# <span id="page-0-0"></span>Classifying finite groups  $G$  with three Aut $(G)$ -orbits

Stephen P. Glasby∗1

<sup>1</sup>Center for the Mathematics of Symmetry and Computation, University of Western Australia, Perth 6009, Australia. [Stephen.Glasby@uwa.edu.au](mailto:Stephen.Glasby@uwa.edu.au)

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#### Abstract

We give a complete and irredundant list of the finite groups  $G$  for which  $Aut(G)$ , acting naturally on  $G$ , has precisely 3 orbits. There are 7 infinite families: one abelian, one non-nilpotent, three families of non-abelian 2-groups and two families of non-abelian p-groups with p odd. The non-abelian 2-group examples were first classified by Bors and Glasby in 2020 and non-abelian p-group examples with p odd were classified independently by Li and Zhu [\[24\]](#page-17-0), and by the author, in March 2024.

Dedication: To Otto H. Kegel on the occasion of his 90th birthday

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#### 1 Introduction

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A group G is called a k-orbit group if  $Aut(G)$ , acting naturally on G, has precisely k orbits. The cyclic group  $C_{p^{k-1}}$ ,  $k \geq 1$ , and the generalized quaternion group  $Q_{2^{k-1}}$ of order  $2^{k-1}$ ,  $k \geqslant 5$ , are both k-orbit p-groups. Although we assume that G is a *finite* group, many of our examples generalize to infinite groups. (For an infinite group  $G$  it is sometimes natural to count orbits under the subgroup of *topological automorphisms*.) Write  $\text{ord}(G) := \{ |g| \mid g \in G \}$  for the set of element orders in G. A 2-orbit group G is elementary abelian and  $\text{ord}(G)$  equals  $\{1,p\}$  where p is a prime. If G is a 3-orbit group, then ord(G) equals  $\{1, p\}$ , with two orbits of elements of order p, or  $\{1, p, p^2\}$ or  $\{1, p, r\}$  where p and r are distinct primes. Thus a 3-orbit group is solvable by Burnside's  $p^{\alpha}r^{\beta}$  theorem. A complete k-orbit group G has  $Aut(G) = \text{Inn}(G) \cong G$ and k equals the number  $k(G)$  of G-conjugacy classes. In general, for a (finite) k-orbit group G we have the bounds  $|\text{ord}(G)| \leq k \leq \min\{k(G), |G|/p\}$  where p is the smallest prime divisor of  $|G|$ . The upper bound  $|G|/p$  is due to [\[21,](#page-17-1) Theorem 1].

This paper gives a complete and irredundant list of finite 3-orbit groups. Clearly  $\{1\}$  is an orbit, and for each characteristic subgroup N of G the set  $N^{\#} := N \setminus \{1\}$ is a union of orbits. If a k-orbit group has a chain  $G = N_1 > \cdots > N_k = \{1\}$  of k characteristic subgroups, then the sets  $N_i \setminus N_{i+1}$  for  $1 \leq i \leq k$  and  $\{1\}$  are the k non-trivial  $Aut(G)$  orbits, see Lemma [2.1](#page-2-0) for more information.

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<span id="page-1-1"></span>Let G be a 3-orbit group and set  $N := \langle G', \Phi(G) \rangle$ . Then  $N \neq 1$  as G is not elementary abelian, and  $N \neq G$  as G is solvable. Hence  $G/N$  and N are both 2-orbit groups and thus elementary abelian. We write  $V = G/N = \mathbb{F}_r^m$  and  $W = N = \mathbb{F}_p^n$ where r, p are primes (possibly equal). Observe that  $Aut(G)$  induces on V a linear subgroup  $A \n\leq \mathrm{GL}_m(r)$ , and induces (via restriction) a linear subgroup  $B \n\leq \mathrm{GL}_n(p)$ . Furthermore, A is transitive on the set  $V \setminus \{0\}$  of non-zero vectors of V, and B is transitive on the set  $W \setminus \{0\}$  of non-zero vectors of W. We use Hering's theorem [\[15,](#page-17-2)  $\S5$ , [\[26,](#page-17-3) p. 512] which classifies the linear subgroups A and B (see Theorem [4.1\)](#page-5-0).

Our main result below agrees with Theorem B of Li and Zhu [\[24,](#page-17-0) Theorem B] as line 7 of Table [1](#page-1-0) corresponds to their families (7) and (8). Hering's theorem is a key tool in [\[2,](#page-16-0) [23,](#page-17-4) [24\]](#page-17-0) and the present paper. Our approach is arguably more geometric than [\[24\]](#page-17-0) motivated by the desire to gain insight into 4-orbit groups. We find it convenient to represent our  $p$ -group (and solvable group) examples as Cartesian products of vector spaces, with specified multiplication rules, and linear actions.

<span id="page-1-0"></span>**Theorem 1.1.** Let G be a finite 3-orbit group with  $N = \langle G', \Phi(G) \rangle$  and  $|N| = p^n$ . *Then* 1 < N < G *and* G *is isomorphic to a group in lines* 1− 7 *of Table* [1](#page-1-0)*. Moreover, the values of*  $V \cong G/N$ ,  $A = Aut(G)<sup>V</sup>$ ,  $W \cong N$ ,  $B = Aut(G)$  ↓ W *are valid, where*  $Aut(G)^V$  and  $Aut(G)\downarrow W$  denote the groups induced on  $G/N$  and N by  $Aut(G)$ .

G	V	$A$ N		$\overline{B}$	Comments	Ref.
1. $(C_{p^2})^n$	$\mathbb{F}_p^n$	$GL_n(p)$	$\mathbb{F}_n^n$	$GL_n(p)$	$p \geqslant 2$ , G abelian	p.4
2. $\mathbb{F}_q^d \rtimes C_r$	$\mathbb{F}_r$	$GL_1(r)$	$\mathbb{F}_a^d$	$\Gamma L_d(q)$	$q = p^{r-1}, p \neq r, d = \frac{n}{r-1}$	6.12
3. $A(n, \theta)$	$\mathbb{F}_2^n$	$\Gamma L_1(2^n)$	$\mathbb{F}_2^n$	$\Gamma L_1(2^n)$	Def. 3.2(a), $n \neq 2^{\ell}$	6.14
4. $B(n)$	$\mathbb{F}_2^{2n}$	$\Gamma L_1(2^{2n})$	$\mathbb{F}_2^n$	$\Gamma L_1(2^n)$	Def. 3.2(b), $n \ge 1$	6.14
5. P	$\mathbb{F}_2^6$	$C_7 \rtimes C_9$	$\mathbb{F}_2^3$	$\Gamma L_1(2^3)$	Def. 3.2(c), $n = 3$	6.14
6. $\mathbb{F}_q^3$ : $\mathbb{F}_q^3$		$\mathbb{F}_q^3$ $\Gamma\mathrm{L}_3(q)$ $\mathbb{F}_q^3$		$\Gamma L_{3}^{+}(q)$	$q=p^{\frac{n}{3}}$ odd, $3 n$	6.9
7. $\mathbb{F}_{p^n}$ : $\mathbb{F}_q$ <sup>b</sup>	$\mathbb{F}_q^{\frac{m}{b}}$	$\mathrm{Sp}_{\frac{m}{2}}(q)\leqslant$	$\mathbb{F}_{p^n}$	$\Gamma L_1(p^n) \leq$	$q = p^b$ odd, $n   b   m$	6.2

Table 1: 3-orbit groups G and  $V \cong G/N$ ,  $A = \text{Aut}(G)^{G/N}$ ,  $B = \text{Aut}(G) \downarrow N$ 

The number  $\omega(G)$  of Aut(G) orbits on G can sometimes be calculated without knowing all of  $Aut(G)$ , see Lemma [2.2.](#page-2-1) Nonsolvable k-orbit groups have been classified for  $1 \leq k \leq 6$ . There are none when  $k \leq 3$ . If G is finite and S is a composition factor of G, then S is simple and  $\omega(G) \geq \omega(S)$ . If G is a finite nonsolvable k-orbit group, then  $G = A_5$  if  $k = 4$  by [\[21\]](#page-17-1),  $G \in \{PSL_2(q) | q \in \{7, 8, 9\}\}\$  if  $k = 5$  by [\[27\]](#page-17-5), and  $G \in \{PSL_3(4), \mathbb{F}_4^2 \rtimes SL_2(4)\}$  if  $k = 6$  by [\[6\]](#page-16-1). By contrast, classifying *solvable* korbit groups is extremely difficult. This paper focuses on classifying (solvable) 3-orbit groups while contemplating 4-orbit groups, see Lemma [6.9](#page-13-0) and Section [7.](#page-15-0)

We note that certain permutation groups of 'rank'  $k$  give rise to  $k$ -orbit groups. A permutation group  $G \leq \text{Sym}(\Omega)$  has *rank* k if it is transitive on  $\Omega$  and the point stabilizer  $G_{\omega}$  has k orbits including  $\{\omega\}$ ; equivalently G has k orbits on  $\Omega \times \Omega$ .

**Lemma 1.2.** If  $\text{Aut}(G)$  has k orbits on G, then the subgroup  $\text{Hol}(G) = G \rtimes \text{Aut}(G)$ *of the symmetric group*  $Sym(G)$  *has rank k and the stabilizer*  $Hol(G)_1$  *of* 1 *is*  $Aut(G)$ *.* 

Although rank 2 permutation groups have been classified, rank 3 groups have only been classified in certain cases e.g. when they are solvable [\[7,](#page-16-2) [10\]](#page-16-3), or quasiprimitive [\[9\]](#page-16-4), <span id="page-2-4"></span>or innately transitive [\[1\]](#page-16-5). Sadly for us, the holomorph<sup>[1](#page-2-2)</sup> Hol(G) of a 3-orbit group G is commonly not solvable, and  $Hol(G)$  is never innately transitive as  $N = \langle \Phi(G), G' \rangle$ is intransitive and is the unique minimal normal subgroup of  $Hol(G) \leqslant Sym(G)$ . For a recent history of the classification of certain low rank groups see [\[11,](#page-17-6) pp. 177– 178]. However, if G is a 3-orbit 2-group, then Aut(G) *is* solvable by [\[2,](#page-16-0) Proposition 3.1. Hence is  $Hol(G)$  also solvable. This generalizes the result [\[5,](#page-16-6) Theorem 2] of Bryukhanova, and is used to prove Theorem [3.3](#page-4-1) when  $p = 2$ . The 3-orbit 2-groups G have been classified in [\[23,](#page-17-4) Corollary 1.3] and [\[2,](#page-16-0) Theorem 1.2]. We find it useful to explicitly specify the (linear) action of  $\text{Aut}(G)$  on the vector space  $N \cong \mathbb{F}_p^n$ .

Our proof of Theorem [1.1](#page-1-0) is divided into three cases. Section [2](#page-2-3) outlines our three-case strategy, and verifies lines 1 and 2 of Table [1.](#page-1-0) Section [3](#page-3-1) considers Case 1 which includes  $p = 2$ , and Sections [4](#page-4-2) and [5](#page-6-0) consider Cases 2 and 3 when p is odd, and  $G$  must be a nonabelian group of exponent  $p$ . Section [6](#page-10-1) devotes considerable effort to constructing examples of infinite and finite  $k$ -orbit groups for small  $k$ . The examples in line 6 of Table [1](#page-1-0) generalize to infinite 3- and 4-orbit groups in Lemma [6.9.](#page-13-0) Similarly, Example [6.12](#page-14-0) generalizes line 2 of Table [1](#page-1-0) and Lemmas [5.1,](#page-7-0) [6.2](#page-10-0) generalize line 8. Finally, Section [7](#page-15-0) investigates the feasibility of classifying 4-orbit groups; as there are so few examples, a classification may be feasible.

#### <span id="page-2-3"></span>2 Preparation for a proof of Theorem [1.1](#page-1-0)

In this section G is a finite 3-orbit group. Let  $N = \langle G', \Phi(G) \rangle$ , and let  $A, B$  be the linear groups induced on  $G/N$  and N by  $Aut(G)$  as in the Introduction. Recall that the *solvable residual*  $A^{\infty}$  of a finite group A, has the property that  $A/A^{\infty}$  is the largest solvable factor group of A. We split the proof of Theorem [1.1](#page-1-0) into cases:

Case 1.  $A^{\infty} = 1$ , Case 2.  $B^{\infty} \neq 1$  (so  $A^{\infty} \neq 1$ ), and Case 3.  $A^{\infty} \neq 1$  and  $B^{\infty} = 1$ .

In Case 1, A is solvable, and so too is  $Aut(G)$  since the kernel of the epimorphism  $\text{Aut}(G) \to A$  is the subgroup  $\text{CAut}(G) \cong \text{Hom}(V, W) \cong (\text{C}_p)^{mn}$  of central automor-phisms, see Lemma [2.3.](#page-3-2) In Theorem [3.3,](#page-4-1) A is solvable or  $p = 2$ .

The following lemmas can sometimes help to compute  $Aut(G)$  and  $\omega(G)$ . It uses the fact: If M is characteristic in G and  $G > M > 1$ , then  $\omega(G) \geq \omega(G/M) + \omega(M) - 1$ .

<span id="page-2-0"></span>**Lemma 2.1.** *Suppose* G *is a finite group and*  $G = M_1 > \cdots > M_k = \{1\}$  *where each*  $M_i$  is characteristic in G. Then  $\omega(G) \geq 2 - k + \sum_{i=1}^{k-1} \omega(M_i/M_{i+1}) \geq k$  for  $k \geq 1$ . If  $\omega(G) = k$ , then G is solvable and has precisely k characteristic subgroups.

*Proof.* The inequality  $\omega(G) \geq 2 - k + \sum_{i=1}^{k-1} \omega(M_i/M_{i+1})$  follows by induction on k from  $\omega(G) \geq \omega(G/M) + \omega(M) - 1$ . This show that  $\omega(G) \geq k$  since  $\omega(M_i/M_{i+1}) \geq 2$ for each i. Suppose that  $\omega(G) = k$ , i.e. equality holds. Then  $\omega(M_i/M_{i+1}) = 2$  for each i, so each section  $M_i/M_{i+1}$  is elementary abelian; whence G is solvable. If G has a characteristic subgroup different to the k subgroups  $M_1, \ldots, M_k$ , then it will refine the characteristic series  $G = M_1 > \cdots > M_k = \{1\}$ , so  $\omega(G) > k$ , a contradiction.  $\Box$ 

<span id="page-2-1"></span>**Lemma 2.2.** *If we know a lower bound*  $\omega(G) \geq \omega_0$  *and a subgroup*  $A_0$  *of*  $\text{Aut}(G)$ *with precisely*  $\omega_0$  *orbits on G, then*  $\omega(G) \leq \omega_0$  *and hence*  $\omega(G) = \omega_0$ *.* 

A finite nonabelian 3-orbit p-group has  $G' \leq \Phi(G)$  so  $N = G' = \mathbb{Z}(G) = \Phi(G)$ . We now show that  $A \le \text{GL}(G/N)$  determines  $B \le \text{GL}(N)$ , and the A orbits on  $V \cong G/N$  determine the Aut(G)-orbits on  $G \setminus N$ .

<span id="page-2-2"></span><sup>&</sup>lt;sup>1</sup>A permutation group H with a regular normal subgroup G satisfies  $G \triangleleft H \leq \text{Hol}(G) \leq \text{Sym}(G)$ .

<span id="page-3-4"></span><span id="page-3-2"></span>Lemma 2.3. *For a nonabelian* 3*-orbit* p*-group* G*, the preimage under the natural map*  $G \to G/Z(G) \cong V$  *of the nonzero A-orbits on* V are the Aut(G)-*orbits on*  $G \setminus Z(G)$ . *Further, the homomorphism*  $\phi$ :  $Aut(G) \to Aut(G/Z(G))$  *has image A and kernel the central automorphisms* CAut(G)  $\cong$  Hom(V, W), and A/K  $\cong$  B *for some* K  $\triangleleft$  A.

*Proof.* By definition,  $\ker(\phi)$  equals  $CAut(G)$  namely the set of all  $\alpha \in Aut(G)$  that act trivially on  $G/Z(G)$ . However,  $G/Z(G) \cong V$  and  $Z(G) \cong W$ , and therefore CAut $(G) \cong \text{Hom}(V, W)$  is the elementary of order  $|W|^{dim(V)} = p^{mn}$ . Furthermore, each  $\alpha \in \text{CAut}(G)$  acts trivially on  $G' = Z(G) \cong W$  since

$$
[g_1z_1, g_2z_2]^{\alpha} = [g_1^{\alpha}z_1^{\alpha}, g_2^{\alpha}z_2^{\alpha}] = [g_1^{\alpha}, g_2^{\alpha}] = [g_1, g_2] \qquad (g_1, g_2 \in G, z_1, z_2 \in Z(G)).
$$

Hence the A-action on V induces an action on  $W$ . The kernel of this action, namely  $K := C_A(W)$ , satisfies CAut(G)  $\leq K$  and  $A/K \cong B$ . The first sentence of the lemma is true since  $g_1 \in g_2Z(G)$  implies  $g_1^{\alpha} = g_2$  for some  $\alpha \in \text{CAut}(G)$ .  $\Box$ 

The following straightforward lemma can be a useful tool for computing  $Aut(G)$ where G is a non-abelian 3-orbit p-group. If we know that  $\mathcal{A}_0 \leq \text{Aut}(G)$  induces a subgroup  $A_0$  of  $A = \text{Aut}(G)^V$ ,  $V = G/\Phi(G)$ , and we show that  $\text{Aut}(G)$  can not induce a larger subgroup of  $GL(V)$ . Then  $A = A_0$ , and hence  $Aut(G) = CAut(G) \cdot A_0$ .

<span id="page-3-0"></span>**Lemma 2.4.** *Suppose we know a subgroup*  $A_0$  *of*  $A \leq \mathrm{GL}_m(p)$ *. If*  $A_0 = \mathrm{GL}_m(p)$  *or*  $A_0$  *is maximal (proper) subgroup of*  $GL_m(p)$  *and*  $A \neq GL_m(p)$ *, then*  $A_0 = A$ *.* 

Let G be a finite 3-orbit group. If G is abelian, then it is easy to see that  $\text{ord}(G) \neq$  $\{1, p\}$  or  $\{1, p, r\}$  where  $p \neq r$ , since  $p, r \in \text{ord}(G)$  implies  $pr \in \text{ord}(G)$ . Hence  $\text{ord}(G) = \{1, p, p^2\}$  so  $G \cong (\mathbb{C}_{p^2})^n \times (\mathbb{C}_p)^k$  where  $\mathbb{C}_{p^2}$  denotes a cyclic group of order  $p^2$  and  $n \geq 1$ . This implies  $k = 0$ , otherwise  $\omega(\overrightarrow{G}) \geq 4$  by Lemma [2.1](#page-2-0) because of the characteristic series  $G > \Omega_1(G) = C_p^{n+k} > \mathcal{O}_1(G) = C_p^n > 1$ . Thus line 1 of Table [1](#page-1-0) is true. Laffey and MacHale [\[21\]](#page-17-1) characterized the 3-orbit groups G with  $\text{ord}(G) = \{1, p, r\}$  where  $p \neq r$ . These groups are Frobenius groups of the form  $W \rtimes V$ where  $m = 1$  and  $r - 1$  divides n, and they are related to projective geometry and 'uniform generation', see [\[13,](#page-17-7) Theorem 1.1]. This verifies line 2 of Table [1.](#page-1-0) Certain k-orbit group generalizations of line 2 (with  $k \geq 3$ ) are given in Example [6.12.](#page-14-0) The 'only' remaining case is when G is a nonabelian  $p$ -group for some prime  $p$ . This difficult case is not considered in [\[21\]](#page-17-1), but is covered in Sections [3–](#page-3-1)[5](#page-6-0) and in [\[24\]](#page-17-0).

<span id="page-3-3"></span>**Hypothesis 2.5.** Let G be a finite nonabelian 3-orbit p-group. Let  $N = \Phi(G)$  and suppose that  $G/N \cong V = (\mathbb{F}_p)^m$  and  $N \cong W = (\mathbb{F}_p)^n$  and  $\text{Aut}(G)$  induces subgroups  $A \leq \mathrm{GL}_m(p)$  and  $B \leq \mathrm{GL}_n(p)$  which act naturally and transitively on  $V \setminus \{0\}$  and  $W \setminus \{0\}$ , respectively. Finally, let  $K \subseteq A$  be such that  $A/K \cong B$  as per Lemma [2.3.](#page-3-2)

We assume Hypothesis [2.5](#page-3-3) in Sections [3–](#page-3-1)[5.](#page-6-0) Thus  $r = p$  and  $N = G' = Z(G)$  $\Phi(G)$  satisfies  $1 < N < G$ . Either  $\exp(G) = p > 2$ ,  $\text{ord}(G) = \{1, p\}$ , and  $\text{Aut}(G)$  has two orbits on elements of order p, or  $exp(G) = p^2$  and  $ord(G) = \{1, p, p^2\}$ . Hering's theorem [\[15\]](#page-17-2) classifies the transitive linear subgroups  $A \le \text{GL}(V)$  and  $B \le \text{GL}(W)$ ; our version is Theorem [4.1.](#page-5-0) The constraint  $B \cong A/K$  (see Lemma [2.3\)](#page-3-2) further restricts the possibilities for B. Our strategy is to compute possibilities for the pair  $(A, B)$  and then hopefully use the pair  $(V, W)$  of modules to reconstruct a unique 3-orbit group G. Lemma [4.3](#page-5-1) describes how (and why) this is possible when  $p > 2$ .

# <span id="page-3-1"></span>3 Case 1 of Theorem [1.1](#page-1-0) when  $A^{\infty} = 1$

In this section we determine  $G, A, B$  for a finite 3-orbit 2-group  $G$ . The only 3-orbit group of order 8 is the quaternion group  $Q_8$  and  $Aut(Q_8) \cong S_4$  is solvable. Our <span id="page-4-4"></span>classification is accelerated by appealing to [\[2,](#page-16-0) Proposition 3.1] which proves that  $Aut(G)$  is solvable for any 3-orbit 2-groups G. This result relies on Theorem [4.1.](#page-5-0)

The possibilities for G when  $p = 2$  were determined first in [\[2,](#page-16-0) Theorem 1.1] and then in [\[23,](#page-17-4) Corollary [1](#page-1-0).3]. The values of  $A$  and  $B$  in lines of 3, 4, 5 of Table 1 of Theorem [3.3](#page-4-1) are proved in Lemma [6.14.](#page-14-1) Certainly A follows from [\[24,](#page-17-0) Table 1]. Cases 2, 3 relate to lines of 6, 7 of Table [1](#page-1-0) and are considered in Sections [4,](#page-4-2) [5.](#page-6-0)

**Remark 3.1.** Let G be a 3-orbit group. As  $[g_1z_1, g_2z_2] = [g_1, g_2]$  for  $z_1, z_2 \in Z(G)$ , commutation gives rise to a bilinear map  $V \times V \to W$ :  $(q_1 Z(G), q_2 Z(G)) \mapsto [q_1, q_2]$ . As  $\omega(G) = 3$ , this map is surjective, so  $|V|^2 \geq |W|$  or  $2m \geq n$ . We now prove the stronger bound  $m \geq n$ . If  $p = 2$ , then the squaring map  $Q: V \to W$  is surjective. Therefore  $2^m = |V| \geq |W| = 2^n$ , and so  $m \geq n$ . Suppose now that  $p > 2$ . If  $n = 1$ , then  $m \geq n$  holds, so suppose that  $n \geq 2$ . Since B is transitive on  $W \setminus \{0\}$  we have

 $p^{n} - 1 = |W \setminus \{0\}|$  divides |B| divides |A| divides  $|GL_m(p)|$ .

Whence  $p^{n} - 1$  divides  $\prod_{i=1}^{m} (p^{i} - 1)$ . Since  $n \geq 2$  and  $p > 2$ , Zsigmondy's theorem implies there exists a primitive prime divisor r of  $p<sup>n</sup> - 1$ . As r has order n modulo p and r divides  $\prod_{i=1}^{m} (p^i - 1)$ , this shows that  $m \geq n$ , as claimed.  $\Box$ 

The history of, and properties of, Suzuki 2-groups is summarized in [\[17,](#page-17-8) VIII.7].

<span id="page-4-0"></span>**Definition 3.2.** (a) Let  $q = 2^n$  and fix  $\theta \in \text{Aut}(\mathbb{F}_q)$  where  $|\theta| > 1$  is odd and  $n \neq 2^{\ell}$ . The set  $A(n, \theta) = \mathbb{F}_q \times \mathbb{F}_q$  with multiplication rule  $(\lambda_1, \zeta_1)(\lambda_2, \zeta_2) = (\lambda_1 + \lambda_2, \zeta_1 + \zeta_2)$  $\zeta_2 + \lambda_1^{\theta} \lambda_2$  defines a group of order  $q^2 = 2^{2n}$  called a *Suzuki* 2*-group of type A*.

(b) Let  $q = 2^n$  $q = 2^n$  $q = 2^n$  where  $n \geq 1$  and  $2$  fix  $\varepsilon \in \mathbb{F}_{q^2}^{\times}$  of order  $q+1$ . The set  $B(n) = \mathbb{F}_{q^2} \times \mathbb{F}_q$ with multiplication rule  $(\lambda_1, \zeta_1)(\lambda_2, \zeta_2) = (\lambda_1^4 + \lambda_2, \zeta_1 + \zeta_2 + \lambda_1 \lambda_2^q)$  ${}_{2}^{q}\varepsilon+(\lambda_{1}\lambda_{2}^{q}% -\lambda_{3}\lambda_{4}^{q})$  $_{2}^{q}\varepsilon)^{q}$  defines a group of order  $q^3$  whose isomorphism type is independent of  $\varepsilon$ , see [\[7,](#page-16-2) Theorem (v)].

(c) Let  $q = 2^3$  and fix  $\varepsilon \in \mathbb{F}_{q^2}^{\times}$  of order  $q^2 - 1$ . The set  $P = \mathbb{F}_{q^2} \times \mathbb{F}_q$  with multiplication rule  $(\lambda_1, \zeta_1)(\lambda_2, \zeta_2) = (\lambda_1 + \lambda_2, \zeta_1 + \zeta_2 + \lambda_1 \lambda_2^2 \varepsilon + (\lambda_1 \lambda_2^2 \varepsilon)^q)$  defines a group of order  $q^3 = 2^9$  with isomorphism type independent of  $\varepsilon$ , see [\[7,](#page-16-2) p. 705].

<span id="page-4-1"></span>**Theorem 3.3.** Let G be a finite nonabelian 3-orbit p-group with  $|\Phi(G)| = p^n$  and let V, A, W, B be as in Hypothesis 2.[5](#page-3-3). If  $p = 2$  or  $A^{\infty} = 1$ , then G, V, A, W, B are as in *lines* 3, 4, 5 *of Table* [1](#page-1-0)*. In particular,*  $p = 2$  *and*  $\text{Aut}(G)$  *is solvable.* 

*Proof.* If  $p = 2$ , then Aut(G) is solvable by [\[2,](#page-16-0) Proposition 3.1]. Assume now that  $A^{\infty} = 1$ , so that A is solvable. Since  $\Phi(G) = \mathbb{Z}(G)$ , the kernel of the homomorphism  $Aut(G) \to A$  is abelian by Lemma [2.3,](#page-3-2) so  $Aut(G)$  is solvable. Thus G is listed in [\[7,](#page-16-2) Theorem]. The nonabelian p-groups in this list are in cases (iv)–(viii) of [\[7,](#page-16-2) Theorem]. The groups in cases (vi), (vii) are excluded because  $Aut(G)^\infty \neq 1$ , see Lemma [6.2.](#page-10-0) The remaining groups in cases (iv), (v), (viii) are those in Definition  $3.2(a,b,c)$  $3.2(a,b,c)$  including the quaternion group  $Q_8$  which is  $B(1)$ . Hence  $p = 2$ . The values of A, B (and  $V, W$ ) for the remaining cases follow from Lemma [6.14\(](#page-14-1)a,b,c).  $\Box$ 

# <span id="page-4-2"></span>4 Case 2 of Theorem [1.1](#page-1-0) when  $B^{\infty} \neq 1$

In this section G is a nonabelian finite 3-orbit  $p$ -group where p is odd, and we assume that B is not solvable. Hence the solvable residual  $B^{\infty}$  of B is nontrivial. Our classification of possible G relies on Hering's theorem [\[15\]](#page-17-2), which is proved in [\[26,](#page-17-3) Appendix A, and classifies the subgroups  $A \leq \mathrm{GL}_m(p)$  which act transitively on the nonzero vectors of the natural module  $V = \mathbb{F}_p^m$ . The following more concise version of Hering's theorem constrains  $A^{\infty}$  instead of A. It follows easily from [\[26,](#page-17-3) Appendix A], and we let the reader check the details. (We used MAGMA  $[3]$  to find H in part  $(a)$ .)

<span id="page-4-3"></span><sup>&</sup>lt;sup>2</sup>Higman [\[16\]](#page-17-9) had  $n > 1$ , but we will allow  $n = 1$  and  $q = 2$  to include the quaternion group  $Q_8$ .

<span id="page-5-4"></span><span id="page-5-0"></span>**Theorem 4.1** (Hering [\[15\]](#page-17-2)). Let  $A \leq \mathrm{GL}_m(p)$  act transitively on the nonzero vectors *of the natural module*  $\mathbb{F}_p^m$ *. Then the solvable residual*  $A^{\infty}$  *of* A *lies in the list:* 

(a)  $A^{\infty} = 1$  *if*  $A \leq \Gamma L_1(p^m)$  *or if*  $m = 2$ ,  $Q_8 \leq A \leq (Q_8.S_3) \circ C_{p-1}$  *and*  $p \in$  $\{5, 7, 11, 23\}$  *or if*  $(m, p) = (4, 3), A = (D_8 \circ Q_8)$ . *H* where  $H \in \{C_5, D_{10}, F_{20}\}$ ,

$$
\text{SL}_{m/b}(p^b) \quad \text{if } 2 \leq m/b \leq m \text{ and } (m/b, p^b) \neq (2, 3),
$$

(b) 
$$
A^{\infty} = \begin{cases} \text{Sp}_{m/b}(p^b) & \text{if } m/b \ge 4 \text{ is even,} \\ \text{G}_2(2^b)' & \text{if } (m, p) \ne (6b, 2), \end{cases}
$$

(c)  $A^{\infty} = SL_2(5)$  *where*  $(m, p) \in \{(4, 3), (2, 11), (2, 19), (2, 29), (2, 59)\},\$ 

(d) 
$$
A^{\infty} = A = \begin{cases} A_6 \text{ or } A_7 & \text{if } (m, p) = (4, 2), \\ SL_2(13) & \text{if } (m, p) = (6, 3). \end{cases}
$$

<span id="page-5-2"></span>Corollary 4.2. If  $A \leq \mathrm{GL}_m(p)$  *is not solvable and acts transitively on the nonzero vectors of the natural module*  $\mathbb{F}_p^m$ , then  $A^{\infty}/\mathbb{Z}(A^{\infty})$  *is a nonabelian simple group. Furthermore, if* B *is not solvable and acts transitively on the nonzero vectors of the natural module*  $\mathbb{F}_p^n$ , then  $B^{\infty}/\mathbb{Z}(B^{\infty}) \cong A^{\infty}/\mathbb{Z}(A^{\infty})$  *is nonabelian and simple.* 

*Proof.* By Theorem [4.1,](#page-5-0)  $A^{\infty}/\mathbb{Z}(A^{\infty})$  is a nonabelian simple group. By Lemma [2.3,](#page-3-2) B is a quotient of A, so  $B^{\infty}$  is a quotient of  $A^{\infty}$ . If  $B^{\infty} \neq 1$  then  $A^{\infty} \neq 1$ . If  $B^{\infty} \neq 1$  is transitive on  $\mathbb{F}_p^n \setminus \{0\}$ , then  $B^{\infty}/\mathbb{Z}(B^{\infty}) \cong A^{\infty}/\mathbb{Z}(A^{\infty})$  follows from Theorem [4.1.](#page-5-0)

<span id="page-5-1"></span>Lemma 4.3. *Suppose* G, p, V, A, W, B *are as in Hypothesis* [2](#page-3-3).5 *where* p *is odd. Then*

- (a) G has exponent p and hence is a factor group of group  $V \ltimes \Lambda^2 V$  and  $n \leq \binom{m}{2}$ .
- (b) *The center*  $Z(A)$  *is cyclic,*  $|Z(A)|$  *divides*  $p^{e_A}-1$  *where*  $e_A | m$  *and*  $|Z(A) \cap K| \leq 2$ *.*
- (c) If  $B^{\infty} \neq 1$ , then  $|Z(A^{\infty})|$  *is odd.*

*Proof.* (a) By Hypothesis [2](#page-3-3).5, G is an m-generated nonabelian 3-orbit p-group. If G has exponent  $p^2$ , then  $Aut(G)$  is transitive on the elements of order p. Since  $p > 2$ , G is abelian by a deep result of Shult  $[29, Corollary 3]$ . This contradiction shows that G has exponent p. Thus G is a factor group of the (universal) m-generated exponent-p class 2 group, the elements of which may be viewed as ordered pairs  $(v, w) \in V \times \Lambda^2 V$ with multiplication rule  $(v_1, w_1)(v_2, w_2) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2)$ . Thus  $n \leq \binom{m}{2}$ .

(b) As  $A$  acts irreducibly on  $V$ , results of Schur and Wedderburn imply that the ring End<sub>F<sub>p</sub>A(V) of  $\mathbb{F}_pA$ -endomorphisms is a finite field, say  $\mathbb{F}_q$  where  $q = p^{e_A}$  depends</sub> on A. Further,  $\mathcal{Z}(A) \leq \mathbb{F}_q^{\times}$  is cyclic and  $|\mathbb{F}_q^{\times}| = p^{e_A} - 1$  where  $e_A | n$ . Hence matrices in  $Z(A)$  are scalars over  $\dot{\mathbb{F}}_q$  and  $\lambda I_n \in K$  precisely when  $\lambda^2 = 1$ , so  $|Z(A) \cap K| \leq 2$ .

(c) Since  $B^{\infty} \neq 1$  and  $A/K \cong B$  (Lemma [2.3\)](#page-3-2), we have  $A^{\infty} \neq 1$ . Hence  $A^{\infty}/Z(A^{\infty}) \cong B^{\infty}/Z(B^{\infty})$  is the unique nonabelian simple composition factor of A and B by Corollary [4.2.](#page-5-2) If  $|Z(A^{\infty})|$  is even, then  $-1 \in Z(A^{\infty})$  by part (b). However -1 acts trivially on  $\Lambda^2 V$  and hence on  $W = \Lambda^2 V/U$ . Therefore  $B^{\infty} \neq A^{\infty}$ , a contradiction. Thus  $|Z(A^{\infty})|$  is odd.  $\Box$ 

We remark that constructing 3-orbit p-groups of (odd) exponent p is the same as finding maximal A-submodules of the exterior square of an A-module.

<span id="page-5-3"></span>**Remark 4.4.** In Lemma [4.3\(](#page-5-1)a), A acts on V and hence on  $\Lambda^2 V$ . As G is a 3orbit group, A acts irreducibly on  $\Lambda^2 V/U$  of  $\Lambda^2 V$ , so U is a maximal A-submodule; and B acts faithfully on  $\Lambda^2 V/U$ . The group  $\mathcal{G} := V \times \Lambda^2 V$  has center  $\{0\} \times \Lambda^2 V$ , hence  $U \triangleleft \mathcal{G}$  and  $G \cong \mathcal{G}/U$ . Thus  $G \cong V \times (\Lambda^2 V/U)$  where  $(v_1, w_1 + U)(v_2, w_2 +$  $U$ ) =  $(v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2 + U)$ . If  $\alpha \in A$  and  $\alpha K \in B$ , then  $\alpha$  acts as  $(v_1, w_1 + U)^{\alpha} = (v_1^{\alpha}, (w_1 + U)^{\alpha K})$ . Thus a 3-orbit group G gives rise to a maximal A-submodule U of  $\Lambda^2 V$ . Conversely, A may not be transitive on the nonzero vectors

<span id="page-6-3"></span>of  $\Lambda^2 V/U$  where U is a maximal A-submodule of  $\Lambda^2 V$ . Interestingly, this is not the case, see Remark [5.6.](#page-9-0)  $\Box$ 

<span id="page-6-1"></span>The following fact, follows from [\[26,](#page-17-3) Line 2, Table 3] when  $d \geq 3$ .

(1) If  $V = \mathbb{F}_q^d$  is the natural module for  $SL_d(q)$ , then  $\Lambda^2 V$  is irreducible.

Now dim $(\Lambda^2 \mathcal{V}) = \binom{d}{2}$  $\binom{d}{2}$ , so  $\Lambda^2 \mathcal{V}$  is a 1-dimensional trivial module if  $d=2$ . If  $d=3$ , then  $\Lambda^2 \mathcal{V}$  is isomorphic to the dual module  $\mathcal{V}^*$  of  $\mathcal{V}$ . If  $q = p^b$ , then the b Galois conjugate modules  $\mathcal{V}^{\theta}$ ,  $\theta \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ , all give rise to the same irreducible db-dimensional  $\mathbb{F}_pSL_d(q)$ -module by [\[17,](#page-17-8) VII.1.16]. This process of changing from V to  $V = \mathcal{V} \downarrow F =$  $\mathbb{F}_p^{bd}$  is sometimes called 'blowing up the dimension' or 'restricting to a subfield'.

<span id="page-6-2"></span>**Theorem 4.5.** *Let*  $G$ *,*  $p$ *,*  $m$ *,*  $V$ *,*  $A$ *,*  $n$ *,*  $W$ *,*  $B$  *be as in Hypothesis* [2](#page-3-3).5*. If*  $B^{\infty} \neq 1$ *, then* p *is an odd prime,*  $m = n$  *is divisible by* 3 *and*  $A^{\infty} \cong B^{\infty} \cong SL_3(p^{n/3})$  *where*  $A^{\infty}$  *acts* as an  $(\mathbb{F}_{p^{n/3}})^3$ -module, and  $B^{\infty}$  acts as its dual. Furthermore, G is isomorphic to the *class* 2 *factor group in Lemma* 6.[9](#page-13-0) *with*  $F = F_0 = \mathbb{F}_{n^{n/3}}$ *, as on line* 6 *of Table* [1](#page-1-0)*.* 

*Proof.* By Corollary [4.2,](#page-5-2)  $B^{\infty}/\mathbb{Z}(B^{\infty}) \cong A^{\infty}/\mathbb{Z}(A^{\infty})$  is a nonabelian simple group. Recall that G is nonabelian by Hypothesis [2](#page-3-3).5. Theorem [3.3](#page-4-1) implies that  $p \neq 2$ otherwise  $A^{\infty} = B^{\infty} = 1$ . Thus  $n \leq \binom{m}{2}$  by Lemma [4.3\(](#page-5-1)a). This shows that  $m \neq 2$ , otherwise  $n = 1$  and  $B^{\infty} = 1$  as  $GL_1(p)$  is cyclic. Hence  $m \ge 3$  and  $p > 2$ . We can rule out case (a) of Theorem [4.1,](#page-5-0) and  $G_2(2^b)'$  in case (b) as  $p \neq 2$ .

Now  $|Z(A^{\infty})|$  is odd by Lemma [4.3\(](#page-5-1)c), so  $A^{\infty} \notin \{SL_2(5), SL_2(13), Sp_{m/b}(p^b)\}\$ in Theorem [4.1.](#page-5-0) This rules out cases (c) and (d) of Theorem 4.1. Thus  $A^{\infty} = SL_{m/b}(p^{b})$ and  $m/b$  is odd because  $|Z(\mathrm{SL}_{m/b}(p^b))| = \gcd(m/b, p^b - 1)$  is odd. Hence  $A^{\infty} \cong B^{\infty}$ by Theorem [4.1,](#page-5-0) so  $m = n$ . Let V be the natural m-dimensional  $A^{\infty}$ -module over  $\mathbb{F}_p$ , and let V be the *d*-dimensional  $A^{\infty}$ -module over  $\mathbb{F}_q$  where  $q = p^b$ . The exterior square  $\Lambda^2 \mathcal{V}$  has dimension  $\binom{d}{2}$ <sup>d</sup><sub>2</sub>) over the field  $\mathbb{F}_q \cong \text{End}_{\mathbb{F}_p A} \otimes \mathcal{V}$  of endomorphisms. However,  $V$  is irreducible and so too is  $\Lambda^2 V$  by [\(1\)](#page-6-1). Hence by [\[17,](#page-17-8) VII.1.16(e)] (see also Remark [5.6\)](#page-9-0), we have  $\binom{m/b}{2} = n/b = m/b$ , so  $m/b = 3$  and  $m = 3b = n$ .

In summary,  $A^{\infty} \cong SL_3(p^b)$  acts faithfully on the 3-space  $\mathcal V$  and  $B^{\infty} \cong SL_3(p^b)$ acts faithfully and irreducibly on the 3-space  $\Lambda^2 \mathcal{V}$ . Adapting the argument in Re-mark [4.4,](#page-5-3) G is isomorphic to the group  $V \times \Lambda^2 V$ . Alternatively, we may identify G with the set  $\mathbb{F}_q^3 \times \mathbb{F}_q^3$  with multiplication given in Lemma [6.9](#page-13-0) (by ignoring the third coordinate). Setting  $F = F_0 = \mathbb{F}_q$  in Lemma [6.9](#page-13-0) shows that A contains  $\mathrm{FL}_3(\mathbb{F}_q) = \mathrm{GL}_3(\mathbb{F}_q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ . If  $q = p^b$ , then  $\mathrm{TL}_3(\mathbb{F}_q)$  is a maximal proper subgroup of  $GL_{3b}(\mathbb{F}_p)$  if  $b > 1$ , so Lemma [2.4](#page-3-0) implies that  $A = \Gamma L_3(\mathbb{F}_q)$  for  $b \geq 1$ . A similar argument shows that  $B = \Gamma L_3^+(\mathbb{F}_q) = \{g \in GL_3(\mathbb{F}_q) \mid \det(g) \in (\mathbb{F}_q^{\times})^2\} \rtimes \text{Aut}(\mathbb{F}_q)$  by Remark [6.8](#page-13-1) and Lemma [6.9.](#page-13-0) This verifies line 6 of Table [1.](#page-1-0)  $\Box$ 

Nonabelian p-groups with precisely 3 characteristic subgroups were called UCS groups (Unique Characteristic Subgroup) by Taunt and were studied in [\[14\]](#page-17-11). A 3 orbit p-group G is a UCS group which is a special group as  $Z(G) = G' = \Phi(G)$ . The structure of a special group G is strongly influenced by the two  $Aut(G)$ -modules  $V = G/Z(G)$  and  $W = Z(G)$ , see [\[14,](#page-17-11) Theorem 6]. The group  $G_4$  in [14, Theorem 8] is an exponent- $p^2$  cousin (with  $B = SO_2(p)$ ) of the exponent-p groups in Theorem [4.5.](#page-6-2)

# <span id="page-6-0"></span>5 Case 3 of Theorem [1.1:](#page-1-0)  $A^{\infty} \neq 1$  and  $B^{\infty} = 1$

In this section G is a 3-orbit p-group where p is odd, and  $A^{\infty} \neq 1$  and  $B^{\infty} = 1$  hold. A prototypical example is an extraspecial p-group as hinted by the following lemma. It is worth examining this case before considering more general examples.

<span id="page-7-3"></span>The 3-orbit group  $Q_8$  was considered in Section [3.](#page-3-1) We show that extraspecial 2-groups are a rich source of 4-orbit groups. We focus primarily on the case  $p > 2$ .

<span id="page-7-0"></span>**Lemma 5.1.** A finite extraspecial p-group G is a 3-orbit group precisely when  $G \cong Q_8$ *or*  $G \cong p_+^{1+m}$  *has odd exponent* p. In these cases Aut(G) *induces on*  $\mathcal{Z}(G)$  *the cyclic*  $subgroup B \cong C_{p-1}$ *. An extraspecial* 2*-group* G with  $G \not\cong Q_8$  *is a* 4*-orbit group.* 

*Proof.* A finite extraspecial p-group G satisfies  $G' = Z(G) = \Phi(G) \cong C_p$ . Suppose that G is a 3-orbit group and the notation in Hypothesis [2.5](#page-3-3) holds. If  $G \setminus Z(G)$ contains elements of orders p and  $p^2$ , then  $Aut(G)$  has at least 4 orbits on G. This is the case if G has odd exponent  $p^2$ , or  $p = 2$  and  $G \not\cong Q_8$ . However, if  $G \cong Q_8$ , then Aut(G) induces  $GL_2(2)$  on  $V = G/Z(G) \cong (C_2)^2$ , and acts trivially on  $Z(G) \cong C_2$ . Hence  $Q_8$  is a 3-orbit group. It follows from [\[30\]](#page-17-12) that an extraspecial p-group of odd exponent  $p$  is a 3-orbit group. This is also proved in Lemma  $6.2$  which also applies to infinite 3-orbit groups. In our case  $B \cong GL_1(p) \cong C_{p-1}$  holds by Lemma [6.2.](#page-10-0)

Suppose now that  $G_{\varepsilon}$  is the extraspecial 2-group  $2^{\frac{1}{\varepsilon}m}$  of order  $2^{m+1}$  and type  $\varepsilon \in \{-,+\}$  where m is even. In  $G_{\varepsilon}$ , squaring induces a (well-defined) quadratic form  $Q_\varepsilon$  on the vector space  $V_\varepsilon = G_\varepsilon / \mathbb{Z}(G_\varepsilon) \cong \mathbb{F}_2^m$ . The preimage in  $G_\varepsilon$  of singular vectors in  $V_{\varepsilon}$  are the noncentral involutions of  $G_{\varepsilon}$ , and the preimage of nonsingular vectors in  $V_{\varepsilon}$  are the elements of order 4 in  $G_{\varepsilon}$ . It is well known that the outer automorphism group  $Out(G_{\varepsilon})$  is isomorphic to the full orthogonal group  $O(Q_{\varepsilon}) \cong O_m^{\varepsilon}(2)$ , see [\[12\]](#page-17-13) or [\[4,](#page-16-8) §2.2.6]. For even  $m \ge 2$  and  $\varepsilon \in \{-, +\}$  the space  $V_{\varepsilon}$  has nonsingular vectors, and it has singular vectors except when  $(\varepsilon, m) = (-, 2)$ . By Witt's theorem  $O_m^{\varepsilon}(2)$  is transitive on the (possibly empty) set of singular vectors and the set of nonsingular vectors. Clearly  $\{1\}$  and  $Z(G_{\varepsilon})\setminus\{1\}$  are  $Aut(G_{\varepsilon})$ -orbits, so that  $G_{\varepsilon}$  is a 3-orbit group if  $(\varepsilon, m) = (-, 2)$ , and a 4-orbit group otherwise. The elements of order 4 form one Aut $(G_{\varepsilon})$ -orbit, and the involutions form two orbits if  $(\varepsilon, m) \neq (-, 2)$  (the central involution is fixed).  $\Box$ 

The following fact from representation theory will guide our proof of Theorem [5.7.](#page-9-1)

<span id="page-7-1"></span>**Remark 5.2.** The symmetric group  $S_n$  acts on  $\mathbb{F}_q^n$  by permuting the elements of a basis  $\{v_1, \ldots, v_n\}$ . Further, the augmentation map  $\phi: V \to \mathbb{F}_q: \sum_{i=1}^n \lambda_i v_i \mapsto \sum_{i=1}^n \lambda_i$ is an  $S_n$ -epimorphism, and  $S_n$  fixes the submodules

$$
W = \ker(\phi) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \sum_{i=1}^{n} \lambda_i = 0 \right\} \text{ and } D = \left\langle \sum_{i=1}^{n} v_i \right\rangle.
$$

The equation  $\sum_{i=1}^{n} i(v_i - v_{i+1}) = (\sum_{i=1}^{n-1} v_i) - (n-1)v_n$  shows that  $D \subseteq W$  if  $p | n$ where  $p = \text{char}(\mathbb{F}_q)$ , and  $D \cap W = \{0\}$  otherwise. We call  $W/(D \cap W)$  the fully deleted permutation module. It can be written over  $\mathbb{F}_p$ , and it is absolutely irreducible.  $\Box$ 

<span id="page-7-2"></span>**Remark 5.3.** Let  $V = \mathbb{F}_q^{2\ell}$  be a symplectic space preserving the nondegenerate alternating bilinear form  $f: \mathcal{V} \times \mathcal{V} \to \mathbb{F}_q$ . Let  $\mathcal{V}$  be the natural module for  $\text{Sp}(\mathcal{V}) \cong \text{Sp}_{2\ell}(q)$ where  $q = p^b$ , with basis  $e_1, \ldots, e_{2\ell}$ . The map  $\phi \colon \Lambda^2 \mathcal{V} \to \mathbb{F}_q$  with  $\sum \lambda_{ij} e_i \wedge e_j \mapsto$  $\sum \lambda_{ij} f(e_i, e_j)$  is a Sp(V)-module epimorphism. Hence  $W := \text{ker}(\phi)$  is an Sp(V)invariant hyperplane. Let  $\langle e_1, e_2 \rangle, \ldots, \langle e_{2\ell-1}, e_{2\ell} \rangle$  be pairwise orthogonal hyperbolic planes. Then  $\phi(\sum_{i\leq j}\lambda_{ij}e_i\wedge e_j)=\sum_{i=1}^{\ell}\lambda_{2i-1,2i}$  and the stabilizer  $Sp_2(q)\wedge S_{\ell}$  of the decomposition  $\langle e_1, e_2 \rangle \oplus \cdots \oplus \langle e_{2l-1}, e_{2l} \rangle$  is a maximal subgroup of  $Sp_{2l}(q)$  which preserves the submodules  $W = \ker(\phi)$  and  $\mathcal{D} = \langle \sum_{i=1}^{\ell} e_{2i-1} \wedge e_{2i} \rangle$  $\mathcal{D} = \langle \sum_{i=1}^{\ell} e_{2i-1} \wedge e_{2i} \rangle$  $\mathcal{D} = \langle \sum_{i=1}^{\ell} e_{2i-1} \wedge e_{2i} \rangle$  in Figure 1 by Re-mark [5.2.](#page-7-1) A symplectic transvection not in  $Sp_2(q) \wr S_\ell$  also preserves these submodules. Hence  $\mathcal{D}$  and  $\mathcal{W}$  are invariant under all of  $Sp_{2\ell}(q)$ .

We show that D and W are A-invariant for A satisfying  $Sp(\mathcal{V}) \leq A \leq \Gamma Sp(\mathcal{V})$ . The notation  $CSp(\mathcal{V})$  and  $\Gamma Sp(\mathcal{V})$  is described in Remark [6.1.](#page-10-2) First,  $CSp_{2\ell}(q)$  =

<span id="page-8-3"></span>

<span id="page-8-0"></span>Figure 1: The A-submodules of  $\Lambda^2 \mathcal{V}$  where  $\mathcal{V} = \mathbb{F}_{n^b}^{2\ell}$  $_{p^{b}}^{2\ell}$  and  $\text{Sp}(\mathcal{V}) \leqslant A \leqslant \Gamma \text{Sp}(\mathcal{V})$ 

 $\langle g_\mu, \text{Sp}_{2\ell}(q) \rangle$  where  $\mathbb{F}_q^{\times} = \langle \mu \rangle$  and  $g_\mu$  satisfies  $e_{2i-1}g_\mu = \mu e_{2i-1}$  and  $e_{2i}g_\mu = e_{2i}$  for  $i \leq$  $\ell$ . Also  $\phi(ug_\mu) = \mu \phi(u)$  for  $u \in \Lambda^2 \mathcal{V}$ , so  $CSp_{2\ell}(q)$  fixes  $\mathcal{W} = \text{ker}(\phi)$  and  $\mathcal{D}$ . Second, if  $g_{\theta}$  satisfies  $\left(\sum_{i=1}^{2\ell} \lambda_i e_i\right)g_{\theta} = \sum_{i=1}^{2\ell} \lambda_i^{\theta} e_i$  for  $\theta \in \text{Aut}(\mathbb{F}_{p^b})$ , then  $\phi(ug_{\theta}) = \phi(u)^{\theta}$  for  $u \in \Lambda^2 \mathcal{V}$ . Hence  $\Gamma \text{Sp}_{2\ell}(p^b)$  fixes  $\mathcal W$  and  $\mathcal D$  and induces  $\Gamma L_1(p^b)$  on  $\mathcal D$ . Finally, the only Sp(V)-submodules of  $\Lambda^2$ V are  $\{0\}$ , W, D,  $\Lambda^2$ V by [\[26,](#page-17-3) Table 5], see also [\[19,](#page-17-14) Hauptsatz 1 when  $p = 2$ . In summary, we have justified Figure [1.](#page-8-0)  $\Box$ 

<span id="page-8-1"></span>**Remark 5.4.** Let  $V = F<sup>m</sup>$  be an m-dimensional vector space over a field F where char(F)  $\neq$  2. Then  $T^2(V) = A^2(V) \oplus S^2(V)$  where  $T^2(V) = V \otimes V$ ,  $A^2(V)$  and  $S^2(V)$  are called the tensor, alternating, and symmetric squares of V, respectively. Set  $A<sup>2</sup>(V) := \{v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V\}$  and  $S<sup>2</sup>(V) := \{v_1 \otimes v_2 + v_2 \otimes v_1 \mid v_1, v_2 \in V\}.$ The identity  $v_1 \otimes v_2 = \frac{1}{2}$  $\frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) + \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$  for  $v_1, v_2 \in V$ implies that  $T^2(V) = A^2(V) \oplus S^2(V)$  holds. The exterior square is isomorphic to the alternating square via  $\Lambda^2(V) \to A^2(V)$ :  $v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$ . The symmetric square is normally defined to be the quotient  $T^2(V)/A^2(V)$  (in all characteristics), and similarly  $\Lambda^2(V)$  is defined to be  $T^2(V)/S^2(V)$  in all characteristics. The isomorphism  $T^2(V)/A^2(V) \to S^2(V)$  with  $v_1 \otimes v_2 + A^2(V) \mapsto v_1 \otimes v_2 + v_2 \otimes v_1$  holds for all F.

<span id="page-8-2"></span>**Remark 5.5.** Let p be an odd prime and set  $q = p^b$ ,  $E = \mathbb{F}_q$ ,  $F = \mathbb{F}_p$ ,  $\mathcal{V} = E^d$ where  $d = 2\ell$  is even and V is an A-module where  $Sp(\mathcal{V}) \leq A \leq \Gamma Sp(\mathcal{V})$ . We view  $V = \mathcal{V} \downarrow F = F^{bd}$  as the A-module  $\mathcal{V} = E^d$  written over F using the inclusions  $A \leqslant \Gamma \operatorname{Sp}_d(E) \leqslant \operatorname{GL}_{bd}(F)$ . The bd-dimensional EA-module  $V \otimes_F E$  is  $\bigoplus_{i=0}^{b-1} \mathcal{V}^{(i)}$  where  $\mathcal{V}^{(i)}$  it the Galois conjugate of  $\mathcal{V}$  by  $\theta^i$  where  $Gal(E : F) = \langle \theta \rangle$  by [\[17,](#page-17-8) VII.1.16(a)]. We shall prove the decomposition  $A^2(V \otimes E) = \bigoplus_i A^2(V^{(i)}) \oplus \bigoplus_{i < j} A_{ij}$  where  $0 \leq i < b$ ,  $0 \leqslant i < j < b$ , and  $\mathcal{A}_{ij}$  is defined below. First,  $T^2(V) \otimes E \cong T^2(V \otimes E)$  equals

$$
\left(\bigoplus_i \mathcal{V}^{(i)}\right)\otimes\left(\bigoplus_j \mathcal{V}^{(j)}\right)=\bigoplus_i T^2(\mathcal{V}^{(i)})\oplus\bigoplus_{i
$$

However,  $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)} \oplus \mathcal{V}^{(j)} \otimes \mathcal{V}^{(i)}$  is a direct sum of submodules say  $\mathcal{A}_{ij} \oplus \mathcal{S}_{ij}$  where

$$
\mathcal{A}_{ij} = \left\{ v_i \otimes v_j - v_j \otimes v_i \mid v_i \in \mathcal{V}^{(i)}, v_j \in \mathcal{V}^{(j)} \right\},
$$
  

$$
\mathcal{S}_{ij} = \left\{ v_i \otimes v_j + v_j \otimes v_i \mid v_i \in \mathcal{V}^{(i)}, v_j \in \mathcal{V}^{(j)} \right\},
$$

 $\mathcal{A}_{ij} \leqslant A^2(V \otimes E)$  and  $\mathcal{S}_{ij} \leqslant S^2(V \otimes E)$  by Remark [5.4.](#page-8-1) Hence

$$
T^2(V \otimes E) = \bigoplus_i A^2(\mathcal{V}^{(i)}) \oplus \bigoplus_i S^2(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} A_{ij} \oplus \bigoplus_{i < j} S_{ij}.
$$

Since  $\mathcal{A}_{ij} \cong \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)} \cong \mathcal{S}_{ij}$  for  $i < j$ , the claimed decomposition follows:

$$
A^{2}(V \otimes E) = \bigoplus_{i} A^{2}(V^{(i)}) \oplus \bigoplus_{i < j} A_{ij} \cong \bigoplus_{i} A^{2}(V^{(i)}) \oplus \bigoplus_{i < j} V^{(i)} \otimes V^{(j)}.
$$

(The containment  $\geq$  holds, and the dimension agree as  $\binom{bd}{2}$  $\binom{bd}{2} = b \binom{d}{2}$  $\binom{d}{2} + \binom{b}{2}$  $_{2}^{b})d^{2}.$  $\Box$  <span id="page-9-2"></span>Recall that  $A^{\infty} \neq 1$  and  $B^{\infty} = 1$  hold in this section.

<span id="page-9-0"></span>**Remark 5.6.** Assume, as in Remark [5.5,](#page-8-2) that  $p > 2$  is prime,  $q = p^b$ ,  $E = \mathbb{F}_q$ ,  $F = \mathbb{F}_p, V = E^d, d = 2\ell$  is even,  $V = V \downarrow F = F^{bd}$  where V is an A-module and  $Sp(V) \leq A \leq \text{PSp}(V)$ . By Remark [5.3,](#page-7-2) E is a b-dimensional FA-module which is irreducible if  $C\text{Sp}(\mathcal{V}) \leq A$  and is trivial if  $A = \text{Sp}(\mathcal{V})$ . Let U be a maximal FAsubmodule of  $A^2(V \otimes E) \downarrow F$ . This remark proves that the quotient FA-module  $(A^2(V \otimes E) \downarrow F)/U$  is isomorphic to a subfield of E containing F (really a quotient FA-module). As  $B^{\infty} = 1$ , we see that  $A^{\infty} = Sp(V)$  acts trivially on this quotient. In the next paragraph, we consider EA-submodules rather than FA-submodules.

Let U be a maximal EA-submodule of  $A^2(V \otimes E)$  such that  $A^{\infty} = Sp(V)$  acts trivially on  $A^2(V \otimes E)/U$ . Remark [5.5](#page-8-2) shows that  $A^2(V \otimes E) = X \oplus Y$  where  $\mathcal{X} \,=\, \bigoplus_i A^2(\mathcal{V}^{(i)})$  and  $\mathcal{Y} \,=\, \bigoplus_{i < j} \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$ . We shall show that  $\mathcal{Y} \, \subseteq \, \mathcal{U}$ . Let  $W = \text{ker}(\phi)$  be as in Remark [5.3](#page-7-2) where  $\Lambda^2(V)/W$  is a 1-dimensional EA-module. If  $i < j$ , then  $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$  is a faithful  $A^{\infty}$ -module which is irreducible if  $j \neq d/2 + i$ by [\[20,](#page-17-15) §5.4], and is a sum of (two isomorphic) faithful irreducible submodules if  $j = d/2 + i$ . Also, the uniserial proper  $A^{\infty}$ -submodule W in Figure [1](#page-8-0) of  $\Lambda^2(\mathcal{V}^{(i)})$ of dimension  $2\ell^2 - \ell - 1$  is nontrivial. Since  $A^{\infty}$  acts trivially on the simple factor module  $A^2(V \otimes E)/\mathcal{U}$ , we see that  $\mathcal{U}$  contains  $\mathcal{Z} := \bigoplus_i \mathcal{W}^{(i)} \oplus \mathcal{Y}$ , as claimed.

Suppose now that U is a maximal FA-submodule of  $A^2(V \otimes E) \downarrow F$  such that  $A^{\infty}$  acts trivially on  $(A^2(V \otimes E) \downarrow F)/U$ . Choose a maximal EA-submodule U of  $A^2(V \otimes E)$  where U contains  $\mathcal{U} \downarrow F$ . Then  $A^{\infty}$  acts trivially on  $A^2(V \otimes E)/\mathcal{U}$ . By the previous paragraph,  $A^2(V \otimes E)/\mathcal{U}$  is a factor of  $A^2(V \otimes E)/\mathcal{Z} = \bigoplus_i (\Lambda^2(V)/\mathcal{W})^{(i)}$ . Now  $(A^2(V \otimes E) \downarrow F)/U$  is an irreducible FA-module and a factor FA-module of  $(A^2(V \otimes E)/\mathcal{U}) \downarrow F \cong (A^2(V \otimes E) \downarrow F)/(\mathcal{U} \downarrow F)$ . The irreducible factor FA-modules are isomorphic to a subfield  $\mathbb{F}_{p^n}$  of  $E = \mathbb{F}_{p^b}$  for some divisor n of b by [\[17,](#page-17-8) VII.1.16(e)]. As  $A$  varies, any divisor  $n$  of  $b$  can arise, see Lemma [6.2.](#page-10-0)  $\Box$ 

<span id="page-9-1"></span>Theorem 5.7. *Let* G *be a finite nonabelian* 3*-orbit* p*-group and let* V *,* A*,* W*,* B *be as in Hypothesis* [2](#page-3-3).5*. If*  $A^{\infty} \neq 1$  *and*  $B^{\infty} = 1$ *, then* p *is odd and* G, V, A, W, B *are as in line* 7 *of Table* [1](#page-1-0) *with*  $|\Phi(G)| = p^n$  *as described in Lemma* [6](#page-10-0).2*.* 

*Proof.* If  $p = 2$ , then  $Aut(G)$  is solvable by Theorem [3.3](#page-4-1) and so  $A^{\infty} = 1$ , a contradiction. Hence  $p > 2$ . If  $n = 1$ , then m must be even and G is the extraspecial group of order  $p^{1+m}$  $p^{1+m}$  $p^{1+m}$ , and exponent p which appears on line 7 of Table 1 with  $b = 1$ . Since  $p > 2$ , we have  $n \leq \binom{m}{2}$  by Lemma [4.3\(](#page-5-1)a). Hence  $m = 2$  implies  $n = 1$ . Suppose now that  $m \ge 3$  and  $n \ge 2$ . Since  $A^{\infty} \ne 1$  and  $B^{\infty} = 1$ , we have  $A^{\infty} \le K$ . If  $H = N_{GL(V)}(A^{\infty})$ , then  $H/A^{\infty} \geqslant A/A^{\infty} \geqslant A/K \cong B$ . We argue using Theorem [4.1](#page-5-0) that  $A^{\infty} \cong Sp_{m/b}(p^b)$ . Since  $p^n-1$  divides  $|B|$ , we see that  $B \neq 1$ . Hence A is strictly larger that  $A^{\infty}$ , so case (d) of Theorem [4.1](#page-5-0) cannot hold, nor can case (a) as  $A^{\infty} \neq 1$ . In case (c), we have  $A^{\infty} = SL_2(5)$  and  $V = \mathbb{F}_3^4$  is an  $A^{\infty}$ -module. The maximal  $A^{\infty}$ submodules of  $\Lambda^2 V$  have codimension 1 and  $\binom{4}{2}$  $2<sup>4</sup>$ ) – 1 = 5 by Remark [5.3.](#page-7-2) Therefore  $n = 1, 5$  by Lemma [4.4.](#page-5-3) But  $n \neq 1$ , so  $n = 5$  and  $242 = p^{n} - 1$  divides |B|. A direct calculation with MAGMA [\[3\]](#page-16-7) shows that  $|H/A^{\infty}| = 8$ . This rules out case (c). Hence we have  $A^{\infty} \in {\{SL_{m/b}(p^b), Sp_{m/b}(p^b)\}}$ . Suppose  $A^{\infty} \cong SL_{m/b}(p^b)$  and  $\mathcal{V} = \mathbb{F}_{p^b}^{m/b}$  $\frac{m}{p^b}$  is its natural module. As  $V$  is irreducible, so too is  $\Lambda^2 V$  by [\(1\)](#page-6-1). Let  $V = V \downarrow \mathbb{F}_p = \mathbb{F}_p^m$ . We claim that  $\Lambda^2 V$  is is a direct sum of faithful irreducible  $\mathbb{F}_p A^\infty$ -modules. The claim follows from Remark [5.5](#page-8-2) as  $\Lambda^2(V\otimes\mathbb{F}_{p^b}) = \bigoplus_i \Lambda^2(V)^{(i)}\oplus \bigoplus_{i< j} \mathcal{V}^{(i)}\otimes\mathcal{V}^{(j)}$  and  $\Lambda^2(V)^{(i)}$ is faithful and irreducible by [\(1\)](#page-6-1), and  $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$  is either a faithful and irreducible  $SL_{m/b}(p^b)$ -modules or a direct sum of two such by [\[20,](#page-17-15) Theorem 5.4.5]. This implies that  $B^{\infty} \cong SL_{m/b}(p^{b}) \neq 1$ , a contradiction. The only remaining possibility in The-orem [4.1](#page-5-0) is  $A^{\infty} \cong Sp_{m/b}(p^b)$  where  $m/b \geq 4$  is even. In this case,  $A \leqslant \Gamma Sp_{m/b}(p^b)$ 

<span id="page-10-3"></span>since the normalizer of  $\text{Sp}_{m/b}(p^b)$  in  $\text{GL}_m(p)$  is  $\text{PSp}_{m/b}(p^b)$ . In summary, we have shown that  $\text{Sp}_{m/b}(p^b) = A^{\infty} \leqslant A \leqslant \Gamma \text{Sp}_{m/b}(p^b) \leqslant \text{GL}_m(p)$ .

We now apply Theorem [4.1\(](#page-5-0)a) to  $B \leq \mathrm{GL}_n(p)$ . First,  $B^{\infty} = 1$  and  $B = A/K$ implies  $\text{Sp}_{m/b}(p^b) = A^{\infty} \leqslant K$ . Thus B is a section of  $\text{PSp}_{m/b}(p^b)/\text{Sp}_{m/b}(p^b) \cong$  $\Gamma L_1(p^b)$ , and so B is metacyclic. Since  $B^{\infty} = 1$ , the choices for B are constrained by Theorem [4.1\(](#page-5-0)a). The extraspecial group  $D_8 \circ Q_8 = 2^{1+4}$  is not metacyclic, and therefore  $(n, p) \neq (4, 3)$ , as subgroups of metacyclic groups are metacyclic. Suppose that  $n = 2$  and  $p \in \{5, 7, 11, 23\}$ . A calculation using MAGMA [\[3\]](#page-16-7) shows that the subgroups of  $GL_2(p)$  with  $p \in \{5, 7, 11, 23\}$  that are both metacyclic and transitive on nonzero vectors, all lie in  $\Gamma L_1(p^2)$ . Therefore  $B \leq \Gamma L_1(p^n)$  as in line 7 of Table [1.](#page-1-0)

By Lemma [4.3\(](#page-5-1)a) the 3-orbit group G is isomorphic to  $(V \rtimes \Lambda^2(V))/U$  where  $V = \mathbb{F}_p^m$  is the natural A-module, and U is a maximal submodule of  $\Lambda^2(V)$  by Remark [4.4.](#page-5-3) The simple quotient A-modules  $\Lambda^2(V)/U$  of  $\Lambda^2(V)$  are the subfields of  $\mathbb{F}_{p^b}$  by Remark [5.6,](#page-9-0) and each subfield gives rise to a 3-orbit group G. Thus G is as described on line 7 of Table [1.](#page-1-0) Large subgroups of  $A$  and  $B$  are described in Lemma [6.2.](#page-10-0) Indeed,  $A \leqslant \Gamma \text{Sp}_{m/b}(p^b)$  and  $B \leqslant \Gamma \text{L}_1(p^b)$  as in line 7 of Table [1.](#page-1-0)  $\Box$ 

### <span id="page-10-1"></span>6 Examples of  $k$ -orbit groups

In this section we give examples of  $k$ -orbit groups for small  $k$ . We focus on 3-orbit groups. Extraspecial p-groups provide examples of both 3-orbit and 4-orbit groups.

If G is a finite extraspecial p-group, or an infinite Heisenberg group, then viewing the elements of G as ordered pairs facilitates a geometric method to construct  $Aut(G)$ . This method, was not used by Winter in [\[30\]](#page-17-12), but is used in Lemma [6.2](#page-10-0) below.

<span id="page-10-2"></span>**Remark 6.1.** We first describe  $\Gamma \text{Sp}(\mathcal{V})$  and  $\text{CSp}(\mathcal{V})$ . Let  $f: \mathcal{V} \times \mathcal{V} \to F$  be a nondegenerate symplectic bilinear form on  $\mathcal{V} = F^d$  where  $d \geq 2$  is even. Let  $\text{PSp}(\mathcal{V})$  be the group of bijective semilinear symplectic similarities on  $\mathcal V$ . These satisfy

$$
(\lambda v)g = \lambda^{\sigma(g)}(vg), (v_1 + v_2)g = v_1g + v_2g, \text{ and } f(v_1g, v_2g) = \delta(g)^{\sigma(g)}f(v_1, v_2)^{\sigma(g)},
$$

for  $g \in \Gamma \text{Sp}(\mathcal{V}), v, v_1, v_2 \in \mathcal{V}, \ \lambda \in F$ , where  $\delta(g) \in F^\times$  and  $\sigma(g) \in \text{Aut}(F)$  depend on g. The map  $\sigma: \Gamma \text{Sp}(\mathcal{V}) \to \text{Aut}(F)$  is an epimorphism. Comparing  $f(v_1(gh), v_2(gh))$ to  $f((v_1g)h, (v_2g)h)$  gives the the 1-cocycle condition  $\delta(gh) = \delta(g)\delta(h)^{\sigma(g^{-1})}$ . The *conformal symplectic group* denoted by  $CSp(\mathcal{V})$  is the kernel of  $\sigma$  *c.f.* [\[4,](#page-16-8) Def. 1.6.14].

We view the elements of  $\mathrm{TL}_1(F)$  as products  $\delta\sigma$  with  $(\delta,\sigma) \in F^\times \times \mathrm{Aut}(F)$  and multiplication rule  $(\delta_1 \sigma_1)(\delta_2 \sigma_2) = \delta_1 \delta_2^{\sigma_1^{-1}} \sigma_1 \sigma_2$ . It follows from the previous paragraph that  $\Gamma \text{Sp}_d(F) = \text{Sp}_d(F) \rtimes \Gamma \text{L}_1(F)$ , see [\[20,](#page-17-15) Table 2.1.C] and Remark [5.3.](#page-7-2)  $\Box$ 

If  $F: F_0$  is a Galois extension of the subfield  $F_0$ , then  $\Gamma L_1(F_0)$  is a factor group of  $\Gamma L_1(F)$ , and hence  $\Gamma L_1(F_0)$  is a factor group of  $\Gamma Sp_d(F) \cong Sp(V) \rtimes \Gamma L_1(F)$ .

<span id="page-10-0"></span>**Lemma 6.2.** Let  $F : F_0$  be a finite Galois field extension where  $char(F) = p \ge 0$ . Let  $f: \mathcal{V} \times \mathcal{V} \rightarrow F$  *be a non-degenerate alternating* F-bilinear form on  $\mathcal{V} = F^d$  where d *is even. Let* Tr *be the trace map*  $F \to F_0$ :  $\lambda \mapsto \sum_{\sigma \in \text{Gal}(F:F_0)} \lambda^{\sigma}$ . The set  $G = \mathcal{V} \times F_0$ *with the multiplication rule*  $(v_1, \zeta_1)(v_2, \zeta_2) = (v_1 + v_2, \zeta_1 + \zeta_2 + \text{Tr}(f(v_1, v_2)))$  *defines a group.* If  $p \neq 2$ , then  $G = G_{F,F_0}$  is a 3-orbit group and

$$
\operatorname{Sp}_d(F) \rtimes (F_0^\times \rtimes \operatorname{Aut}(F)) \leqslant \operatorname{Aut}(G)^{G/G'} \qquad \text{and} \qquad \operatorname{GL}_1(F_0) \leqslant \operatorname{Aut}(G) \downarrow G'.
$$

If  $F = F_0 = \mathbb{F}_p$ , where p is an odd prime, then  $A = \text{CSp}_d(p)$  and  $B = \text{GL}_1(p)$ .

<span id="page-11-0"></span>*Proof.* Since the map  $V \times V \to F_0$ :  $(v_1, v_2) \mapsto \text{Tr}(f(v_1, v_2))$  is biadditive, the multiplication on G is associative. Hence G is a group where  $(0,0)$  is the identity element and  $(v,\zeta)^{-1} = (-v,-\zeta)$  as  $f(v,v) = 0$ . If  $p > 0$ , then the exponent of G is p since  $(v,\zeta)^k = (kv,k\zeta)$  for  $k \in \mathbb{Z}$ . The commutator  $[(v_1,\zeta_1),(v_2,\zeta_2)]$  equals  $(0, 2\text{Tr}(f(v_1, v_2))$ . Thus G is abelian if  $p = 2$ . Suppose now that  $p \neq 2$ . As f and Tr are surjective functions, it follows that  $G' = \{0\} \times F_0$ .

Let A be the subgroup of  $\text{PSp}(\mathcal{V})$  (see Remark [6.1\)](#page-10-2) comprising all g satisfying  $f(v_1g, v_2g) = \delta(g)f(v_1, v_2)^{\sigma(g)}$  with  $\delta(g) \in F_0^{\times}$ . Then the structure of A is  $\text{Sp}_d(F) \rtimes$  $(F_0^{\times} \rtimes \text{Aut}(F))$ . Using  $\text{Tr}(\delta \lambda^{\sigma}) = \delta \text{Tr}(\lambda)$  for  $\delta \in F_0$ ,  $\lambda \in F$ ,  $\sigma \in \text{Aut}(F)$ , we show below that  $(v, \zeta)^g = (vg, \delta(g)\zeta)$  defines an action of  $g \in \mathcal{A}$  on G:

$$
(v_1, \zeta_1)^g (v_2, \zeta_2)^g = (v_1g, \delta(g)\zeta_1)(v_2g, \delta(g)\zeta_2)
$$
  
=  $(v_1g + v_2g, \delta(g)\zeta_1 + \delta(g)\zeta_2 + \text{Tr}(\delta(g)f(v_1, v_2)^{\sigma(g)}))$   
=  $((v_1 + v_2)g, \delta(g)(\zeta_1 + \zeta_2 + \text{Tr}(f(v_1, v_2))))$   
=  $(v_1 + v_2, \zeta_1 + \zeta_2 + \text{Tr}(f(v_1, v_2))^g = ((v_1, \zeta_1)(v_2, \zeta_2))^g$ .

Thus q is a bijective endomorphism of G, i.e. an automorphism of G. Moreover,  $A$ acts on G since  $((v,\zeta)^g)^h = (v,\zeta)^{gh}$ . Therefore Aut(G) has 3 orbits on G, namely  $\{(0,0)\},\{0\}\times F_0^{\times},\{V\setminus\{0\}\}\times F_0$ , that is  $1,G'\setminus\{1\},G\setminus G'$ . We have therefore shown that  $A \leq \text{Aut}(G)^{G/G'}$  and  $\text{GL}_1(F_0) \leq \text{Aut}(G) \downarrow G'$ , as claimed.

Finally, suppose that  $F = F_0 = \mathbb{F}_p$ , where p is an odd prime. In this case G is an extraspecial p-group of order  $p^{1+d}$  and exponent p. It follows from [\[30\]](#page-17-12) that  $\text{Out}(p^{1+d}) = \text{CSp}_d(p)$  and hence  $A = \text{CSp}_d(p)$  and  $B = \text{GL}_1(p)$  as claimed.  $\Box$ 

**Remark 6.3.** The subgroup U in Remark [4.4](#page-5-3) is the kernel of the map  $\Lambda^2 V \to F_0$ defined by  $v_1 \wedge v_2 \mapsto \text{Tr}(f(v_1, v_2))$  where f and Tr are as in Lemma [6.2.](#page-10-0)

Lemma 6.4. *The group* B(n) *in Definition* [3.2\(](#page-4-0)b) *is isomorphic to the Suzuki* 2*-group*  $B(n, 1, \xi)$  *defined by* [\[16,](#page-17-9) *Column III*] where  $\xi \neq \tau + \tau^{-1}$  for all  $\tau \in \mathbb{F}_{2^n}^{\times}$ .

*Proof.* Let  $q = 2^n$ . The polynomial  $t^2 + \xi t + 1$  is irreducible in  $\mathbb{F}_q[t]$  since  $\tau + \tau^{-1} = \xi$ has no solutions for  $\tau \in \mathbb{F}_q^{\times}$ . Let  $\varepsilon$  be a root of  $t^2 + \xi t + 1$ . Then  $\varepsilon + \varepsilon^q = \xi$  and  $\varepsilon^{q+1} = 1$ . Hence  $\mathbb{F}_q[\varepsilon] = \mathbb{F}_{q^2}$  and the norm map  $\mathbb{F}_{q^2}^{\times} \to \mathbb{F}_q^{\times}$  sends  $\alpha + \beta \varepsilon \in \mathbb{F}_{q^2}^{\times}$  to  $(\alpha + \beta \varepsilon)(\alpha + \beta \varepsilon^q) = \alpha^2 + \xi \alpha \beta + \beta^2$ . The Suzuki 2-group  $B(n, 1, \xi)$  can, by Higman [\[16,](#page-17-9) Column V, be identified with the set  $\mathbb{F}_q^3$  with the following multiplication rule

 $(\alpha_1, \beta_1, \zeta_1)(\alpha_2, \beta_2, \zeta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \zeta_1 + \zeta_2 + \alpha_1\alpha_2 + \xi\alpha_1\beta_2 + \beta_1\beta_2).$ 

The third coordinate is related to the 'bilinearized' form of the norm map

$$
(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q = (\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon^{-1}) = \alpha_1 \alpha_2 + \xi \alpha_1 \beta_2 + \beta_1 \beta_2.
$$

Therefore,  $(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \in \mathbb{F}_q$  and hence

$$
(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \varepsilon + ((\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \varepsilon)^q = (\alpha_1 \alpha_2 + \xi \alpha_1 \beta_2 + \beta_1 \beta_2)(\varepsilon + \varepsilon^q).
$$

Since  $\varepsilon + \varepsilon^q = \xi$ , the map  $B(n, 1, \xi) \to B(n)$  defined by  $(\alpha, \beta, \zeta) \mapsto (\alpha + \beta \varepsilon, \zeta \xi)$  is an isomorphism. Consequently, the isomorphism type of  $B(n, 1, \xi)$  is independent of the choice of  $\xi$  for which  $t^2 + \xi t + 1$  is irreducible.  $\Box$ 

We will construct examples of 3- and 4-orbit groups using the exterior algebra  $\Lambda(\mathcal{V})$  of a vector space V. If  $\dim(\mathcal{V}) = d$ , then  $\Lambda(\mathcal{V}) = \bigoplus_{k=0}^{\dim(V)} \Lambda^k(\mathcal{V})$  is a graded algebra with  $\dim(\Lambda^k(\mathcal{V})) = \binom{d}{k}$ <sup>d</sup><sub>k</sub>) and hence  $\dim(\Lambda(V)) = 2^d$ . The following preliminary lemma exploits the action of  $GL(V)$  on  $\Lambda^k(V)$ , see [\[22,](#page-17-16) XIX].

<span id="page-12-3"></span><span id="page-12-2"></span>**Lemma 6.5.** Let  $V = F^d$  be an d-dimensional vector space over a field F. Suppose *that*  $1 < k \leq d$  *and*  $n = \binom{d}{k}$  $\mathcal{L}_k^{d}$ ). The action of  $\text{GL}(\mathcal{V})$  on  $\Lambda^k(\mathcal{V})$  induces a homomorphism  $\phi_{d,k} : GL_d(F) \to GL_n(F)$  *of matrix groups. The kernel of*  $\phi_{d,k}$  *is*  $GL_d(F)$  *if*  $k > d$ ,  $SL_d(F)$  *if*  $k = d$ , and  $\{\lambda I_d \mid \lambda \in F$  and  $\lambda^k = 1\}$  *if*  $k < d$ .

*Proof.* If  $k > d$ , then  $\Lambda^k(\mathcal{V}) = \{0\}$  so ker  $\phi_{d,k} = GL_d(F)$ . Let  $\mathcal{V} = \langle e_1, \ldots, e_d \rangle$ . If  $k = d$ , then  $\Lambda^d(\mathcal{V}) = \langle e_1 \wedge \cdots \wedge e_d \rangle$  and  $g\phi_{d,d} = \big(\det(g)\big)$ , so that ker  $\phi_{d,d} = SL_d(F)$ .

Let  $\langle v_1, \ldots, v_k \rangle$  be a typical k-subspace of V where  $k < d$ . As  $g \in \text{ker } \phi_{d,k}$  fixes  $v_1 \wedge \cdots \wedge v_k$ , it also fixes the k-subspace  $\langle v_1, \ldots, v_k \rangle$  by [\[28,](#page-17-17) Lemma 12.6]. As  $k < d$  we may choose a vector  $v_{k+1}$  in  $\mathcal{V} \setminus \langle v_1, \ldots, v_k \rangle$ . Since g fixes the k-subspaces  $\langle v_1, \ldots, v_k \rangle$ and  $\langle v_2, \ldots, v_{k+1} \rangle$ , it fixes their intersection, *viz.*  $\langle v_2, \ldots, v_k \rangle$ . Thus g fixes all  $(k-1)$ subspaces. By induction, g fixes all 1-subspaces of  $V$  and hence g is a scalar matrix. However,  $\lambda I_d \in \text{ker } \phi_{d,k}$  precisely when  $\lambda^k = 1$ . This completes the proof.  $\Box$ 

<span id="page-12-1"></span>**Remark 6.6.** If  $F = \mathbb{F}_q$ , then  $\{\lambda \in \mathbb{F}_q^{\times} \mid \lambda^k = 1\}$  is cyclic of order  $gcd(k, q - 1)$ .

**Lemma 6.7.** Let  $\Lambda(V)$  be the exterior algebra of the F-vector space  $V = F^3$  where  $char(F) \neq 2$ . Then the set  $G = G_F = V \times \Lambda^2 V \times \Lambda^3 V$  with the multiplication rule

$$
(v_1, w_1, x_1)(v_2, w_2, x_2) = (v_1 + v_2, w_1 + w_1 + v_1 \wedge v_2, x_1 + x_2 + v_1 \wedge w_2 + w_1 \wedge v_2)
$$

*defines a* 4*-orbit group.* Also  $Aut(G)$  *induces on*  $G/\gamma_2(G)$ ,  $\gamma_2(G)/\gamma_3(G)$  *and*  $\gamma_3(G)$ *subgroups* A, B, C respectively where  $\Gamma L(\mathcal{V}) \leq A$ ,  $\{g \wedge g \mid g \in \Gamma L(\mathcal{V})\} \leq B$  and  ${g \wedge g \wedge g \mid g \in \Gamma\mathbb{L}(\mathcal{V})} \leq C$ *. In particular,*  $\gamma_3(G) = \mathbb{Z}(G)$ *,*  $\gamma_2(G) = C_G(\gamma_2(G))$  *and*  $G/\gamma_3(G)$  *is a* 3*-orbit group.* If  $|F| = q$  *is odd, then*  $|G| = q^7$  *and*  $|G/\gamma_3(G)| = q^6$ *.* 

*Proof.* The exterior algebra  $\Lambda(V)$  equals  $\bigoplus_{i=0}^{3} \Lambda^{i}(V)$  where  $\dim(\Lambda^{i}(V)) = \binom{3}{i}$  $i$ ). A basis  $(e_1, e_2, e_3)$  for  $\Lambda^1(\mathcal{V}) = \mathcal{V}$  gives bases  $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$  for  $\Lambda^2(\mathcal{V})$  and  $(e_1 \wedge e_2 \wedge e_3)$ for  $\Lambda^3(\mathcal{V})$ . Relative to these bases a  $3 \times 3$  matrix  $g \in GL(\mathcal{V})$  induces the  $3 \times 3$  matrix  $\det(g)g^{-T} = g \wedge g \in GL(\Lambda^2 \mathcal{V})$  and the  $1 \times 1$  matrix  $(\det(g)) = g \wedge g \wedge g \in GL(\Lambda^3 \mathcal{V})$ . Hence the action of  $GL(V)$  on  $\Lambda^2(V)$  is different from the 'natural' and 'dual' actions.

The group of units  $\Lambda(\mathcal{V})^{\times}$  a has normal subgroup  $M = \{1\} \times \mathcal{V} \times \Lambda^2 \mathcal{V} \times \Lambda^3 \mathcal{V}$ , and

$$
(1 + v1 + w1 + x1) \wedge (1 + v2 + w2 + x2)
$$
  
= 1 + (v<sub>1</sub> + v<sub>2</sub>) + (w<sub>1</sub> + v<sub>1</sub> \wedge v<sub>2</sub> + w<sub>2</sub>) + (x<sub>1</sub> + w<sub>1</sub> \wedge v<sub>2</sub> + v<sub>1</sub> \wedge w<sub>2</sub> + x<sub>2</sub>).

Therefore the stated multiplication rule of triples in  $G = V \times \Lambda^2 V \times \Lambda^3 V$  defines an isomorphism  $G \to M$ :  $(v, w, x) \mapsto 1 + v + w + x$ . In particular, G is a group.

The identity element of G is  $(0,0,0)$  and  $(v,w,x)^{-1} = (-v,-w,-x)$  since  $\wedge$  is antisymmetric and  $w \wedge v + v \wedge w = 0$ . Since  $(v, w, x)^k = (kv, kw, kx)$  for  $k \in \mathbb{Z}$  it follows that G is torsion free if  $char(F) = 0$ , and has (odd) exponent  $p = char(F)$ otherwise. In both cases  $\gamma_2(G) = \{0\} \times \Lambda^2 \mathcal{V} \times \Lambda^3 \mathcal{V}$  holds because

<span id="page-12-0"></span>(2) 
$$
[(v_1, w_1, x_1), (v_2, w_2, x_2)] = (0, 2v_1 \wedge v_2, 2(v_1 \wedge w_2 + w_1 \wedge v_2)).
$$

Setting  $[(v_1, w_1, x_1), (v_2, w_2, x_2)] = (0, w', x')$  in [\(2\)](#page-12-0) where  $w' = 2v_1 \wedge v_2$  gives that

 $[[ (v_1, w_1, x_1), (v_2, w_2, x_2) ], (v_3, w_3, x_3) ] = (0, 0, 2w' \wedge v_3) = (0, 0, 4v_1 \wedge v_1 \wedge v_3).$ 

Hence  $\gamma_3(G) = \{0\} \times \{0\} \times \Lambda^3 \mathcal{V}$  as  $char(F) \neq 2$ . Observe that if  $v \in \mathcal{V}$  satisfies  $v \wedge w = 0$  for all  $w \in \Lambda^2 \mathcal{V}$ , then  $v = 0$ . Hence [\(2\)](#page-12-0) implies that  $\gamma_3(G) = Z(G)$ , and  $C_G(\gamma_2(G)) = \gamma_2(G).$ 

Now  $g \in GL(V)$  acts on G is via  $(v, w, x)^g = (vg, w(g \wedge g), x(g \wedge g \wedge g))$  as described above. Hence G is transitive on the nonzero vectors of  $G/\gamma_2(G) = V$ ,  $\gamma_2(G)/\gamma_3(G) =$  $\Lambda^2 V$ ,  $\gamma_3(G)/\gamma_4(G) = \Lambda^3 V$ , so G is a 4-orbit group. Further,  $\sigma \in \text{Aut}(F)$  acts to G via  $(v, w, x)$ <sup> $\sigma$ </sup> =  $(v^{\sigma}, w^{\sigma}, x^{\sigma})$  by applying  $\sigma$  to the coordinates of v, w, x relative the stated bases. This shows that  $\Gamma\mathsf{L}(\mathcal{V})$  is a subgroup of  $\text{Aut}(G)$  which induces  $\Gamma\mathsf{L}(\mathcal{V}) \leq A$ ,  ${g \land g \mid g \in \Gamma\mathbb{L}(\mathcal{V})} \leq B$  and  ${g \land g \land g \mid g \in \Gamma\mathbb{L}(\mathcal{V})} \leq C$  as claimed.  $\Box$  <span id="page-13-3"></span><span id="page-13-1"></span>**Remark 6.8.** The group  $G_F$  in Lemma [6.7](#page-12-1) is abelian if  $char(F) = 2$ . If  $g \in GL_3(F)$ , then  $g \wedge g = \det(g)g^{-T}$  so that  $\det(g \wedge g) = \det(g)^3 \det(g^{-T}) = \det(g)^2 \in (F^{\times})^2$ . The homomorphism  $GL(V) \to GL(\Lambda^2 V)$  has kernel  $\langle -1 \rangle$ , and  $GL(V) \to GL(\Lambda^3 V)$ has kernel  $SL(V)$  by Lemma [6.5.](#page-12-2)  $\Box$ 

<span id="page-13-0"></span>**Lemma 6.9.** Let  $F : F_0$  be a finite separable field extension where  $char(F) \neq 2$ . *Then the trace map*  $\text{Tr}: F \to F_0$  *is surjective, and the set*  $G = G_{F,F_0} = F^3 \times F^3 \times F_0$ *endowed with the multiplication rule*

 $(v_1, w_1, x_1)(v_2, w_2, x_2) = (v_1 + v_2, w_1 + w_1 + v_1 \wedge v_2, x_1 + x_2 + \text{Tr}(v_1 \wedge w_2 + w_1 \wedge v_2))$ 

*defines a group.* Let  $H = \{g \in GL_3(F) \mid \det(g) \in F_0^{\times}\}\$  and let  $H^+$  be the subgroup  $H^+ = \{g \in GL_3(F) \mid \det(g) \in (F^\times)^2\}$ . If  $\text{Aut}(G)$  *induces on*  $G/\gamma_2(G)$ ,  $\gamma_2(G)/\gamma_3(G)$ *and*  $\gamma_3(G)$  *subgroups* A, B, C respectively, then  $H \rtimes \text{Aut}(F) \leq A$ ,  $H^+ \rtimes \text{Aut}(F) \leq B$ and  $F_0^{\times} \leq C$ . Moreover, G is a 4-orbit group and  $G/\gamma_3(G)$  is a 3-orbit group.

*Proof.* If  $\sigma_1, \ldots, \sigma_{|F:F_0|}$  are the  $F_0$ -linear embeddings  $F \to \overline{F}$  into the algebraic closure  $\overline{F}$  of F, then  $\text{Tr}(x) = \sum_{i=1}^{|F:F_0|} \sigma_i(x)$ . Since  $\sigma_1, \ldots, \sigma_n$  are linearly independent over  $F_0$ , the  $F_0$ -linear map Tr is nonzero, and hence is surjective. Let M be the kernel of the trace map Tr:  $F \to F_0$ . The First Isomorphism Theorem gives  $F^+ / M \cong F_0^+$ . We will show that  $G_{F,F_0}$  is a factor group of the group  $G_F$  in Lemma [6.7.](#page-12-1) Indeed, the map  $\phi: G_F \to G_{F,F_0}: (v, w, x) \mapsto (v, w, x + M)$  preserves multiplication and has kernel  $\{(0,0,x) \mid x \in M\} \leq Z(G_F)$  where  $x + M$  is viewed as an element of  $F_0$  via the isomorphism  $F^+/M \cong F_0^+$ . Therefore  $G_F/M \cong G_{F,F_0}$ .

The epimorphism  $\phi$  maps  $\gamma_i(G_F)$  to  $\gamma_i(G_{F,F_0})$  for  $1 \leq i \leq 3$ . Further, if  $g \in H$ , then  $g \wedge g \in H^+$  by Remark [6.8.](#page-13-1) The remaining claims follow from Lemma [6.7.](#page-12-1)  $\Box$ 

Subgroups  $G_1, G_2 \leq \text{Sym}(\Omega)$  with the same orbits on  $\Omega$  are called *orbit-equivalent*.

<span id="page-13-2"></span>**Lemma 6.10.** Let F be a division ring, and  $C \leq F^{\times}$  a finite subgroup. Suppose *that*  $A \leq \text{Aut}(F)$  *fixes* C *setwise and is orbit equivalent to*  $\text{Aut}(C) \leq \text{Sym}(C)$ *. Let*  $\mathcal{V} = F^d$  be a d-space over F. Then the set  $G = C \times V$  endowed with the multiplication *rule*  $(\lambda, v)(\mu, w) = (\lambda\mu, \mu v + w)$  *defines a group. Further,* Aut(G) *has one more orbit on* G than Aut(C) has on C, i.e.  $\omega(G) = \omega(C) + 1$ .

*Proof.* We now show that the set  $G = C \times V$  is a group. Associativity holds as

$$
((\lambda, v)(\mu, w))(\nu, x) = (\lambda \mu \nu, \mu \nu v + \nu w + x) = (\lambda, v)((\mu, w)(\nu, x))
$$

holds for all  $(\lambda, v), (\mu, w), (\nu, x) \in C \times V$ . The identity element of G is (1,0). Also  $(\lambda, v)$  has inverse  $(\lambda^{-1}, -\lambda^{-1}v)$  and  $(\lambda, v)^n = (\lambda^n, (\lambda^{n-1} + \cdots + \lambda + 1)v)$  for  $n \ge 0$ .

We now show that  $M := \{1\} \times V$  is characteristic in G. If char(F) = 0, then this follows since elements of  $G \setminus M$  have finite order (as  $\lambda^n = 1$  implies  $(\lambda, v)^n = (1, 0)$ ), while nontrivial elements of M have infinite order. If  $char(F) > 0$ , then C is contained in the multiplicative group of a finite field by the proof of  $[18,$  Theorem 6. Hence M is a normal Sylow p-subgroup of  $G$  and thus characteristic in  $G$ . We next show that  $\omega(G) = \omega(C) + \omega(M) - 1$ . Clearly  $G/M \cong C$ . First, Aut $(M)$  has two orbits on M. Note that an invertible F-linear map  $g \in Aut_F(\mathcal{V})$  acts on G via  $(\mu, v)^g = (\mu, vg)$ . Hence  ${\rm Aut}_F(V)$  has one nontrivial orbit on M, so  $\omega(M) = 2$ . Also  $\alpha \in A \leq {\rm Aut}(F)$ acts coordinatewise on  $M \cong F^d$ , and hence A acts on G via  $(\lambda, v)^\alpha = (\lambda^\alpha, v^\alpha)$ . Since A is orbit-equivalent to  $Aut(C) \leqslant Sym(C)$ , both A and  $Aut(C)$  have  $\omega(C)$  orbits on C. These two types of automorphisms of G generate a subgroup of  $Aut(G)$  with  $\omega(C) + 1$  orbits. This proves that  $\omega(G) = \omega(C) + 1$  by Lemma [2.1.](#page-2-0)  $\Box$  <span id="page-14-2"></span>**Example 6.11.** Let  $F = \mathbb{H}$  be the real quaternions. Then  $\mathbb{H}^{\times}$  contains the quaternion subgroup  $C = {\pm 1, \pm i, \pm j, \pm k} \cong Q_8$ . Set  $r = \frac{i+j}{\sqrt{2}}, s = \frac{1+i+j+k}{2}$  $\frac{+j+k}{2}$  and  $t = \frac{1+i}{\sqrt{2}}$  $\frac{1}{2}$ . The binary octahedral group  $BO = \langle r, s, t \mid r^2 = s^3 = t^4 = rst \rangle$  satisfies  $|BO| = 48$ ,  $Z(BO) = \langle rst \rangle = \langle -1 \rangle$ , and  $C \leqslant BO$ . The subgroup A of Aut(F) comprising the inner automorphisms  $F \to F: \lambda \mapsto \alpha^{-1} \lambda \alpha, \alpha \in BO$ , fixes  $BO' = Q_8 = C$  setwise. Furthermore  $\overline{A} \cong \overline{BO}/\langle -1 \rangle \cong S_4 \cong \text{Aut}(Q_8)$ , so  $\overline{A}$  is orbit-equivalent to  $\text{Aut}(C)$ . Lemma [6.10](#page-13-2) shows that  $G = C \times \mathbb{H}^d$  satisfies  $\omega(G) = \omega(Q_8) + 1 = 4$  for  $d \ge 1$ .  $\Box$ 

<span id="page-14-0"></span>**Example 6.12.** Let  $p, r$  be distinct primes. Set  $e = r^{\ell}$  where  $\ell \geq 1$ . Then  $p \nmid e$ and the cyclotomic polynomial  $\Phi_e(t)$  is irreducible over the finite field  $\mathbb{F}_p$  precisely p has order deg( $\Phi_e(t)$ ) =  $\phi(e)$  modulo e by [\[25,](#page-17-19) Ex. 3.42, p. 124]; in this case  $\mathbb{F}_q$  where  $q = p^{\phi(e)}$  is the splitting field of  $\Phi_e(t)$  over  $\mathbb{F}_p$ . Hence  $C_e \leq \mathbb{F}_q^{\times}$  and  $Gal(\mathbb{F}_q/\mathbb{F}_p)$  $C_{\phi(e)} \cong \text{Aut}(C_e)$ . Set  $C = C_e$ ,  $F = \mathbb{F}_q$  and  $A = \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$  in Lemma [6.10](#page-13-2) noting that the orbits of  $\text{Aut}(C)$  and A on C are the elements of C of the same order. Thus the set  $G = C_e \times V$  is a group  $C_e \ltimes \mathbb{F}_q^d$  with  $\omega(G) = \omega(C_{r^{\ell}}) + 1 = \ell + 2$ . Setting  $\ell = 1$ gives  $e = r$ ,  $q = p^{r-1}$ ,  $G = C_r \ltimes (C_p)^{d(r-1)}$ , and  $\omega(G) = 3$  as in line 2 of Table [1.](#page-1-0)

**Lemma 6.13.** If  $p \neq r$  are prime and G is an  $(\ell + 2)$ -orbit group with  $|G| = p^m r^{\ell}$ ,  $G' \cong C_p^m$  and  $G/G' \cong C_{r^{\ell}}$ , then G is isomorphic to the group in Example 6.[12](#page-14-0).

*Proof.* By assumption,  $Aut(G)$  has precisely  $\ell + 2$  orbits on G. As  $|ord(G)| = \ell + 2$ , these orbits are the sets  $O_1, O_p, O_r, \ldots, O_{r^{\ell}}$  where  $O_n = \{g \in G \mid |g| = n\}$ . Let R be a Sylow r-subgroup, and let  $P = G'$  be the normal p-subgroup. Now  $Z(G) \cap P$  is trivial, otherwise G has at least  $\ell + 3$  orbits. Hence R acts fixed-point-freely on P. By Maschke's theorem  $P = P_1 \oplus \cdots \oplus P_d$  where each  $P_i$  is an irreducible R-module. The  $P_i$  must be pairwise isomorphic R-modules, otherwise Aut(G) has at least 3 orbits on P, a contradiction. Let  $|P_1| = \cdots = |P_d| = p^b = q$ . Hence each  $\lambda \in R$  may be viewed as acting as a  $d \times d$  scalar matrix over  $\mathbb{F}_q$ . Thus  $G \cong \mathbb{F}_q^d \rtimes C_{r^{\ell}}$  in Example [6.12.](#page-14-0)

<span id="page-14-1"></span>**Lemma 6.14.** Let  $q = 2^n$  and let  $\text{Tr}: \mathbb{F}_{q^2} \to \mathbb{F}_q: \mu \mapsto \mu + \mu^q$  denote the trace map.

- (a) *If*  $\theta \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_2)$  *has*  $|\theta| > 1$  *odd, then*  $n \neq 2^{\ell}$  *and the group*  $A(n, \theta)$  *in Definition* [3](#page-4-0).2(a) *is a* 3*-orbit* 2*-group of order*  $q^2$  *with*  $A \cong B \cong \Gamma L_1(q)$ *.*
- (b) If  $n \geq 1$  and  $\varepsilon \in \mathbb{F}_{q^2}^{\times}$  have order  $q + 1$ , then  $B(n) = B_{\varepsilon}(n)$  in Definition [3](#page-4-0).2(b) *is a* 3*-orbit* 2*-group of order*  $q^3$  *with*  $A \cong \Gamma L_1(q^2)$  *and*  $B \cong \Gamma L_1(q)$ *.*
- (c) Let  $q = 8$  and  $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle \cong C_{63}$  $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle \cong C_{63}$  $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle \cong C_{63}$ . The group  $P = P(\varepsilon)$  in Definition 3.2(c) is *a* 3*-orbit* 2*-group isomorphic to* SmallGroup(2 9 ,10 494 213) *in* Magma [\[3\]](#page-16-7) *with*  $A \cong C_7 \rtimes C_9 < \Gamma L_1(\mathbb{F}_{64})$  *of order* 63 *and*  $B = \Gamma L_1(\mathbb{F}_8)$  *of order* 21*.*

*Proof.* (a) A simple calculation shows that  $G := A(n, \theta)$  defined by Definition [3.2\(](#page-4-0)a) is a group with  $Z(G) = G' = \Phi(G) = \{0\} \times \mathbb{F}_q$ , see also [\[7,](#page-16-2) Theorem (iv), p. 704]. The group  $\Gamma L_1(q)$  acts faithfully on G via  $(\mu, \zeta)^{(\alpha, \lambda)} = (\mu^{\alpha} \lambda, \zeta^{\alpha} \lambda \lambda^{\theta})$ . Hence G is a 3-orbit 2-group of order  $q^2$  with  $m = n$  and  $\Gamma\mathrm{L}_1(q) \leqslant A$  and  $\Gamma\mathrm{L}_1(q) \leqslant B$ . No solvable group of  $GL_n(2)$  properly contains  $\Gamma L_1(2^n)$  by Theorem [4.1\(](#page-5-0)a). Therefore  $A = \Gamma L_1(2^n)$ . It follows from [\[16\]](#page-17-9) that  $\Gamma L_1(q) \leq B$ . Similar reasoning shows that  $B = \Gamma L_1(2^n)$ , so line 3 of Table [1](#page-1-0) is valid.

(b) The group  $B_{\varepsilon}(n)$  appears in [\[7,](#page-16-2) Theorem (v)]. Since  $B_{\varepsilon}(n) \cong B_{\varepsilon'}(n)$  when  $\langle \varepsilon \rangle = \langle \varepsilon' \rangle$  has order  $q+1$  we write  $B(n)$  instead of  $B_{\varepsilon}(n)$ . The multiplication rule in [\[7,](#page-16-2) Theorem (v)] can be rewritten as  $(\mu_1, \zeta_1)(\mu_2, \zeta_2) = (\mu_1 + \mu_2, \zeta_1 + \zeta_2 + \text{Tr}(\varepsilon \mu_1 \mu_2^q))$  $\binom{q}{2}$ ) since  $\varepsilon^q = \varepsilon^{-1}$ . The group  $\Gamma L_1(q^2)$  acts faithfully on  $B(n)$  via  $(\mu, \zeta)^{(\alpha, \lambda)} = (\mu^{\alpha} \lambda, \zeta^{\alpha} \lambda \lambda^{\theta})$ . Arguing as in part (a) we have  $A \cong \Gamma L_1(q^2)$  and  $B \cong \Gamma L_1(q)$  as in line 4 of Table [1.](#page-1-0)

(c) The group  $P(\varepsilon)$  appears in [\[7,](#page-16-2) Theorem (vi)]. If  $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle = \langle \varepsilon' \rangle$ , then  $P(\varepsilon) \cong P(\varepsilon')$  so we write P rather than  $P(\varepsilon)$ . The maps  $\psi, \phi: P \to P$  defined by

$$
(\alpha, \zeta)^{\psi} = (\varepsilon^3 \alpha, \varepsilon^9 \zeta)
$$
 and  $(\alpha, \zeta)^{\phi} = (\varepsilon \alpha^4, \zeta^4)$ 

<span id="page-15-2"></span>can be shown to be homomorphisms of P that satisfy  $\psi^{21} = \phi^9 = 1$ ,  $\psi^{\phi} = \psi^4$  and  $\psi^7 = \phi^3$ . Hence  $\psi, \phi \in \text{Aut}(P)$  and  $\langle \psi, \phi \rangle = \langle \psi^3, \phi \rangle = C_7 \rtimes C_9$ . A computation with MAGMA [\[3\]](#page-16-7) shows that  $|\text{Aut}(P)| = 2^{18} \cdot 63$ . There are  $2^{18}$  central automorphisms so  $A \cong C_{21} \cdot C_3 \cong C_7 \rtimes C_9 < \Gamma L_1(\mathbb{F}_{64})$  and  $B = C_7 \rtimes C_3 = \Gamma L_1(\mathbb{F}_8)$  as  $\mathbb{F}_8^{\times} = \langle \varepsilon^9 \rangle$ .

# <span id="page-15-0"></span>7 Examples of 4-orbit groups

"A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one." PAUL HALMOS

In this section, we consider the feasibility of classifying finite k-orbit groups for  $k \leq 6$ . The nonsolvable k-orbit groups have been classified for  $k = 4, 5, 6$ , see §1 for details. To assess the feasibility of classifying the solvable k-orbit groups for  $k =$ 4, 5, 6, we employ Halmos' strategy, and seek a large stock of examples, particularly when  $k = 4$ . Using MAGMA [\[3\]](#page-16-7), we studied the 1265679 groups of order less than  $2^{10}$ excluding  $2^9$ . Only 86 of these are nonabelian solvable 4-orbit groups! It appears that these groups belong to a small number of infinite (and finite) families. For brevity, we list (without proof of correctness) most of these below and in Table [2.](#page-15-1) Given the difficulty of computing automorphism groups, classifying the solvable 4-orbit groups may just be feasible. The most difficult case will be when G does not have four 'obvious' characteristic subset (e.g. determined by element orders or characteristic subgroups). For  $k = 5, 6$  a complete classification may involve too many possibilities.

The Aut(G) orbit lengths for a 3-orbit group G follow from Theorem [1.1.](#page-1-0) They are  $1, p^{n} - 1, p^{n}(r^{m} - 1)$  $1, p^{n} - 1, p^{n}(r^{m} - 1)$  where  $p = r$  except for line 2 of Table 1, and the respective orbitelement orders are  $1, p, p^2$  in lines  $1, 3, 4, 5$  and  $1, p, r$  otherwise. For k-orbit groups with  $k \geq 4$  the orbit lengths and orders are less obvious. Clearly the sum of the orbit lengths is  $|G|$  and some orders in  $\text{ord}(G)$  may be be duplicated, see Table [2.](#page-15-1) If G is a solvable 4-orbit group with precisely 4 characteristic subgroups, arranged as  $\diamondsuit$ or  $\frac{1}{2}$ , then the four Aut(G) orbits are obvious. As minimal characteristic subgroups are elementary abelian, the *abelian* 4-orbit groups are  $(C_{p^3})^m$ ;  $(C_{p^2})^k \times (C_p)^{m-k}$  for  $1 \leq k < m$  and  $(\mathrm{C}_{pr})^m$  where  $p \neq r$  are prime and  $m \geq 1$ . The Aut(G) orbit lengths are obvious, and the orders are  $1, p, p^2, p^3$ ;  $1, p, p, p^2$  and  $1, p, r, pr$  respectively.

By Lemma [2.1,](#page-2-0) a nonabelian solvable 4-orbit group  $G$  either has four characteristic subgroups  $G > M_1 > M_2 > 1$  where  $G', Z(G) \in \{M_1, M_2\}$ , or is a UCS group (see [\[14\]](#page-17-11)) with  $G > G' = \mathbb{Z}(G) > 1$ . Hence a nilpotent 4-orbit group is a p-group of

	$ G $ $G$ $Aut(G)$ orbit lengths	Orders Conditions, action
	$3p \t C_p \rtimes C_3 \t 1, p-1, p, p$	$1, p, 3, 3 \quad p \equiv 1 \pmod{6}$ , cubing
	$3p^2$ $(C_p)^2 \rtimes C_3$ $1, p^2-1, p^2, p^2$	$1, p, 3, 3 \quad p \equiv 1 \pmod{6}, \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right)$
	$3p^2$ $(C_p)^2 \rtimes C_3$ $1, 2(p-1), (p-1)^2, 2p^2$	$1, p, p, 3 \quad p \equiv 1 \pmod{6}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$
	$3p^4$ $(C_{p^2})^2 \rtimes C_3$ $1, q-1, q^2-q, 2q^2; q=p^2$	$1, p, p^2, 3 \quad p \equiv 2 \pmod{3}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$
	$2p^4$ $(C_{p^2})^2 \rtimes C_2$ $1, p-1, p^4-p, p^4$	$1, p, p^2, 2 \quad p \geqslant 2$ , inversion
	$2p^2$ $C_{p^2} \rtimes C_2$ $1, p-1, p^2-p, p^2$	$1, p, p^2, 2 \quad p \geqslant 2$ , inversion $D_{2p^2}$
	$8p^2$ $(C_p)^2 \rtimes Q_8$ $1, p^2-1, p^2, 6p^2$	$1, p, 2, 4 \quad p \in \{3, 5, 7, 11, 23\}$

<span id="page-15-1"></span>Table 2: Examples of solvable (non-nilpotent) 4-orbit groups where  $p$  is a prime

<span id="page-16-10"></span>exponent dividing  $p^3$  and class at most 3. We list below some infinite families of 4orbit  $p$ -groups. Some 4-orbit solvable non- $p$ -groups are listed in Table [2;](#page-15-1) the first three lines are UCS groups. Dornhoff  $[8]$  studies groups N for which  $Aut(N)$  has a solvable subgroup, say A, with four orbits on N (i.e. three orbits on  $N \setminus \{1\}$ ). He lists N in [\[8,](#page-16-9) Theorems 1.1, 2.1] and constrains the structure of N in [8, Theorems 3.1, 4.1]. (No constraints are given in the case that N has exponent  $p$ .) The permutation groups  $N \rtimes A \leq \text{Sym}(N)$  are not always guaranteed to have rank 4. Hence obtaining complete and irredundant list of 4-orbit groups G may be quite difficult, especially in the case that each *solvable* subgroup of  $Aut(G)$  has more than 4 orbits on G.

There are many infinite families of 4-orbit p-groups. First, if  $H$  is a nonabelian 3-orbit p-group, and E is an elementary abelian p-group, then  $G = H \times E$  is a 4-orbit group. Next, if p is an odd prime and  $q = p^b$ , then the group  $G = \mathbb{F}_q^3 \times \mathbb{F}_q \times \mathbb{F}_q$  in Example [6.7](#page-12-1) is a 4-orbit group of exponent p with  $\text{Aut}(G)$  orbit lengths  $1, q-1, q^4$  –  $q, q^7 - q^4$ . Also, the extraspecial 2-groups  $2^{1+2k}_\varepsilon$  with  $\varepsilon = \pm$  and  $(k, \varepsilon) \neq (1, +)$  are 4-orbit groups of exponent 4 by Lemma [5.1.](#page-7-0) These have automorphism orbit lengths  $1, 1, q(q - \varepsilon), q^2 + \varepsilon q - 2$  and element orders 1, 2, 2, 4, and Aut(G) is not solvable.

Nonabelian solvable 4-orbit groups that are not p-groups are listed in Table [2.](#page-15-1) This list omits some families such as the groups  $(C_p)^{d(r-1)} \rtimes C_{r^2}$  with p, r distinct primes, and  $\text{ord}_{r^2}(p) = \phi(r^2) = r(r-1)$ , see Example [6.12.](#page-14-0) See also [\[21,](#page-17-1) Theorem 4].

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