = \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 1 & k & 0 & \cdots & 0 \\ 0 & 1 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}_{N \times N}$$

Then we introduce a lemma as follows.

Lemma 1. For the Sylvester equation

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^{T}, \tag{A.2}$$

where  $\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{r}, \boldsymbol{s}^T$  are defined as

$$\boldsymbol{K} = \boldsymbol{J}^{[N]}[k], \quad \boldsymbol{L}^{T} = \boldsymbol{J}^{[M]}[l], \quad \boldsymbol{R} = \boldsymbol{F}_{k}^{[N]}[\rho(k)], \quad \boldsymbol{S} = \boldsymbol{F}_{l}^{[M]}[\sigma(l)], \quad (A.3a)$$

$$\boldsymbol{r} = \boldsymbol{R} \mathbf{e}_N = \boldsymbol{F}_k^{[N]}[\rho(k)] \mathbf{e}_N, \quad \boldsymbol{s} = \boldsymbol{S} \mathbf{e}_M = \boldsymbol{F}_l^{[M]}[\sigma(l)] \mathbf{e}_M, \quad \mathbf{e}_N = (\underbrace{1, 0, 0, \cdots, 0}_{N-\text{dimensional}})^T, \quad (A.3b)$$

its solution matrix M can be formulated as

$$\boldsymbol{M} = \boldsymbol{R}\boldsymbol{G}\boldsymbol{S}^{T} = \boldsymbol{F}_{k}^{[N]}[\rho(k)] \cdot \boldsymbol{G} \cdot \left(\boldsymbol{F}_{l}^{[M]}[\sigma(l)]\right)^{T},$$
(A.4)

where G is a matrix determined by K and L:

$$\boldsymbol{G} = (g_{i,j})_{N \times M}, \quad g_{i,j} = \begin{pmatrix} i+j-2\\ i-1 \end{pmatrix} \frac{(-1)^{i-1}}{(k-l)^{i+j-1}}, \quad (A.5)$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad n \ge m, \quad m, n \in \mathbb{Z}^+.$$
(A.6)

Proof. Firstly, with (A.4) one can rewrite the Sylvester equation as

$$KRGS^{T} - RGS^{T}L = Re_{N}e_{M}^{T}S^{T}, \Rightarrow R(KG - GL)S^{T} = Re_{N}e_{M}^{T}S^{T}.$$
 (A.7)

Thus G satisfies the following Sylvester equation

$$KG - GL = \mathbf{e}_N \mathbf{e}_M^T, \tag{A.8}$$

which has been solved in [30], where the solution is given by (A.5).

Then we have the following theorem, which devotes to constructing the KP-type multiplepole solution of pKN(-1) system.

Theorem 6. To derive multiple-pole solution via Cauchy matrix approach, we set

$$\boldsymbol{K} = \boldsymbol{J}^{[N]}[k], \quad \boldsymbol{L}^{T} = \boldsymbol{J}^{[N]}[l], \quad \boldsymbol{R}_{j} = \boldsymbol{F}_{k}^{[N]}[\rho_{j}(k)], \quad \boldsymbol{S}_{j} = \boldsymbol{F}_{l}^{[N]}[\sigma_{j}(l)], \quad j = 1, 2, \quad (A.9a)$$

$$\boldsymbol{r}_{j} = \boldsymbol{R}_{j} \boldsymbol{e}_{N} = \boldsymbol{F}_{k}^{[N]}[\rho_{j}(k)]\boldsymbol{e}_{N}, \quad \boldsymbol{s}_{j} = \boldsymbol{S}_{j} \boldsymbol{e}_{N} = \boldsymbol{F}_{l}^{[N]}[\sigma_{j}(l)]\boldsymbol{e}_{N}, \quad (A.9b)$$

where  $\rho_i(k)$  and  $\sigma_i(l)$  are defined as in (3.36). Then **M** can be constructed as:

$$\boldsymbol{M} = \boldsymbol{R}_1 \boldsymbol{G} \boldsymbol{S}_1^T + \boldsymbol{R}_2 \boldsymbol{G} \boldsymbol{S}_2^T, \qquad (A.10)$$

where G follows the expression (A.5). By definition (3.24) and (A.9), (u, v) solves the pKN(-1) system.

Remark 8. For the Sylvester equation

$$KG + GL = \mathbf{e}_N \mathbf{e}_M^T$$

where  $\mathbf{K} = \mathbf{J}^{[N]}[k]$  and  $\mathbf{L}^T = \mathbf{J}^{[M]}[l]$ , solution  $\mathbf{G}$  is constructed as

$$\boldsymbol{G} = (g_{i,j})_{N \times M}, \quad g_{i,j} = \begin{pmatrix} i+j-2\\ i-1 \end{pmatrix} \frac{(-1)^{i+j}}{(k+l)^{i+j-1}}.$$
 (A.11)

If one replace G in Theorem 6 with (A.11), then M solves the Sylvester equation

$$KM + ML = rs^T$$
.

As for the explicit formula of multiple-pole solution of the FL equation, we have the following theorem.

**Theorem 7.** To derive multiple-pole solution of the FL equation, we set

$$K = J^{[N]}[k], \quad R_j = F_k^{[N]}[\rho_j(k)], \quad r_j = R_j \mathbf{e}_N, \quad j = 1, 2,$$
 (A.12)

where  $\rho_i(k)$  is defined as in (4.5a). The reduced Cauchy matrix **M** is constructed by

$$\boldsymbol{M} = \boldsymbol{R}_1 \boldsymbol{G} \boldsymbol{R}_1^{\dagger} - \mathrm{i} \boldsymbol{R}_2 \boldsymbol{G} \boldsymbol{R}_2^{\dagger} \boldsymbol{K}^{\dagger}, \qquad (A.13)$$

where

$$\boldsymbol{G} = (g_{i,j})_{N \times N}, \qquad g_{i,j} = \binom{i+j-2}{i-1} \frac{(-1)^{i+j}}{k+k^*}.$$
 (A.14)

Then (4.6) provides multiple-pole solution for the FL equation (1.2).

**Remark 9.** Notice that we use the expression (A.11) with  $l = k^*$  instead of (A.5) with  $l = -k^*$  in this theorem. In fact, if we started from

$$\boldsymbol{K}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{L} = \boldsymbol{r}\boldsymbol{s}^{T}, \quad \boldsymbol{L} = -\boldsymbol{K}^{\dagger}, \quad \boldsymbol{K} = \boldsymbol{J}^{[N]}[k], \quad (A.15)$$

we could not use Theorem 6. In this case, matrix L will be of the form

$$\boldsymbol{L}^{T} = \begin{pmatrix} -k^{*} & 0 & 0 & \cdots & 0 \\ -1 & -k^{*} & 0 & \cdots & 0 \\ 0 & -1 & -k^{*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -k^{*} \end{pmatrix}_{N \times N}$$
(A.16)

which is not the standard Jordan matrix in our definition. To tackle the problem, we consider L' = -L and the following set:

$$\boldsymbol{K}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{L}' = \boldsymbol{r}\boldsymbol{s}^{T}, \quad \boldsymbol{L}' = \boldsymbol{K}^{\dagger}, \quad \boldsymbol{K} = \boldsymbol{J}^{[N]}[k], \quad (A.17)$$

which indicates  $(\mathbf{L}')^T = \mathbf{J}^{[N]}[k^*]$ . Thus one can use the results in Remark 8 to construct the multiple-pole solution.

# **B** Correspondence to the Matsuno solution

In the literature [20], for the FL equation (to avoid misunderstandings, we use u rather than u in the following FL equation):

$$u_{xt} - u + 2i|u|^2 u_x = 0, (B.1)$$

the author (Y. Matsuno) introduced a bilinear transformation u = g/f and rewrote (B.1) as

$$D_x D_t g \cdot f = gf, \tag{B.2a}$$

$$D_t f \cdot f^* = igg^*, \tag{B.2b}$$

$$D_x D_t f \cdot f^* = i D_x g \cdot g^*, \tag{B.2c}$$

where D is the Hirota's bilinear operator defined as [45]

$$D_x^m D_t^n f(x,t) \cdot g(x,t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x,t'=t}.$$
 (B.2d)

Then, by Theorem 3.1 of [20], the following constructions give rise to the following bright N-soliton solution of the FL equation:

$$f = |W|, \quad g = \begin{vmatrix} W & \mathbf{z}_t^T \\ \mathbf{1} & 0 \end{vmatrix}, \quad W = (d_{ij})_{N \times N}, \quad d_{ij} = \frac{z_i z_j^* - \mathbf{i} p_j^*}{p_i + p_j^*}, \tag{B.3a}$$

$$z_i = \exp\left(p_i x + \frac{1}{p_i} t + \zeta_{i0}\right), \quad \mathbf{z}_t = (z_1/p_1, \cdots, z_N/p_N), \quad \mathbf{1} = (1, 1, \cdots, 1).$$
 (B.3b)

Now we consider transformations:

$$p_i \to -\frac{1}{k_i}, \quad z_i \to \exp\left(-\frac{1}{k_i}x - k_it + \zeta_{i0}\right), \quad \mathbf{z}_t \to (-k_1z_1, \cdots, -k_Nz_N), \quad (B.4)$$

which lead to

$$d_{ij} = -\frac{k_i z_i z_j^* k_j^* + ik_i}{k_i + k_j^*}, \quad d_{ji}^* = -\frac{k_i z_i z_j^* k_j^* - ik_j^*}{k_i + k_j^*}.$$
 (B.5)

Let  $\rho_2(k_i) = k_i z_i$  and  $\rho_1(k_i) = 1$  in (4.5c). Then we have  $W = -\mathbf{M}^{\dagger}$ ,  $\mathbf{1} = \mathbf{r}_1^{\dagger}$  and  $\mathbf{z}_t^T = -\mathbf{r}_2$  in Theorem 4. Thus we can rewrite Matsuno's solution as

$$\mathbf{u} = \frac{\begin{vmatrix} W & \mathbf{z}_t^T \\ \mathbf{1} & 0 \end{vmatrix}}{|W|} = -\mathbf{1}W^{-1}\mathbf{z}_t^T = -\mathbf{r}_1^{\dagger}\mathbf{M}^{-\dagger}\mathbf{r}_2 = -(\mathbf{r}_2^{\dagger}\mathbf{M}^{-1}\mathbf{r}_1)^{\dagger} = -u_{[\mathrm{KP}]}^*, \quad (B.6)$$

where  $u_{[KP]}$  follows the KP-type construction in (4.6) with (4.5c).

Matsuno also indicated in [20] that there is an alternative expression u = g'/f':

$$f' = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g' = \begin{vmatrix} A & I & \mathbf{y}_t^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & \mathbf{1} & 0 \end{vmatrix}, \quad (B.7)$$

where

$$A = (a_{ij})_{N \times N}, \quad a_{ij} = \frac{y_i y_j^*}{q_i + q_j^*}, \quad y_i = \exp\left(q_i x + \frac{1}{q_i} t + \eta_{i0}\right),$$
(B.8a)

$$B = (b_{ij})_{N \times N}, \quad b_{ij} = \frac{1q_j}{q_i^* + q_j}, \quad \mathbf{y}_t = (y_1/q_1, \cdots, y_N/q_N).$$
(B.8b)

Similarly, we take the following transformations:

$$q_i \to \frac{1}{k_i}, \quad y_i \to \exp\left(\frac{1}{k_i}x + k_it + \eta_{i0}\right), \quad \mathbf{y}_t = (k_1y_1, \cdots, k_Ny_N),$$
(B.9)

which lead to

$$a_{ij} = \frac{k_i y_i y_j^* k_j^*}{k_i + k_j^*}, \quad b_{ij} = \frac{\mathrm{i}k_i^*}{k_i^* + k_j}.$$
 (B.10)

Let  $\rho_1(k_i) = k_i y_i$  and  $\sigma_1(k_i) = 1$  in (4.13b), which implies  $A = M_1$ ,  $B = -M_2$ ,  $\mathbf{y}_t = \mathbf{r}_1^T$ ,  $\mathbf{1} = \mathbf{s}_1^T$  in Theorem 5. Then we have

$$f' = |AB + I| = |I_N - M_1 M_2|,$$
(B.11)

$$g' = \begin{vmatrix} I + AB & \mathbf{y}_t^T \\ \mathbf{1} & 0 \end{vmatrix} = \begin{vmatrix} I_N - M_1 M_2 & r_1 \\ s_1^T & 0 \end{vmatrix},$$
(B.12)

which shows

$$\mathbf{u} = \frac{g'}{f'} = -\boldsymbol{s}_1^T (\boldsymbol{I}_N - \boldsymbol{M}_1 \boldsymbol{M}_2)^{-1} \boldsymbol{r}_1 = -\boldsymbol{u}_{[\text{AKNS}]}, \qquad (B.13)$$

where  $u_{[AKNS]}$  follows the AKNS-type construction (4.11) with (4.13b).

In Proposition 3.1 and 3.2 of [20], the equivalence of (f, g) and (f', g') is established, which coincides with our discovery in section 3.2.3. Note that we have proved a more general case for pKN(-1) system.

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