Classifying finite groups G with three Aut(G)-orbits

Stephen P. Glasby*1

¹Center for the Mathematics of Symmetry and Computation, University of Western Australia, Perth 6009, Australia. Stephen.Glasby@uwa.edu.au

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Abstract

We give a complete and irredundant list of the finite groups G for which Aut(G), acting naturally on G, has precisely 3 orbits. There are 7 infinite families: one abelian, one non-nilpotent, three families of non-abelian 2-groups and two families of non-abelian p-groups with p odd. The non-abelian 2-group examples were first classified by Bors and Glasby in 2020 and non-abelian p-group examples with p odd were classified independently by Li and Zhu [24], and by the author, in March 2024.

Dedication: To Otto H. Kegel on the occasion of his 90th birthday

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1 Introduction

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A group G is called a k-orbit group if $\operatorname{Aut}(G)$, acting naturally on G, has precisely k orbits. The cyclic group $\operatorname{C}_{p^{k-1}}, \, k \geqslant 1$, and the generalized quaternion group $Q_{2^{k-1}}$ of order $2^{k-1}, \, k \geqslant 5$, are both k-orbit p-groups. Although we assume that G is a finite group, many of our examples generalize to infinite groups. (For an infinite group G it is sometimes natural to count orbits under the subgroup of topological automorphisms.) Write $\operatorname{ord}(G) := \{|g| \mid g \in G\}$ for the set of element orders in G. A 2-orbit group G is elementary abelian and $\operatorname{ord}(G)$ equals $\{1,p\}$ where p is a prime. If G is a 3-orbit group, then $\operatorname{ord}(G)$ equals $\{1,p\}$, with two orbits of elements of order p, or $\{1,p,p^2\}$ or $\{1,p,r\}$ where p and r are distinct primes. Thus a 3-orbit group is solvable by Burnside's $p^{\alpha}r^{\beta}$ theorem. A complete k-orbit group G has $\operatorname{Aut}(G) = \operatorname{Inn}(G) \cong G$ and K equals the number K(G) of G-conjugacy classes. In general, for a (finite) K-orbit group G we have the bounds $|\operatorname{ord}(G)| \leq K \leq \min\{k(G), |G|/p\}$ where F is the smallest prime divisor of F. The upper bound F is due to F is due to F is the smallest prime divisor of F is a F in F in F is due to F in F is due to F in F

This paper gives a complete and irredundant list of finite 3-orbit groups. Clearly $\{1\}$ is an orbit, and for each characteristic subgroup N of G the set $N^{\#} := N \setminus \{1\}$ is a union of orbits. If a k-orbit group has a chain $G = N_1 > \cdots > N_k = \{1\}$ of k characteristic subgroups, then the sets $N_i \setminus N_{i+1}$ for $1 \le i < k$ and $\{1\}$ are the k non-trivial $\mathrm{Aut}(G)$ orbits, see Lemma 2.1 for more information.

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Let G be a 3-orbit group and set $N := \langle G', \Phi(G) \rangle$. Then $N \neq 1$ as G is not elementary abelian, and $N \neq G$ as G is solvable. Hence G/N and N are both 2-orbit groups and thus elementary abelian. We write $V = G/N = \mathbb{F}_r^m$ and $W = N = \mathbb{F}_p^n$ where r, p are primes (possibly equal). Observe that $\operatorname{Aut}(G)$ induces on V a linear subgroup $A \leq \operatorname{GL}_m(r)$, and induces (via restriction) a linear subgroup $B \leq \operatorname{GL}_n(p)$. Furthermore, A is transitive on the set $V \setminus \{0\}$ of non-zero vectors of V, and B is transitive on the set $W \setminus \{0\}$ of non-zero vectors of W. We use Hering's theorem [15, §5], [26, p. 512] which classifies the linear subgroups A and B (see Theorem 4.1).

Our main result below agrees with Theorem B of Li and Zhu [24, Theorem B] as line 7 of Table 1 corresponds to their families (7) and (8). Hering's theorem is a key tool in [2, 23, 24] and the present paper. Our approach is arguably more geometric than [24] motivated by the desire to gain insight into 4-orbit groups. We find it convenient to represent our p-group (and solvable group) examples as Cartesian products of vector spaces, with specified multiplication rules, and linear actions.

Theorem 1.1. Let G be a finite 3-orbit group with $N = \langle G', \Phi(G) \rangle$ and $|N| = p^n$. Then 1 < N < G and G is isomorphic to a group in lines 1 - 7 of Table 1. Moreover, the values of $V \cong G/N$, $A = \operatorname{Aut}(G)^V$, $W \cong N$, $B = \operatorname{Aut}(G) \downarrow W$ are valid, where $\operatorname{Aut}(G)^V$ and $\operatorname{Aut}(G) \downarrow W$ denote the groups induced on G/N and N by $\operatorname{Aut}(G)$.

Table 1: 3-orbit groups G and $V \cong G/N$, $A = \operatorname{Aut}(G)^{G/N}$, $B = \operatorname{Aut}(G) \downarrow N$

G	V	A	N	B	Comments	Ref.
1. $(C_{p^2})^n$	\mathbb{F}_p^{n}	$\mathrm{GL}_n(p)$	\mathbb{F}_p^{n}	$\mathrm{GL}_n(p)$	$p \geqslant 2$, G abelian	p. 4
2. $\mathbb{F}_q^d \rtimes \mathcal{C}_r$	\mathbb{F}_r	$\mathrm{GL}_1(r)$	\mathbb{F}_q^{d}	$\Gamma L_d(q)$	$q = p^{r-1}, p \neq r, d = \frac{n}{r-1}$	6.12
3. $A(n,\theta)$	\mathbb{F}_2^{n}	$\Gamma L_1(2^n)$	\mathbb{F}_2^{n}	$\Gamma L_1(2^n)$	Def. 3.2(a), $n \neq 2^{\ell}$	6.14
4. $B(n)$	\mathbb{F}_2^{2n}	$\Gamma L_1(2^{2n})$	\mathbb{F}_2^{n}	$\Gamma L_1(2^n)$	Def. 3.2(b), $n \ge 1$	6.14
5. P	\mathbb{F}_2^{6}	$C_7 \rtimes C_9$	\mathbb{F}_2^{3}	$\Gamma L_1(2^3)$	Def. $3.2(c), n = 3$	6.14
6. $\mathbb{F}_q^3:\mathbb{F}_q^3$	\mathbb{F}_q^{3}	$\Gamma L_3(q)$	\mathbb{F}_q^{3}	$\Gamma L_3^+(q)$	$q = p^{\frac{n}{3}} \text{ odd, } 3 \mid n$	6.9
7. \mathbb{F}_{p^n} : $\mathbb{F}_q^{\frac{m}{b}}$	$\mathbb{F}_q^{rac{m}{b}}$	$\operatorname{Sp}_{\frac{m}{b}}(q) \leqslant$	\mathbb{F}_{p^n}	$\Gamma L_1(p^n) \leqslant$	$q = p^b \text{ odd}, n \mid b \mid m$	6.2

The number $\omega(G)$ of $\operatorname{Aut}(G)$ orbits on G can sometimes be calculated without knowing all of $\operatorname{Aut}(G)$, see Lemma 2.2. Nonsolvable k-orbit groups have been classified for $1 \leq k \leq 6$. There are none when $k \leq 3$. If G is finite and S is a composition factor of G, then S is simple and $\omega(G) \geqslant \omega(S)$. If G is a finite nonsolvable k-orbit group, then $G = A_5$ if k = 4 by [21], $G \in \{\operatorname{PSL}_2(q) \mid q \in \{7, 8, 9\}\}$ if k = 5 by [27], and $G \in \{\operatorname{PSL}_3(4), \mathbb{F}_4^2 \rtimes \operatorname{SL}_2(4)\}$ if k = 6 by [6]. By contrast, classifying solvable k-orbit groups is extremely difficult. This paper focuses on classifying (solvable) 3-orbit groups while contemplating 4-orbit groups, see Lemma 6.9 and Section 7.

We note that certain permutation groups of 'rank' k give rise to k-orbit groups. A permutation group $G \leq \operatorname{Sym}(\Omega)$ has $\operatorname{rank} k$ if it is transitive on Ω and the point stabilizer G_{ω} has k orbits including $\{\omega\}$; equivalently G has k orbits on $\Omega \times \Omega$.

Lemma 1.2. If $\operatorname{Aut}(G)$ has k orbits on G, then the subgroup $\operatorname{Hol}(G) = G \rtimes \operatorname{Aut}(G)$ of the symmetric group $\operatorname{Sym}(G)$ has rank k and the stabilizer $\operatorname{Hol}(G)_1$ of 1 is $\operatorname{Aut}(G)$.

Although rank 2 permutation groups have been classified, rank 3 groups have only been classified in certain cases e.g. when they are solvable [7, 10], or quasiprimitive [9],

or innately transitive [1]. Sadly for us, the holomorph 1 Hol(G) of a 3-orbit group G is commonly not solvable, and Hol(G) is never innately transitive as $N = \langle \Phi(G), G' \rangle$ is intransitive and is the unique minimal normal subgroup of Hol $(G) \leq \operatorname{Sym}(G)$. For a recent history of the classification of certain low rank groups see [11, pp. 177–178]. However, if G is a 3-orbit 2-group, then $\operatorname{Aut}(G)$ is solvable by [2, Proposition 3.1]. Hence is Hol(G) also solvable. This generalizes the result [5, Theorem 2] of Bryukhanova, and is used to prove Theorem 3.3 when p=2. The 3-orbit 2-groups G have been classified in [23, Corollary 1.3] and [2, Theorem 1.2]. We find it useful to explicitly specify the (linear) action of $\operatorname{Aut}(G)$ on the vector space $N \cong \mathbb{F}_p^n$.

Our proof of Theorem 1.1 is divided into three cases. Section 2 outlines our three-case strategy, and verifies lines 1 and 2 of Table 1. Section 3 considers Case 1 which includes p=2, and Sections 4 and 5 consider Cases 2 and 3 when p is odd, and G must be a nonabelian group of exponent p. Section 6 devotes considerable effort to constructing examples of infinite and finite k-orbit groups for small k. The examples in line 6 of Table 1 generalize to infinite 3- and 4-orbit groups in Lemma 6.9. Similarly, Example 6.12 generalizes line 2 of Table 1 and Lemmas 5.1, 6.2 generalize line 8. Finally, Section 7 investigates the feasibility of classifying 4-orbit groups; as there are so few examples, a classification may be feasible.

2 Preparation for a proof of Theorem 1.1

In this section G is a finite 3-orbit group. Let $N = \langle G', \Phi(G) \rangle$, and let A, B be the linear groups induced on G/N and N by $\operatorname{Aut}(G)$ as in the Introduction. Recall that the solvable residual A^{∞} of a finite group A, has the property that A/A^{∞} is the largest solvable factor group of A. We split the proof of Theorem 1.1 into cases:

Case 1.
$$A^{\infty} = 1$$
, Case 2. $B^{\infty} \neq 1$ (so $A^{\infty} \neq 1$), and Case 3. $A^{\infty} \neq 1$ and $B^{\infty} = 1$.

In Case 1, A is solvable, and so too is $\operatorname{Aut}(G)$ since the kernel of the epimorphism $\operatorname{Aut}(G) \to A$ is the subgroup $\operatorname{CAut}(G) \cong \operatorname{Hom}(V, W) \cong (\operatorname{C}_p)^{mn}$ of central automorphisms, see Lemma 2.3. In Theorem 3.3, A is solvable or p = 2.

The following lemmas can sometimes help to compute $\operatorname{Aut}(G)$ and $\omega(G)$. It uses the fact: If M is characteristic in G and G > M > 1, then $\omega(G) \ge \omega(G/M) + \omega(M) - 1$.

Lemma 2.1. Suppose G is a finite group and $G = M_1 > \cdots > M_k = \{1\}$ where each M_i is characteristic in G. Then $\omega(G) \ge 2 - k + \sum_{i=1}^{k-1} \omega(M_i/M_{i+1}) \ge k$ for $k \ge 1$. If $\omega(G) = k$, then G is solvable and has precisely k characteristic subgroups.

Proof. The inequality $\omega(G) \geq 2 - k + \sum_{i=1}^{k-1} \omega(M_i/M_{i+1})$ follows by induction on k from $\omega(G) \geq \omega(G/M) + \omega(M) - 1$. This show that $\omega(G) \geq k$ since $\omega(M_i/M_{i+1}) \geq 2$ for each i. Suppose that $\omega(G) = k$, i.e. equality holds. Then $\omega(M_i/M_{i+1}) = 2$ for each i, so each section M_i/M_{i+1} is elementary abelian; whence G is solvable. If G has a characteristic subgroup different to the k subgroups M_1, \ldots, M_k , then it will refine the characteristic series $G = M_1 > \cdots > M_k = \{1\}$, so $\omega(G) > k$, a contradiction. \square

Lemma 2.2. If we know a lower bound $\omega(G) \geqslant \omega_0$ and a subgroup A_0 of $\operatorname{Aut}(G)$ with precisely ω_0 orbits on G, then $\omega(G) \leqslant \omega_0$ and hence $\omega(G) = \omega_0$.

A finite nonabelian 3-orbit p-group has $G' \leq \Phi(G)$ so $N = G' = \operatorname{Z}(G) = \Phi(G)$. We now show that $A \leq \operatorname{GL}(G/N)$ determines $B \leq \operatorname{GL}(N)$, and the A orbits on $V \cong G/N$ determine the $\operatorname{Aut}(G)$ -orbits on $G \setminus N$.

¹A permutation group H with a regular normal subgroup G satisfies $G \leq H \leq \text{Hol}(G) \leq \text{Sym}(G)$.

Lemma 2.3. For a nonabelian 3-orbit p-group G, the preimage under the natural map $G \to G/\mathbb{Z}(G) \cong V$ of the nonzero A-orbits on V are the $\operatorname{Aut}(G)$ -orbits on $G \setminus \mathbb{Z}(G)$. Further, the homomorphism $\phi \colon \operatorname{Aut}(G) \to \operatorname{Aut}(G/\mathbb{Z}(G))$ has image A and kernel the central automorphisms $\operatorname{CAut}(G) \cong \operatorname{Hom}(V, W)$, and $A/K \cong B$ for some $K \bowtie A$.

Proof. By definition, $\ker(\phi)$ equals $\mathrm{CAut}(G)$ namely the set of all $\alpha \in \mathrm{Aut}(G)$ that act trivially on $G/\mathrm{Z}(G)$. However, $G/\mathrm{Z}(G) \cong V$ and $\mathrm{Z}(G) \cong W$, and therefore $\mathrm{CAut}(G) \cong \mathrm{Hom}(V,W)$ is the elementary of order $|W|^{\dim(V)} = p^{mn}$. Furthermore, each $\alpha \in \mathrm{CAut}(G)$ acts trivially on $G' = \mathrm{Z}(G) \cong W$ since

$$[g_1z_1, g_2z_2]^{\alpha} = [g_1^{\alpha}z_1^{\alpha}, g_2^{\alpha}z_2^{\alpha}] = [g_1^{\alpha}, g_2^{\alpha}] = [g_1, g_2] \qquad (g_1, g_2 \in G, z_1, z_2 \in \mathcal{Z}(G)).$$

Hence the A-action on V induces an action on W. The kernel of this action, namely $K := C_A(W)$, satisfies $\mathrm{CAut}(G) \leqslant K$ and $A/K \cong B$. The first sentence of the lemma is true since $g_1 \in g_2\mathrm{Z}(G)$ implies $g_1^{\alpha} = g_2$ for some $\alpha \in \mathrm{CAut}(G)$.

The following straightforward lemma can be a useful tool for computing $\operatorname{Aut}(G)$ where G is a non-abelian 3-orbit p-group. If we know that $\mathcal{A}_0 \leq \operatorname{Aut}(G)$ induces a subgroup A_0 of $A = \operatorname{Aut}(G)^V$, $V = G/\Phi(G)$, and we show that $\operatorname{Aut}(G)$ can not induce a larger subgroup of $\operatorname{GL}(V)$. Then $A = A_0$, and hence $\operatorname{Aut}(G) = \operatorname{CAut}(G).\mathcal{A}_0$.

Lemma 2.4. Suppose we know a subgroup A_0 of $A \leq \operatorname{GL}_m(p)$. If $A_0 = \operatorname{GL}_m(p)$ or A_0 is maximal (proper) subgroup of $\operatorname{GL}_m(p)$ and $A \neq \operatorname{GL}_m(p)$, then $A_0 = A$.

Let G be a finite 3-orbit group. If G is abelian, then it is easy to see that $\operatorname{ord}(G) \neq \{1,p\}$ or $\{1,p,r\}$ where $p \neq r$, since $p,r \in \operatorname{ord}(G)$ implies $pr \in \operatorname{ord}(G)$. Hence $\operatorname{ord}(G) = \{1,p,p^2\}$ so $G \cong (C_{p^2})^n \times (C_p)^k$ where C_{p^2} denotes a cyclic group of order p^2 and $n \geqslant 1$. This implies k = 0, otherwise $\omega(G) \geqslant 4$ by Lemma 2.1 because of the characteristic series $G > \Omega_1(G) = C_p^{n+k} > \mho_1(G) = C_p^n > 1$. Thus line 1 of Table 1 is true. Laffey and MacHale [21] characterized the 3-orbit groups G with $\operatorname{ord}(G) = \{1,p,r\}$ where $p \neq r$. These groups are Frobenius groups of the form $W \rtimes V$ where m=1 and r-1 divides n, and they are related to projective geometry and 'uniform generation', see [13, Theorem 1.1]. This verifies line 2 of Table 1. Certain k-orbit group generalizations of line 2 (with $k \geqslant 3$) are given in Example 6.12. The 'only' remaining case is when G is a nonabelian p-group for some prime p. This difficult case is not considered in [21], but is covered in Sections 3–5 and in [24].

Hypothesis 2.5. Let G be a finite nonabelian 3-orbit p-group. Let $N = \Phi(G)$ and suppose that $G/N \cong V = (\mathbb{F}_p)^m$ and $N \cong W = (\mathbb{F}_p)^n$ and $\operatorname{Aut}(G)$ induces subgroups $A \leq \operatorname{GL}_m(p)$ and $B \leq \operatorname{GL}_n(p)$ which act naturally and transitively on $V \setminus \{0\}$ and $W \setminus \{0\}$, respectively. Finally, let $K \leq A$ be such that $A/K \cong B$ as per Lemma 2.3.

We assume Hypothesis 2.5 in Sections 3–5. Thus r=p and $N=G'=\mathrm{Z}(G)=\Phi(G)$ satisfies 1< N< G. Either $\exp(G)=p>2$, $\operatorname{ord}(G)=\{1,p\}$, and $\operatorname{Aut}(G)$ has two orbits on elements of order p, or $\exp(G)=p^2$ and $\operatorname{ord}(G)=\{1,p,p^2\}$. Hering's theorem [15] classifies the transitive linear subgroups $A\leqslant \operatorname{GL}(V)$ and $B\leqslant \operatorname{GL}(W)$; our version is Theorem 4.1. The constraint $B\cong A/K$ (see Lemma 2.3) further restricts the possibilities for B. Our strategy is to compute possibilities for the pair (A,B) and then hopefully use the pair (V,W) of modules to reconstruct a unique 3-orbit group G. Lemma 4.3 describes how (and why) this is possible when p>2.

3 Case 1 of Theorem 1.1 when $A^{\infty} = 1$

In this section we determine G, A, B for a finite 3-orbit 2-group G. The only 3-orbit group of order 8 is the quaternion group Q_8 and $\operatorname{Aut}(Q_8) \cong S_4$ is solvable. Our

classification is accelerated by appealing to [2, Proposition 3.1] which proves that Aut(G) is solvable for any 3-orbit 2-groups G. This result relies on Theorem 4.1.

The possibilities for G when p=2 were determined first in [2, Theorem 1.1] and then in [23, Corollary 1.3]. The values of A and B in lines of 3, 4, 5 of Table 1 of Theorem 3.3 are proved in Lemma 6.14. Certainly A follows from [24, Table 1]. Cases 2, 3 relate to lines of 6, 7 of Table 1 and are considered in Sections 4, 5.

Remark 3.1. Let G be a 3-orbit group. As $[g_1z_1, g_2z_2] = [g_1, g_2]$ for $z_1, z_2 \in \mathbf{Z}(G)$, commutation gives rise to a bilinear map $V \times V \to W \colon (g_1\mathbf{Z}(G), g_2\mathbf{Z}(G)) \mapsto [g_1, g_2]$. As $\omega(G) = 3$, this map is surjective, so $|V|^2 \geqslant |W|$ or $2m \geqslant n$. We now prove the stronger bound $m \geqslant n$. If p = 2, then the squaring map $Q \colon V \to W$ is surjective. Therefore $2^m = |V| \geqslant |W| = 2^n$, and so $m \geqslant n$. Suppose now that p > 2. If n = 1, then $m \geqslant n$ holds, so suppose that $n \geqslant 2$. Since B is transitive on $W \setminus \{0\}$ we have

$$p^n - 1 = |W \setminus \{0\}|$$
 divides $|B|$ divides $|A|$ divides $|\operatorname{GL}_m(p)|$.

Whence $p^n - 1$ divides $\prod_{i=1}^m (p^i - 1)$. Since $n \ge 2$ and p > 2, Zsigmondy's theorem implies there exists a primitive prime divisor r of $p^n - 1$. As r has order n modulo p and r divides $\prod_{i=1}^m (p^i - 1)$, this shows that $m \ge n$, as claimed.

The history of, and properties of, Suzuki 2-groups is summarized in [17, VIII.7].

Definition 3.2. (a) Let $q = 2^n$ and fix $\theta \in \operatorname{Aut}(\mathbb{F}_q)$ where $|\theta| > 1$ is odd and $n \neq 2^{\ell}$. The set $A(n,\theta) = \mathbb{F}_q \times \mathbb{F}_q$ with multiplication rule $(\lambda_1,\zeta_1)(\lambda_2,\zeta_2) = (\lambda_1 + \lambda_2,\zeta_1 + \zeta_2 + \lambda_1^{\theta}\lambda_2)$ defines a group of order $q^2 = 2^{2n}$ called a *Suzuki 2-group of type A*.

- $\zeta_2 + \lambda_1^{\theta} \lambda_2$) defines a group of order $q^2 = 2^{2n}$ called a *Suzuki 2-group of type A*. (b) Let $q = 2^n$ where $n \ge 1$ and $f(x) \in \mathbb{F}_{q^2}^{\times}$ of order q+1. The set $f(x) = \mathbb{F}_{q^2} \times \mathbb{F}_{q^2}$ with multiplication rule $f(x) = (\lambda_1, \zeta_1)(\lambda_2, \zeta_2) = (\lambda_1 + \lambda_2, \zeta_1 + \zeta_2 + \lambda_1 \lambda_2^q \varepsilon + (\lambda_1 \lambda_2^q \varepsilon)^q)$ defines a group of order $f(x) = (\lambda_1, \zeta_1)(\lambda_2, \zeta_2)$ is independent of $f(x) = (\lambda_1, \zeta_1)(\lambda_2, \zeta_2)$.
- (c) Let $q=2^3$ and fix $\varepsilon \in \mathbb{F}_{q^2}^{\times}$ of order q^2-1 . The set $P=\mathbb{F}_{q^2}\times\mathbb{F}_q$ with multiplication rule $(\lambda_1,\zeta_1)(\lambda_2,\zeta_2)=(\lambda_1+\lambda_2,\zeta_1+\zeta_2+\lambda_1\lambda_2^2\varepsilon+(\lambda_1\lambda_2^2\varepsilon)^q)$ defines a group of order $q^3=2^9$ with isomorphism type independent of ε , see [7, p. 705].

Theorem 3.3. Let G be a finite nonabelian 3-orbit p-group with $|\Phi(G)| = p^n$ and let V, A, W, B be as in Hypothesis 2.5. If p = 2 or $A^{\infty} = 1$, then G, V, A, W, B are as in lines 3, 4, 5 of Table 1. In particular, p = 2 and $\operatorname{Aut}(G)$ is solvable.

Proof. If p=2, then $\operatorname{Aut}(G)$ is solvable by [2, Proposition 3.1]. Assume now that $A^{\infty}=1$, so that A is solvable. Since $\Phi(G)=\operatorname{Z}(G)$, the kernel of the homomorphism $\operatorname{Aut}(G)\to A$ is abelian by Lemma 2.3, so $\operatorname{Aut}(G)$ is solvable. Thus G is listed in [7, Theorem]. The nonabelian p-groups in this list are in cases (iv)–(viii) of [7, Theorem]. The groups in cases (vi), (vii) are excluded because $\operatorname{Aut}(G)^{\infty}\neq 1$, see Lemma 6.2. The remaining groups in cases (iv), (v), (viii) are those in Definition 3.2(a,b,c) including the quaternion group Q_8 which is B(1). Hence p=2. The values of A,B (and V,W) for the remaining cases follow from Lemma 6.14(a,b,c).

4 Case 2 of Theorem 1.1 when $B^{\infty} \neq 1$

In this section G is a nonabelian finite 3-orbit p-group where p is odd, and we assume that B is not solvable. Hence the solvable residual B^{∞} of B is nontrivial. Our classification of possible G relies on Hering's theorem [15], which is proved in [26, Appendix A], and classifies the subgroups $A \leq \operatorname{GL}_m(p)$ which act transitively on the nonzero vectors of the natural module $V = \mathbb{F}_p^m$. The following more concise version of Hering's theorem constrains A^{∞} instead of A. It follows easily from [26, Appendix A], and we let the reader check the details. (We used MAGMA [3] to find H in part (a).)

²Higman [16] had n > 1, but we will allow n = 1 and q = 2 to include the quaternion group Q_8 .

Theorem 4.1 (Hering [15]). Let $A \leq \operatorname{GL}_m(p)$ act transitively on the nonzero vectors of the natural module \mathbb{F}_p^m . Then the solvable residual A^{∞} of A lies in the list:

(a)
$$A^{\infty} = 1$$
 if $A \leq \Gamma L_1(p^m)$ or if $m = 2$, $Q_8 \triangleleft A \leq (Q_8.S_3) \circ C_{p-1}$ and $p \in \{5, 7, 11, 23\}$ or if $(m, p) = (4, 3)$, $A = (D_8 \circ Q_8).H$ where $H \in \{C_5, D_{10}, F_{20}\}$,

$$\text{(b)} \ \ A^{\infty} = \begin{cases} \operatorname{SL}_{m/b}(p^b) & \text{if } 2 \leqslant m/b \leqslant m \text{ and } (m/b, p^b) \neq (2, 3), \\ \operatorname{Sp}_{m/b}(p^b) & \text{if } m/b \geqslant 4 \text{ is even,} \\ \operatorname{G}_2(2^b)' & \text{if } (m, p) \neq (6b, 2), \end{cases}$$

(c)
$$A^{\infty} = \mathrm{SL}_2(5)$$
 where $(m, p) \in \{(4, 3), (2, 11), (2, 19), (2, 29), (2, 59)\},\$

(d)
$$A^{\infty} = A = \begin{cases} A_6 \text{ or } A_7 & \text{if } (m, p) = (4, 2), \\ \operatorname{SL}_2(13) & \text{if } (m, p) = (6, 3). \end{cases}$$

Corollary 4.2. If $A \leq \operatorname{GL}_m(p)$ is not solvable and acts transitively on the nonzero vectors of the natural module \mathbb{F}_p^m , then $A^{\infty}/\mathbb{Z}(A^{\infty})$ is a nonabelian simple group. Furthermore, if B is not solvable and acts transitively on the nonzero vectors of the natural module \mathbb{F}_p^n , then $B^{\infty}/\mathbb{Z}(B^{\infty}) \cong A^{\infty}/\mathbb{Z}(A^{\infty})$ is nonabelian and simple.

Proof. By Theorem 4.1, $A^{\infty}/Z(A^{\infty})$ is a nonabelian simple group. By Lemma 2.3, B is a quotient of A, so B^{∞} is a quotient of A^{∞} . If $B^{\infty} \neq 1$ then $A^{\infty} \neq 1$. If $B^{\infty} \neq 1$ is transitive on $\mathbb{F}_p^n \setminus \{0\}$, then $B^{\infty}/Z(B^{\infty}) \cong A^{\infty}/Z(A^{\infty})$ follows from Theorem 4.1. \square

Lemma 4.3. Suppose G, p, V, A, W, B are as in Hypothesis 2.5 where p is odd. Then

- (a) G has exponent p and hence is a factor group of group $V \ltimes \Lambda^2 V$ and $n \leqslant \binom{m}{2}$.
- (b) The center Z(A) is cyclic, |Z(A)| divides $p^{e_A}-1$ where $e_A \mid m$ and $|Z(A)\cap K| \leq 2$.
- (c) If $B^{\infty} \neq 1$, then $|Z(A^{\infty})|$ is odd.

Proof. (a) By Hypothesis 2.5, G is an m-generated nonabelian 3-orbit p-group. If G has exponent p^2 , then $\operatorname{Aut}(G)$ is transitive on the elements of order p. Since p > 2, G is abelian by a deep result of Shult [29, Corollary 3]. This contradiction shows that G has exponent p. Thus G is a factor group of the (universal) m-generated exponent-p class 2 group, the elements of which may be viewed as ordered pairs $(v, w) \in V \times \Lambda^2 V$ with multiplication rule $(v_1, w_1)(v_2, w_2) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2)$. Thus $n \leq {m \choose 2}$.

- (b) As A acts irreducibly on V, results of Schur and Wedderburn imply that the ring $\operatorname{End}_{\mathbb{F}_pA}(V)$ of \mathbb{F}_pA -endomorphisms is a finite field, say \mathbb{F}_q where $q=p^{e_A}$ depends on A. Further, $\operatorname{Z}(A) \leqslant \mathbb{F}_q^{\times}$ is cyclic and $|\mathbb{F}_q^{\times}| = p^{e_A} 1$ where $e_A \mid n$. Hence matrices in $\operatorname{Z}(A)$ are scalars over \mathbb{F}_q and $\lambda I_n \in K$ precisely when $\lambda^2 = 1$, so $|\operatorname{Z}(A) \cap K| \leqslant 2$.
- (c) Since $B^{\infty} \neq 1$ and $A/K \cong B$ (Lemma 2.3), we have $A^{\infty} \neq 1$. Hence $A^{\infty}/Z(A^{\infty}) \cong B^{\infty}/Z(B^{\infty})$ is the unique nonabelian simple composition factor of A and B by Corollary 4.2. If $|Z(A^{\infty})|$ is even, then $-1 \in Z(A^{\infty})$ by part (b). However -1 acts trivially on Λ^2V and hence on $W = \Lambda^2V/U$. Therefore $B^{\infty} \neq A^{\infty}$, a contradiction. Thus $|Z(A^{\infty})|$ is odd.

We remark that constructing 3-orbit p-groups of (odd) exponent p is the same as finding maximal A-submodules of the exterior square of an A-module.

Remark 4.4. In Lemma 4.3(a), A acts on V and hence on Λ^2V . As G is a 3-orbit group, A acts irreducibly on Λ^2V/U of Λ^2V , so U is a maximal A-submodule; and B acts faithfully on Λ^2V/U . The group $\mathcal{G} := V \times \Lambda^2V$ has center $\{0\} \times \Lambda^2V$, hence $U \lhd \mathcal{G}$ and $G \cong \mathcal{G}/U$. Thus $G \cong V \times (\Lambda^2V/U)$ where $(v_1, w_1 + U)(v_2, w_2 + U) = (v_1 + v_2, w_1 + w_2 + v_1 \wedge v_2 + U)$. If $\alpha \in A$ and $\alpha K \in B$, then α acts as $(v_1, w_1 + U)^{\alpha} = (v_1^{\alpha}, (w_1 + U)^{\alpha K})$. Thus a 3-orbit group G gives rise to a maximal A-submodule U of Λ^2V . Conversely, A may not be transitive on the nonzero vectors

of $\Lambda^2 V/U$ where U is a maximal A-submodule of $\Lambda^2 V$. Interestingly, this is not the case, see Remark 5.6.

The following fact, follows from [26, Line 2, Table 3] when $d \ge 3$.

(1) If $\mathcal{V} = \mathbb{F}_q^d$ is the natural module for $\mathrm{SL}_d(q)$, then $\Lambda^2 \mathcal{V}$ is irreducible.

Now dim($\Lambda^2 \mathcal{V}$) = $\binom{d}{2}$, so $\Lambda^2 \mathcal{V}$ is a 1-dimensional trivial module if d=2. If d=3, then $\Lambda^2 \mathcal{V}$ is isomorphic to the dual module \mathcal{V}^* of \mathcal{V} . If $q=p^b$, then the b Galois conjugate modules \mathcal{V}^θ , $\theta \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)$, all give rise to the same irreducible db-dimensional $\mathbb{F}_p\operatorname{SL}_d(q)$ -module by [17, VII.1.16]. This process of changing from \mathcal{V} to $V=\mathcal{V}\downarrow F=\mathbb{F}_p^{bd}$ is sometimes called 'blowing up the dimension' or 'restricting to a subfield'.

Theorem 4.5. Let G, p, m, V, A, n, W, B be as in Hypothesis 2.5. If $B^{\infty} \neq 1$, then p is an odd prime, m = n is divisible by 3 and $A^{\infty} \cong B^{\infty} \cong \operatorname{SL}_3(p^{n/3})$ where A^{∞} acts as an $(\mathbb{F}_{p^{n/3}})^3$ -module, and B^{∞} acts as its dual. Furthermore, G is isomorphic to the class 2 factor group in Lemma 6.9 with $F = F_0 = \mathbb{F}_{p^{n/3}}$, as on line 6 of Table 1.

Proof. By Corollary 4.2, $B^{\infty}/Z(B^{\infty}) \cong A^{\infty}/Z(A^{\infty})$ is a nonabelian simple group. Recall that G is nonabelian by Hypothesis 2.5. Theorem 3.3 implies that $p \neq 2$ otherwise $A^{\infty} = B^{\infty} = 1$. Thus $n \leq {m \choose 2}$ by Lemma 4.3(a). This shows that $m \neq 2$, otherwise n = 1 and $B^{\infty} = 1$ as $GL_1(p)$ is cyclic. Hence $m \geq 3$ and p > 2. We can rule out case (a) of Theorem 4.1, and $G_2(2^b)'$ in case (b) as $p \neq 2$.

Now $|\mathbf{Z}(A^{\infty})|$ is odd by Lemma 4.3(c), so $A^{\infty} \notin \{\mathrm{SL}_2(5), \mathrm{SL}_2(13), \mathrm{Sp}_{m/b}(p^b)\}$ in Theorem 4.1. This rules out cases (c) and (d) of Theorem 4.1. Thus $A^{\infty} = \mathrm{SL}_{m/b}(p^b)$ and m/b is odd because $|\mathbf{Z}(\mathrm{SL}_{m/b}(p^b))| = \gcd(m/b, p^b - 1)$ is odd. Hence $A^{\infty} \cong B^{\infty}$ by Theorem 4.1, so m = n. Let V be the natural m-dimensional A^{∞} -module over \mathbb{F}_p , and let \mathcal{V} be the d-dimensional A^{∞} -module over \mathbb{F}_q where $q = p^b$. The exterior square $\Lambda^2 \mathcal{V}$ has dimension $\binom{d}{2}$ over the field $\mathbb{F}_q \cong \mathrm{End}_{\mathbb{F}_p A^{\infty}} \mathcal{V}$ of endomorphisms. However, \mathcal{V} is irreducible and so too is $\Lambda^2 \mathcal{V}$ by (1). Hence by [17, VII.1.16(e)] (see also Remark 5.6), we have $\binom{m/b}{2} = n/b = m/b$, so m/b = 3 and m = 3b = n. In summary, $A^{\infty} \cong \mathrm{SL}_3(p^b)$ acts faithfully on the 3-space \mathcal{V} and $B^{\infty} \cong \mathrm{SL}_3(p^b)$

In summary, $A^{\infty} \cong \operatorname{SL}_3(p^b)$ acts faithfully on the 3-space \mathcal{V} and $B^{\infty} \cong \operatorname{SL}_3(p^b)$ acts faithfully and irreducibly on the 3-space $\Lambda^2 \mathcal{V}$. Adapting the argument in Remark 4.4, G is isomorphic to the group $\mathcal{V} \times \Lambda^2 \mathcal{V}$. Alternatively, we may identify G with the set $\mathbb{F}_q^3 \times \mathbb{F}_q^3$ with multiplication given in Lemma 6.9 (by ignoring the third coordinate). Setting $F = F_0 = \mathbb{F}_q$ in Lemma 6.9 shows that A contains $\Gamma L_3(\mathbb{F}_q) = \operatorname{GL}_3(\mathbb{F}_q) \times \operatorname{Aut}(\mathbb{F}_q)$. If $q = p^b$, then $\Gamma L_3(\mathbb{F}_q)$ is a maximal proper subgroup of $\operatorname{GL}_{3b}(\mathbb{F}_p)$ if b > 1, so Lemma 2.4 implies that $A = \Gamma L_3(\mathbb{F}_q)$ for $b \geqslant 1$. A similar argument shows that $B = \Gamma L_3^+(\mathbb{F}_q) = \{g \in \operatorname{GL}_3(\mathbb{F}_q) \mid \det(g) \in (\mathbb{F}_q^{\times})^2\} \times \operatorname{Aut}(\mathbb{F}_q)$ by Remark 6.8 and Lemma 6.9. This verifies line 6 of Table 1.

Nonabelian p-groups with precisely 3 characteristic subgroups were called UCS groups (Unique Characteristic Subgroup) by Taunt and were studied in [14]. A 3-orbit p-group G is a UCS group which is a special group as $Z(G) = G' = \Phi(G)$. The structure of a special group G is strongly influenced by the two Aut(G)-modules V = G/Z(G) and W = Z(G), see [14, Theorem 6]. The group G_4 in [14, Theorem 8] is an exponent- p^2 cousin (with $B = SO_2(p)$) of the exponent-p groups in Theorem 4.5.

5 Case 3 of Theorem 1.1: $A^{\infty} \neq 1$ and $B^{\infty} = 1$

In this section G is a 3-orbit p-group where p is odd, and $A^{\infty} \neq 1$ and $B^{\infty} = 1$ hold. A prototypical example is an extraspecial p-group as hinted by the following lemma. It is worth examining this case before considering more general examples.

The 3-orbit group Q_8 was considered in Section 3. We show that extraspecial 2-groups are a rich source of 4-orbit groups. We focus primarily on the case p > 2.

Lemma 5.1. A finite extraspecial p-group G is a 3-orbit group precisely when $G \cong Q_8$ or $G \cong p_+^{1+m}$ has odd exponent p. In these cases Aut(G) induces on Z(G) the cyclic subgroup $B \cong C_{p-1}$. An extraspecial 2-group G with $G \not\cong Q_8$ is a 4-orbit group.

Proof. A finite extraspecial p-group G satisfies $G' = \operatorname{Z}(G) = \Phi(G) \cong \operatorname{C}_p$. Suppose that G is a 3-orbit group and the notation in Hypothesis 2.5 holds. If $G \setminus \operatorname{Z}(G)$ contains elements of orders p and p^2 , then $\operatorname{Aut}(G)$ has at least 4 orbits on G. This is the case if G has odd exponent p^2 , or p=2 and $G \not\cong Q_8$. However, if $G \cong Q_8$, then $\operatorname{Aut}(G)$ induces $\operatorname{GL}_2(2)$ on $V = G/\operatorname{Z}(G) \cong (\operatorname{C}_2)^2$, and acts trivially on $\operatorname{Z}(G) \cong \operatorname{C}_2$. Hence Q_8 is a 3-orbit group. It follows from [30] that an extraspecial p-group of odd exponent p is a 3-orbit group. This is also proved in Lemma 6.2 which also applies to infinite 3-orbit groups. In our case $B \cong \operatorname{GL}_1(p) \cong \operatorname{C}_{p-1}$ holds by Lemma 6.2.

Suppose now that G_{ε} is the extraspecial 2-group 2_{ε}^{1+m} of order 2^{m+1} and type $\varepsilon \in \{-, +\}$ where m is even. In G_{ε} , squaring induces a (well-defined) quadratic form Q_{ε} on the vector space $V_{\varepsilon} = G_{\varepsilon}/\mathrm{Z}(G_{\varepsilon}) \cong \mathbb{F}_2^m$. The preimage in G_{ε} of singular vectors in V_{ε} are the noncentral involutions of G_{ε} , and the preimage of nonsingular vectors in V_{ε} are the elements of order 4 in G_{ε} . It is well known that the outer automorphism group $\mathrm{Out}(G_{\varepsilon})$ is isomorphic to the full orthogonal group $\mathrm{O}(Q_{\varepsilon}) \cong \mathrm{O}_m^{\varepsilon}(2)$, see [12] or [4, §2.2.6]. For even $m \geq 2$ and $\varepsilon \in \{-, +\}$ the space V_{ε} has nonsingular vectors, and it has singular vectors except when $(\varepsilon, m) = (-, 2)$. By Witt's theorem $\mathrm{O}_m^{\varepsilon}(2)$ is transitive on the (possibly empty) set of singular vectors and the set of nonsingular vectors. Clearly $\{1\}$ and $\mathrm{Z}(G_{\varepsilon}) \setminus \{1\}$ are $\mathrm{Aut}(G_{\varepsilon})$ -orbits, so that G_{ε} is a 3-orbit group if $(\varepsilon, m) = (-, 2)$, and a 4-orbit group otherwise. The elements of order 4 form one $\mathrm{Aut}(G_{\varepsilon})$ -orbit, and the involutions form two orbits if $(\varepsilon, m) \neq (-, 2)$ (the central involution is fixed).

The following fact from representation theory will guide our proof of Theorem 5.7.

Remark 5.2. The symmetric group S_n acts on \mathbb{F}_q^n by permuting the elements of a basis $\{v_1, \ldots, v_n\}$. Further, the augmentation map $\phi \colon V \to \mathbb{F}_q \colon \sum_{i=1}^n \lambda_i v_i \mapsto \sum_{i=1}^n \lambda_i$ is an S_n -epimorphism, and S_n fixes the submodules

$$W = \ker(\phi) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \sum_{i=1}^{n} \lambda_i = 0 \right\} \quad \text{and} \quad D = \left\langle \sum_{i=1}^{n} v_i \right\rangle.$$

The equation $\sum_{i=1}^{n} i(v_i - v_{i+1}) = (\sum_{i=1}^{n-1} v_i) - (n-1)v_n$ shows that $D \subseteq W$ if $p \mid n$ where $p = \operatorname{char}(\mathbb{F}_q)$, and $D \cap W = \{0\}$ otherwise. We call $W/(D \cap W)$ the fully deleted permutation module. It can be written over \mathbb{F}_p , and it is absolutely irreducible. \square

Remark 5.3. Let $\mathcal{V} = \mathbb{F}_q^{2\ell}$ be a symplectic space preserving the nondegenerate alternating bilinear form $f \colon \mathcal{V} \times \mathcal{V} \to \mathbb{F}_q$. Let \mathcal{V} be the natural module for $\operatorname{Sp}(\mathcal{V}) \cong \operatorname{Sp}_{2\ell}(q)$ where $q = p^b$, with basis $e_1, \ldots, e_{2\ell}$. The map $\phi \colon \Lambda^2 \mathcal{V} \to \mathbb{F}_q$ with $\sum \lambda_{ij} e_i \wedge e_j \mapsto \sum \lambda_{ij} f(e_i, e_j)$ is a $\operatorname{Sp}(\mathcal{V})$ -module epimorphism. Hence $\mathcal{W} := \ker(\phi)$ is an $\operatorname{Sp}(\mathcal{V})$ -invariant hyperplane. Let $\langle e_1, e_2 \rangle, \ldots, \langle e_{2\ell-1}, e_{2\ell} \rangle$ be pairwise orthogonal hyperbolic planes. Then $\phi(\sum_{i < j} \lambda_{ij} e_i \wedge e_j) = \sum_{i=1}^{\ell} \lambda_{2i-1,2i}$ and the stabilizer $\operatorname{Sp}_2(q) \wr S_\ell$ of the decomposition $\langle e_1, e_2 \rangle \oplus \cdots \oplus \langle e_{2\ell-1}, e_{2\ell} \rangle$ is a maximal subgroup of $\operatorname{Sp}_{2\ell}(q)$ which preserves the submodules $\mathcal{W} = \ker(\phi)$ and $\mathcal{D} = \langle \sum_{i=1}^{\ell} e_{2i-1} \wedge e_{2i} \rangle$ in Figure 1 by Remark 5.2. A symplectic transvection not in $\operatorname{Sp}_2(q) \wr S_\ell$ also preserves these submodules. Hence \mathcal{D} and \mathcal{W} are invariant under all of $\operatorname{Sp}_{2\ell}(q)$.

We show that \mathcal{D} and \mathcal{W} are A-invariant for A satisfying $\operatorname{Sp}(\mathcal{V}) \leqslant A \leqslant \Gamma \operatorname{Sp}(\mathcal{V})$. The notation $\operatorname{CSp}(\mathcal{V})$ and $\operatorname{\GammaSp}(\mathcal{V})$ is described in Remark 6.1. First, $\operatorname{CSp}_{2\ell}(q) =$

$$\begin{array}{c|c}
\Lambda^{2}V & 1 \\
W & 2\ell^{2} - \ell - 2 \\
0 & 1 \\
p \text{ divides } \ell
\end{array}$$

$$\begin{array}{c|c}
\Lambda^{2}V \\
W \\
2\ell^{2} - \ell - 1 \\
1 & \{0\} \\
p \text{ coprime to } \ell$$

Figure 1: The A-submodules of $\Lambda^2 \mathcal{V}$ where $\mathcal{V} = \mathbb{F}_{p^b}^{2\ell}$ and $\operatorname{Sp}(\mathcal{V}) \leqslant A \leqslant \Gamma \operatorname{Sp}(\mathcal{V})$

 $\langle g_{\mu}, \operatorname{Sp}_{2\ell}(q) \rangle$ where $\mathbb{F}_q^{\times} = \langle \mu \rangle$ and g_{μ} satisfies $e_{2i-1}g_{\mu} = \mu e_{2i-1}$ and $e_{2i}g_{\mu} = e_{2i}$ for $i \leq \ell$. Also $\phi(ug_{\mu}) = \mu\phi(u)$ for $u \in \Lambda^2 \mathcal{V}$, so $\operatorname{CSp}_{2\ell}(q)$ fixes $\mathcal{W} = \ker(\phi)$ and \mathcal{D} . Second, if g_{θ} satisfies $(\sum_{i=1}^{2\ell} \lambda_i e_i) g_{\theta} = \sum_{i=1}^{2\ell} \lambda_i^{\theta} e_i$ for $\theta \in \operatorname{Aut}(\mathbb{F}_{p^b})$, then $\phi(ug_{\theta}) = \phi(u)^{\theta}$ for $u \in \Lambda^2 \mathcal{V}$. Hence $\operatorname{\GammaSp}_{2\ell}(p^b)$ fixes \mathcal{W} and \mathcal{D} and induces $\operatorname{\GammaL}_1(p^b)$ on \mathcal{D} . Finally, the only $\operatorname{Sp}(\mathcal{V})$ -submodules of $\Lambda^2 \mathcal{V}$ are $\{0\}$, \mathcal{W} , \mathcal{D} , $\Lambda^2 \mathcal{V}$ by [26, Table 5], see also [19, Hauptsatz 1] when p = 2. In summary, we have justified Figure 1.

Remark 5.4. Let $V = F^m$ be an m-dimensional vector space over a field F where $\operatorname{char}(F) \neq 2$. Then $T^2(V) = A^2(V) \oplus S^2(V)$ where $T^2(V) = V \otimes V$, $A^2(V)$ and $S^2(V)$ are called the tensor, alternating, and symmetric squares of V, respectively. Set $A^2(V) := \{v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V\}$ and $S^2(V) := \{v_1 \otimes v_2 + v_2 \otimes v_1 \mid v_1, v_2 \in V\}$. The identity $v_1 \otimes v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) + \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$ for $v_1, v_2 \in V$ implies that $T^2(V) = A^2(V) \oplus S^2(V)$ holds. The exterior square is isomorphic to the alternating square via $\Lambda^2(V) \to A^2(V)$: $v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$. The symmetric square is normally defined to be the quotient $T^2(V)/A^2(V)$ (in all characteristics), and similarly $\Lambda^2(V)$ is defined to be $T^2(V)/S^2(V)$ in all characteristics. The isomorphism $T^2(V)/A^2(V) \to S^2(V)$ with $v_1 \otimes v_2 + A^2(V) \mapsto v_1 \otimes v_2 + v_2 \otimes v_1$ holds for all F. \square

Remark 5.5. Let p be an odd prime and set $q = p^b$, $E = \mathbb{F}_q$, $F = \mathbb{F}_p$, $\mathcal{V} = E^d$ where $d = 2\ell$ is even and \mathcal{V} is an A-module where $\operatorname{Sp}(\mathcal{V}) \leqslant A \leqslant \Gamma \operatorname{Sp}(\mathcal{V})$. We view $V = \mathcal{V} \downarrow F = F^{bd}$ as the A-module $\mathcal{V} = E^d$ written over F using the inclusions $A \leqslant \Gamma \operatorname{Sp}_d(E) \leqslant \operatorname{GL}_{bd}(F)$. The bd-dimensional EA-module $V \otimes_F E$ is $\bigoplus_{i=0}^{b-1} \mathcal{V}^{(i)}$ where $\mathcal{V}^{(i)}$ it the Galois conjugate of \mathcal{V} by θ^i where $\operatorname{Gal}(E:F) = \langle \theta \rangle$ by [17, VII.1.16(a)]. We shall prove the decomposition $A^2(V \otimes E) = \bigoplus_i A^2(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} \mathcal{A}_{ij}$ where $0 \leqslant i < b$, $0 \leqslant i < j < b$, and \mathcal{A}_{ij} is defined below. First, $T^2(V) \otimes E \cong T^2(V \otimes E)$ equals

$$\left(\bigoplus_{i} \mathcal{V}^{(i)}\right) \otimes \left(\bigoplus_{j} \mathcal{V}^{(j)}\right) = \bigoplus_{i} T^{2}(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} \left(\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)} \oplus \mathcal{V}^{(j)} \otimes \mathcal{V}^{(i)}\right).$$

However, $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)} \oplus \mathcal{V}^{(j)} \otimes \mathcal{V}^{(i)}$ is a direct sum of submodules say $\mathcal{A}_{ij} \oplus \mathcal{S}_{ij}$ where

$$\mathcal{A}_{ij} = \left\{ v_i \otimes v_j - v_j \otimes v_i \mid v_i \in \mathcal{V}^{(i)}, v_j \in \mathcal{V}^{(j)} \right\},$$

$$\mathcal{S}_{ij} = \left\{ v_i \otimes v_j + v_j \otimes v_i \mid v_i \in \mathcal{V}^{(i)}, v_j \in \mathcal{V}^{(j)} \right\},$$

 $\mathcal{A}_{ij} \leqslant A^2(V \otimes E)$ and $\mathcal{S}_{ij} \leqslant S^2(V \otimes E)$ by Remark 5.4. Hence

$$T^{2}(V \otimes E) = \bigoplus_{i} A^{2}(\mathcal{V}^{(i)}) \oplus \bigoplus_{i} S^{2}(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} \mathcal{A}_{ij} \oplus \bigoplus_{i < j} \mathcal{S}_{ij}.$$

Since $A_{ij} \cong \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)} \cong \mathcal{S}_{ij}$ for i < j, the claimed decomposition follows:

$$A^{2}(V \otimes E) = \bigoplus_{i} A^{2}(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} \mathcal{A}_{ij} \cong \bigoplus_{i} A^{2}(\mathcal{V}^{(i)}) \oplus \bigoplus_{i < j} \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}.$$

(The containment \geqslant holds, and the dimension agree as $\binom{bd}{2} = b\binom{d}{2} + \binom{b}{2}d^2$.)

Recall that $A^{\infty} \neq 1$ and $B^{\infty} = 1$ hold in this section.

Remark 5.6. Assume, as in Remark 5.5, that p > 2 is prime, $q = p^b$, $E = \mathbb{F}_q$, $F = \mathbb{F}_p$, $V = E^d$, $d = 2\ell$ is even, $V = \mathcal{V} \downarrow F = F^{bd}$ where \mathcal{V} is an A-module and $\mathrm{Sp}(\mathcal{V}) \leqslant A \leqslant \Gamma \mathrm{Sp}(\mathcal{V})$. By Remark 5.3, E is a b-dimensional FA-module which is irreducible if $\mathrm{CSp}(\mathcal{V}) \leqslant A$ and is trivial if $A = \mathrm{Sp}(\mathcal{V})$. Let U be a maximal FA-submodule of $A^2(V \otimes E) \downarrow F$. This remark proves that the quotient FA-module $(A^2(V \otimes E) \downarrow F)/U$ is isomorphic to a subfield of E containing F (really a quotient FA-module). As $B^{\infty} = 1$, we see that $A^{\infty} = \mathrm{Sp}(\mathcal{V})$ acts trivially on this quotient. In the next paragraph, we consider EA-submodules rather than FA-submodules.

Let \mathcal{U} be a maximal EA-submodule of $A^2(V \otimes E)$ such that $A^{\infty} = \operatorname{Sp}(\mathcal{V})$ acts trivially on $A^2(V \otimes E)/\mathcal{U}$. Remark 5.5 shows that $A^2(V \otimes E) = \mathcal{X} \oplus \mathcal{Y}$ where $\mathcal{X} = \bigoplus_i A^2(\mathcal{V}^{(i)})$ and $\mathcal{Y} = \bigoplus_{i < j} \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$. We shall show that $\mathcal{Y} \subseteq \mathcal{U}$. Let $\mathcal{W} = \ker(\phi)$ be as in Remark 5.3 where $\Lambda^2(\mathcal{V})/\mathcal{W}$ is a 1-dimensional EA-module. If i < j, then $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$ is a faithful A^{∞} -module which is irreducible if $j \neq d/2 + i$ by [20, §5.4], and is a sum of (two isomorphic) faithful irreducible submodules if j = d/2 + i. Also, the uniserial proper A^{∞} -submodule \mathcal{W} in Figure 1 of $\Lambda^2(\mathcal{V}^{(i)})$ of dimension $2\ell^2 - \ell - 1$ is nontrivial. Since A^{∞} acts trivially on the simple factor module $A^2(V \otimes E)/\mathcal{U}$, we see that \mathcal{U} contains $\mathcal{Z} := \bigoplus_i \mathcal{W}^{(i)} \oplus \mathcal{Y}$, as claimed.

Suppose now that U is a maximal FA-submodule of $A^2(V \otimes E) \downarrow F$ such that A^{∞} acts trivially on $(A^2(V \otimes E) \downarrow F)/U$. Choose a maximal EA-submodule \mathcal{U} of $A^2(V \otimes E)$ where U contains $\mathcal{U} \downarrow F$. Then A^{∞} acts trivially on $A^2(V \otimes E)/\mathcal{U}$. By the previous paragraph, $A^2(V \otimes E)/\mathcal{U}$ is a factor of $A^2(V \otimes E)/\mathcal{Z} = \bigoplus_i (\Lambda^2(V)/\mathcal{W})^{(i)}$. Now $(A^2(V \otimes E) \downarrow F)/U$ is an irreducible FA-module and a factor FA-module of $(A^2(V \otimes E)/\mathcal{U}) \downarrow F \cong (A^2(V \otimes E) \downarrow F)/(\mathcal{U} \downarrow F)$. The irreducible factor FA-modules are isomorphic to a subfield \mathbb{F}_{p^n} of $E = \mathbb{F}_{p^b}$ for some divisor n of p by [17, VII.1.16(e)]. As p varies, any divisor p of p can arise, see Lemma 6.2.

Theorem 5.7. Let G be a finite nonabelian 3-orbit p-group and let V, A, W, B be as in Hypothesis 2.5. If $A^{\infty} \neq 1$ and $B^{\infty} = 1$, then p is odd and G, V, A, W, B are as in line 7 of Table 1 with $|\Phi(G)| = p^n$ as described in Lemma 6.2.

Proof. If p=2, then $\operatorname{Aut}(G)$ is solvable by Theorem 3.3 and so $A^{\infty}=1$, a contradiction. Hence p > 2. If n = 1, then m must be even and G is the extraspecial group of order p^{1+m} , and exponent p which appears on line 7 of Table 1 with b=1. Since p>2, we have $n\leqslant {m\choose 2}$ by Lemma 4.3(a). Hence m=2 implies n=1. Suppose now that $m \geqslant 3$ and $n \geqslant 2$. Since $A^{\infty} \neq 1$ and $B^{\infty} = 1$, we have $A^{\infty} \leqslant K$. If $H = N_{\mathrm{GL}(V)}(A^{\infty})$, then $H/A^{\infty} \geqslant A/A^{\infty} \geqslant A/K \cong B$. We argue using Theorem 4.1 that $A^{\infty} \cong \operatorname{Sp}_{m/b}(p^b)$. Since $p^n - 1$ divides |B|, we see that $B \neq 1$. Hence A is strictly larger that A^{∞} , so case (d) of Theorem 4.1 cannot hold, nor can case (a) as $A^{\infty} \neq 1$. In case (c), we have $A^{\infty} = \mathrm{SL}_2(5)$ and $V = \mathbb{F}_3^4$ is an A^{∞} -module. The maximal A^{∞} submodules of $\Lambda^2 V$ have codimension 1 and $\binom{4}{2} - 1 = 5$ by Remark 5.3. Therefore n=1,5 by Lemma 4.4. But $n\neq 1$, so n=5 and $242=p^n-1$ divides |B|. A direct calculation with MAGMA [3] shows that $|H/A^{\infty}| = 8$. This rules out case (c). Hence we have $A^{\infty} \in \{\mathrm{SL}_{m/b}(p^b), \mathrm{Sp}_{m/b}(p^b)\}$. Suppose $A^{\infty} \cong \mathrm{SL}_{m/b}(p^b)$ and $\mathcal{V} = \mathbb{F}_{p^b}^{m/b}$ is its natural module. As \mathcal{V} is irreducible, so too is $\Lambda^2 \mathcal{V}$ by (1). Let $V = \mathcal{V} \downarrow \mathbb{F}_p = \mathbb{F}_p^m$. We claim that $\Lambda^2 V$ is is a direct sum of faithful irreducible $\mathbb{F}_p A^{\infty}$ -modules. The claim follows from Remark 5.5 as $\Lambda^2(\mathcal{V} \otimes \mathbb{F}_{p^b}) = \bigoplus_i \Lambda^2(\mathcal{V})^{(i)} \oplus \bigoplus_{i < j} \mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$ and $\Lambda^2(\mathcal{V})^{(i)}$ is faithful and irreducible by (1), and $\mathcal{V}^{(i)} \otimes \mathcal{V}^{(j)}$ is either a faithful and irreducible $SL_{m/b}(p^b)$ -modules or a direct sum of two such by [20, Theorem 5.4.5]. This implies that $B^{\infty} \cong \mathrm{SL}_{m/b}(p^b) \neq 1$, a contradiction. The only remaining possibility in Theorem 4.1 is $A^{\infty} \cong \operatorname{Sp}_{m/b}(p^b)$ where $m/b \geqslant 4$ is even. In this case, $A \leqslant \Gamma \operatorname{Sp}_{m/b}(p^b)$

since the normalizer of $\operatorname{Sp}_{m/b}(p^b)$ in $\operatorname{GL}_m(p)$ is $\operatorname{\GammaSp}_{m/b}(p^b)$. In summary, we have shown that $\operatorname{Sp}_{m/b}(p^b) = A^{\infty} \leqslant A \leqslant \operatorname{\GammaSp}_{m/b}(p^b) \leqslant \operatorname{GL}_m(p)$.

We now apply Theorem 4.1(a) to $B \leqslant \operatorname{GL}_n(p)$. First, $B^{\infty} = 1$ and B = A/K implies $\operatorname{Sp}_{m/b}(p^b) = A^{\infty} \leqslant K$. Thus B is a section of $\operatorname{\GammaSp}_{m/b}(p^b)/\operatorname{Sp}_{m/b}(p^b) \cong \operatorname{\GammaL}_1(p^b)$, and so B is metacyclic. Since $B^{\infty} = 1$, the choices for B are constrained by Theorem 4.1(a). The extraspecial group $D_8 \circ Q_8 = 2^{1+4}_-$ is not metacyclic, and therefore $(n,p) \neq (4,3)$, as subgroups of metacyclic groups are metacyclic. Suppose that n=2 and $p \in \{5,7,11,23\}$. A calculation using MAGMA [3] shows that the subgroups of $\operatorname{GL}_2(p)$ with $p \in \{5,7,11,23\}$ that are both metacyclic and transitive on nonzero vectors, all lie in $\operatorname{\GammaL}_1(p^2)$. Therefore $B \leqslant \operatorname{\GammaL}_1(p^n)$ as in line 7 of Table 1. By Lemma 4.3(a) the 3-orbit group G is isomorphic to $(V \rtimes \Lambda^2(V))/U$ where

By Lemma 4.3(a) the 3-orbit group G is isomorphic to $(V \rtimes \Lambda^2(V))/U$ where $V = \mathbb{F}_p^m$ is the natural A-module, and U is a maximal submodule of $\Lambda^2(V)$ by Remark 4.4. The simple quotient A-modules $\Lambda^2(V)/U$ of $\Lambda^2(V)$ are the subfields of \mathbb{F}_{p^b} by Remark 5.6, and each subfield gives rise to a 3-orbit group G. Thus G is as described on line 7 of Table 1. Large subgroups of A and B are described in Lemma 6.2. Indeed, $A \leq \Gamma \operatorname{Sp}_{m/b}(p^b)$ and $B \leq \Gamma \operatorname{L}_1(p^b)$ as in line 7 of Table 1.

6 Examples of k-orbit groups

In this section we give examples of k-orbit groups for small k. We focus on 3-orbit groups. Extraspecial p-groups provide examples of both 3-orbit and 4-orbit groups.

If G is a finite extraspecial p-group, or an infinite Heisenberg group, then viewing the elements of G as ordered pairs facilitates a geometric method to construct Aut(G). This method, was not used by Winter in [30], but is used in Lemma 6.2 below.

Remark 6.1. We first describe $\Gamma \operatorname{Sp}(\mathcal{V})$ and $\operatorname{CSp}(\mathcal{V})$. Let $f: \mathcal{V} \times \mathcal{V} \to F$ be a non-degenerate symplectic bilinear form on $\mathcal{V} = F^d$ where $d \geq 2$ is even. Let $\Gamma \operatorname{Sp}(\mathcal{V})$ be the group of bijective semilinear symplectic similarities on \mathcal{V} . These satisfy

$$(\lambda v)g = \lambda^{\sigma(g)}(vg), (v_1 + v_2)g = v_1g + v_2g, \text{ and } f(v_1g, v_2g) = \delta(g)^{\sigma(g)}f(v_1, v_2)^{\sigma(g)},$$

for $g \in \Gamma \operatorname{Sp}(\mathcal{V})$, $v, v_1, v_2 \in \mathcal{V}$, $\lambda \in F$, where $\delta(g) \in F^{\times}$ and $\sigma(g) \in \operatorname{Aut}(F)$ depend on g. The map $\sigma \colon \Gamma \operatorname{Sp}(\mathcal{V}) \to \operatorname{Aut}(F)$ is an epimorphism. Comparing $f(v_1(gh), v_2(gh))$ to $f((v_1g)h, (v_2g)h)$ gives the 1-cocycle condition $\delta(gh) = \delta(g)\delta(h)^{\sigma(g^{-1})}$. The conformal symplectic group denoted by $\operatorname{CSp}(\mathcal{V})$ is the kernel of σ c.f. [4, Def. 1.6.14].

We view the elements of $\Gamma L_1(F)$ as products $\delta \sigma$ with $(\delta, \sigma) \in F^{\times} \times \operatorname{Aut}(F)$ and multiplication rule $(\delta_1 \sigma_1)(\delta_2 \sigma_2) = \delta_1 \delta_2^{\sigma_1^{-1}} \sigma_1 \sigma_2$. It follows from the previous paragraph that $\Gamma \operatorname{Sp}_d(F) = \operatorname{Sp}_d(F) \rtimes \Gamma L_1(F)$, see [20, Table 2.1.C] and Remark 5.3.

If $F: F_0$ is a Galois extension of the subfield F_0 , then $\Gamma L_1(F_0)$ is a factor group of $\Gamma L_1(F)$, and hence $\Gamma L_1(F_0)$ is a factor group of $\Gamma \operatorname{Sp}_d(F) \cong \operatorname{Sp}(\mathcal{V}) \rtimes \Gamma L_1(F)$.

Lemma 6.2. Let $F: F_0$ be a finite Galois field extension where $\operatorname{char}(F) = p \geqslant 0$. Let $f: \mathcal{V} \times \mathcal{V} \to F$ be a non-degenerate alternating F-bilinear form on $\mathcal{V} = F^d$ where d is even. Let Tr be the trace map $F \to F_0: \lambda \mapsto \sum_{\sigma \in \operatorname{Gal}(F:F_0)} \lambda^{\sigma}$. The set $G = \mathcal{V} \times F_0$ with the multiplication rule $(v_1, \zeta_1)(v_2, \zeta_2) = (v_1 + v_2, \zeta_1 + \zeta_2 + \operatorname{Tr}(f(v_1, v_2)))$ defines a group. If $p \neq 2$, then $G = G_{F,F_0}$ is a 3-orbit group and

$$\operatorname{Sp}_d(F) \rtimes (F_0^{\times} \rtimes \operatorname{Aut}(F)) \leqslant \operatorname{Aut}(G)^{G/G'}$$
 and $\operatorname{GL}_1(F_0) \leqslant \operatorname{Aut}(G) \downarrow G'$.

If $F = F_0 = \mathbb{F}_p$, where p is an odd prime, then $A = \mathrm{CSp}_d(p)$ and $B = \mathrm{GL}_1(p)$.

Proof. Since the map $\mathcal{V} \times \mathcal{V} \to F_0$: $(v_1, v_2) \mapsto \operatorname{Tr}(f(v_1, v_2))$ is biadditive, the multiplication on G is associative. Hence G is a group where (0,0) is the identity element and $(v,\zeta)^{-1} = (-v,-\zeta)$ as f(v,v) = 0. If p > 0, then the exponent of G is p since $(v,\zeta)^k = (kv,k\zeta)$ for $k \in \mathbb{Z}$. The commutator $[(v_1,\zeta_1),(v_2,\zeta_2)]$ equals $(0,2\operatorname{Tr}(f(v_1,v_2)))$. Thus G is abelian if p=2. Suppose now that $p \neq 2$. As f and f are surjective functions, it follows that $G' = \{0\} \times F_0$.

Let \mathcal{A} be the subgroup of $\Gamma \operatorname{Sp}(\mathcal{V})$ (see Remark 6.1) comprising all g satisfying $f(v_1g, v_2g) = \delta(g)f(v_1, v_2)^{\sigma(g)}$ with $\delta(g) \in F_0^{\times}$. Then the structure of \mathcal{A} is $\operatorname{Sp}_d(F) \rtimes (F_0^{\times} \rtimes \operatorname{Aut}(F))$. Using $\operatorname{Tr}(\delta \lambda^{\sigma}) = \delta \operatorname{Tr}(\lambda)$ for $\delta \in F_0$, $\lambda \in F$, $\sigma \in \operatorname{Aut}(F)$, we show below that $(v, \zeta)^g = (vg, \delta(g)\zeta)$ defines an action of $g \in \mathcal{A}$ on G:

$$(v_1, \zeta_1)^g (v_2, \zeta_2)^g = (v_1 g, \delta(g) \zeta_1) (v_2 g, \delta(g) \zeta_2)$$

$$= (v_1 g + v_2 g, \delta(g) \zeta_1 + \delta(g) \zeta_2 + \operatorname{Tr}(\delta(g) f(v_1, v_2)^{\sigma(g)}))$$

$$= ((v_1 + v_2) g, \delta(g) (\zeta_1 + \zeta_2 + \operatorname{Tr}(f(v_1, v_2))))$$

$$= (v_1 + v_2, \zeta_1 + \zeta_2 + \operatorname{Tr}(f(v_1, v_2))^g = ((v_1, \zeta_1) (v_2, \zeta_2))^g.$$

Thus g is a bijective endomorphism of G, i.e. an automorphism of G. Moreover, \mathcal{A} acts on G since $((v,\zeta)^g)^h = (v,\zeta)^{gh}$. Therefore $\operatorname{Aut}(G)$ has 3 orbits on G, namely $\{(0,0)\}, \{0\} \times F_0^{\times}, (\mathcal{V} \setminus \{0\}) \times F_0$, that is $1, G' \setminus \{1\}, G \setminus G'$. We have therefore shown that $\mathcal{A} \leq \operatorname{Aut}(G)^{G/G'}$ and $\operatorname{GL}_1(F_0) \leq \operatorname{Aut}(G) \downarrow G'$, as claimed.

Finally, suppose that $F = F_0 = \mathbb{F}_p$, where p is an odd prime. In this case G is an extraspecial p-group of order p^{1+d} and exponent p. It follows from [30] that $\operatorname{Out}(p^{1+d}) = \operatorname{CSp}_d(p)$ and hence $A = \operatorname{CSp}_d(p)$ and $B = \operatorname{GL}_1(p)$ as claimed. \square

Remark 6.3. The subgroup U in Remark 4.4 is the kernel of the map $\Lambda^2 V \to F_0$ defined by $v_1 \wedge v_2 \mapsto \text{Tr}(f(v_1, v_2))$ where f and Tr are as in Lemma 6.2.

Lemma 6.4. The group B(n) in Definition 3.2(b) is isomorphic to the Suzuki 2-group $B(n,1,\xi)$ defined by [16, Column III] where $\xi \neq \tau + \tau^{-1}$ for all $\tau \in \mathbb{F}_{2^n}^{\times}$.

Proof. Let $q=2^n$. The polynomial $t^2+\xi t+1$ is irreducible in $\mathbb{F}_q[t]$ since $\tau+\tau^{-1}=\xi$ has no solutions for $\tau\in\mathbb{F}_q^{\times}$. Let ε be a root of $t^2+\xi t+1$. Then $\varepsilon+\varepsilon^q=\xi$ and $\varepsilon^{q+1}=1$. Hence $\mathbb{F}_q[\varepsilon]=\mathbb{F}_{q^2}$ and the norm map $\mathbb{F}_{q^2}^{\times}\to\mathbb{F}_q^{\times}$ sends $\alpha+\beta\varepsilon\in\mathbb{F}_{q^2}^{\times}$ to $(\alpha+\beta\varepsilon)(\alpha+\beta\varepsilon^q)=\alpha^2+\xi\alpha\beta+\beta^2$. The Suzuki 2-group $B(n,1,\xi)$ can, by Higman [16, Column V], be identified with the set \mathbb{F}_q^3 with the following multiplication rule

$$(\alpha_1, \beta_1, \zeta_1)(\alpha_2, \beta_2, \zeta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \zeta_1 + \zeta_2 + \alpha_1\alpha_2 + \xi\alpha_1\beta_2 + \beta_1\beta_2).$$

The third coordinate is related to the 'bilinearized' form of the norm map

$$(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q = (\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon^{-1}) = \alpha_1 \alpha_2 + \xi \alpha_1 \beta_2 + \beta_1 \beta_2.$$

Therefore, $(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \in \mathbb{F}_q$ and hence

$$(\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \varepsilon + ((\alpha_1 + \beta_1 \varepsilon)(\alpha_2 + \beta_2 \varepsilon)^q \varepsilon)^q = (\alpha_1 \alpha_2 + \xi \alpha_1 \beta_2 + \beta_1 \beta_2)(\varepsilon + \varepsilon^q).$$

Since $\varepsilon + \varepsilon^q = \xi$, the map $B(n, 1, \xi) \to B(n)$ defined by $(\alpha, \beta, \zeta) \mapsto (\alpha + \beta \varepsilon, \zeta \xi)$ is an isomorphism. Consequently, the isomorphism type of $B(n, 1, \xi)$ is independent of the choice of ξ for which $t^2 + \xi t + 1$ is irreducible.

We will construct examples of 3- and 4-orbit groups using the exterior algebra $\Lambda(\mathcal{V})$ of a vector space \mathcal{V} . If $\dim(\mathcal{V}) = d$, then $\Lambda(\mathcal{V}) = \bigoplus_{k=0}^{\dim(\mathcal{V})} \Lambda^k(\mathcal{V})$ is a graded algebra with $\dim(\Lambda^k(\mathcal{V})) = \binom{d}{k}$ and hence $\dim(\Lambda(\mathcal{V})) = 2^d$. The following preliminary lemma exploits the action of $GL(\mathcal{V})$ on $\Lambda^k(\mathcal{V})$, see [22, XIX].

Lemma 6.5. Let $V = F^d$ be an d-dimensional vector space over a field F. Suppose that $1 < k \le d$ and $n = \binom{d}{k}$. The action of GL(V) on $\Lambda^k(V)$ induces a homomorphism $\phi_{d,k} \colon GL_d(F) \to GL_n(F)$ of matrix groups. The kernel of $\phi_{d,k}$ is $GL_d(F)$ if k > d, $SL_d(F)$ if k = d, and $\{\lambda I_d \mid \lambda \in F \text{ and } \lambda^k = 1\}$ if k < d.

Proof. If k > d, then $\Lambda^k(\mathcal{V}) = \{0\}$ so $\ker \phi_{d,k} = \operatorname{GL}_d(F)$. Let $\mathcal{V} = \langle e_1, \dots, e_d \rangle$. If k = d, then $\Lambda^d(\mathcal{V}) = \langle e_1 \wedge \dots \wedge e_d \rangle$ and $g\phi_{d,d} = (\det(g))$, so that $\ker \phi_{d,d} = \operatorname{SL}_d(F)$.

Let $\langle v_1, \ldots, v_k \rangle$ be a typical k-subspace of \mathcal{V} where k < d. As $g \in \ker \phi_{d,k}$ fixes $v_1 \wedge \cdots \wedge v_k$, it also fixes the k-subspace $\langle v_1, \ldots, v_k \rangle$ by [28, Lemma 12.6]. As k < d we may choose a vector v_{k+1} in $\mathcal{V} \setminus \langle v_1, \ldots, v_k \rangle$. Since g fixes the k-subspaces $\langle v_1, \ldots, v_k \rangle$ and $\langle v_2, \ldots, v_{k+1} \rangle$, it fixes their intersection, viz. $\langle v_2, \ldots, v_k \rangle$. Thus g fixes all (k-1)-subspaces. By induction, g fixes all 1-subspaces of \mathcal{V} and hence g is a scalar matrix. However, $\lambda I_d \in \ker \phi_{d,k}$ precisely when $\lambda^k = 1$. This completes the proof.

Remark 6.6. If $F = \mathbb{F}_q$, then $\{\lambda \in \mathbb{F}_q^{\times} \mid \lambda^k = 1\}$ is cyclic of order $\gcd(k, q - 1)$.

Lemma 6.7. Let $\Lambda(\mathcal{V})$ be the exterior algebra of the F-vector space $\mathcal{V} = F^3$ where $\operatorname{char}(F) \neq 2$. Then the set $G = G_F = \mathcal{V} \times \Lambda^2 \mathcal{V} \times \Lambda^3 \mathcal{V}$ with the multiplication rule

$$(v_1, w_1, x_1)(v_2, w_2, x_2) = (v_1 + v_2, w_1 + w_1 + v_1 \wedge v_2, x_1 + x_2 + v_1 \wedge w_2 + w_1 \wedge v_2)$$

defines a 4-orbit group. Also $\operatorname{Aut}(G)$ induces on $G/\gamma_2(G)$, $\gamma_2(G)/\gamma_3(G)$ and $\gamma_3(G)$ subgroups A, B, C respectively where $\Gamma L(\mathcal{V}) \leqslant A$, $\{g \land g \mid g \in \Gamma L(\mathcal{V})\} \leqslant B$ and $\{g \land g \land g \mid g \in \Gamma L(\mathcal{V})\} \leqslant C$. In particular, $\gamma_3(G) = \operatorname{Z}(G)$, $\gamma_2(G) = C_G(\gamma_2(G))$ and $G/\gamma_3(G)$ is a 3-orbit group. If |F| = q is odd, then $|G| = q^7$ and $|G/\gamma_3(G)| = q^6$.

Proof. The exterior algebra $\Lambda(\mathcal{V})$ equals $\bigoplus_{i=0}^{3} \Lambda^{i}(\mathcal{V})$ where $\dim(\Lambda^{i}(\mathcal{V})) = \binom{3}{i}$. A basis (e_{1}, e_{2}, e_{3}) for $\Lambda^{1}(\mathcal{V}) = \mathcal{V}$ gives bases $(e_{2} \wedge e_{3}, e_{3} \wedge e_{1}, e_{1} \wedge e_{2})$ for $\Lambda^{2}(\mathcal{V})$ and $(e_{1} \wedge e_{2} \wedge e_{3})$ for $\Lambda^{3}(\mathcal{V})$. Relative to these bases a 3×3 matrix $g \in GL(\mathcal{V})$ induces the 3×3 matrix $\det(g)g^{-T} = g \wedge g \in GL(\Lambda^{2}\mathcal{V})$ and the 1×1 matrix $(\det(g)) = g \wedge g \wedge g \in GL(\Lambda^{3}\mathcal{V})$. Hence the action of $GL(\mathcal{V})$ on $\Lambda^{2}(\mathcal{V})$ is different from the 'natural' and 'dual' actions.

The group of units $\Lambda(\mathcal{V})^{\times}$ a has normal subgroup $M = \{1\} \times \mathcal{V} \times \Lambda^2 \mathcal{V} \times \Lambda^3 \mathcal{V}$, and

$$(1 + v_1 + w_1 + x_1) \wedge (1 + v_2 + w_2 + x_2)$$

= 1 + (v₁ + v₂) + (w₁ + v₁ \land v₂ + w₂) + (x₁ + w₁ \land v₂ + v₁ \land w₂ + x₂).

Therefore the stated multiplication rule of triples in $G = \mathcal{V} \times \Lambda^2 \mathcal{V} \times \Lambda^3 \mathcal{V}$ defines an isomorphism $G \to M \colon (v, w, x) \mapsto 1 + v + w + x$. In particular, G is a group.

The identity element of G is (0,0,0) and $(v,w,x)^{-1}=(-v,-w,-x)$ since \wedge is antisymmetric and $w \wedge v + v \wedge w = 0$. Since $(v,w,x)^k=(kv,kw,kx)$ for $k \in \mathbb{Z}$ it follows that G is torsion free if $\operatorname{char}(F)=0$, and has (odd) exponent $p=\operatorname{char}(F)$ otherwise. In both cases $\gamma_2(G)=\{0\}\times\Lambda^2\mathcal{V}\times\Lambda^3\mathcal{V}$ holds because

(2)
$$[(v_1, w_1, x_1), (v_2, w_2, x_2)] = (0, 2v_1 \land v_2, 2(v_1 \land w_2 + w_1 \land v_2)).$$

Setting $[(v_1, w_1, x_1), (v_2, w_2, x_2)] = (0, w', x')$ in (2) where $w' = 2v_1 \wedge v_2$ gives that

$$[[(v_1, w_1, x_1), (v_2, w_2, x_2)], (v_3, w_3, x_3)] = (0, 0, 2w' \land v_3) = (0, 0, 4v_1 \land v_1 \land v_3).$$

Hence $\gamma_3(G) = \{0\} \times \{0\} \times \Lambda^3 \mathcal{V}$ as $\operatorname{char}(F) \neq 2$. Observe that if $v \in \mathcal{V}$ satisfies $v \wedge w = 0$ for all $w \in \Lambda^2 \mathcal{V}$, then v = 0. Hence (2) implies that $\gamma_3(G) = \operatorname{Z}(G)$, and $C_G(\gamma_2(G)) = \gamma_2(G)$.

Now $g \in GL(\mathcal{V})$ acts on G is via $(v, w, x)^g = (vg, w(g \land g), x(g \land g \land g))$ as described above. Hence G is transitive on the nonzero vectors of $G/\gamma_2(G) = V$, $\gamma_2(G)/\gamma_3(G) = \Lambda^2 V$, $\gamma_3(G)/\gamma_4(G) = \Lambda^3 V$, so G is a 4-orbit group. Further, $\sigma \in Aut(F)$ acts to G via $(v, w, x)^{\sigma} = (v^{\sigma}, w^{\sigma}, x^{\sigma})$ by applying σ to the coordinates of v, w, x relative the stated bases. This shows that $\Gamma L(\mathcal{V})$ is a subgroup of Aut(G) which induces $\Gamma L(\mathcal{V}) \leqslant A$, $\{g \land g \mid g \in \Gamma L(\mathcal{V})\} \leqslant B$ and $\{g \land g \land g \mid g \in \Gamma L(\mathcal{V})\} \leqslant C$ as claimed.

Remark 6.8. The group G_F in Lemma 6.7 is abelian if $\operatorname{char}(F) = 2$. If $g \in \operatorname{GL}_3(F)$, then $g \wedge g = \det(g)g^{-T}$ so that $\det(g \wedge g) = \det(g)^3 \det(g^{-T}) = \det(g)^2 \in (F^{\times})^2$. The homomorphism $\operatorname{GL}(\mathcal{V}) \to \operatorname{GL}(\Lambda^2 \mathcal{V})$ has kernel $\langle -1 \rangle$, and $\operatorname{GL}(\mathcal{V}) \to \operatorname{GL}(\Lambda^3 \mathcal{V})$ has kernel $\operatorname{SL}(\mathcal{V})$ by Lemma 6.5.

Lemma 6.9. Let $F: F_0$ be a finite separable field extension where $\operatorname{char}(F) \neq 2$. Then the trace map $\operatorname{Tr}: F \to F_0$ is surjective, and the set $G = G_{F,F_0} = F^3 \times F^3 \times F_0$ endowed with the multiplication rule

$$(v_1, w_1, x_1)(v_2, w_2, x_2) = (v_1 + v_2, w_1 + w_1 + v_1 \wedge v_2, x_1 + x_2 + \operatorname{Tr}(v_1 \wedge w_2 + w_1 \wedge v_2))$$

defines a group. Let $H = \{g \in \operatorname{GL}_3(F) \mid \det(g) \in F_0^{\times}\}$ and let H^+ be the subgroup $H^+ = \{g \in \operatorname{GL}_3(F) \mid \det(g) \in (F^{\times})^2\}$. If $\operatorname{Aut}(G)$ induces on $G/\gamma_2(G)$, $\gamma_2(G)/\gamma_3(G)$ and $\gamma_3(G)$ subgroups A, B, C respectively, then $H \rtimes \operatorname{Aut}(F) \leqslant A$, $H^+ \rtimes \operatorname{Aut}(F) \leqslant B$ and $F_0^{\times} \leqslant C$. Moreover, G is a 4-orbit group and $G/\gamma_3(G)$ is a 3-orbit group.

Proof. If $\sigma_1, \ldots, \sigma_{|F:F_0|}$ are the F_0 -linear embeddings $F \to \overline{F}$ into the algebraic closure \overline{F} of F, then $\operatorname{Tr}(x) = \sum_{i=1}^{|F:F_0|} \sigma_i(x)$. Since $\sigma_1, \ldots, \sigma_n$ are linearly independent over F_0 , the F_0 -linear map Tr is nonzero, and hence is surjective. Let M be the kernel of the trace map $\operatorname{Tr}: F \to F_0$. The First Isomorphism Theorem gives $F^+/M \cong F_0^+$. We will show that G_{F,F_0} is a factor group of the group G_F in Lemma 6.7. Indeed, the map $\phi\colon G_F \to G_{F,F_0}\colon (v,w,x) \mapsto (v,w,x+M)$ preserves multiplication and has kernel $\{(0,0,x) \mid x \in M\} \leqslant \operatorname{Z}(G_F)$ where x+M is viewed as an element of F_0 via the isomorphism $F^+/M \cong F_0^+$. Therefore $G_F/M \cong G_{F,F_0}$.

The epimorphism ϕ maps $\gamma_i(G_F)$ to $\gamma_i(G_{F,F_0})$ for $1 \leq i \leq 3$. Further, if $g \in H$, then $g \wedge g \in H^+$ by Remark 6.8. The remaining claims follow from Lemma 6.7. \square

Subgroups $G_1, G_2 \leq \operatorname{Sym}(\Omega)$ with the same orbits on Ω are called *orbit-equivalent*.

Lemma 6.10. Let F be a division ring, and $C \leq F^{\times}$ a finite subgroup. Suppose that $A \leq \operatorname{Aut}(F)$ fixes C setwise and is orbit equivalent to $\operatorname{Aut}(C) \leq \operatorname{Sym}(C)$. Let $\mathcal{V} = F^d$ be a d-space over F. Then the set $G = C \times \mathcal{V}$ endowed with the multiplication rule $(\lambda, v)(\mu, w) = (\lambda \mu, \mu v + w)$ defines a group. Further, $\operatorname{Aut}(G)$ has one more orbit on G than $\operatorname{Aut}(C)$ has on C, i.e. $\omega(G) = \omega(C) + 1$.

Proof. We now show that the set $G = C \times \mathcal{V}$ is a group. Associativity holds as

$$((\lambda, v)(\mu, w))(\nu, x) = (\lambda \mu \nu, \mu \nu v + \nu w + x) = (\lambda, v)((\mu, w)(\nu, x))$$

holds for all (λ, v) , (μ, w) , $(\nu, x) \in C \times \mathcal{V}$. The identity element of G is (1, 0). Also (λ, v) has inverse $(\lambda^{-1}, -\lambda^{-1}v)$ and $(\lambda, v)^n = (\lambda^n, (\lambda^{n-1} + \cdots + \lambda + 1)v)$ for $n \ge 0$.

We now show that $M:=\{1\}\times \mathcal{V}$ is characteristic in G. If $\operatorname{char}(F)=0$, then this follows since elements of $G\setminus M$ have finite order (as $\lambda^n=1$ implies $(\lambda,v)^n=(1,0)$), while nontrivial elements of M have infinite order. If $\operatorname{char}(F)>0$, then C is contained in the multiplicative group of a finite field by the proof of [18, Theorem 6]. Hence M is a normal Sylow p-subgroup of G and thus characteristic in G. We next show that $\omega(G)=\omega(C)+\omega(M)-1$. Clearly $G/M\cong C$. First, $\operatorname{Aut}(M)$ has two orbits on M. Note that an invertible F-linear map $g\in\operatorname{Aut}_F(\mathcal{V})$ acts on G via $(\mu,v)^g=(\mu,vg)$. Hence $\operatorname{Aut}_F(\mathcal{V})$ has one nontrivial orbit on M, so $\omega(M)=2$. Also $\alpha\in A\leqslant\operatorname{Aut}(F)$ acts coordinatewise on $M\cong F^d$, and hence A acts on G via $(\lambda,v)^\alpha=(\lambda^\alpha,v^\alpha)$. Since A is orbit-equivalent to $\operatorname{Aut}(C)\leqslant\operatorname{Sym}(C)$, both A and $\operatorname{Aut}(C)$ have $\omega(C)$ orbits on C. These two types of automorphisms of G generate a subgroup of $\operatorname{Aut}(G)$ with $\omega(C)+1$ orbits. This proves that $\omega(G)=\omega(C)+1$ by Lemma 2.1.

Example 6.11. Let $F = \mathbb{H}$ be the real quaternions. Then \mathbb{H}^{\times} contains the quaternion subgroup $C = \{\pm 1, \pm i, \pm j, \pm k\} \cong Q_8$. Set $r = \frac{i+j}{\sqrt{2}}$, $s = \frac{1+i+j+k}{2}$ and $t = \frac{1+i}{\sqrt{2}}$. The binary octahedral group $\mathrm{BO} = \langle r, s, t \mid r^2 = s^3 = t^4 = rst \rangle$ satisfies $|\mathrm{BO}| = 48$, $\mathrm{Z(BO)} = \langle rst \rangle = \langle -1 \rangle$, and $C \leq \mathrm{BO}$. The subgroup A of $\mathrm{Aut}(F)$ comprising the inner automorphisms $F \to F \colon \lambda \mapsto \alpha^{-1}\lambda\alpha$, $\alpha \in \mathrm{BO}$, fixes $\mathrm{BO}' = Q_8 = C$ setwise. Furthermore $A \cong \mathrm{BO}/\langle -1 \rangle \cong S_4 \cong \mathrm{Aut}(Q_8)$, so A is orbit-equivalent to $\mathrm{Aut}(C)$. Lemma 6.10 shows that $G = C \times \mathbb{H}^d$ satisfies $\omega(G) = \omega(Q_8) + 1 = 4$ for $d \geqslant 1$. \square

Example 6.12. Let p, r be distinct primes. Set $e = r^{\ell}$ where $\ell \geqslant 1$. Then $p \nmid e$ and the cyclotomic polynomial $\Phi_e(t)$ is irreducible over the finite field \mathbb{F}_p precisely p has order $\deg(\Phi_e(t)) = \phi(e)$ modulo e by [25, Ex. 3.42, p. 124]; in this case \mathbb{F}_q where $q = p^{\phi(e)}$ is the splitting field of $\Phi_e(t)$ over \mathbb{F}_p . Hence $C_e \leqslant \mathbb{F}_q^{\times}$ and $\operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p) = C_{\phi(e)} \cong \operatorname{Aut}(C_e)$. Set $C = C_e$, $F = \mathbb{F}_q$ and $A = \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ in Lemma 6.10 noting that the orbits of $\operatorname{Aut}(C)$ and A on C are the elements of C of the same order. Thus the set $G = C_e \times \mathcal{V}$ is a group $C_e \ltimes \mathbb{F}_q^d$ with $\omega(G) = \omega(C_{r^\ell}) + 1 = \ell + 2$. Setting $\ell = 1$ gives e = r, $q = p^{r-1}$, $G = C_r \ltimes (C_p)^{d(r-1)}$, and $\omega(G) = 3$ as in line 2 of Table 1. \square

Lemma 6.13. If $p \neq r$ are prime and G is an $(\ell + 2)$ -orbit group with $|G| = p^m r^\ell$, $G' \cong \mathbb{C}_p^m$ and $G/G' \cong \mathbb{C}_{r^\ell}$, then G is isomorphic to the group in Example 6.12.

Proof. By assumption, $\operatorname{Aut}(G)$ has precisely $\ell+2$ orbits on G. As $|\operatorname{ord}(G)|=\ell+2$, these orbits are the sets $O_1,O_p,O_r,\ldots,O_{r^\ell}$ where $O_n=\{g\in G\mid |g|=n\}$. Let R be a Sylow r-subgroup, and let P=G' be the normal p-subgroup. Now $\operatorname{Z}(G)\cap P$ is trivial, otherwise G has at least $\ell+3$ orbits. Hence R acts fixed-point-freely on P. By Maschke's theorem $P=P_1\oplus\cdots\oplus P_d$ where each P_i is an irreducible R-module. The P_i must be pairwise isomorphic R-modules, otherwise $\operatorname{Aut}(G)$ has at least R0 orbits on R1, a contradiction. Let R2 orbits on R3, a contradiction. Let R4 orbits over R5. Thus R6 or R6 may be viewed as acting as a R6 decay are also scalar matrix over R7. Thus R6 orbits on R8 in Example 6.12.

Lemma 6.14. Let $q = 2^n$ and let $\operatorname{Tr}: \mathbb{F}_{q^2} \to \mathbb{F}_q: \mu \mapsto \mu + \mu^q$ denote the trace map.

- (a) If $\theta \in \operatorname{Gal}(\mathbb{F}_q/\mathbb{F}_2)$ has $|\theta| > 1$ odd, then $n \neq 2^{\ell}$ and the group $A(n,\theta)$ in Definition 3.2(a) is a 3-orbit 2-group of order q^2 with $A \cong B \cong \Gamma L_1(q)$.
- (b) If $n \ge 1$ and $\varepsilon \in \mathbb{F}_{q^2}^{\times}$ have order q+1, then $B(n) = B_{\varepsilon}(n)$ in Definition 3.2(b) is a 3-orbit 2-group of order q^3 with $A \cong \Gamma L_1(q^2)$ and $B \cong \Gamma L_1(q)$.
- (c) Let q = 8 and $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle \cong C_{63}$. The group $P = P(\varepsilon)$ in Definition 3.2(c) is a 3-orbit 2-group isomorphic to SmallGroup(2⁹,10 494 213) in MAGMA [3] with $A \cong C_7 \rtimes C_9 < \Gamma L_1(\mathbb{F}_{64})$ of order 63 and $B = \Gamma L_1(\mathbb{F}_8)$ of order 21.

Proof. (a) A simple calculation shows that $G := A(n, \theta)$ defined by Definition 3.2(a) is a group with $Z(G) = G' = \Phi(G) = \{0\} \times \mathbb{F}_q$, see also [7, Theorem (iv), p. 704]. The group $\Gamma L_1(q)$ acts faithfully on G via $(\mu, \zeta)^{(\alpha, \lambda)} = (\mu^{\alpha} \lambda, \zeta^{\alpha} \lambda \lambda^{\theta})$. Hence G is a 3-orbit 2-group of order q^2 with m = n and $\Gamma L_1(q) \leq A$ and $\Gamma L_1(q) \leq B$. No solvable group of $GL_n(2)$ properly contains $\Gamma L_1(2^n)$ by Theorem 4.1(a). Therefore $A = \Gamma L_1(2^n)$. It follows from [16] that $\Gamma L_1(q) \leq B$. Similar reasoning shows that $B = \Gamma L_1(2^n)$, so line 3 of Table 1 is valid.

- (b) The group $B_{\varepsilon}(n)$ appears in [7, Theorem (v)]. Since $B_{\varepsilon}(n) \cong B_{\varepsilon'}(n)$ when $\langle \varepsilon \rangle = \langle \varepsilon' \rangle$ has order q+1 we write B(n) instead of $B_{\varepsilon}(n)$. The multiplication rule in [7, Theorem (v)] can be rewritten as $(\mu_1, \zeta_1)(\mu_2, \zeta_2) = (\mu_1 + \mu_2, \zeta_1 + \zeta_2 + \text{Tr}(\varepsilon \mu_1 \mu_2^q))$ since $\varepsilon^q = \varepsilon^{-1}$. The group $\Gamma L_1(q^2)$ acts faithfully on B(n) via $(\mu, \zeta)^{(\alpha, \lambda)} = (\mu^{\alpha} \lambda, \zeta^{\alpha} \lambda \lambda^{\theta})$. Arguing as in part (a) we have $A \cong \Gamma L_1(q^2)$ and $B \cong \Gamma L_1(q)$ as in line 4 of Table 1.
- (c) The group $P(\varepsilon)$ appears in [7, Theorem (vi)]. If $\mathbb{F}_{q^2}^{\times} = \langle \varepsilon \rangle = \langle \varepsilon' \rangle$, then $P(\varepsilon) \cong P(\varepsilon')$ so we write P rather than $P(\varepsilon)$. The maps $\psi, \phi \colon P \to P$ defined by

$$(\alpha,\zeta)^{\psi} = (\varepsilon^3 \alpha, \varepsilon^9 \zeta)$$
 and $(\alpha,\zeta)^{\phi} = (\varepsilon \alpha^4, \zeta^4)$

can be shown to be homomorphisms of P that satisfy $\psi^{21} = \phi^9 = 1$, $\psi^{\phi} = \psi^4$ and $\psi^7 = \phi^3$. Hence $\psi, \phi \in \operatorname{Aut}(P)$ and $\langle \psi, \phi \rangle = \langle \psi^3, \phi \rangle = \operatorname{C}_7 \rtimes \operatorname{C}_9$. A computation with MAGMA [3] shows that $|\operatorname{Aut}(P)| = 2^{18} \cdot 63$. There are 2^{18} central automorphisms so $A \cong \operatorname{C}_{21}.\operatorname{C}_3 \cong \operatorname{C}_7 \rtimes \operatorname{C}_9 < \Gamma \operatorname{L}_1(\mathbb{F}_{64})$ and $B = \operatorname{C}_7 \rtimes \operatorname{C}_3 = \Gamma \operatorname{L}_1(\mathbb{F}_8)$ as $\mathbb{F}_8^{\times} = \langle \varepsilon^9 \rangle$. \square

7 Examples of 4-orbit groups

"A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one."

PAUL HALMOS

In this section, we consider the feasibility of classifying finite k-orbit groups for $k \leq 6$. The nonsolvable k-orbit groups have been classified for k = 4, 5, 6, see §1 for details. To assess the feasibility of classifying the solvable k-orbit groups for k = 4, 5, 6, we employ Halmos' strategy, and seek a large stock of examples, particularly when k = 4. Using MAGMA [3], we studied the 1265679 groups of order less than 2^{10} excluding 2^9 . Only 86 of these are nonabelian solvable 4-orbit groups! It appears that these groups belong to a small number of infinite (and finite) families. For brevity, we list (without proof of correctness) most of these below and in Table 2. Given the difficulty of computing automorphism groups, classifying the solvable 4-orbit groups may just be feasible. The most difficult case will be when G does not have four 'obvious' characteristic subset (e.g. determined by element orders or characteristic subgroups). For k = 5, 6 a complete classification may involve too many possibilities.

The $\operatorname{Aut}(G)$ orbit lengths for a 3-orbit group G follow from Theorem 1.1. They are $1, p^n - 1, p^n(r^m - 1)$ where p = r except for line 2 of Table 1, and the respective orbit-element orders are $1, p, p^2$ in lines 1, 3, 4, 5 and 1, p, r otherwise. For k-orbit groups with $k \geq 4$ the orbit lengths and orders are less obvious. Clearly the sum of the orbit lengths is |G| and some orders in $\operatorname{ord}(G)$ may be be duplicated, see Table 2. If G is a solvable 4-orbit group with precisely 4 characteristic subgroups, arranged as \circlearrowleft or $\overset{\bullet}{\mathbf{i}}$, then the four $\operatorname{Aut}(G)$ orbits are obvious. As minimal characteristic subgroups are elementary abelian, the abelian 4-orbit groups are $(C_{p^3})^m$; $(C_{p^2})^k \times (C_p)^{m-k}$ for $1 \leq k < m$ and $(C_{pr})^m$ where $p \neq r$ are prime and $m \geq 1$. The $\operatorname{Aut}(G)$ orbit lengths are obvious, and the orders are $1, p, p^2, p^3$; $1, p, p, p^2$ and 1, p, r, pr respectively.

By Lemma 2.1, a nonabelian solvable 4-orbit group G either has four characteristic subgroups $G > M_1 > M_2 > 1$ where $G', Z(G) \in \{M_1, M_2\}$, or is a UCS group (see [14]) with G > G' = Z(G) > 1. Hence a nilpotent 4-orbit group is a p-group of

Table 2: Examples of solvable (non-nilpotent) 4-orbit groups where p is a prime

G	G	Aut(G) orbit lengths	Orders	Conditions, action
3p	$C_p \rtimes C_3$	1, p-1, p, p	1, p, 3, 3	$p \equiv 1 \pmod{6}$, cubing
$3p^2$	$(\mathbf{C}_p)^2 \rtimes \mathbf{C}_3$	$1, p^2 - 1, p^2, p^2$	1, p, 3, 3	$p \equiv 1 \pmod{6}, \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$
$3p^2$	$(\mathbf{C}_p)^2 \rtimes \mathbf{C}_3$	$1, 2(p-1), (p-1)^2, 2p^2$	1,p,p,3	$p \equiv 1 \pmod{6}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
$3p^4$	$(\mathbf{C}_{p^2})^2 \rtimes \mathbf{C}_3$	$1, q - 1, q^2 - q, 2q^2; q = p^2$	$1, p, p^2, 3$	$p \equiv 2 \pmod{3}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$
$2p^4$	$(\mathbf{C}_{p^2})^2 \rtimes \mathbf{C}_2$	$1, p-1, p^4-p, p^4$	$1, p, p^2, 2$	$p \geqslant 2$, inversion
$2p^2$	$C_{p^2} \rtimes C_2$	$1, p-1, p^2-p, p^2$	$1, p, p^2, 2$	$p \geqslant 2$, inversion D_{2p^2}
$8p^2$	$(\mathbf{C}_p)^2 \rtimes Q_8$	$1, p^2 - 1, p^2, 6p^2$	1,p,2,4	$p \in \{3, 5, 7, 11, 23\}$

exponent dividing p^3 and class at most 3. We list below some infinite families of 4-orbit p-groups. Some 4-orbit solvable non-p-groups are listed in Table 2; the first three lines are UCS groups. Dornhoff [8] studies groups N for which $\operatorname{Aut}(N)$ has a solvable subgroup, say A, with four orbits on N (i.e. three orbits on $N \setminus \{1\}$). He lists N in [8, Theorems 1.1, 2.1] and constrains the structure of N in [8, Theorems 3.1, 4.1]. (No constraints are given in the case that N has exponent p.) The permutation groups $N \times A \leq \operatorname{Sym}(N)$ are not always guaranteed to have rank 4. Hence obtaining complete and irredundant list of 4-orbit groups G may be quite difficult, especially in the case that each $\operatorname{solvable}$ subgroup of $\operatorname{Aut}(G)$ has more than 4 orbits on G.

There are many infinite families of 4-orbit p-groups. First, if H is a nonabelian 3-orbit p-group, and E is an elementary abelian p-group, then $G = H \times E$ is a 4-orbit group. Next, if p is an odd prime and $q = p^b$, then the group $G = \mathbb{F}_q^3 \times \mathbb{F}_q^3 \times \mathbb{F}_q$ in Example 6.7 is a 4-orbit group of exponent p with $\operatorname{Aut}(G)$ orbit lengths $1, q - 1, q^4 - q, q^7 - q^4$. Also, the extraspecial 2-groups 2_{ε}^{1+2k} with $\varepsilon = \pm$ and $(k, \varepsilon) \neq (1, +)$ are 4-orbit groups of exponent 4 by Lemma 5.1. These have automorphism orbit lengths $1, 1, q(q - \varepsilon), q^2 + \varepsilon q - 2$ and element orders 1, 2, 2, 4, and $\operatorname{Aut}(G)$ is not solvable.

Nonabelian solvable 4-orbit groups that are not p-groups are listed in Table 2. This list omits some families such as the groups $(C_p)^{d(r-1)} \rtimes C_{r^2}$ with p, r distinct primes, and $\operatorname{ord}_{r^2}(p) = \phi(r^2) = r(r-1)$, see Example 6.12. See also [21, Theorem 4].

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