HAMILTONIAN STATIONARY LAGRANGIAN SURFACES WITH HARMONIC MEAN CURVATURE IN COMPLEX SPACE FORMS

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ABSTRACT. We completely classify Hamiltonian stationary Lagrangian surfaces with harmonic mean curvature and constant curvature in complex space forms.

1. INTRODUCTION

Let $\tilde{M}^n$ be a complex n-dimensional Kähler manifold with the complex structure J and the Kähler metric $\langle\, ,\rangle$. An n-dimensional submanifold M of $\tilde{M}^n$ is called Lagrangian if $\langle X, JY\rangle = 0$ for all tangent vector fields X and Y of M. A normal vector field ξ of a Lagrangian submanifold M is called a Hamiltonian variation if $\xi = J\nabla f$ for some compactly supported function f on M, where ∇ is the gradient on M. A Lagrangian submanifold is said to be Hamiltonian stationary if it is a critical point of the volume functional for all deformations with Hamiltonian variation vector fields. A Lagrangian submanifold M in $\tilde{M}^n$ is Hamiltonian stationary if and only if its mean curvature vector H satisfies

$$\mathrm{div}(JH) = 0 \tag{1.1}$$

on M (cf. [7]), where div is the divergence on M. This implies that any Lagrangian submanifold with parallel mean curvature is Hamiltonian stationary.

It is a fundamental and interesting problem to construct and classify Hamiltonian stationary Lagrangian submanifolds with non-parallel mean curvature in a specific Kähler manifold. A Hamiltonian stationary Lagrangian surface in a complex space form has constant mean curvature if and only if its mean curvature vector is parallel. Motivated by this fact, we investigate the case where the mean curvature is a non-constant harmonic function.

In this paper, we completely classify Hamiltonian stationary Lagrangian surfaces with non-constant harmonic mean curvature and constant curvature in complex space forms.

2. PRELIMINARIES

Let $\tilde{M}^n(4\epsilon)$ be a complete and simply connected complex space form of complex dimension n and constant holomorphic sectional curvature 4ϵ, that is, $\tilde{M}^n(4\epsilon)$ is the complex Euclidean space $\mathbb{C}^n$, the complex projective space $\mathbb{C}P^n(4\epsilon)$ or the complex hyperbolic space $\mathbb{C}H^n(4\epsilon)$ according as $\epsilon = 0$, $\epsilon > 0$ or $\epsilon < 0$.

Let M be a Lagrangian submanifold of $\tilde{M}^n(4\epsilon)$. We denote the Levi-Civita connections on M^n and $\tilde{M}^n(4\epsilon)$ by ∇ and $\tilde{\nabla}$, respectively. The Gauss and Weingarten

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formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for tangent vector fields X, Y and normal vector field ξ, where h, A and D are the second fundamental form, the shape operator and the normal connection. The mean curvature vector field H is defined by $H = (1/n)\operatorname{trace} h$. The function $|H|$ is called the mean curvature. We have (cf. [5])

$$D_X JY = J(\nabla_X Y), \tag{2.1}$$

$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle . \tag{2.2}$$

Denote by R the Riemann curvature tensor of ∇. Then the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle \\ &\quad + \epsilon(\langle Y,Z\rangle \langle X,W\rangle - \langle X,Z\rangle \langle Y,W\rangle),\end{aligned} \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where X, Y, Z, W are vectors tangent to M, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

3. Hamiltonian stationary Lagrangian surfaces

Let M be a Hamiltonian stationary Lagrangian surface in $\tilde{M}^2(4\epsilon)$, where $\epsilon \in \{-1,0,1\}$. Suppose that $H \neq 0$ everywhere. Denote by K the curvature of M. Let $\{e_1, e_2\}$ be a local orthonormal basis of M such that Je_1 is parallel to H. It follows from (2.2) that second fundamental form takes the form

$$\begin{aligned} h(e_1,e_1) &= (a-c)Je_1 + bJe_2, \\ h(e_1,e_2) &= bJe_1 + cJe_2, \\ h(e_2,e_2) &= cJe_1 - bJe_2 \end{aligned} \tag{3.1}$$

for some functions a, b and c. Putting $\omega_i^j(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle$, by (2.1) and (3.1) we have

$$\begin{aligned}(\bar{\nabla}_{e_1} h)(e_2,e_2) &= (e_1 c + 3b\omega_1^2(e_1))Je_1 - (e_1 b - 3c\omega_1^2(e_1))Je_2, \\ (\bar{\nabla}_{e_2} h)(e_1,e_2) &= \{e_2 b + (a-3c)\omega_1^2(e_2)\}Je_1 + (e_2 c + 3b\mu)Je_2, \\ (\bar{\nabla}_{e_1} h)(e_1,e_2) &= \{e_1 b + (a-3c)\omega_1^2(e_1)\}Je_1 + (e_1 c + 3b\omega_1^2(e_1)Je_2, \\ (\bar{\nabla}_{e_2} h)(e_1,e_1) &= \{e_2(a-c) - 3b\omega_1^2(e_2)\}Je_1 + \{e_2 b + (a-3c)\mu\}Je_2.\end{aligned}$$

Therefore, the equation (2.4) of Codazzi implies

$$e_1 c + 3b\omega_1^2(e_1) = e_2 b + (a-3c)\omega_1^2(e_2), \tag{3.2}$$

$$-e_1 b + 3c\omega_1^2(e_1) = e_2 c + 3b\omega_1^2(e_2), \tag{3.3}$$

$$e_2(a-c) - 3b\omega_1^2(e_2) = e_1 b + (a-3c)\omega_1^2(e_1). \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$e_2 a - a\omega_1^2(e_1) = 0. \tag{3.5}$$

The Hamiltonian stationary condition (1.1) is equivalent to

$$e_1 a + a\omega_1^2(e_2) = 0. \tag{3.6}$$

Using (3.5) and (3.6), we obtain

$$[a^{-1}e_1, a^{-1}e_2] = 0.$$

Therefore, there exists a local coordinate system $\{u, v\}$ such that $e_1 = a\partial_u$, $e_2 = a\partial_v$. Hence, the metric tensor is given by

$$g = a^{-2}(du^2 + dv^2), \tag{3.7}$$

which implies that

$$\omega_1^2(e_1) = a_v, \quad \omega_1^2(e_2) = -a_u, \tag{3.8}$$

$$K = -(a_u)^2 - (\alpha_v)^2 + a(a_{uu} + a_{vv}). \tag{3.9}$$

Equations (3.2) and (3.3) are rewritten as

$$ac_u + 3ba_v = ab_v - (a - 3c)a_u, \tag{3.10}$$

$$-ab_u + 3ca_v = ac_v - 3ba_u. \tag{3.11}$$

Put $G = \epsilon - K$. Then the equation (2.3) of Gauss and (3.1) yield that

$$G = 2b^2 - ac + 2c^2. \tag{3.12}$$

Remark 3.1. From (2.1), (3.8) and (3.9), it follows that if M has constant mean curvature, then its mean curvature vector is parallel and $K \equiv 0$.

4. Main theorem

Let $\mathbb{C}_1^3$ be the complex 3-space endowed with the inner product

$$\langle (z_1, z_2, z_3), (w_1, w_2, w_3) \rangle = \mathrm{Re}(-z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3).$$

Put $H_1^5(-1) = \{z \in \mathbb{C}_1^3 : \langle z, z \rangle = -1\}$. Let $\pi : H_1^5(-1) \subset \mathbb{C}_1^3 \to \mathbb{C}H^2(-4)$ be the Hopf fibration.

The main result of this paper is the following classification theorem.

Theorem 4.1. *Let M be a Hamiltonian stationary Lagrangian surface in $\tilde{M}^2(4\epsilon)$, where $\epsilon \in \{-1, 0, 1\}$. Suppose that H is nowhere vanishing. If M has non-constant harmonic mean curvature and constant curvature K, then $K = \epsilon = -1$ and M is locally congruent to the image of $\pi \circ \phi$, where $\phi : M \to H_1^5(-1) \subset \mathbb{C}_1^3$ is given by one of the following immersions*:

(1)

$$\phi(x, y) = \left(me^y + \frac{e^{-y} + 2im^2xe^y}{2m}, me^{ix+y}, \frac{e^{-y} + 2im^2xe^y}{2m} \right);$$

(2)

$$\phi(x, y) = \left(1 - \frac{i(1 + m^2)}{m^2x + y}, \frac{m\sqrt{1 + m^2}e^{ix}}{m^2x + y}, \frac{\sqrt{1 + m^2}e^{iy}}{m^2x + y} \right),$$

where m is a positive real number.

Proof. We shall use the same notation as in Section 3. Suppose that $|H|$ is a harmonic function on M. Then by (3.7) we have

$$a_{uu} + a_{vv} = 0. \tag{4.1}$$

Moreover, suppose that $|H|$ is non-constant and K is constant. Then, combining (3.9) and (4.1) shows that $K < 0$ and

$$a_u = \sqrt{-K}\cos\theta, \quad a_v = \sqrt{-K}\sin\theta. \tag{4.2}$$

for some function θ on M. Substituting (4.2) into (4.1), we have

$$-(\sin\theta)\theta_u + (\cos\theta)\theta_v = 0. \tag{4.3}$$

On the other hand, since $a_{uv} - a_{vu} = 0$ holds, by (4.2) we obtain

$$(\cos\theta)\theta_u + (\sin\theta)\theta_v = 0. \tag{4.4}$$

It follows (4.3) from (4.4) that $\theta_u = \theta_v = 0$, that is, θ is constant. Solving (4.2), we conclude that up to translations, a is given by

$$a = \sqrt{-K}\{(\cos\theta)u + (\sin\theta)v\}. \tag{4.5}$$

Case (i): $b = 0$ on an open subset $\mathcal{U}$. In this case, (3.10) and (3.11) reduce respectively to

$$(a - 3c)a_u + ac_u = 0, \tag{4.6}$$

$$3ca_v - ac_v = 0. \tag{4.7}$$

Differentiating (3.12) with respect to v, we have

$$ca_v + (a - 4c)c_v = 0. \tag{4.8}$$

Combining (4.7) and (4.8) gives

$$(a - c)c_v = 0.$$

If $c_v \neq 0$ on an open subset in $\mathcal{U}$, then $a = c$ on that open subset. From (3.12) we see that a is constant, which contradicts our assumption. Hence we have $c_v = 0$ on $\mathcal{U}$. Thus, by (4.7) we get

$$ca_v = 0.$$

Case (i.1): $c = 0$ on an open subset $\mathcal{U}_1 \subset \mathcal{U}$. In this case, from (3.12) we have $K = \epsilon = -1$. It follows from (4.6) that $a_u = 0$ on $\mathcal{U}_1$. Hence, using (4.5) and $K = -1$ yields that $a^2 = v^2$. Applying the coordinate transformation: $y = -\int v^{-1}dv$, we see that the metric tensor (3.7) becomes

$$g = m^2 e^{2y} du^2 + dy^2$$

for some positive constant m, and the second fundamental form satisfies

$$h(\partial_u, \partial_u) = J\partial_u, \quad h(\partial_u, \partial_y) = h(\partial_y, \partial_y) = 0.$$

Rewriting u with x and according to [2, page 3475], we conclude that $\mathcal{U}_1$ is congruent to the Lagrangian surface obtained from (1).

Case (i.2): $a_v = 0$ on an open subset $\mathcal{U}_2 \subset \mathcal{U}$. In this case, $a_u \neq 0$. Differentiating (3.12) with respect to u leads to

$$ca_u + (a - 4c)c_u = 0. \tag{4.9}$$

Eliminating c_u from (4.6) and (4.9) gives

$$(a - 2c)(a - 6c)a_u = 0.$$

If $a \neq 2c$ on an open set in $\mathcal{U}_2$, then $a = 6c$ and it follows from (3.12) that a is constant, which contradicts our assumption. Hence $a = 2c$, which together with (3.12) implies $K = \epsilon = -1$. Thus, by (4.5) we obtain $a^2 = u^2$. Applying the coordinate transformation: $x = (u + v)/2$ and $y = (u - v)/2$, we see that the metric tensor (3.7) becomes

$$g = \frac{2}{(x + y)^2}(dx^2 + dy^2)$$

and the second fundamental form satisfies

$$h(\partial_x,\partial_x)=J\partial_x,\quad h(\partial_x,\partial_y)=0,\quad h(\partial_y,\partial_y)=J\partial_y.$$

According to [3, page 124], $\mathcal{U}_2$ is congruent to the Lagrangian surface obtained from (2) with $m=1$.

Case (ii): $b\neq 0$ on an open subset $\mathcal{V}$. We put $A=a_u$ and $B=a_v$, which are constants. Differentiating (3.12) with respect to u and v, we obtain

$$b_u=\frac{cA+(a-4c)c_u}{4b}, \tag{4.10}$$

$$b_v=\frac{cB+(a-4c)c_v}{4b}. \tag{4.11}$$

Substituting (4.10) and (4.11) into (3.10) and (3.11) yields

$$4abc_u-a(a-4c)c_v=4b(3c-a)A-(12b^2-ac)B, \tag{4.12}$$

$$a(a-4c)c_u+4abc_v=(12b^2-ac)A+12bcB. \tag{4.13}$$

Solving (4.12) and (4.13) for c_u and c_v, we have

$$c_u=\frac{Af_1+Bf_2}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.14}$$

$$c_v=\frac{Af_3+Bf_4}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.15}$$

where

$$\begin{aligned}f_1&=-4ab^2-a^2c+4ac^2,\\ f_2&=-48b^3+16abc-48bc^2,\\ f_3&=4a^2b+48b^3-32abc+48bc^2,\\ f_4&=12ab^2-a^2c+4ac^2.\end{aligned}$$

Substituting (4.14) and (4.15) into (4.10) and (4.11) gives

$$b_u=\frac{Ag_1+Bg_2}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.16}$$

$$b_v=\frac{Ag_3+Bg_4}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.17}$$

where

$$\begin{aligned}g_1&=-a^2b+8abc,\\ g_2&=-12ab^2+4a^2c+48b^2c-28ac^2+48c^3,\\ g_3&=a^3+12ab^2-12a^2c-48b^2c+44ac^2-48c^3,\\ g_4&=3a^2b-8abc.\end{aligned}$$

Differentiating (4.14) and (4.15) with respect to v and u respectively, using (4.14)-(4.17), we get

$$c_{uv}=\frac{A^2h_1+ABh_2+B^2h_3}{a^2(a^2+16b^2-8ac+16c^2)^2}, \tag{4.18}$$

$$c_{vu}=\frac{A^2h_4+ABh_5+B^2h_6}{a^2(a^2+16b^2-8ac+16c^2)^2}, \tag{4.19}$$

where

$$\begin{aligned}
h_1 =& -12a^4b-144a^2b^3+160a^3bc+768ab^3c-656a^2bc^2+768abc^3,\\
h_2 =& -108a^3b^2-960ab^4+18a^4c+1168a^2b^2c+2304b^4c-252a^3c^2\\
&-3840ab^2c^2+1296a^2c^3+4608b^2c^3-2880ac^4+2304c^5,\\
h_3 =& -96a^2b^3+768b^5-384ab^3c+32a^2bc^2+1536b^3c^2-384abc^3+768bc^4,\\
h_4 =& -8a^4b-96a^2b^3-768b^5+128a^3bc+1152ab^3c-608a^2bc^2\\
&-1536b^3c^2+1152abc^3-768bc^4,\\
h_5 =& -92a^3b^2-192ab^4+18a^4c+784a^2b^2c+2304b^4c-252a^3c^2\\
&-3072ab^2c^2+1296a^2c^3+4608b^2c^3-2880ac^4+2304c^5,\\
h_6 =& -240a^2b^3+80a^3bc+768ab^3c-496a^2bc^2+768abc^3.
\end{aligned}$$

Since $c_{uv}-c_{vu}$=0 holds, from (4.18), (4.19) and $b\neq 0$, it follows that

$$A^2k_1+ABk_2+B^2k_3=0, \tag{4.20}$$

where

$$\begin{aligned}
k_1 =& a^4+12a^2b^2-192b^4-8a^3c+96ab^2c+12a^2c^2\\
&-384b^2c^2+96ac^3-192c^4,\\
k_2 =& 4a^3b+192ab^3-96a^2bc+192abc^2,\\
k_3 =& -36a^2b^2-192b^4+20a^3c+288ab^2c\\
&-132a^2c^2-384b^2c^2+288ac^3-192c^4.
\end{aligned}$$

Using (4.14)-(4.17), we get

$$(A^2k_1+ABk_2+B^2k_3)_u=\frac{4(A^3P_1+A^2BP_2+AB^2P_3+B^3P_4)}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.21}$$

$$(A^2k_1+ABk_2+B^2k_3)_v=\frac{8(A^3P_5+A^2BP_6+AB^2P_7+B^3P_8)}{a(a^2+16b^2-8ac+16c^2)}, \tag{4.22}$$

where

$$\begin{aligned}
P_1 =& -a^5-24a^3b^2-192ab^4+12a^4c+168a^2b^2c+384b^4c\\
&-56a^3c^2-576ab^2c^2+168a^2c^3+768b^2c^3-384ac^4+384c^5,\\
P_2 =& -2a^4b-72a^2b^3-1920b^5+24a^3bc+1344ab^3c\\
&-264a^2bc^2-3840b^3c^2+1344abc^3-1920bc^4,\\
P_3 =& 32a^3b^2+960ab^4-14a^4c-888a^2b^2c-2688b^4c+224a^3c^2\\
&+4032ab^2c^2-1272a^2c^3-5376b^2c^3+3072ac^4-2688c^5,\\
P_4 =& 24a^2b^3+1152b^5-8a^3bc-960ab^3c\\
&+216a^2bc^2+2304b^3c^2-960abc^3+1152bc^4,\\
P_5 =& a^4b+60a^2b^3+576b^5-32a^3bc-672ab^3c\\
&+252a^2bc^2+1152b^3c^2-672abc^3+576bc^4,\\
P_6 =& -a^5-38a^3b^2-192ab^4+24a^4c+444a^2b^2c+1344b^4c-218a^3c^2\\
&-2016ab^2c^2+924a^2c^3+2688b^2c^3-1824ac^4+1344c^5,
\end{aligned}$$

$$\begin{aligned}P_7 =& -4a^4b-180a^2b^3-960b^5+96a^3bc+1248ab^3c\\ &-564a^2bc^2-1920b^3c^2+1248abc^3-960bc^4,\\ P_8 =& 6a^3b^2-5a^4c-84a^2b^2c-192b^4c+50a^3c^2\\ &+288ab^2c^2-180a^2c^3-384b^2c^3+288ac^4-192c^5.\end{aligned}$$

Thus, it follows from (4.20), (4.21) and (4.22) that

$$A^3P_1+A^2BP_2+AB^2P_3+B^3P_4=0, \tag{4.23}$$

$$A^3P_5+A^2BP_6+AB^2P_7+B^3P_8=0. \tag{4.24}$$

Using a computer algebra system, we check that the resultant of the left-hand sides of (4.23) and (4.24) with respect to A is, up to a constant factor,

$$B^9(2b^2-ac+2c^2)^3(a^2+16b^2-8ac+16c^2)^7(a^2+48b^2-24ac+48c^2)^2Q, \tag{4.25}$$

where

$$\begin{aligned}Q =& 5a^6+120a^4b^2+720a^2b^4+2304b^6-60a^5c-720a^3b^2c\\ &-3456ab^4c+300a^4c^2+3168a^2b^2c^2+6912b^4c^2-1008a^3c^3\\ &-6912ab^2c^3+2448a^2c^4+6912b^2c^4-3456ac^5+2304c^6.\end{aligned}$$

Combining (3.12) and (4.25), we find that (4.25) can be simplified to

$$B^9G^3(a^2+8G)^7(a^2+24G)^2(5a^6+60a^4G+180a^2G^2+288G^3), \tag{4.26}$$

which must vanish on $\mathcal{V}$. Taking into account $\nabla a \neq 0$, we have

$$BG=0.$$

If $G\neq 0$, then $B=0$, which implies that (4.20) and (4.24) reduce respectively to

$$\begin{aligned}&a^4+12a^2b^2-192b^4-8a^3c+96ab^2c+12a^2c^2\\ &-384b^2c^2+96ac^3-192c^4=0,\end{aligned} \tag{4.27}$$

$$\begin{aligned}&a^4+60a^2b^2+576b^4-32a^3c-672ab^2c\\ &+252a^2c^2+1152b^2c^2-672ac^3+576c^4=0.\end{aligned} \tag{4.28}$$

Subtracting (4.27) from (4.28), we derive

$$24(2b^2-ac+2c^2)(a^2+16b^2-8ac+16c^2)=0. \tag{4.29}$$

Combining (3.12) and (4.29) implies

$$a^2+8G=0,$$

which contradicts $\nabla a\neq 0$. Hence $G=0$, that is, $K=\epsilon=-1$.

From (3.12) we see that $c\neq 0$. Changing the sign of e_1 if necessary, we may assume that $c>0$. Put

$$f=\frac{\sqrt{b^2+c^2-b\sqrt{b^2+c^2}}}{\sqrt{2}(b^2+c^2)},\quad k=\frac{\sqrt{b^2+c^2+b\sqrt{b^2+c^2}}}{\sqrt{2}(b^2+c^2)}, \tag{4.30}$$

which are non-zero and unequal everywhere. It follows from (3.12) and (4.30) that

$$a=\frac{\sqrt{f^2+k^2}}{fk},\quad b=\frac{k^2-f^2}{(f^2+k^2)^{\frac{3}{2}}},\quad c=\frac{2fk}{(f^2+k^2)^{\frac{3}{2}}}. \tag{4.31}$$

We make the following change of the basis:

$$\tilde{e}_1 = \frac{k}{\sqrt{f^2+k^2}}e_1 + \frac{f}{\sqrt{f^2+k^2}}e_2, \quad \tilde{e}_2 = \frac{f}{\sqrt{f^2+k^2}}e_1 - \frac{k}{\sqrt{f^2+k^2}}e_2. \tag{4.32}$$

Then, using (3.1), (4.31) and (4.32), we find

$$h(\tilde{e}_1, \tilde{e}_1) = f^{-1}J\tilde{e}_1, \quad h(\tilde{e}_1, \tilde{e}_2) = 0, \quad h(\tilde{e}_2, \tilde{e}_2) = k^{-1}J\tilde{e}_2.$$

Therefore, according to [3, Theorem 4.1], there exists a local coordinate system $\{x, y\}$ such that $\partial_x = f\tilde{e}_1$ and $\partial_y = k\tilde{e}_2$, where f and k satisfy

$$\frac{f_y}{k} = \frac{k_x}{f}, \quad \Big(\frac{f_y}{k}\Big)_y + \Big(\frac{k_x}{f}\Big)_x = fk. \tag{4.33}$$

The Hamiltonian stationary condition (1.1) is equivalent to (cf. [6])

$$\Big(\frac{k}{f}\Big)_x + \Big(\frac{f}{k}\Big)_y = 0. \tag{4.34}$$

By Theorem 3.1 of [1], up to translations and sign, the exact solutions of the over-determined PDE system (4.33)-(4.34) are given by

$$f = \lambda m \operatorname{csch}\Big(\frac{\lambda(m^2x+y)}{\sqrt{1+m^2}}\Big), \quad k = \lambda \operatorname{csch}\Big(\frac{\lambda(m^2x+y)}{\sqrt{1+m^2}}\Big); \tag{4.35}$$

$$f = \lambda m \sec\Big(\frac{\lambda(m^2x+y)}{\sqrt{1+m^2}}\Big), \quad k = \lambda \sec\Big(\frac{\lambda(m^2x+y)}{\sqrt{1+m^2}}\Big); \tag{4.36}$$

$$f = \frac{m\sqrt{1+m^2}}{m^2x+y}, \quad k = \frac{\sqrt{1+m^2}}{m^2x+y}, \tag{4.37}$$

where λ and m are positive real numbers.

We note that $f = mk$. By the first equation of (4.31) we have

$$a = \frac{\sqrt{1+m^2}}{mk}.$$

Furthermore, it follows from (4.32) that

$$\partial_u = \frac{1}{1+m^2}(\partial_x + m^2\partial_y), \quad \partial_v = \frac{m}{1+m^2}(\partial_x - \partial_y).$$

Since a_u and a_v are constant, the solutions (4.35) and (4.36) are excluded. Considering also that $a_u^2 + a_v^2 = 1$ and $f \neq k$, we see that f and g are given by (4.37) with $m \neq 1$. In [4, Section 7], it is proved that the corresponding surface is congruent to the Lagrangian surface obtained from case (2) with $m \neq 1$. This completes the proof. ∎

In view of Remark 3.1, it is interesting to investigate the possible values of K for Hamiltonian stationary Lagrangian surfaces with non-constant mean curvature and constant curvature K in $\tilde{M}^2(4\epsilon)$. There are no known examples of such surfaces with $K \neq \epsilon$. Hence, we pose the following problem:

Problem 4.1. *Does there exist a Hamiltonian stationary Lagrangian surface with non-constant mean curvature and constant curvature $K \neq \epsilon$ in $\tilde{M}^2(4\epsilon)$?*

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