

# On differentiability of reward functionals corresponding to Markovian randomized stopping times

Boy Schultz\*

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## Abstract

We conduct an investigation of the differentiability and continuity of reward functionals associated to Markovian randomized stopping times. Our focus is mostly on the differentiability, which is a crucial ingredient for a common approach to derive analytic expressions for the reward function.

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## 1. Introduction

A common problem in stochastic calculus is to derive analytic expressions for reward functionals

$$J_\tau(x) := \mathbb{E}_x[e^{-r\tau}g(X_\tau)],$$

with a given stopping time  $\tau$ . In the literature this problem has been discussed extensively for a variety processes  $X$ , functions  $g$  and stopping times  $\tau$ , see e.g. [8, 11, 12, 14]. Here, let us first consider the case where  $X = (X_t)_{t \in [0, \infty)}$  is a linear diffusion with values in an interval  $I$ ,  $r \geq 0$  is a discount factor and  $g$  a Hölder continuous payoff function. The solution to our problem is most well known if  $\tau$  is a first exit time of some open interval  $(a, b) \subset I$ , see e.g. [12, Section 9].

Due to their connection to subgame perfect Nash equilibria, so called *Markovian randomized stopping times*<sup>1</sup> are of particular interest in the context of games of stopping and were recently considered in [2, 3, 5, 6, 9, 16]. This class of stopping times will be formally introduced in Section 2 and contains, among others, all stopping times of the form

$$\tau = \inf \left\{ t \geq 0 : \int_0^t \psi(X_s) ds \geq E \text{ or } X_t \notin (a, b) \right\} \quad (1)$$

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\*Kiel University, Mathematical Department  
email: [schultz@math.uni-kiel.de](mailto:schultz@math.uni-kiel.de)

<sup>1</sup>The origin of the name is explained in [6, Section 3.2]

with  $a < b \in I$ , a Hölder continuous function  $\psi : I \rightarrow [0, \infty)$  and an exponentially distributed random variable  $E$ . Here,  $\psi$  plays the role of an infinitesimal stopping rate. Note that first exit times of open intervals are contained as a special case. For  $\tau$  given by (1) the corresponding reward function  $J_\tau$  is typically determined as the unique solution to the ordinary differential equation

$$(A - r - \psi)J_\tau = -g\psi \quad (2)$$

on  $(a, b)$  with boundary conditions

$$J_\tau(a) = g(a), \quad J_\tau(b) = g(b) \quad (3)$$

where  $A$  denotes the differential generator of the diffusion  $X$ , see [8, Theorem 13.16]. Provided that the coefficients of the differential operator  $A$  are sufficiently nice, this allows to determine an analytic expression for  $J_\tau$ .

However, if  $\psi$  is merely piecewise Hölder continuous  $J_\tau$  must no longer be  $C^2$  in the jumps  $\psi$ . Thus (2) cannot hold as a second order ODE on  $(a, b)$  in the classical sense. This situation is encountered in [3], where equilibrium stopping rates  $\psi$  are piecewise Hölder continuous by the nature of the problem. Generally, discontinuous stopping rates  $\psi$  could be considered in any kind of stopping game in which Markovian randomized stopping times are equilibrium candidates, such as [2] or [4].

Now let  $\psi$  only satisfy Hölder conditions on  $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ ,  $x_0 := a, x_n := b$ . General theory still provides that on each interval  $(x_{i-1}, x_i)$ ,  $i = 1, \dots, n$  the function  $J_\tau$  is twice continuously differentiable and satisfies (2), see Theorem 1, (ii). To salvage the approach and recover an analytic expression for  $J_\tau$  from (2) posed on  $(x_{i-1}, x_i)$ ,  $i = 1, \dots, n$  we need appropriate boundary conditions on each interval. Since (2) is a second order differential equation, as a rule of thumb, we need two boundary conditions for each interval  $(x_{i-1}, x_i)$ ,  $i = 1, \dots, n$ . Taking into account (3) we still need  $2n - 2$  conditions. It is comparatively simple to show that  $J_\tau$  is continuous on  $(a, b)$  which yields the first  $n - 1$  conditions, see Theorem 1, (i). The challenging part is to show that  $J_\tau$  is even differentiable on  $(a, b)$  which gives the remaining  $n - 1$  conditions, see Theorem 1, (iii).

In fact it is surprisingly simple to put that idea in more precise terms. For that let  $\tau^{(a,b)} := \inf\{t \geq 0 : X_t \notin (a, b)\}$  and assume  $\mathbb{E}_x[\tau^{(a,b)}] < \infty$ . Now, if  $\tilde{J}$  is  $C^0$  on  $[a, b]$ ,  $C^1$  on  $(a, b)$ ,  $C^2$  on  $\bigcup_{i \in \{1, \dots, n\}} (x_{i-1}, x_i)$  and satisfies (2) on  $(x_{i-1}, x_i)$ ,  $i = 1, \dots, n$  as well as (3), then  $\tilde{J} = J_\tau$ . This follows from the subsequent computation, which will be justified step by step below. We have

$$\begin{aligned} \tilde{J}(x) &= \mathbb{E}_x \left[ e^{-r\tau^{(a,b)} - \int_0^{\tau^{(a,b)}} \psi(X_t) dt} J(X_{\tau^{(a,b)}}) \right] \\ &\quad - \mathbb{E}_x \left[ \int_0^{\tau^{(a,b)}} \mathbb{1}_{\{X_t \notin \{x_1, \dots, x_{n-1}\}\}} e^{-rt - \int_0^t \psi(X_s) ds} (A - r - \psi) \tilde{J}(X_t) dt \right] \\ &= \mathbb{E}_x \left[ e^{-r\tau^{(a,b)} - \int_0^{\tau^{(a,b)}} \psi(X_t) dt} g(X_{\tau^{(a,b)}}) \right] + \mathbb{E}_x \left[ \int_0^{\tau^{(a,b)}} e^{-rt - \int_0^t \psi(X_s) ds} \psi(X_t) g(X_t) dt \right] \\ &= J_\tau(x). \end{aligned}$$

In the first step we use a version of Dynkin's formula based on the generalized Itô formula [15, Chapter IV, Theorem 71]. This formula requires that  $\tilde{J}$  is  $C^1$  with absolutely continuous derivative. Here, the absolute continuity of  $\tilde{J}$  is implied by the piecewise  $C^2$  assumption. Note that Dynkin's formula also involves the assumption  $\mathbb{E}_x[\tau^{(a,b)}] < \infty$ . Besides that we use that  $\mathbb{E}_x[\int_0^t \mathbb{1}_{\{X_s \in \{x_1, \dots, x_{n-1}\}\}} ds] = 0$  for all  $t \geq 0$ . In the second step we apply (2) and (3). The last equality follows from (19) which will be shown later on.

The main contribution of the present paper is a proof of the  $C^1$  property of  $J_\tau$  with  $\tau$  given by (1), but with  $\psi$  being merely piecewise Hölder continuous, which is a crucial part of the previous argument. Additionally, we show continuity of  $J_\tau$  for general Markovian randomized stopping times  $\tau$  under very mild conditions. Both results are stated in Theorem 1 next to a known result which provides sufficient conditions for  $J_\tau$  to be  $C^2$ .

## 2. General framework

We consider a regular linear Itô diffusion  $X = (X_t)_{t \in [0, \infty)}$  taking values in an interval  $I \subset \mathbb{R}$  and defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$  satisfying the usual hypotheses. Generally we assume that the behavior of  $X$  in the interior  $I^\circ$  of  $I$  is governed by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in I^\circ \quad (4)$$

with an  $(\mathcal{F}_t)_{t \in [0, \infty)}$ -adapted, real valued standard Brownian motion  $W = (W_t)_{t \in [0, \infty)}$  and Lipschitz continuous coefficients  $\mu : I \rightarrow \mathbb{R}$ ,  $\sigma : I \rightarrow (0, \infty)$ . A jointly continuous version of the local time process of  $X$  at  $y \in I$  will be denoted by  $(l_t^y)_{t \in [0, \infty)}$ . The existence of such a version follows e.g. from [17, Theorem (44.2)]. Let  $E \sim \text{Exp}(1)$  be a random variable on  $(\Omega, \mathcal{F}, \mathbb{P})$  which is independent from  $X$ . We set  $\mathcal{F}_\infty^X := \sigma(X_t : t \geq 0)$  and denote the canonical shift operator associated to  $X$  by  $\theta$ . If  $Y$  is any random variable on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$  with values in some metric space, we denote its distribution by  $\mathbb{P}^Y$ . As usual, we write  $\mathbb{P}_x$  for the conditional distribution of  $\mathbb{P}$  given  $X_0 = x$  and  $\mathbb{E}_x$  for the corresponding expectation operator. We write  $\mathbb{E}[\dots]$  for the function  $I \ni x \mapsto \mathbb{E}_x[\dots]$ .

For open or closed  $D \subset I$  we denote the Borel  $\sigma$ -algebra on  $D$  by  $\mathcal{B}(D)$ , the space of Radon measures on  $D$  by  $\text{RM}(D)$  and set  $\text{B}(D) := \{f : D \rightarrow \mathbb{R} : f \text{ is bounded and measurable}\}$ ,  $C^0(D) := \{f : D \rightarrow \mathbb{R} : f \text{ is continuous}\}$ ,  $C^m(D) := \{f : D \rightarrow \mathbb{R} : f \text{ is } m \text{ times continuously differentiable}\}$ . If  $f : D \rightarrow [0, \infty)$  is Lebesgue integrable, we denote the measure that maps  $\Gamma \in \mathcal{B}(D)$  to  $\int_\Gamma f dx$  by  $f dx$ . For  $D' \subset D \subset I$  and functions  $f : D \rightarrow \mathbb{R}$  or measures  $\lambda \in \text{RM}(D)$  we denote the restrictions of  $f$  and  $\lambda$  to  $D'$  by  $f|_{D'}$  and  $\lambda|_{D'}$  respectively. We call  $f : D \rightarrow \mathbb{R}$  piecewise Hölder continuous, if there are  $\inf D = x_0 < x_1 < \dots < x_n = \sup D$ ,  $n \in \mathbb{N}$  such that  $f|_{D \cap (x_{i-1}, x_i)}$  is Hölder continuous (possibly with different exponents) for each  $i \in \{1, \dots, n\}$ .

We denote the first exit time from open  $D \subset I$  by

$$\tau^D := \inf\{t \geq 0 : X_t \notin D\}$$

For  $D \subset I$  open (in  $I$ ) and  $\lambda \in \text{RM}(D)$ . We set the additive functional  $A^{D, \lambda} = (A_t^{D, \lambda})_{t \in [0, \infty)}$  generated by  $X, D$  and  $\lambda$  to be given by

$$A_t^{D, \lambda}(\omega) := \int_D l_t^y(\omega) \lambda(dy) + \infty \mathbb{1}_{\{\tau^D(\omega) \leq t\}}, \quad t \geq 0. \quad (5)$$

We define the *Markovian randomized (stopping) time* generated by  $D$ ,  $\lambda$  and  $E$  as

$$\tau^{D,\lambda} := \inf\{t \geq 0 : A_t^{D,\lambda} \geq E\}. \quad (6)$$

The space of all Markovian randomized stopping times based on the random variable  $E$  is denoted by

$$\mathcal{Z} := \mathcal{Z}(E) := \{\tau^{D,\lambda} : D \subset I \text{ open (in } I), \lambda \in \text{RM}(D)\}.$$

Markovian randomized stopping times  $\tau \in \mathcal{Z}$  feature a Markov property of the following type: With a natural extension of the shift operator  $\theta$  it holds that

$$\mathbb{1}_{\{\tau \geq \sigma\}}\tau = \mathbb{1}_{\{\tau \geq \sigma\}}\theta_\sigma \circ \tau + \sigma$$

in distribution for all  $\mathcal{F}_\infty^X$ -measurable  $\sigma$ , see [3]. More detailed discussions can be found in [6, Section 3].

We fix a measurable function  $g : I \rightarrow \mathbb{R}$  and a constant  $r \geq 0$ . Based on that define the *reward functional* corresponding to  $\tau \in \mathcal{Z}$  via

$$J_\tau(x) := J(x, \tau) := \mathbb{E}_x[e^{-r\tau}g(X_\tau)], \quad x \in I$$

whenever the right hand side exists possibly taking value  $\pm\infty$ . Note that

$$J_{\tau^{D,\lambda}}(x) = g(x)$$

for all  $\tau^{D,\lambda} \in \mathcal{Z}$  and all  $x \in I \setminus D$ , so the behavior of  $J_{\tau^{D,\lambda}}$  on  $I \setminus D$  is predetermined by  $g$ . Thus, in the following we are only concerned with differentiability on  $D$  and with continuity on the closure  $\overline{D}$  of  $D$  (in  $I$ ).

### 3. Main result

**Theorem 1.** Let  $\tau = \tau^{D,\lambda} \in \mathcal{Z}$ ,  $A := A^{D,\lambda}$  and  $\inf I \leq a < b \leq \sup I$  such that  $(a, b) \subset D$  and  $\mathbb{E}_x[\tau^{(a,b)}] < \infty$  for all  $x \in (a, b)$ .

- (i) If  $g$  is bounded on  $(a, b)$ ,  $\lambda((a, b)) < \infty$  and  $J_\tau(y) \in \mathbb{R}$  for all  $y \in I \cap \{a, b\}$  then  $J_\tau|_{[a,b]} \in C^0([a, b])$ .
- (ii) Let  $\psi : (a, b) \rightarrow [0, \infty)$  be Hölder continuous. Suppose that  $\lambda|_{(a,b)} = \frac{\psi}{\sigma}dx$ , that  $g$  is Hölder continuous on  $(a, b)$  and that  $J_\tau(y) \in \mathbb{R}$  for all  $y \in I \cap \{a, b\}$ . Then  $J_\tau|_{(a,b)} \in C^2((a, b))$ .
- (iii) Let  $\psi : (a, b) \rightarrow [0, \infty)$  be piecewise Hölder continuous. Suppose that  $\lambda|_{(a,b)} = \frac{\psi}{\sigma}dx$ , that  $g$  is Hölder continuous on  $(a, b)$  and that  $J_\tau(y) \in \mathbb{R}$  for all  $y \in I \cap \{a, b\}$ . Then  $J_\tau|_{(a,b)} \in C^1((a, b))$ .

**Remark.** In fact our main contribution is (iii). The other two statements are mostly for completeness, with (i) being less involved on our part and (ii) a direct consequence of a result from [8]. Continuity of the functional  $J$  as well as related functionals including those treated in

Lemma 2 has been studied extensively, e.g. by [7, 8, 10]. Assembling suitable results from the various sources already leads us most of the way towards (i).

The proof of (iii) consists of two major parts. The first one is to show that  $J_\tau$  is in the domain of the characteristic operator of  $X$  (defined according to [7]). The second part is to put this together with (i) and (ii) to prove the claim. The first part follows a line of arguments from [7, 8]. Lemma 3, Lemma 4 and Lemma 5 are adaptations of theorems from [7, 8] that accommodate for our deviating general assumptions, in particular that we allow discontinuity of  $\psi$ .

In the following we use the terms infinitesimal operator, weak infinitesimal operator, characteristic operator, resolvent, potential, standard process, continuous homogeneous multiplicative functional, (standardized)  $\alpha$ -subprocess, canonical diffusion and weak convergence according to the definitions in [7]. We merely note that for open  $D \subset I$  this means weak convergence of a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $B(D)$  to some  $f \in B(D)$  is defined via the property  $\int f_n d\mu \rightarrow \int f d\mu$  for all finite measures  $\mu$  on  $D$ . We denote the domain of a (weak) infinitesimal operator  $A$  by  $D_A$ . Similarly, for the characteristic operator  $\mathfrak{A}$  of a Markov process with values in  $J \subset I$  we denote the set of all measurable functions such that  $\mathfrak{A}f(x)$  exists for a fixed  $x \in J$  by  $D_{\mathfrak{A}}(x)$ . Moreover, the domain of  $\mathfrak{A}$ , i.e. the set of all measurable functions such that  $\mathfrak{A}f(y)$  exists for all  $y \in J$ , is denoted by  $D_{\mathfrak{A}}$ . We will also use the notation for Markov processes from [7, Subsection 3.1]. There, a Markov process is denoted as a quadruple  $Y = (Y_t, \zeta, \mathcal{M}_t, Q_x) = ((Y_t)_{t \in [0, \infty)}, \zeta, (\mathcal{M}_t)_{t \in [0, \infty)}, (Q_x)_{x \in I})$  consisting of paths  $(Y_t)_{t \in [0, \infty)}$ , lifetime  $\zeta$ , filtration at lifetime  $(\mathcal{M}_t)_{t \in [0, \infty)}$ , i.e.  $\mathcal{M}_t$  is a  $\sigma$ -algebra on  $\{\omega \in \Omega : \zeta(\omega) > t\}$  with  $A \in \mathcal{M}_s$  implying  $A \cap \{\zeta > s'\} \in \mathcal{M}_{s'}$  for all  $s \leq s'$  and transition probabilities  $Q_x$ ,  $x \in I$  with  $I$  denoting the state space of  $Y$ . In particular, the process  $X$  from Section 2 is a standard process which reads  $X = (X_t, \infty, \mathcal{F}_t, \mathbb{P}_x)$ .

The next lemma is concerned with the continuity of two expectations which are closely related to  $J_\tau$  from Theorem 1. It will not only lead us most of the way towards the proof of Theorem 1 (i) but also be used to show that Lemma 3 can be applied in the proofs of Lemma 4 and Lemma 5.

**Lemma 2.** Let  $\tau^{D, \lambda} \in \mathcal{Z}$ ,  $A := A^{D, \lambda}$  the corresponding additive functional given by (5),  $\tilde{\lambda} \in \text{RM}(D)$ ,  $\tilde{A} := A^{D, \tilde{\lambda}}$  the functional corresponding to  $D, \lambda$  given by (5),  $J \subset D$  an open interval,  $\tau := \tau^J$  and  $h : I \rightarrow \mathbb{R}$  a bounded measurable function. Additionally, we set  $\tilde{A}_{t-} := \lim_{s \nearrow t} \tilde{A}_s$  with convention  $\tilde{A}_{0-} := 0$ .

(i) If  $\tau < \infty$   $\mathbb{P}_x$ -a.s. for all  $x \in J$ , then the function  $f : I \rightarrow \mathbb{R}$ ,

$$x \mapsto \mathbb{E}_x[e^{-A_\tau} h(X_\tau)]$$

is continuous on the closure  $\bar{J}$  of  $J$  (in  $I$ ).

(ii) If  $\mathbb{E}_x[\tau] < \infty$  for all  $x \in J$ ,  $\mu|_J, \sigma|_J$  are bounded and  $\tilde{\lambda}(J) < \infty$ , then the function  $F : I \rightarrow \mathbb{R}$ ,

$$x \mapsto \mathbb{E}_x \left[ \int_0^\tau e^{-A_t} h(X_t) d\tilde{A}_t \right]$$

is continuous on  $I$ . We use the convention  $\int_0^\tau := \int_{[0, \tau)}$ .

*Proof.* (i) Since  $h$  is bounded,  $f$  is clearly well defined. Set  $J := (a, b)$  for  $a < b \in [-\infty, \infty]$ . Note that  $a, b$  are not necessarily in  $I$ .

For  $y \in \mathbb{R}$  let  $\eta_y := \inf\{t \geq 0 : X_t = y\}$ , with convention  $\inf \emptyset := \infty$ . The strong Markov property yields

$$\begin{aligned} f(x) &= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-(\theta_{\eta_y} \circ A_\tau + A_{\eta_y})} h(X_{\theta_{\eta_y} \circ A_\tau + A_{\eta_y}})] + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}} \mathbb{E}_x[e^{-\theta_{\eta_y} \circ A_\tau} h(X_{\theta_{\eta_y} \circ A_\tau + A_{\eta_y}}) | \mathcal{F}_{\eta_y}]] + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}} \mathbb{E}_{X_{\eta_y}}[e^{-A_\tau} h(X_\tau)]] + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}}] \mathbb{E}_y[e^{-A_\tau} h(X_\tau)] + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] \\ &= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}}] f(y) + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] \end{aligned}$$

for all  $x, y \in I$ . Thus

$$\begin{aligned} |f(x) - f(y)| &= |\mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}}] f(y) + \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} e^{-A_\tau} h(X_\tau)] - f(y)| \\ &\leq |f(y)| (1 - \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}}]) + \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}}] \end{aligned} \quad (7)$$

for all  $x, y \in I$ . By symmetry it suffices to show that both summands on the right hand side of (7) go to 0, whenever  $J \ni x \rightarrow y \in \bar{J}$ .

We start with the first summand. Since  $X$  is a regular diffusion,  $\lim_{J \ni x \rightarrow y} \mathbb{P}_x(\eta_y < \varepsilon) = 1$  for all  $\varepsilon > 0$ , cf. [10, Section 3.3, 10c)]. By continuity of paths  $\mathbb{P}_x(\tau > 0) = 1$  for all  $x \in J$ . By construction  $t \mapsto A_t$  is continuous on  $[0, \tau^D)$  with  $A_0 = 0$ . Putting that together we find that  $\mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} e^{-A_{\eta_y}}] \xrightarrow{J \ni x \rightarrow y} 0$ .

We treat the second summand next. If  $y \in \{a, b\}$ , then  $\mathbb{E}_y[\mathbb{1}_{\{\eta_y > \tau\}}] = 0$  and we are done. For  $y \in (a, b)$  we have  $\mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}}] = \mathbb{P}_x(\eta_y > \eta_b)$  if  $y \leq x \leq b$  and  $\mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}}] = \mathbb{P}_x(\eta_y > \eta_a)$  if  $a \leq x \leq y$ . By [10, Section 4.4] the functions  $[y, b] \ni x \mapsto \mathbb{P}_x(\eta_y > \eta_b)$  and  $[a, y] \ni x \mapsto \mathbb{P}_x(\eta_y > \eta_a)$  are continuous with  $\mathbb{P}_x(\eta_x > \eta_b) = 0$  and  $\mathbb{P}_x(\eta_x > \eta_a) = 0$ , which proves the claim.

(ii) Let  $\psi : J \rightarrow \mathbb{R}$ ,  $x \mapsto \tilde{\lambda}((a, x])$  and  $\Psi : J \rightarrow \mathbb{R}$ ,  $x \mapsto \int_0^x \psi(y) dy$ . Note that  $\tilde{\lambda}(J) < \infty$  implies that  $\psi$  is bounded and  $\Psi$  is continuous and bounded. For  $x \in J$  and  $t < \tau^D$  the Itô-Meyer formula, cf. [15, Chapter 3, Theorem 70], provides

$$\begin{aligned} \tilde{A}_t &= \int_J l_t^y \tilde{\lambda}(dy) = 2 \left( \Psi(X_t) - \Psi(x) - \int_0^t \psi(X_s) dX_s \right) \\ &= 2\Psi(X_t) - 2\Psi(x) - \int_0^t 2\psi(X_s) \mu(X_s) ds - \int_0^t 2\psi(X_s) \sigma(X_s) dW_s. \end{aligned}$$

$\mathbb{P}_x$ -a.s. By continuity of the paths of  $A$  on  $[0, \tau^D)$  in particular

$$\tilde{A}_{t-} = 2\Psi(X_t) - 2\Psi(x) - \int_0^t 2\psi(X_s) \mu(X_s) ds - \int_0^t 2\psi(X_s) \sigma(X_s) dW_s \quad (8)$$

$\mathbb{P}_x$ -a.s. for all  $x \in J$  and all  $t \leq \tau^D$ . We find that

$$\begin{aligned} F(x) &\leq \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x \left[ \int_0^\tau d\tilde{A}_\tau \right] = \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\tilde{A}_{\tau-}] \\ &= \left( \sup_{z \in I} |h(z)| \right) \left( \mathbb{E}_x[2\Psi(X_\tau)] - 2\Psi(x) - 2\mathbb{E}_x \left[ \int_0^\tau 2\psi(X_s) \mu(X_s) ds \right] - \mathbb{E}_x \left[ \int_0^\tau 2\psi(X_s) \sigma(X_s) dW_s \right] \right) \end{aligned}$$

for all  $x \in I$ . Since  $h, \psi, \Psi, \mu|_J$  and  $\sigma|_J$  are bounded and  $\mathbb{E}_x[\tau] < \infty$  the function  $F$  is well defined.

For all  $x, y \in I$  the strong Markov property yields

$$\begin{aligned}
F(x) &= \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \int_0^{\eta_y} e^{-A_t} h(X_t) d\tilde{A}_t \right] + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \mathbb{E}_x \left[ \int_{\theta_{\eta_y \circ \eta_y}}^{\theta_{\eta_y \circ \tau}} e^{-A_t} h(X_t) d\tilde{A}_t \middle| \mathcal{F}_{\eta_y} \right] \right] \\
&\quad + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y > \tau\}} \int_0^\tau e^{-A_t} h(X_t) d\tilde{A}_t \right] \\
&= \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \int_0^{\eta_y} e^{-A_t} h(X_t) d\tilde{A}_t \right] + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \mathbb{E}_{X_{\eta_y}} \left[ \int_0^{\eta_y} e^{-A_t} h(X_t) d\tilde{A}_t \right] \right] \\
&\quad + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y > \tau\}} \int_0^\tau e^{-A_t} h(X_t) d\tilde{A}_t \right] \\
&= \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}}] F(y) + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \int_0^{\eta_y} e^{-A_t} h(X_t) d\tilde{A}_t \right] \\
&\quad + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y > \tau\}} \int_0^\tau e^{-A_t} h(X_t) d\tilde{A}_t \right]. \tag{9}
\end{aligned}$$

Applying (8) to (9) we obtain

$$\begin{aligned}
&|F(x) - F(y)| \\
&= \left| F(y)(1 - \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}}]) + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y \leq \tau\}} \int_0^{\eta_y} e^{-A_t} h(X_t) d\tilde{A}_t \right] + \mathbb{E}_x \left[ \mathbb{1}_{\{\eta_y > \tau\}} \int_0^\tau e^{-A_t} h(X_t) d\tilde{A}_t \right] \right| \\
&\leq |F(y)|(1 - \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}}]) + \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\mathbb{1}_{\{\eta_y \leq \tau\}} \tilde{A}_{\eta_y-}] + \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\mathbb{1}_{\{\eta_y > \tau\}} \tilde{A}_{\tau-}] \\
&\leq |F(y)|(1 - \mathbb{P}_x(\eta_y \leq \tau)) + \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\tilde{A}_{(\eta_y \wedge \tau)-}] + \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\tilde{A}_{(\eta_y \wedge \tau)-}] \\
&= |F(y)|(1 - \mathbb{P}_x(\eta_y \leq \tau)) + 2 \left( \sup_{z \in I} |h(z)| \right) (2\mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})] - 2\Psi(x)) \\
&\quad - 2 \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x \left[ \int_0^{\eta_y \wedge \tau} 2\psi(X_s) \mu(X_s) ds - \int_0^{\eta_y \wedge \tau} 2\psi(X_s) \sigma(X_s) dW_s \right] \\
&\leq |F(y)|(1 - \mathbb{P}_x(\eta_y \leq \tau)) + 4 \left( \sup_{z \in I} |h(z)| \right) |\mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})] - \Psi(x)| \\
&\quad + 4 \left( \sup_{z \in I} |h(z)| \right) \mathbb{E}_x[\eta_y \wedge \tau] \left( \sup_{z \in J} |\psi(z) \mu(z)| \right). \tag{10}
\end{aligned}$$

for all  $x \in J$  and all  $y \in I$ . For the last step note that  $Y = (Y_t)_{t \in [0, \infty)}$ ,  $Y_t := \int_0^t \mathbb{1}_{\{s < \eta_y \wedge \tau\}} \psi(X_s) \sigma(X_s) dW_s$ ,  $t \geq 0$  is a true martingale. Thus, invoking  $\mathbb{E}_x[\tau] < \infty$  the expectation vanishes by the optional sampling theorem.

By symmetry it suffices to show that all the summands on the right hand side of (10) go to 0, whenever  $J \ni x \rightarrow y \in \bar{J}$ . The first summand has already been dealt with.

We denote the scale function of  $X$  by  $s$ . We have

$$\begin{aligned} \mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})] &= \begin{cases} \Psi(y)\mathbb{P}_x(X_{\eta_y \wedge \tau} = y) + \Psi(a)\mathbb{P}_x(X_{\eta_y \wedge \tau} = a), & \text{if } a \leq x \leq y, \\ \Psi(y)\mathbb{P}_x(X_{\eta_y \wedge \tau} = y) + \Psi(b)\mathbb{P}_x(X_{\eta_y \wedge \tau} = b), & \text{if } y \leq x \leq b \end{cases} \\ &= \begin{cases} \Psi(y)\frac{s(x)-s(a)}{s(y)-s(a)} + \Psi(a)\frac{s(y)-s(x)}{s(y)-s(a)}, & \text{if } a \leq x \leq y \\ \Psi(y)\frac{s(b)-s(x)}{s(b)-s(y)} + \Psi(b)\frac{s(x)-s(y)}{s(b)-s(y)}, & \text{if } y \leq x \leq b \end{cases} \end{aligned}$$

for all  $x \in \bar{J}$ .  $s$  is continuous, cf. [10, Section 4.2]. Thus, the function  $\bar{J} \ni x \mapsto \mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})]$  is continuous with  $\lim_{J \ni x \rightarrow y} \mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})] = \Psi(y)$ , i.e.  $\lim_{J \ni x \rightarrow y} \mathbb{E}_x[\Psi(X_{\eta_y \wedge \tau})] - \Psi(x) = 0$  by continuity of  $\Psi$ . This means the second summand in (10) vanishes for  $J \ni x \rightarrow y$ .

For  $x \in \bar{J}$  and  $y \in J$  we either have  $\mathbb{E}_x[\eta_y \wedge \tau] = \mathbb{E}_x[\eta_y \wedge \eta_a]$  or  $\mathbb{E}_x[\eta_y \wedge \tau] = \mathbb{E}_x[\eta_y \wedge \eta_b]$  depending on whether  $y \leq x \leq b$  or  $a \leq x \leq y$ . The mappings  $[y, b] \ni x \mapsto \mathbb{E}_x[\eta_y \wedge \eta_a]$  and  $[a, y] \ni x \mapsto \mathbb{E}_x[\eta_y \wedge \eta_a]$  are continuous with  $\mathbb{E}_y[\eta_y \wedge \eta_a] = 0$  and  $\mathbb{E}_y[\eta_y \wedge \eta_a]$  by [10, Section 4.2, 20a), 20b)] since  $X$  is a regular diffusion. Thus the second and third summand vanish for  $y \rightarrow x$  which proves the claim.  $\square$

**Lemma 3.** Let  $Y = (Y_t)_{t \in [0, \infty)}$  be a standard process with continuous paths and values in an interval  $J \subset \mathbb{R}$ . Let  $A$  be the weak infinitesimal operator of  $Y$  and  $\psi : J \rightarrow [0, \infty)$  a bounded measurable function with the property  $\mathbb{E}[\psi(Y_t)] \rightarrow \psi$  weakly for  $t \searrow 0$ . We set  $\varphi := (\varphi_t)_{t \in [0, \infty)}$ ,

$$\varphi_t := \int_0^t \psi(Y_s) ds, \quad t \geq 0.$$

Let  $\tilde{A}$  denote the weak infinitesimal operator of the transition function  $\tilde{\mathbb{P}}$  given by

$$\tilde{\mathbb{P}}(t, x, \Gamma) := \mathbb{E}_x[\mathbb{1}_\Gamma(Y_t)e^{-\varphi_t}], \quad t \geq 0, x \in J, \Gamma \in \mathcal{B}(J).$$

Now

$$\mathcal{D}_{\tilde{A}} \cap C^0(J) \cap \mathcal{B}(J) = \mathcal{D}_A \cap C^0(J) \cap \mathcal{B}(J)$$

and

$$\tilde{A}f = Af - \psi f$$

for all  $f \in \mathcal{D}_{\tilde{A}} \cap C^0(J) \cap \mathcal{B}(J)$ .

**Remark.** This lemma is an adaptation of [7, Theorem 9.7] in the weak formulation outlined in subsequent remark, cf. [7, p. 299]. The original weak formulation of the theorem yields  $\mathcal{D}_{\tilde{A}} = \mathcal{D}_A$  provided that the additional condition (9.62) from [7, p. 299] is satisfied.

*Proof.* Clearly  $\sup_{x \in J} \mathbb{E}_x[\varphi_t] \rightarrow 0$  for  $t \searrow 0$ . Thus the weak formulation of [7, Theorem 9.6] from the subsequent Remark 1, [7, p. 297] yields

$$cE - \tilde{A} = (c \text{id} - A)(\text{id} + S_c) \tag{11}$$



for some  $c > 0$ , where  $\text{id}$  denotes the identity operator and  $S_c$  maps a measurable function  $f : J \rightarrow \mathbb{R}$  to the function  $S_c f(x) := \mathbb{E}_x[\int_0^\infty e^{-ct} f(Y_t) \psi(Y_t) dt]$ ,  $x \in J$ , whenever the right hand side is well defined and finite. We denote the resolvent operator of the semigroup generated by (the transition function of)  $Y$  by  $R_c$ ,  $c > 0$ , cf. [7, Subsection 5.1]. Now by definition

$$S_c f(x) = R_c(f\psi)(x), \quad x \in J.$$

Together with (11) this implies

$$\mathcal{D}_{\tilde{A}} = \{f \in \mathcal{B}(J) : f + R_c(f\psi) \in \mathcal{D}_A\}. \quad (12)$$

Let  $f \in C^0(J) \cap \mathcal{B}(J)$ . By [7, Theorem 1.7]  $R_c(f\psi) \in \mathcal{D}_A$  if  $\mathbb{E} \cdot [f(Y_t)\psi(Y_t)] \rightarrow f\psi$  weakly for  $t \searrow 0$ . However, for each  $x \in J$

$$\begin{aligned} & |\mathbb{E}_x[f(Y_t)\psi(Y_t)] - f(x)\psi(x)| \leq |\mathbb{E}_x[(f(Y_t) - f(x))\psi(Y_t)]| + |f(x)(\mathbb{E}_x[\psi(Y_t)] - \psi(x))| \\ & \leq \left( \sup_{y \in I} \psi(y) \right) \mathbb{E}_x[(f(Y_t) - f(x))] + |f(x)(\mathbb{E}_x[\psi(Y_t)] - \psi(x))| \xrightarrow{t \searrow 0} 0 \end{aligned}$$

by dominated convergence using  $f \in C^0(J) \cap \mathcal{B}(J)$  and due to the assumption on  $\psi$ . As  $f$  and  $\psi$  are both bounded, dominated convergence also yields  $\mathbb{E} \cdot [f(Y_t)\psi(Y_t)] \rightarrow f\psi$  weakly for  $t \searrow 0$  as desired.

If  $f \in \mathcal{D}_{\tilde{A}} \cap C^0(J) \cap \mathcal{B}(J)$ , then  $f + R_c(f\psi) \in \mathcal{D}_A$  by (12).  $R_c(f\psi) \in \mathcal{D}_A$  has already been shown and so by linearity  $f = (f + R_c(f\psi)) - R_c(f\psi) \in \mathcal{D}_A$ .

If  $f \in \mathcal{D}_A \cap C^0(J) \cap \mathcal{B}(J)$ , then since  $R_c(f\psi) \in \mathcal{D}_A$  also  $f + R_c(f\psi) \in \mathcal{D}_A$ . By (12) this implies  $f \in \mathcal{D}_{\tilde{A}}$ .

To prove the final claim let  $f \in \mathcal{D}_{\tilde{A}} \cap C^0(J) \cap \mathcal{B}(J)$ . Once again by [7, Theorem 9.6]

$$(c \text{id} - \tilde{A})f = (c \text{id} - A)(\text{id} + S_c)f = (c \text{id} - A)f + (c \text{id} - A)R_c(\psi f) = (c \text{id} - A)f + \psi f,$$

$$\text{i.e. } \tilde{A}f = Af - \psi f. \quad \square$$

**Lemma 4.** Let  $\mathfrak{A}$  denote the characteristic operator of  $X$ . Moreover, let  $J \subset I$  be an open interval with  $\tau^J < \infty$   $\mathbb{P}_x$ -a.s. for all  $x \in J$  and  $\psi : J \rightarrow [0, \infty)$  a bounded measurable function. Assume that  $\mathbb{E} \cdot [\psi(X_{t \wedge \tau^J})] \big|_J \rightarrow \psi$  weakly for  $t \searrow 0$ . We extend  $\psi$  to  $I$  by 0 outside of  $J$ , set  $\tau := \tau^J$  and

$$f(x) := \mathbb{E}_x \left[ e^{-\int_0^\tau \psi(X_s) ds} g(X_\tau) \right]$$

for all  $x \in I$ . Then  $f$  is continuous with

$$\mathfrak{A}f(x) - \psi(x)f(x) = 0$$

and in particular  $f \in \mathcal{D}_{\mathfrak{A}}(y)$  for all  $y \in J$ .

**Remark.** The proof of this result is analogous to the proof of [8, Theorem 13.11] up to the part where we apply Lemma 3 instead of its counterpart [7, Theorem 9.7].

*Proof.* Note that  $f$  is continuous by Lemma 2 (i).

Let  $\hat{X} = (\hat{X}_t)_{t \in [0, \infty)}$ , be given by  $\hat{X}_t(\omega) := X_{t \wedge \tau}(\omega)$ ,  $t \in [0, \infty)$ . By [7, Theorem 10.3]  $\hat{X} = (\hat{X}_t, \infty, \mathcal{F}_t, \mathbb{P}_x)$  is a standard process. Let  $\alpha = (\alpha_t)_{t \in [0, \infty)}$  be given by

$$\alpha_t := e^{-\int_0^t \psi(\hat{X}) ds}, \quad t \geq 0.$$

Clearly  $\alpha$  is a continuous homogeneous multiplicative functional of  $\hat{X}$ . We let  $\tilde{X} = (\tilde{X}_t, \tilde{\zeta}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}_x)$  denote the standardized  $\alpha$ -subprocess of  $\hat{X}$ , cf. [7, Subsection 10.10, Subsection 10.12, Subsection 10.17]. In particular this means that  $\tilde{X}$  is a process on  $(\Omega \times [0, \infty], \tilde{\mathcal{F}} \otimes \mathcal{B}([0, \infty]))$  with  $\tilde{\mathcal{F}}$  denoting the  $\sigma$ -algebra generated by  $\tilde{\mathcal{F}}_t, t \geq 0$ ,  $\tilde{\zeta}(\omega, t) := t$ ,  $\tilde{X}_t := \hat{X}_t(\omega), t < \zeta(\omega, t)$  and  $\tilde{\mathbb{P}}_x$  are such that

$$\tilde{\mathbb{P}}_x(\tilde{X}_t \in \Gamma) = \mathbb{E}_x[\mathbb{1}_\Gamma(\hat{X}_t)\alpha_t] \quad (13)$$

for all  $\Gamma \in \mathcal{B}(I)$ . Moreover,  $\tilde{X}$  is a standard process, cf. [7, Theorem 10.7]. Now let  $F : I \rightarrow \mathbb{R}$  be bounded and measurable and  $x \in J$ . We denote the expectation corresponding to  $\tilde{\mathbb{P}}_x$  by  $\tilde{\mathbb{E}}_x$ . Now (13) extends to

$$\tilde{\mathbb{E}}_x[F(\tilde{X}_t)] = \mathbb{E}_x[F(\hat{X}_t)\alpha_t].$$

Using  $\hat{X}_t = X_{t \wedge \tau}$  and  $\alpha_t = \alpha_{t \wedge \tau}$  for all  $t \geq 0$  since  $\psi = 0$  outside of  $J$ , we find that

$$\tilde{\mathbb{E}}_x[F(\tilde{X}_t)] = \mathbb{E}_x[\mathbb{1}_{\{t < \tau\}} F(\hat{X}_t)\alpha_t] + \mathbb{E}_x[\mathbb{1}_{\{t \geq \tau\}} F(X_\tau)\alpha_\tau].$$

Since we assumed  $\tau < \infty$   $\mathbb{P}_x$ -a.s. and that  $F$  is bounded, dominated convergence yields

$$\lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_x[F(\tilde{X}_t)] = \mathbb{E}_x[F(X_\tau)\alpha_\tau]. \quad (14)$$

From now on let  $F$  be bounded and measurable such that  $F = g$  on the boundary  $\partial J \subset I$  of  $J$ . By definition of  $f$ , followed by dominated convergence applied to (14) we infer

$$f(x) = \mathbb{E}_x[g(X_\tau)\alpha_\tau] = \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_x[F(\tilde{X}_t)].$$

Thus, for the weak infinitesimal generator  $\tilde{A}$  of  $\tilde{X}$  we have  $\tilde{A}f = 0$  and in particular  $f \in \mathcal{D}_{\tilde{A}}$ .

We denote the weak infinitesimal operator of  $\hat{X}$  by  $\hat{A}$ . Note that  $\hat{X}$  is a standard process with continuous paths and since  $\mathbb{E}_y[\psi(X_{t \wedge \tau})] = \psi(y)$  for all  $y \in I \setminus (a, b)$  we still have  $\mathbb{E}[\psi(X_{t \wedge \tau})] \rightarrow \psi$  weakly for  $t \searrow 0$  despite extending  $\psi$ . Due to (13), Lemma 3 can be applied to the process  $\hat{X}$ ,  $\tilde{\mathbb{P}}_x$  and the extension of  $\psi$  to provide  $\hat{A}G = \tilde{A}G - \psi G$  for all  $G \in \mathcal{D}_{\tilde{A}} \cap C^0(I) \cap B(I) = \mathcal{D}_{\hat{A}} \cap C^0(I) \cap B(I)$ . In particular since  $f$  is continuous,

$$\hat{A}f - \psi f = \tilde{A}f = 0 \quad (15)$$

and  $f \in \mathcal{D}_{\hat{A}}$ . If we denote the characteristic operator of  $\hat{X}$  by  $\hat{\mathfrak{A}}$ , [7, Lemma 5.6] provides  $f \in \hat{\mathfrak{A}}(x)$  and  $\hat{\mathfrak{A}}f(x) = \psi(x)f(x)$  due to (15). As  $J$  is open, by definition of the characteristic operator  $\mathcal{D}_{\hat{\mathfrak{A}}}(x) = \mathcal{D}_{\hat{\mathfrak{A}}}(x)$  and  $\hat{\mathfrak{A}}f(x) = \hat{\mathfrak{A}}f(x)$ . In particular  $f \in \mathcal{D}_{\hat{\mathfrak{A}}}$  and  $\hat{\mathfrak{A}}f(x) - \psi(x)f(x) = 0$ , which proves the claim.  $\square$

**Lemma 5.** Let  $\mathfrak{A}$  denote the characteristic operator of  $X$ . Moreover, let  $J \subset I$  be an open interval with  $\mathbb{E}_x[\tau^J] < \infty$  for all  $x \in J$ ,  $\psi : J \rightarrow [0, \infty)$  bounded and measurable and  $h : J \rightarrow \mathbb{R}$  bounded and measurable. Assume that  $\mathbb{E}[\psi(X_{t \wedge \tau^J})] \rightarrow \psi$  weakly for  $t \searrow 0$  and  $\mathbb{E}[h(X_{t \wedge \tau^J})e^{-\int_0^{t \wedge \tau^J} \psi(X_s)ds}] \rightarrow h$  weakly for  $t \searrow 0$  respectively. We extend the functions  $\psi, h$  to  $I$  by 0 outside of  $J$ , set  $\tau := \tau^J$  and

$$F(x) := \mathbb{E}_x \left[ \int_0^\tau e^{-\int_0^s \psi(X_s)ds} h(X_s) dt \right]$$

for all  $x \in I$ . Then  $F$  is continuous with

$$\mathfrak{A}F(x) - \psi(x)F(x) = -h(x)$$

and in particular  $F \in \mathcal{D}_{\mathfrak{A}}(y)$  for all  $x \in J$ .

**Remark.** Once again, the proof of this lemma is mostly analogous to its counterpart [8, Theorem 13.12]. The first difference is that here  $\tilde{\mathbb{E}} \cdot [h(\tilde{X}_t)] \rightarrow h$  weakly for  $t \searrow 0$  is assumed directly and not inferred from continuity. The other is that, just as for Lemma 4, we rely on Lemma 3 instead of its counterpart [8, Theorem 9.7].

*Proof.* Note that  $F$  is continuous by Lemma 2 (ii).

Let  $\hat{X} = (\hat{X}_t, \infty, \mathcal{F}_t, \mathbb{P}_x)$ ,  $\alpha = (\alpha_t)_{t \in [0, \infty)}$  and  $\tilde{X} = (\tilde{X}_t, \tilde{\zeta}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}_x)$  be the processes constructed in the proof of Lemma 4,  $\hat{A}, \tilde{A}$  their weak infinitesimal generators and  $\hat{\mathfrak{A}}, \tilde{\mathfrak{A}}$  their characteristic operators.

Let  $x \in I$ . Once again, we denote the expectation corresponding to  $\tilde{\mathbb{P}}_x$  by  $\tilde{\mathbb{E}}_x$ . Using the definitions of  $\tilde{X}_t$  and  $\hat{X}_t$ , [7, Theorem 10.5, (10.20)] and the assumption  $h = 0$  outside of  $J$  we obtain

$$\tilde{\mathbb{E}}_x[h(\tilde{X}_t)] = \tilde{\mathbb{E}}_x[h(\hat{X}_t)\mathbb{1}_{\{\tilde{\zeta} > t\}}] = \mathbb{E}_x[h(\hat{X}_t)\alpha_t] = \mathbb{E}_x[\mathbb{1}_{\{\tau > t\}}h(X_t)\alpha_t]. \quad (16)$$

for all  $t \geq 0$ . Let  $\tilde{R}$  denote the potential of  $\tilde{X}$ . Using the definition of the potential in the first step, (16) in the second, Fubini's theorem in the third and the definition of  $F$  is the last step we obtain

$$\begin{aligned} \tilde{R}h(x) &= \int_0^\infty \tilde{\mathbb{E}}_x[h(\tilde{X}_t)] dt = \int_0^\infty \mathbb{E}_x[\mathbb{1}_{\{\tau > t\}}h(X_t)\alpha_t] dt = \mathbb{E}_x \left[ \int_0^\infty \mathbb{1}_{\{\tau > t\}}h(X_t)\alpha_t dt \right] \\ &= \mathbb{E}_x \left[ \int_0^\tau h(X_t)\alpha_t dt \right] = F(x). \end{aligned}$$

Thus, since  $x \in I$  was arbitrary,  $\tilde{R}h = F$ .

We have  $\tilde{\mathbb{E}}_y[h(\tilde{X}_t)] = \mathbb{E}_y[h(\hat{X}_t)\alpha_t] = 0$  for all  $t \geq 0$  and all  $y \in I \setminus J$  since  $h = 0$  outside  $J$  and  $\hat{X}_t = y$   $\mathbb{P}_y$ -a.s. Combined with  $\tilde{\mathbb{E}} \cdot [h(\tilde{X}_t)] = \mathbb{E} \cdot [h(\hat{X}_t)\alpha_t] = \mathbb{E} \cdot [h(X_{t \wedge \tau^J})e^{-\int_0^{t \wedge \tau^J} \psi(X_s)ds}]$  and the assumption on  $h$ , this implies  $\tilde{\mathbb{E}} \cdot [h(\tilde{X}_t)] \rightarrow h$  weakly for  $t \searrow 0$ . Now [7, Theorem 1.7'] provides  $F \in \mathcal{D}_{\tilde{A}}$  and  $\tilde{A}F = -h$ .

As mentioned in the proof of Lemma 4,  $\tilde{X}$  is a standard process and due to (13), the pre-conditions of Lemma 3 are met for  $\hat{X}$ ,  $\tilde{\mathbb{P}}_x$  and  $\psi$ . The lemma yields  $\tilde{A}G = \hat{A}G - \psi G$  for all  $G \in \mathcal{D}_{\hat{A}} \cap C^0(I) \cap B(I) = \mathcal{D}_{\tilde{A}} \cap C^0(I) \cap B(I)$ . In particular since  $F$  is continuous

$$\hat{A}F - \psi F = \tilde{A}F = -h \quad (17)$$

and  $F \in \mathcal{D}_{\hat{A}}$ .

[7, Lemma 5.6] provides  $F \in \hat{\mathfrak{A}}(x)$  and  $\hat{\mathfrak{A}}F(x) = \psi(x)F(x) - h(x)$  due to (17). As  $J$  is open, by definition of the characteristic operator  $\mathcal{D}_{\hat{\mathfrak{A}}}(x) = \mathcal{D}_{\mathfrak{A}}(x)$  and  $\hat{\mathfrak{A}}F(x) = \mathfrak{A}F(x)$ . In particular  $F \in \mathcal{D}_{\mathfrak{A}}$  and  $\mathfrak{A}F(x) - \psi(x)f(x) - h(x) = 0$  which proves the claim.  $\square$

*Proof.* (of Theorem 1)

(i) We first calculate the conditional distribution of  $\tau$  under  $\mathbb{P}_x$  given  $\mathcal{F}_{\infty}^X$ . Since  $A_t$  is  $\mathcal{F}_{\infty}^X$ -measurable and  $E$  is independent from  $X$  we have

$$\mathbb{P}_x(\tau_2 > t | \mathcal{F}_{\infty}^X) = \mathbb{P}_x(A_t < E | \mathcal{F}_{\infty}^X) = e^{-A_t}.$$

for all  $t \geq 0$ . Thus  $d(-e^{-A_\cdot})$  (i.e. the Markov kernel  $\Omega \times \mathcal{B}([0, \infty]) \ni (\omega, (a, b]) \mapsto e^{-A_a(\omega)} - e^{-A_b(\omega)}$ ) is a regular version of the conditional distribution of  $\tau$  given  $\mathcal{F}_{\infty}^X$ . Applying a change of variables for finite variation processes, cf. [15, p. 42], and using that  $A$  only jumps in  $\tau^D$ , we find

$$e^{-A_a(\omega)} - e^{-A_b(\omega)} = \int_{(a, b \wedge \tau^D)} e^{-A_s} dA_\cdot(s) + e^{-A_{\tau^D-}} - e^{-A_{\tau^D}}.$$

With the distribution of  $\tau$ , the fact  $\tau^{(a,b)} \leq \tau^D$  and the Markov property we obtain

$$\begin{aligned} & J_\tau(x) \\ &= \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} > \tau\}} e^{-r\tau} g(X_\tau)] + \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} \leq \tau\}} e^{-r(\theta_{\tau^{(a,b)}} \circ \tau + \tau^{(a,b)})} g(X_{\theta_{\tau^{(a,b)}} \circ \tau + \tau^{(a,b)}})] \\ &= \mathbb{E}_x[\mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} > \tau\}} e^{-r\tau} g(X_\tau) | \mathcal{F}_{\infty}^X]] + \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} \leq \tau\}} e^{-r\tau^{(a,b)}} \mathbb{E}_x[e^{-r\theta_{\tau^{(a,b)}} \circ \tau} g(X_{\theta_{\tau^{(a,b)}} \circ \tau + \tau^{(a,b)}}) | \mathcal{F}_{\tau^{(a,b)}}]] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt} g(X_t) d(-e^{-A_t}) \right] + \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} \leq \tau\}} e^{-r\tau^{(a,b)}} \mathbb{E}_{X_{\tau^{(a,b)}}}[e^{-r\tau} g(X_\tau)]] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt} g(X_t) e^{-A_t} dA_t \right] + \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} \leq \tau\}} e^{-r\tau^{(a,b)}} J_\tau(X_{\tau^{(a,b)}})] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt-A_t} g(X_t) dA_t \right] + \mathbb{E}_x[e^{-r\tau^{(a,b)}} J_\tau(X_{\tau^{(a,b)}}) \mathbb{E}_x[\mathbb{1}_{\{\tau^{(a,b)} \leq \tau\}} | \mathcal{F}_{\infty}^X]] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt-A_t} g(X_t) dA_t \right] + \mathbb{E}_x[e^{-r\tau^{(a,b)}} J_\tau(X_{\tau^{(a,b)}}) e^{-A_{\tau^{(a,b)}-}}] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt-A_t} g(X_t) dA_t \right] + \mathbb{E}_x[e^{-r\tau^{(a,b)} - A_{\tau^{(a,b)}-}} (\mathbb{1}_{\{a,b\}} J_\tau)(X_{\tau^{(a,b)}})] \end{aligned} \quad (18)$$

for all  $x \in [a, b]$ . With that the claim follows from Lemma 2.

(ii) By the occupation density formula [17, p. 104],  $\lambda|_{(a,b)} = \frac{\psi}{\sigma} dx$  implies  $A_t = \int_0^t \psi(X_s) ds$  for all  $t \in [0, \tau^{(a,b)})$ ,  $\mathbb{P}_x$ -a.s. for all  $x \in I$ . Setting  $\tilde{\psi} := \psi + r$  by (18) we infer

$$\begin{aligned} J_\tau(x) &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-rt - \int_0^t \psi(X_s) ds} \psi(X_t) g(X_t) dt \right] + \mathbb{E}_x \left[ e^{-r\tau^{(a,b)} - \int_0^{\tau^{(a,b)}} \psi(X_s) ds} (\mathbb{1}_{\{a,b\}} J_\tau)(X_{\tau^{(a,b)}}) \right] \\ &= \mathbb{E}_x \left[ \int_{[0, \tau^{(a,b)})} e^{-\int_0^t \tilde{\psi}(X_s) ds} \psi(X_t) g(X_t) dt \right] + \mathbb{E}_x \left[ e^{-\int_0^{\tau^{(a,b)}} \tilde{\psi}(X_s) ds} (\mathbb{1}_{\{a,b\}} J_\tau)(X_{\tau^{(a,b)}}) \right] \end{aligned} \quad (19)$$

for all  $x \in (a, b)$ . Whenever  $a, b \in I$ , the functions  $\tilde{\psi}$  and  $\psi \cdot g|_{(a,b)}$  are Hölder continuous on  $(a, b)$  and the dynamics of  $X$  on  $[a, b]$  coincide with the dynamics of a canonical diffusion. If  $a \notin I$  or  $b \notin I$ , we cover  $(a, b)$  by subintervals that are bounded away from the boundary of  $I$  and argue separately for each subinterval. Now the claim follows from [8, Theorem 13.16].

(iii) We denote the characteristic operator of  $X$  by  $\mathfrak{A}$ . By (i)  $J_\tau|_{(a,b)} \in C^0([a, b])$ . By (ii)  $J_\tau|_{(x_{i-1}, x_i)} \in C^2((x_{i-1}, x_i))$  for all  $i \in \{1, \dots, n\}$ . Set  $x := x_i$  for some  $i \in \{1, \dots, n-1\}$  and  $\tau_h := \tau^{(x-h, x+h)}$ . By a generalized Itô formula, cf. [13], we obtain

$$\begin{aligned} & \frac{\mathbb{E}_x[J_\tau(X_{\tau_h})] - J_\tau(x)}{\mathbb{E}_x[\tau_h]} \\ &= \frac{\mathbb{E}_x \left[ \int_0^{\tau_h} \mathbb{1}_{\{X_s \neq x\}} \mathfrak{A} J_\tau(X_s) ds + \frac{1}{2} \int_0^{\tau_h} \partial_x J_\tau(x+) - \partial_x J_\tau(x-) dl_s^x \right]}{\mathbb{E}_x[\tau_h]} \\ &= \frac{\mathbb{E}_x \left[ \int_0^{\tau_h} \mathbb{1}_{\{X_s \neq x\}} \mathfrak{A} J_\tau(X_s) ds \right]}{\mathbb{E}_x[\tau_h]} + \frac{1}{2} (\partial_x J_\tau(x+) - \partial_x J_\tau(x-)) \frac{\mathbb{E}_x[l_{\tau_h}^x]}{\mathbb{E}_x[\tau_h]} \end{aligned} \quad (20)$$

for all  $h \in (0, (x - x_{i-1}) \wedge (x_{i+1} - x))$  with  $\partial_x J_\tau(x+)$  and  $\partial_x J_\tau(x-)$  denoting the right- and left derivative of  $J_\tau$  at  $x$ . By (19), Lemma 4 and Lemma 5

$$\begin{aligned} & \mathfrak{A} J_\tau(y) \\ &= 2\mathfrak{A} \mathbb{E}_y \left[ \int_{[0, \tau^{(x_{i-1}, x)})} e^{-\int_0^t \tilde{\psi}(X_s) ds} \psi(X_t) g(X_t) dt \right] + 2\mathfrak{A} \mathbb{E}_y \left[ e^{-\int_0^{\tau^{(x_{i-1}, x)}} \tilde{\psi}(X_s) ds} (\mathbb{1}_{\{x_{i-1}, x\}} J_\tau)(X_{\tau^{(x_{i-1}, x)}}) \right] \\ &= \psi(y) \mathbb{E}_y \left[ \int_{[0, \tau^{(x_{i-1}, x)})} e^{-\int_0^t \tilde{\psi}(X_s) ds} \psi(X_t) g(X_t) dt \right] - g(y) \\ & \quad + \psi(y) \mathbb{E}_y \left[ e^{-\int_0^{\tau^{(x_{i-1}, x)}} \tilde{\psi}(X_s) ds} (\mathbb{1}_{\{x_{i-1}, x\}} J_\tau)(X_{\tau^{(x_{i-1}, x)}}) \right] \\ &= \psi(y) J_\tau(y) - g(y) \end{aligned}$$

for all  $y \in (x_{i-1}, x)$ . Due to the Hölder conditions on  $g, \psi$  and continuity of  $J_\tau$  we find that  $\lim_{y \nearrow x} \mathfrak{A} J_\tau(y)$  is finite. Analogously we obtain that  $\lim_{y \searrow x} \mathfrak{A} J_\tau(y)$  is finite. Thus by [3, Lemma A.4] the first summand of the right hand side of (20) has a finite limit for  $h \searrow 0$ . By [2, Lemma 26] we have  $\lim_{h \searrow 0} \frac{\mathbb{E}_x[l_{\tau_h}^x]}{\mathbb{E}_x[\tau_h]} = \infty$ . Thus, the right hand side has a finite limit for  $h \searrow 0$  if and only if  $\partial_x J_\tau(x+) - \partial_x J_\tau(x-) = 0$ . We finish the proof by showing that  $\lim_{h \searrow 0} \frac{\mathbb{E}_x[J_\tau(X_{\tau_h})] - J_\tau(x)}{\mathbb{E}_x[\tau_h]}$  exists in  $\mathbb{R}$ . For that set

$$\hat{\psi}(y) = \frac{\tilde{\psi}(y+) + \tilde{\psi}(y-)}{2}$$

for all  $y \in (a, b)$ . By [1, Lemma 5.5] and dominated convergence

$$\begin{aligned} & \left| \mathbb{E}_y[\mathbb{1}_{\{\tau^{(a,b)} > t\}} \hat{\psi}(X_t)] - \hat{\psi}(y) \right| = \left| \mathbb{E}_y[\mathbb{1}_{\{\tau^{(a,b)} > t\}} \hat{\psi}(X_t)] - \frac{\psi(y+) + \psi(y-)}{2} \right| \\ & \leq \left| \mathbb{E}_y[\mathbb{1}_{\{\tau^{(a,b)} > t\}} \hat{\psi}(X_t)] - \mathbb{E}_y[\hat{\psi}(X_t)] \right| + \left| \mathbb{E}_x[\mathbb{1}_{\{X_t > y\}}(\hat{\psi}(X_t) - \psi(y+))] \right| \\ & \quad + \left| \mathbb{E}_y[\mathbb{1}_{\{X_t < y\}}(\hat{\psi}(X_t) - \psi(y-))] \right| + \left| \psi(y+) \left( \mathbb{E}_y[\mathbb{1}_{\{X_t > y\}}] - \frac{1}{2} \right) \right| \\ & \quad + \left| \psi(y-) \left( \mathbb{E}_y[\mathbb{1}_{\{X_t < y\}}] - \frac{1}{2} \right) \right| \xrightarrow{t \searrow 0} 0. \end{aligned}$$

Thus also  $\int_{(a,b)} \mathbb{E}_y[\mathbb{1}_{\{\tau^{(a,b)} > t\}} \hat{\psi}(X_t)] d\mu(y) \rightarrow \int_{(a,b)} \hat{\psi}(y) d\mu(y)$  for all finite measures  $\mu$  on  $(a, b)$ , i.e.  $\mathbb{E}[\mathbb{1}_{\{\tau^{(a,b)} > t\}} \hat{\psi}(X_t)] \rightarrow \hat{\psi}$  weakly for  $t \searrow 0$ . Using  $\mathbb{E}_y[\int_0^t \mathbb{1}_{\{X_s \in \{x_1, \dots, x_{n-1}\}\}} ds] = 0$  for all  $t \geq 0$ , (19) implies

$$J_\tau(y) = \mathbb{E}_y \left[ \int_{[0, \tau^{(a,b)})} e^{-\int_0^t \hat{\psi}(X_s) ds} g(X_t) dt \right] + \mathbb{E}_y \left[ e^{-\int_0^{\tau^{(a,b)}} \hat{\psi}(X_s) ds} (\mathbb{1}_{\{a,b\}} J_\tau)(X_{\tau^{(a,b)}}) \right]$$

for all  $y \in (a, b)$ . Thus by Lemma 4 and Lemma 5 we have  $J_\tau \in \mathcal{D}_{\mathfrak{A}}(y)$  for all  $y \in (a, b)$ . By definition this means that  $\lim_{h \searrow 0} \frac{\mathbb{E}_x[J_\tau(X_{\tau_h})] - J_\tau(x)}{\mathbb{E}_x[\tau_h]}$  exists, which finishes the proof.  $\square$

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