

FIRST MEMOIR ON THE ASYMPTOTICS OF CERTAIN INFINITE PRODUCTS

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To Mourad Ismail, with admiration and warm wishes

ABSTRACT. The product sides of the Rogers–Ramanujan identities and alike often appear to be ‘transparently modular’ (functions). The old work by Rogers (1894) and recent work by Rosengren makes use (somewhat implicitly) of this fact for proving the identities with the help of underlying modular equations — the main challenge is verifying the latter for the sum sides. Here we speculate on the potentials of using the asymptotics of such q -identities for proving them, in particular, when a product side is ‘asymmetric’, hence not modular.

This note is inspired by several developments around the famous Rogers–Ramanujan identities including the old memoir [7] of L. J. Rogers, which made an influence on the present title. It would be also not mistaken to acknowledge that my personal interest in, taste for and education on q -series and Rogers–Ramanujan identities (in particular) were tremendously influenced by the work of Mourad Ismail. Particular examples of his fine work on the latter subject, again through my personal views, are [1, 3, 4]. I find it appropriate to dedicate this piece to Mourad on the occasion of his round birthday.

To prepare the stage, we list the standard q -notation (when q is viewed as a complex parameter we assume it to satisfy $|q| < 1$): the q -Pochhammer symbol is given by

$$(a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j)$$

and its multiple version is $(a_1, \dots, a_s; q)_k = (a_1; q)_k \cdots (a_s; q)_k$. These are used below for k a non-negative integer as well as for $k = \infty$.

1. ROGERS–RAMANUJAN AND ROGERS–SELBERG IDENTITIES

One proof of the celebrated Rogers–Ramanujan identities

$$G(q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{1}{(q, q^4; q^5)_{\infty}}, \quad H(q) = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q; q)_k} = \frac{1}{(q^2, q^3; q^5)_{\infty}},$$

based on the original methodology of Rogers [7] was recently given by Rosengren [8]. It is based on the ‘self-replicating’ equations satisfied by (both sum and product sides of) the two functions:

$$\begin{aligned} G(q) &= \frac{(q^8; q^8)_{\infty}}{(q^2; q^2)_{\infty}} (qH(-q^4) + G(q^{16})), \\ H(q) &= \frac{(q^8; q^8)_{\infty}}{(q^2; q^2)_{\infty}} (G(-q^4) + q^3H(q^{16})). \end{aligned} \tag{1}$$

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Similar machinery was used by Rosengren in [9] to prove the identities

$$\begin{aligned} A(q) &= (-q; q)_\infty \sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q^4; q^4)_k (-q; q^2)_k} = \frac{1}{(q, q^2, q^5, q^6; q^7)_\infty}, \\ B(q) &= (-q; q)_\infty \sum_{k=0}^{\infty} \frac{q^{2k(k+1)}}{(q^4; q^4)_k (-q; q^2)_k} = \frac{1}{(q, q^3, q^4, q^6; q^7)_\infty}, \\ C(q) &= (-q; q)_\infty \sum_{k=0}^{\infty} \frac{q^{2k(k+1)}}{(q^4; q^4)_k (-q; q^2)_{k+1}} = \frac{1}{(q^2, q^3, q^4, q^5; q^7)_\infty} \end{aligned}$$

by showing that

$$\begin{aligned} A(q) &= \frac{1}{(q^2; q^4)_\infty^2} (qC(-q^2) + A(q^8)), & B(q) &= \frac{1}{(q^2; q^4)_\infty^2} (A(-q^2) + qB(q^8)), \\ C(q) &= \frac{1}{(q^2; q^4)_\infty^2} (B(-q^2) + q^3C(q^8)), \end{aligned} \tag{2}$$

for both the sum and product sides.

The first thing to point out is that the product sides of all the identities above are *modular* functions after an appropriate normalisation (which makes identities (1) and (2) particular examples of modular equations for participating modular functions). Namely, for $b > a > 0$ integers, the product $(q^a, q^{b-a}; q^b)_\infty$ transforms into a modular function

$$q^{bB_2(a/b)/2} (q^a, q^{b-a}; q^b)_\infty,$$

where $B_2(t) = t^2 - t + 1/6$ is the second Bernoulli polynomial; also $q^{b/24} (q^b; q^b)_\infty$ is a (weight $1/2$) modular function (a ‘scaled’ Dedekind eta function). For example, if we write

$$\begin{aligned} \hat{A}(q) &= q^{-1/42} A(q) = \frac{q^{-1/42}}{(q, q^2, q^5, q^6; q^7)_\infty}, & \hat{B}(q) &= q^{5/42} B(q) = \frac{q^{5/42}}{(q, q^3, q^4, q^6; q^7)_\infty}, \\ \hat{C}(q) &= q^{17/42} C(q) = \frac{q^{17/42}}{(q^2, q^3, q^4, q^5; q^7)_\infty}, \end{aligned}$$

then equations (2) assume a symmetric form

$$\begin{aligned} \hat{A}(q) &= \frac{q^{1/6}}{(q^2; q^4)_\infty^2} (\pm \hat{C}(-q^2) + \hat{A}(q^8)), & \hat{B}(q) &= \frac{q^{1/6}}{(q^2; q^4)_\infty^2} (\pm \hat{A}(-q^2) + \hat{B}(q^8)), \\ \hat{C}(q) &= \frac{q^{1/6}}{(q^2; q^4)_\infty^2} (\pm \hat{B}(-q^2) + \hat{C}(q^8)) \end{aligned}$$

(we ignore the choice of sign), in which $q^{1/6}$ corresponds to the modular normalisation of the eta-type product $1/(q^2; q^4)_\infty^2$. In fact, the sign variation leads to companion identities

$$\begin{aligned} A(q) &= \frac{(-q, -q^5, -q^9, -q^{13}; q^{14})_\infty \cdot (-qC(-q^2) + A(q^8))}{(q^2; q^4)_\infty^2 (q, q^5, q^9, q^{13}; q^{14})_\infty}, \\ B(q) &= \frac{(-q, -q^3, -q^{11}, -q^{13}; q^{14})_\infty \cdot (A(-q^2) - qB(q^8))}{(q^2; q^4)_\infty^2 (q, q^3, q^{11}, q^{13}; q^{14})_\infty}, \\ C(q) &= \frac{(-q^3, -q^5, -q^9, -q^{11}; q^{14})_\infty \cdot (B(-q^2) - q^3C(q^8))}{(q^2; q^4)_\infty^2 (q^3, q^5, q^9, q^{11}; q^{14})_\infty} \end{aligned}$$

of (2), and we also have similar closed forms for

$$G(q) = \frac{-qH(-q^4) + G(q^{16})}{(q^4; q^8)_\infty (q, q^9; q^{10})_\infty^2 (q^6, q^{10}, q^{14}; q^{20})_\infty},$$

$$H(q) = \frac{G(-q^4) - q^3 H(q^{16})}{(q^4; q^8)_\infty (q^3, q^7; q^{10})_\infty^2 (q^2, q^{10}, q^{18}; q^{20})_\infty}$$

complementing (1).

Notice however that these alternative modular equations for $A(q), B(q), C(q)$ and for $G(q), H(q)$ seem to be harder to establish directly for the sum sides of the corresponding Rogers–Ramanujan-type identities.

2. KANADE–RUSSELL MODULO 9 IDENTITIES

One can also perform similar modular normalisation of the product sides of the Kanade–Russell modulo 9 conjectural identities [5, 11]

$$\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2}}{(q; q)_m (q^3; q^3)_n} \stackrel{?}{=} K_1(q) = \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty},$$

$$\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+m+3n}}{(q; q)_m (q^3; q^3)_n} \stackrel{?}{=} K_2(q) = \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty},$$

$$\sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+2m+3n}}{(q; q)_m (q^3; q^3)_n} \stackrel{?}{=} K_3(q) = \frac{1}{(q^3, q^4, q^5, q^6; q^9)_\infty},$$

to check (routinely!) that these product sides satisfy the equations

$$K_1(q) = \frac{qK_2(-q^2) + K_1(q^8)}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (-q^5, -q^{13}; q^{18})_\infty},$$

$$K_2(q) = \frac{K_3(-q^2) + qK_2(q^8)}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (-q, -q^{17}; q^{18})_\infty},$$

$$K_3(q) = \frac{K_1(-q^2) + q^5 K_3(q^8)}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (-q^7, -q^{11}; q^{18})_\infty}.$$

They also satisfy

$$K_1(q) = \frac{(-q, -q^{17}; q^{18})_\infty \cdot (-qK_2(-q^2) + K_1(q^8))}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (q, q^5, q^{13}, q^{17}; q^{18})_\infty},$$

$$K_2(q) = \frac{(-q^7, -q^{11}; q^{18})_\infty \cdot (K_3(-q^2) - qK_2(q^8))}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (q, q^7, q^{11}, q^{17}; q^{18})_\infty},$$

$$K_3(q) = \frac{(-q^5, -q^{13}; q^{18})_\infty \cdot (K_1(-q^2) - q^5 K_3(q^8))}{(q^2; q^4)_\infty (q^6; q^{12})_\infty (q^3, q^{15}; q^{18})_\infty (q^5, q^7, q^{11}, q^{13}; q^{18})_\infty}.$$

The techniques in [8, 9] are hardly applicable to the sum sides of either identities for $K_1(q), K_2(q), K_3(q)$: this time we deal with double summations.

3. ANDREWS–GORDON IDENTITIES

The Rogers–Ramanujan identities are first entries in a general family of the Andrews–Gordon identities, which feature for each $k \geq 2$ the product sides. Similarly, the product

parts of modulo 7 Andrews–Gordon identities,

$$P_i(q) = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty}, \quad \text{where } i = 1, \dots, k,$$

while the sum sides correspond to $(k-1)$ -fold summations. For $k=3$, the eta-type products coincide with those for $A(q), B(q), C(q)$ which we have already treated in Section 1. Similar treatment for the sum sides is not known.

Unfortunately, there seems to be no simple-looking extension of the story to modulo 9, 13 (and higher). For modulo 11 and the products

$$P_i(q) = \frac{(q^i, q^{11-i}, q^{11}; q^{11})_\infty}{(q; q)_\infty}, \quad \text{where } i = 1, \dots, 5,$$

we find some ‘surrogate’ versions:

$$\begin{aligned} P_1(q^2) &= \frac{P_5(-q^2) - q^6 P_3(q^8)}{(q^2; q^4)_\infty^2 (q^{18}, q^{26}; q^{44})_\infty}, \\ P_2(q^2) &= \frac{-q^2 P_1(-q^2) + P_5(q^8)}{(q^2; q^4)_\infty^2 (q^{14}, q^{30}; q^{44})_\infty}, \\ P_3(q^2) &= \frac{P_4(-q^2) - q^4 P_2(q^8)}{(q^2; q^4)_\infty^2 (q^{10}, q^{34}; q^{44})_\infty}, \\ P_4(q^2) &= \frac{P_2(-q^2) - q^6 P_1(q^8)}{(q^2; q^4)_\infty^2 (q^6, q^{38}; q^{44})_\infty}, \\ P_5(q^2) &= q^2 \frac{-P_3(-q^2) + P_4(q^8)}{(q^2; q^4)_\infty^2 (q^2, q^{42}; q^{44})_\infty}; \end{aligned}$$

and there are no alternative plus-sign companions.

4. ASYMMETRIC KANADE–RUSSELL IDENTITIES

There are more Kanade–Russell modulo 9 identities [5,11] which are not modular functions. They include

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+m+2n}}{(q; q)_m (q^3; q^3)_n} &\stackrel{?}{=} K_4(q) = \frac{1}{(q^2, q^3, q^5, q^8; q^9)_\infty}, \\ \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+m+4n}(1+q^{m+1})}{(q; q)_m (q^3; q^3)_n} &\stackrel{?}{=} K_5(q) = \frac{1}{(q, q^4, q^6, q^7; q^9)_\infty}, \end{aligned}$$

where the sum form of the last identity was simplified from its original by Hickerson [2], who also complemented the set with the identities

$$\begin{aligned} \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+m+n}(1-\omega q^{m+3n+1})}{(q; q)_m (q^3; q^3)_n} &\stackrel{?}{=} K_6(q) = \frac{(q^6; q^9)_\infty (\omega q, \bar{\omega} q^3; q^3)_\infty}{(q^2; q^3)_\infty}, \\ \sum_{m,n \geq 0} \frac{q^{m^2+3mn+3n^2+2n}(1-\omega q^{3m+3n+2})}{(q; q)_m (q^3; q^3)_n} &\stackrel{?}{=} K_7(q) = \frac{(q^3; q^9)_\infty (\omega q^2, \bar{\omega} q^3; q^3)_\infty}{(q; q^3)_\infty}, \end{aligned}$$

with ω a primitive cubic root of unity and $\bar{\omega}$ its conjugate, and their conjugations (so that there are four additional conjectural identities in total). It is hard to expect any modular-type functional equations for these new instances, though the product sides may possess some modular-like behaviour.

Consider a real number a from the interval $0 < a \leq 1$. In his final Example 4 in [12] Zagier outlines the asymptotics

$$\begin{aligned} -\ln \prod_{m=0}^{\infty} (1 - e^{-(m+a)x}) &\sim \frac{\zeta(2)}{x} - \ln x^{\zeta(0,a)} + \ln \zeta'(0, a) - \sum_{n=1}^{\infty} \frac{B_n}{n \cdot n!} \frac{B_{n+1}(a)}{n+1} (-x)^n \\ &= \frac{\pi^2}{6x} + \ln x^{a-\frac{1}{2}} + \ln \frac{\Gamma(a)}{\sqrt{2\pi}} - \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{n \cdot (n+1)!} B_{n+1}(a) x^n \end{aligned} \quad (3)$$

as $x \rightarrow 0$, in which $\zeta(s, a)$ denotes the Hurwitz zeta function, $B_n(t)$ and $B_n = B_n(1)$ are the Bernoulli polynomials and numbers respectively. Note that $B_n = B_n(1) = 0$ for $n > 1$ odd. More generally, $B_n(1-a) = -B_n(a)$ for $n > 1$ odd implying that the asymptotics

$$-\ln \prod_{m=0}^{\infty} (1 - e^{-(m+a)x})(1 - e^{-(m+1-a)x}) \sim \frac{\pi^2}{3x} + \ln \frac{\sqrt{\pi}}{\sqrt{2} \sin \pi a} - \frac{1}{2} \left(a^2 - a + \frac{1}{6} \right) x + o(x^N)$$

as $x \rightarrow 0$, where $N > 1$ can be taken arbitrary. In the case of rational a , writing it as a/b with $0 < a < b$ integers to the lowest terms and taking $2\pi b x$ for x , the asymptotics reads

$$-\ln(q^a, q^{b-a}; q^b)_{\infty} \Big|_{q=e^{-2\pi x}} \sim \frac{\pi}{6bx} + \ln \frac{\sqrt{\pi}}{\sqrt{2} \sin(\pi a/b)} - bB_2(a/b)\pi x + o(x^N)$$

and further

$$q^{bB_2(a/b)/2} (q^a, q^{b-a}; q^b)_{\infty} \Big|_{q=e^{-2\pi x}} \sim \frac{\sqrt{2} \sin(\pi a/b)}{\sqrt{\pi}} e^{-\pi/(6bx)} (1 + o(x^N))$$

as $x \rightarrow 0$ for any $N > 1$. This agrees with the modular behaviour (hinted in Section 1) of the symmetric products $q^{bB_2(a/b)/2} (q^a, q^{b-a}; q^b)_{\infty}$. At the same time the asymptotics of an individual product $(q^a; q^b)_{\infty}$ for $a \neq b/2$ and $a \neq b$ as $q = e^{-2\pi x} \rightarrow 1$ is clearly very different, since the terms for even n in (3) contribute to it.

There are however (infinitely) many examples of asymmetric products compiled from $(q^a; q^b)_{\infty}$ that exhibit a modular-like asymptotics as $q \rightarrow 1$ (but also—with a bit of extra analysis outlined in [12]—at q approaching other roots of unity). We illustrate this on examples of the Kanade–Russell products $K_4(q)$ and $K_5(q)$. Using, for $r = 3$, the summation formula

$$\sum_{k=0}^{r-1} B_{n+1} \left(t + \frac{k}{r} \right) = r^{-n} B_{n+1}(rt)$$

for the Bernoulli polynomials and multiplication formula

$$\sum_{k=0}^{r-1} \Gamma \left(t + \frac{k}{r} \right) = (2\pi)^{(r-1)/2} r^{-rt+1/2} \Gamma(rt)$$

for the gamma function we see that both $1/(q, q^4, q^7; q^9)_{\infty} \Big|_{q=e^{-2\pi x}}$ and $1/(q; q^3)_{\infty} \Big|_{q=e^{-2\pi x}}$ exhibit the *same* asymptotic behaviour

$$\frac{\pi}{36x} + \ln(6\pi x)^{-\frac{1}{6}} + \ln \frac{\Gamma(\frac{1}{3})}{\sqrt{2\pi}} - \sum_{n=1}^{\infty} (-1)^n \frac{B_n}{n \cdot (n+1)!} B_{n+1}(\frac{1}{3}) (6\pi x)^n \quad \text{as } x \rightarrow 0,$$

and a similar complete match takes place for the asymptotics of $1/(q^2, q^5, q^8; q^9)_{\infty}$ and $1/(q^2; q^3)_{\infty}$ as $q \rightarrow 1$. In particular, the asymptotics of both $q^{1/12} K_4(q)$ and $q^{1/12} K_5(q)$ as $q \rightarrow 1$ coincide with that of $q^{1/12}/(q, q^2; q^3)_{\infty}$, which is a modular function. The functions $\hat{K}_4(q) = q^{1/12} K_4(q)$ and $\hat{K}_5(q) = q^{1/12} K_5(q)$ are however *not* modular in the classical sense but, perhaps, are components of a vector-valued modular function, hence have a controllable mock-theta-like behaviour [6, 13]. What we could check numerically is that the

functions $\hat{K}_4(\pm q^k)$ and $\hat{K}_5(\pm q^k)$ for various k (and a particular sign choice for each k) do not seem to be linearly related to each other with coefficients from the field of modular functions.

Echoing [14], the asymptotics at roots of unity for a particular q -sum-to-product identity can serve as a ground for its proof. Though it looks like a doable task for the product side, there seem to be no efficient strategies to make it work for the sum side. One still needs a creative work in the spirit of Rogers and Rosengren [7–9].

5. FINITE IDENTITIES

In fact, it is quite suggestive that manipulations of the type $q \mapsto \pm q^k$ are natural at the level of *finite* (q -polynomial) versions of the sum sides of Rogers–Ramanujan(-type) identities. These usually originate from combinatorial interpretations, and many are recorded in the literature; we limit ourselves by citing [10] and [11] for a historical overview and references provided there. In spite of this personal belief there seems to be no evidence for existence of finite versions of equations (1) and (2).

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