

NODAL COUNTS FOR THE ROBIN PROBLEM ON LIPSCHITZ DOMAINS

KATIE GITTINS, ASMA HASSANNEZHAD, CORENTIN LÉNA, AND DAVID SHER

ABSTRACT. We consider the Courant-sharp eigenvalues of the Robin Laplacian for bounded, connected, open sets in \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary. We prove Pleijel's theorem which implies that there are only finitely many Courant-sharp eigenvalues in this setting as well as an improved version of Pleijel's theorem, extending previously known results that required more regularity of the boundary. In addition, we obtain an upper bound for the number of Courant-sharp Robin eigenvalues of a bounded, connected, convex, open set in \mathbb{R}^n with C^2 boundary that is explicit in terms of the geometric quantities of the set and the norm sup of the negative part of the Robin parameter.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with Lipschitz boundary. Let $h \in L^\infty(\partial\Omega, \mathbb{R})$. We consider the eigenvalue problem for the Robin Laplacian

$$\begin{cases} \Delta u = \mu u & \text{in } \Omega, \\ \partial_\nu u + hu = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Delta = -\operatorname{div} \operatorname{grad}$ is the positive Laplacian and ν is the unit outward-pointing normal along $\partial\Omega$. The eigenvalues of the Robin Laplacian on $L^2(\Omega)$, counted with multiplicity, can be written as

$$\mu_1(\Omega, h) \leq \mu_2(\Omega, h) \leq \cdots \leq \mu_k(\Omega, h) \leq \cdots \nearrow +\infty.$$

We use the notation $\Delta_\Omega^{R,h}$ to refer to the Robin Laplacian in (1). The case where $h \equiv 0$ corresponds to the Neumann Laplacian and the case where $h \rightarrow +\infty$ corresponds to the Dirichlet Laplacian.

Let u_k be an eigenfunction of $\Delta_\Omega^{R,h}$ corresponding to the k -th eigenvalue $\mu_k(\Omega, h)$. The connected components of $\Omega \setminus \overline{\{x \in \Omega : u_k(x) = 0\}}$ are called nodal domains of u_k . Courant's Nodal Domain theorem asserts that any eigenfunction u_k corresponding to the k -th eigenvalue $\mu_k(\Omega, h)$ has at most k nodal domains. In the case where $\mu_k(\Omega, h)$ has an eigenfunction with exactly k nodal domains, we say that $\mu_k(\Omega, h)$ is a Courant-sharp eigenvalue and u_k is a Courant-sharp eigenfunction.

Let $\mathcal{N}_\Omega^h(k)$ denote the number of nodal domains of the eigenfunction of $\Delta_\Omega^{R,h}$ corresponding to the k -th eigenvalue (counted with multiplicities). Pleijel's theorem [Ple56, BM82] asserts that for a bounded, connected, open set $\Omega \subset \mathbb{R}^n$,

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{N}_\Omega^{+\infty}(k)}{k} \leq \gamma(n), \quad (2)$$

where $\gamma(n) := (2\pi)^n / \omega_n^2 j_{\frac{n-2}{2}}^n < 1$, ω_n is the Lebesgue measure of a ball in \mathbb{R}^n of radius 1 and $j_{\frac{n-2}{2}}$ is the smallest positive zero of the Bessel function $J_{\frac{n-2}{2}}$. Pleijel's theorem for the Dirichlet Laplacian also holds in the Riemannian setting, see [Pee57, BM82].

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It has been shown that the upper bound in (2) is not sharp. Improved versions of Pleijel's theorem that obtain a smaller constant (depending only on n) in the right-hand side of (2) have been obtained in [Don14, Ste14, Bou15]. Pleijel's theorem implies that there are only finitely many Courant-sharp Dirichlet eigenvalues of Ω . This result, called the weak Pleijel theorem, holds for any open set $\Omega \subset \mathbb{R}^n$ of finite Lebesgue measure (see, e.g., [vdBG16]).

The extension of Pleijel's theorem to the Neumann and Robin eigenvalue problems, as well as investigating geometric bounds on Courant-sharp eigenvalues and the number of them, has been the subject of several studies summarised below over the last few years. In this paper, we further these developments by focusing on the Robin eigenvalue problem on a Lipschitz domain Ω with an arbitrary Robin parameter $h \in L^\infty(\partial\Omega)$.

De Ponti, Farinelli, and Violo [DFV24] proved Pleijel's theorem in the setting of metric-measure spaces, in particular, for the Neumann problem on a so-called uniform domain in the non-smooth setting of RCD spaces. Since Lipschitz domains are examples of uniform domains in \mathbb{R}^n , see e.g. [Jon81, MS79, Väi88], their results imply Pleijel's theorem for the Neumann problem on Lipschitz domains in \mathbb{R}^n . Pleijel's theorem for the Neumann problem was first proved for compact surfaces with piecewise real-analytic boundary [Pol09] and then extended to bounded domains in \mathbb{R}^n with $C^{1,1}$ boundary [Lén19]. In [BCM24], Pleijel's theorem was shown to hold for chain domains (roughly speaking, a collection of bounded, disjoint, planar domains and a collection of thin necks joining these domains) and it was shown that the Courant-sharp Neumann eigenvalues are bounded uniformly in terms of the geometric quantities of a family of such domains. Pleijel's theorem for the Robin problem on $C^{1,1}$ domains in \mathbb{R}^n in the case where $h \geq 0$ was proven in [Lén19], while in [HS24] it was shown that an improved version of Pleijel's theorem for the Robin problem on $C^{1,1}$ domains holds for any $h \in L^\infty(\partial\Omega)$.

We extend the result in [DFV24] for the Neumann problem on Lipschitz domains to the Robin problem on Lipschitz domains for any $h \in L^\infty(\Omega)$ (see Theorem 3.1), relaxing the boundary regularity required in [HS24].

The proof of Pleijel's theorem for the Dirichlet Laplacian relies on two main ingredients: the Faber-Krahn inequality and Weyl's law. The primary challenge in extending this proof to other boundary conditions lies in adapting the argument involving the Faber-Krahn inequality. A key step in the proof is to obtain an upper bound for the Neumann Rayleigh quotient of a Robin eigenfunction on a nodal domain in terms of the corresponding eigenvalue, the geometry of the underlying domain and the Robin parameter. To obtain such a bound, we make use of the fact that any Lipschitz domain has an outward-pointing vector field (see Section 2). Moreover, in this setting, we improve the upper bound in Pleijel's theorem (see Theorem 3.3). The proof of an improved version of Pleijel's theorem for the Robin problem on a Lipschitz domain is rather intricate and requires first establishing a quantitative version of certain results proved in [DFV24], including a quantitative form of the Faber-Krahn inequality for mixed Dirichlet-Neumann eigenvalues on domains with 'small' volume (see Proposition 3.4).

Although in the paper we only consider Lipschitz domains in \mathbb{R}^n , one can show that Pleijel's theorem and its improved version for the Robin problem also hold in the Riemannian setting. This is because the results of [DFV24] are valid in a very general context, and other techniques used in the proof can be adapted to the Riemannian setting, as discussed in [HS24, Theorem 2.5] and [Don14, BM82].

With Pleijel's theorem in hand, it is natural to investigate how many Courant-sharp eigenvalues there are and how the geometry of the underlying domain and the Robin parameter h

can be used to quantify this number. Roughly speaking, while Weyl’s law plays a key role in the final step of the proof of Pleijel’s theorem, the remainder in Weyl’s law plays an important role in getting more information on the count of the Courant-sharp eigenvalues. Hence, the bound is sensitive to the geometry of the domain and its boundary. Employing bounds on the remainder has been successfully used in [vdBG16, BH16]. In [BH16], an upper bound for the number of Courant-sharp Dirichlet eigenvalues of a bounded, open set in \mathbb{R}^2 with C^2 boundary is obtained in terms of the area, the perimeter, bounds on the principal curvatures and the cut-distance to the boundary. In [vdBG16], several upper bounds for the number of Courant-sharp Dirichlet eigenvalues of various Euclidean domains are obtained. In particular, an upper bound for the number of Courant-sharp Dirichlet eigenvalues of a bounded, open, convex set in \mathbb{R}^n is obtained in terms of the $(n - 1)$ -dimensional Hausdorff measure of the boundary and the volume of the set. To extend the results for the Dirichlet Laplacian to the Neumann and Robin problems, more regularity and geometric assumptions are needed. For the Neumann problem and the Robin problem with $h \geq 0$, upper bounds for the number of Courant-sharp eigenvalues of a bounded, open, convex set in \mathbb{R}^n , $n \geq 2$, with C^2 boundary are obtained in [GL20] in terms of the volume of the set, the isoperimetric ratio and the principal curvatures of the boundary.

We obtain an upper bound for the number of Courant-sharp Robin eigenvalues of a bounded, open, connected, convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with C^2 boundary, that is explicit in terms of the geometric quantities of Ω and the Robin parameter (see Theorem 4.7). To do this, we first obtain an inequality comparing the Robin counting function to a shifted Neumann counting function and employ a result from [GL20, Appendix A] which gives an upper bound on the Neumann counting function for such domains. We then obtain an upper bound for the largest Courant-sharp Robin eigenvalue (see Theorem 4.4), for which the convexity of the domain is not required. Our result then follows by substituting the bound for the eigenvalue into the bound for the counting function.

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In Section 2, we obtain an upper bound for the Neumann Rayleigh quotient of a Robin eigenfunction on a nodal domain. We then employ this bound together with techniques from [DFV24] to prove Pleijel’s theorem in Section 3, where we also prove an improved version of Pleijel’s theorem. However, the proof of the latter is elaborate and entails additional technical steps. For the convenience of the reader and to illustrate the additional steps required to obtain an improved version, we present the proof of Pleijel’s theorem first, followed by the improved version. In order to obtain the improved version of Pleijel’s theorem, we in particular require a quantitative version of the Faber-Krahn inequality for domains with small volume in this setting, which we prove in the Appendix. In Section 4, we obtain an explicit upper bound on the number of Courant-sharp Robin eigenvalues of a bounded, open, connected, convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with C^2 boundary.

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2. NEUMANN RAYLEIGH QUOTIENTS OF ROBIN EIGENFUNCTIONS ON NODAL DOMAINS

Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, that is, an open, bounded and connected Lipschitz set.

Definition 2.1. Let Ω be a Lipschitz domain. We call the mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ an *outward-pointing vector field* relative to Ω if

- (i) F is of class C^∞ with compact support;
- (ii) there exists $\gamma_F(\Omega) > 0$ such that, for a.e. $x \in \partial\Omega$, we have

$$F(x) \cdot \nu(x) \geq \gamma_F(\Omega).$$

The following proposition is due to Mitrea-Taylor [MT99, Appendix A] and to Verchota [Ver82]. For the convenience of the reader, we provide a proof.

Proposition 2.2. *Any Lipschitz domain Ω has an outward-pointing vector field.*

Proof. For any $x_0 \in \partial\Omega$, we call $F_{x_0} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a *local outward-pointing vector field* at x_0 , relative to Ω , if

- (i) F_{x_0} is of class C^∞ with compact support;
- (ii) for all $x \in \partial\Omega$ such that $\nu(x)$ exists, $F_{x_0}(x) \cdot \nu(x) \geq 0$;
- (iii) there exists $\gamma_{x_0} > 0$ and an open neighborhood U_{x_0} of x_0 such that, for all $x \in U_{x_0} \cap \partial\Omega$ at which $\nu(x)$ exists,

$$F_{x_0}(x) \cdot \nu(x) \geq \gamma_{x_0}.$$

If, relative to Ω , there exists a local outward-pointing vector field at every point of $\partial\Omega$, then there exists a (global) outward-pointing vector field. Indeed, by compactness of $\partial\Omega$, we can find a finite family of points $\{x_1, \dots, x_N\} \subset \partial\Omega$ such that the open sets U_{x_1}, \dots, U_{x_N} cover $\partial\Omega$. Then,

$$F := F_{x_1} + \dots + F_{x_N}$$

is an outward-pointing vector field in the sense of Definition 2.1.

To conclude the proof, we note that the existence of a local outward-pointing vector field at every point $x_0 \in \partial\Omega$ follows easily from the definition of a Lipschitz domain. Up to a suitable choice of coordinates, we can assume that $x_0 = 0$ and that

$$\Omega \cap (B_r^{n-1} \times (-M, M)) = \{(x', y) : y < f(x')\},$$

where (with r and M some positive constants) B_r^{n-1} is the ball centered at 0 of radius r in \mathbb{R}^{n-1} , and $f : B_r^{n-1} \rightarrow (-M, M)$ is a Lipschitz function such that $f(0) = 0$. We set

$$F_0(x) := \chi(x) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

with χ a smooth non-negative function such that $0 \leq \chi \leq 1$ pointwise, the support of χ is contained in $B_{r/2}^{n-1}$, and $\chi = 1$ pointwise in $B_{r/4}^{n-1}$. The vector field F_0 is then locally outward-pointing at 0. \square

Remark 2.3. By scaling, the outward-pointing vector field F may be chosen to be unitary in a neighborhood of $\partial\Omega$, see e.g. [MT99, Appendix A].

The existence of an outward-pointing vector field allows us to prove the fundamental inequalities used in this paper, which are a generalization of [HS24, Proposition 2.2].

Proposition 2.4. *Suppose that Ω is a Lipschitz domain with outward-pointing vector field F . Suppose that $h \in L^\infty(\partial\Omega, \mathbb{R})$, and that μ and u are an eigenvalue and a corresponding eigenfunction respectively of the Robin Laplacian $\Delta_\Omega^{R,h}$. Finally, suppose that D is a nodal domain of u . Then*

$$\frac{\int_D |\nabla u|^2 dx}{\int_D u^2 dx} \leq (\sqrt{\mu + \Gamma_1(\Omega, F)H} + \Gamma_2(\Omega, F)H)^2, \quad (3)$$

where

$$H = \|\min\{0, h(x)\}\|_{L^\infty(\partial\Omega)} = \|\max\{-h(x), 0\}\|_{L^\infty(\partial\Omega)}$$

and where

$$\Gamma_1(\Omega, F) := \frac{1}{\gamma_F(\Omega)} \sup_\Omega |\nabla \cdot F|, \quad (4)$$

$$\Gamma_2(\Omega, F) := \frac{2}{\gamma_F(\Omega)} \sup_\Omega |F|. \quad (5)$$

Proof. We assume for simplicity that ∂D is Lipschitz and that $\partial D \cap \partial\Omega$ is an $(n-2)$ -dimensional submanifold of ∂D . If not, one uses an approximation argument via Sard's theorem as in [BM82, Lén19, HS24].

Since u vanishes on $\partial D \cap \Omega$, Green's identity implies

$$\begin{aligned} \int_{\partial D \cap \partial\Omega} u^2 ds &\leq \frac{1}{\gamma_F(\Omega)} \int_{\partial D \cap \partial\Omega} u^2 (F \cdot \nu) ds = \frac{1}{\gamma_F(\Omega)} \int_{\partial D} (u^2 F) \cdot \nu ds \\ &= \frac{1}{\gamma_F(\Omega)} \int_D (2u \nabla u \cdot F + \operatorname{div}(F) u^2) dx \\ &\leq \frac{1}{\gamma_F(\Omega)} \int_D (2|u| |\nabla u| |F| + |\operatorname{div}(F)| u^2) dx \\ &\leq \int_D (\Gamma_2(\Omega, F) |u| |\nabla u| + \Gamma_1(\Omega, F) u^2) dx, \end{aligned}$$

and from the Cauchy-Schwarz inequality, we obtain

$$\int_{\partial D \cap \partial\Omega} u^2 ds \leq \Gamma_1(\Omega, F) \int_D u^2 dx + \Gamma_2(\Omega, F) \left(\int_D u^2 dx \right)^{1/2} \left(\int_D |\nabla u|^2 dx \right)^{1/2}. \quad (6)$$

The bound (3) follows from (6) precisely as in [HS24, Proposition 2.2]. Namely, applying Green's identity to u in the nodal domain D ,

$$\begin{aligned} \int_D |\nabla u|^2 dx &= \int_D u \Delta u dx + \int_{\partial D} u (\nabla u \cdot \nu) ds \\ &= \mu \int_D u^2 dx - \int_{\partial D \cap \partial\Omega} h u^2 ds \\ &\leq \mu \int_D u^2 dx + H \int_{\partial D \cap \partial\Omega} u^2 ds, \end{aligned}$$

where we have used the facts that $u = 0$ on $\partial D \cap \Omega$ and that $\nabla u \cdot \nu = -hu$ on $\partial\Omega$. Inserting the bound (6) here and rearranging yields (3). \square

A particularly useful example of an outward-pointing vector field, for a domain Ω with *smooth* boundary, is the gradient of the distance function to the boundary, see Example 2.7. For more general Ω , we may sometimes take the outward-pointing vector field to be the gradient of a replacement for the distance function, which we define here:

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a C^2 function $g = g(x)$ defined on an open neighborhood of $\overline{\Omega}$ is an *outward-pointing function* if there exists $\gamma_g > 0$ for which $\nabla g \cdot \nu \geq \gamma_g$ almost everywhere on $\partial\Omega$.

Remark 2.6. If $g = g(x)$ is a smooth outward-pointing function, then its gradient $F = \nabla g$ is an outward-pointing vector field, with

$$\Gamma_1(\Omega, F) = \frac{1}{\gamma_F(\Omega)} \sup_{\Omega} |\Delta g|,$$

$$\Gamma_2(\Omega, F) = \frac{2}{\gamma_F(\Omega)} \sup_{\Omega} |\nabla g|.$$

This can be useful in special cases to analyze the geometric dependence of the constants $\Gamma_1(\Omega, F)$ and $\Gamma_2(\Omega, F)$.

It is an intriguing question whether every bounded Lipschitz domain admits an outward-pointing function. We highlight a few important examples that admit an outward-pointing function, where the geometric dependency of these constants can be expressed more explicitly.

Example 2.7. a) Let Ω be an open, bounded, connected subset of \mathbb{R}^n , $n \geq 2$, with C^2 or $C^{1,1}$ boundary. Then we can define the outward-pointing function g to be any C^2 function that coincides with the distance function $d(\cdot, \partial\Omega)$ on a neighborhood of the boundary staying within the positive reach of the boundary, that is a neighborhood of the boundary where there exists a unique nearest point in $\partial\Omega$. When the boundary is C^2 , the positive reach of $\partial\Omega$ is a neighborhood of the boundary defined by the cut-distance (the distance between $\partial\Omega$ and its cut locus). When the boundary is $C^{1,1}$, the fact that it has a positive reach is established in [Fed59, Theorem 4.12]. In both cases, the distance function from the boundary is C^2 or $C^{1,1}$, respectively. For the latter, see [HS24, Lemma 2.3]. Then $\gamma_{\nabla g}(\Omega) = 1$, and $\Gamma_1(\Omega, \nabla g)$ and $\Gamma_2(\Omega, \nabla g)$ can be expressed in terms of bounds on the Laplacian and the gradient of the distance function. See Lemma 4.5 for an explicit bound for C^2 domains.

In Section 4, we use these types of bounds in order to obtain an upper bound for the largest Courant-sharp Robin eigenvalue for a C^2 convex domain, extending the result in [GL20] to the case of the Robin problem where the parameter can be negative.

b) Another important family of examples are curvilinear polygons. Let Ω be a curvilinear polygon (i.e., a planar domain with smooth boundary except for a finite number of vertices). In this case, we can also construct an outward-pointing function g for which $\Gamma_1(\Omega, \nabla g)$ and $\Gamma_2(\Omega, \nabla g)$ depend only on a list of geometric quantities described in [BCM24].

In [BCM24], Beck, Canzani, and Marzuola considered a family of *chain domains*. They defined a chain domain as a collection of domains with smooth boundaries except for finitely many vertices that are joined by thin necks. They proved Pleijel's theorem for the Courant-sharp Neumann eigenvalues of chain domains as well as an upper bound for Courant-sharp Neumann eigenvalues of such domains. Curvilinear polygons are a subfamily of chain domains that have no necks.

We outline the approach for constructing an outward-pointing function for curvilinear polygons. For some $\delta > 0$ which depends on some explicit geometric quantities, we take the (local) outward-pointing function to be $d(\cdot, \partial\Omega)$, in a δ -neighborhood of each smooth arc of the boundary. In a δ -neighborhood of each of the vertices, we view the angle as the graph $\Gamma f = \{(x, y) : y = f(x)\}$ of a function f , where the vertex is located at $(0, f(0))$, and the y -axis is directed along the bisection of the angle. We define the

(local) outward-pointing function to be $g(x, y) = y$. Then, in this neighborhood, $\gamma_{\nabla g}$ can be bounded from below by some explicit geometric quantities.

We can glue these functions together using a partition of unity. Roughly speaking, this gluing occurs in the region where the distance to each vertex is between $\delta/2$ and δ . We arrange the partition of unity so that it depends only on the tangential coordinate along the boundary and not on the distance to the boundary. Consequently, the gradient of the partition of unity is parallel to the boundary and is bounded by $C\delta^{-1}$, while the Laplacian of the partition of unity is bounded by $C\delta^{-2}$, where C is a universal constant. Moreover, since the normal derivatives of the partition functions vanish, the normal derivative of the outward-pointing function is bounded from below by 1 away from the vertices. Near the vertices, the lower bound remains away from zero and can be explicitly given in terms of geometric quantities.

By using Proposition 2.4 it is possible to extend the results of [BCM24] for the Robin problem on curvilinear polygons. However, in the next section, we provide a different approach to prove Pleijel's theorem for the Robin problem in the more general setting of Lipschitz domains in \mathbb{R}^n , using Proposition 2.4 and adapting the techniques from [DFV24].

3. PLEIJEL'S THEOREM FOR THE ROBIN PROBLEM FOR LIPSCHITZ DOMAINS IN \mathbb{R}^n

The proof of Pleijel's theorem for the Robin problem with an arbitrary Robin parameter $h \in L^\infty(\partial\Omega, \mathbb{R})$ in [HS24] relies on the techniques from [Lén19], which require the $C^{1,1}$ regularity of the domain, and an estimate relating the Robin eigenvalue on a nodal domain to the first mixed Dirichlet-Neumann eigenvalue of that domain. By using the techniques used in [DFV24] together with Proposition 2.4, we obtain Pleijel's theorem for Lipschitz domains. However, to obtain an improved version, the proof becomes more complex because we also need to establish a quantitative version of the Faber-Krahn inequality for the first mixed Dirichlet-Neumann eigenvalue on domains with small volume. Therefore, we include the proofs of both Pleijel's theorem and the improved version of Pleijel's theorem.

We begin with Pleijel's theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{N}_\Omega^h(k)}{k} \leq \gamma(n) < 1,$$

where $\gamma(n) := \frac{(2\pi)^n}{\omega_n^2 \gamma_{\frac{n-2}{2}}^n}$.

We first state a simplified version of the results of [DFV24, Theorem 5.1, and Theorem 5.3] for open subsets of a Lipschitz domain in \mathbb{R}^n in the following lemma. Note that Lipschitz domains are examples of uniform domains in \mathbb{R}^n for which the results of [DFV24] hold. These results give a version of the Faber-Krahn inequality for the mixed Dirichlet-Neumann problem on domains with small volume.

Let Ω be an open Lipschitz domain in \mathbb{R}^n . For any open set $U \subset \Omega$, we consider the following mixed Dirichlet-Neumann problem.

$$\begin{cases} \Delta f = \lambda f & \text{in } U, \\ \partial_\nu f = 0 & \text{on } \partial U \cap \partial\Omega, \\ f = 0 & \text{on } \partial U \cap \Omega. \end{cases}$$

We denote its first eigenvalue by $\lambda_1(U)$. When U is compactly contained in Ω , it is the Dirichlet eigenvalue problem on U .

Lemma 3.2. a) *There exist positive constants $c_1 = c_1(\Omega)$ and $c_2 = c_2(\Omega, n)$ such that for every open set $U \subset \Omega$ with $|U| \leq c_1$, we have*

$$\lambda_1(U)|U|^{2/n} \geq c_2.$$

b) *For every $\epsilon \in (0, 1)$ and $\delta > 0$, there exist an open subset Ω_δ of Ω containing $\partial\Omega$ with $|\Omega_\delta| < \delta$, and constants $\theta_0 = \theta_0(\Omega, n, \epsilon, \delta)$ and $\theta_1 = \theta_1(\Omega, n, \epsilon)$ such that for any open set $U \subset \Omega$ with*

$$|U| \leq \theta_0 \quad \text{and} \quad \frac{|U \cap \Omega_\delta|}{|U|} \leq \theta_1,$$

the following holds

$$\lambda_1(U)|U|^{2/n} \geq (1 - \epsilon)\lambda_1^D(\mathbb{B})|\mathbb{B}|^{2/n}, \quad (7)$$

where \mathbb{B} is the ball of radius 1 in \mathbb{R}^n .

We assume without loss of generality that $\theta_0 \leq c_1$ throughout this section and in the Appendix.

Proof of Theorem 3.1. Let D be a nodal domain of u_k , an eigenfunction corresponding to the Robin eigenvalue μ_k . Then combining with Inequality (3), we have

$$\lambda_1(D) \leq \frac{\int_D |\nabla u_k|^2}{\int_D u_k^2} \leq (\sqrt{\mu_k + \Gamma_1(\Omega, F)H} + \Gamma_2(\Omega, F)H)^2 = \mu_k + o(\mu_k). \quad (8)$$

The last identity is for k large enough so that $\mu_k > 0$. Throughout the proof, we can assume this is the case. Let $\{D_j\}_{j=1}^{N_\Omega^h(k)}$ be the nodal domains of u_k . For given $\epsilon \in (0, 1)$, $\delta \in (0, c_1)$, let c_1 , Ω_δ , θ_0 and θ_1 be as in Lemma 3.2. We now proceed as in [DFV24], categorising the nodal domains into three disjoint classes as follows. For the reader's convenience, we include the details here.

- I. $|D_j| > \theta_0$;
- II. $|D_j| \leq \theta_0$ and $|D_j \cap \Omega_\delta| > \theta_1|D_j|$;
- III. $|D_j| \leq \theta_0$ and $|D_j \cap \Omega_\delta| \leq \theta_1|D_j|$,

Let N_I , N_{II} and N_{III} denote the number of nodal domains in each family respectively. Note that $N_\Omega^h(k) = N_I + N_{II} + N_{III}$.

Type I nodal domains. We clearly have $N_I \leq |\Omega|/\theta_0$. Therefore,

$$\frac{N_I}{k} \leq \frac{|\Omega|}{\theta_0 k}. \quad (9)$$

Type II nodal domains. By Lemma 3.2 and inequality (8), we obtain the following inequality for N_{II} .

$$\begin{aligned} (\mu_k + o(\mu_k))^{n/2} \delta &\geq (\mu_k + o(\mu_k))^{n/2} |\Omega_\delta| \\ &\geq \sum_{j \in II} \lambda_1(D_j)^{n/2} |D_j \cap \Omega_\delta| \\ &\geq \theta_1 \sum_{j \in II} \lambda_1(D_j)^{n/2} |D_j| \\ &\geq \theta_1 c_2^{n/2} N_{II}; \end{aligned}$$

Therefore,

$$\frac{N_{II}}{k} \leq \frac{(\mu_k + o(\mu_k))^{n/2} \delta}{\theta_1 c_2^{n/2} k}. \quad (10)$$

Type III nodal domains. Again, by Lemma 3.2 and inequality (8), we get

$$(\mu_k + o(\mu_k))^{n/2} |\Omega| \geq \sum_{j \in \mathbb{III}} \lambda_1(D_j)^{n/2} |D_j| \geq N_{\mathbb{III}} (1 - \epsilon)^{n/2} \lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}|.$$

Hence,

$$\frac{N_{\mathbb{III}}}{k} \leq \frac{(\mu_k + o(\mu_k))^{n/2} |\Omega|}{(1 - \epsilon)^{n/2} \lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}| k}. \quad (11)$$

Taking the limit of (9), (10), and (11) as $k \rightarrow \infty$ and using the Weyl asymptotics for the Robin problem (see e.g. [BS80, FG12]):

$$\lim_{k \rightarrow \infty} \frac{\mu_k^{n/2} |\Omega|}{k} = \frac{(2\pi)^n}{\omega_n}, \quad (12)$$

we get

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{N}_{\Omega}^h(k)}{k} = \frac{N_I + N_{\mathbb{II}} + N_{\mathbb{III}}}{k} \leq \frac{\delta (2\pi)^n}{\theta_1 c_2^{n/2} \omega_n |\Omega|} + \frac{\gamma(n)}{(1 - \epsilon)^{n/2}}.$$

Note that $\omega_n j_{\frac{n-2}{2}}^n = \lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}|$. We conclude by first sending $\delta \rightarrow 0$ and then sending $\epsilon \rightarrow 0$. Note that θ_1 is independent of δ . \square

We now prove an improved version of Pleijel's theorem. See [Ste14, Bou15] for improvements for the Dirichlet problem and [HS24] for an improvement for the Robin problem on $C^{1,1}$ domains.

Theorem 3.3 (Pleijel's theorem - Improved). *There exists a constant $\varepsilon = \varepsilon(n) > 0$ depending only on the dimension, such that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ we have*

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{N}_{\Omega}^h(k)}{k} \leq \gamma(n) - \varepsilon.$$

We discuss the strategy of the proof as the details are rather intricate. To improve the upper bound $\gamma(n)$ in Theorem 3.1, we first need to prove a quantitative Faber-Krahn inequality for the mixed Dirichlet-Neumann eigenvalue for nodal domains with 'small' volume. The results in [DFV24] do not yield a quantitative version. We prove this important component of the proof in the Appendix. Next, we categorize the nodal domains into disjoint families and use a sphere-packing argument to show that the family of nodal domains satisfying a quantitative version of the Faber-Krahn inequality constitutes a nontrivial proportion of the total volume. This allows the ε improvement.

We first state a quantitative version of the Faber-Krahn inequality for which we introduce a slightly modified version of the Fraenkel asymmetry.

Proposition 3.4. *For any $\epsilon \in (0, 1)$, $\delta > 0$ sufficiently small, and any $C_0 < \infty$, there exist a neighbourhood Ω_{δ} of $\partial\Omega$ with $|\Omega_{\delta}| < \delta$, a positive constant $\theta_0 = \theta_0(\Omega, \epsilon, \delta)$, and another positive constant $\theta_1 = \theta_1(\Omega, n, \epsilon)$, such that for any open set $D \subset \overline{\Omega}$ satisfying*

$$|D| \leq \theta_0, \quad \frac{|D \cap \Omega_{\delta}|}{|D|} \leq \theta_1, \quad \lambda_1(D) |D|^{2/n} \leq C_0,$$

we have

$$\lambda_1(D) \geq (1 - \epsilon + C \tilde{A}(D)^4) \lambda_1^D(D^*).$$

Here D^* is a ball with the same volume as D and $\tilde{A}(D) = \inf_U A(D \cap U)$, where A is the Fraenkel asymmetry and the infimum is taken over all $U \subseteq \Omega$ containing $\Omega_{\delta}^c := \Omega \setminus \Omega_{\delta}$.

We refer to the quantity $\tilde{A}(D)$ as a *modified Fraenkel asymmetry* of D . The proof of Proposition 3.4 is deferred to the Appendix.

Proof of Theorem 3.3. Let $\{\mu_k\}$ be the Robin eigenvalues with a corresponding basis $\{u_k\}$ of Robin eigenfunctions. Abusing notation, we let C refer to any positive constant, depending only on n unless otherwise specified. Throughout, for k large enough so that $\mu_k > 0$, we let B_k be a ball in \mathbb{R}^n , chosen so that its first Dirichlet eigenvalue is μ_k :

$$\lambda_1^D(B_k) = \mu_k.$$

Lemma 3.5. *There exist $k_0 = k_0(\Omega) \in \mathbb{N}$ and $C > 0$ for which, for all $k \geq k_0$,*

$$|B_k| \leq C \frac{|\Omega|}{k}. \quad (13)$$

Proof. By (12), there exists k_0 depending on Ω such that, for $k \geq k_0$,

$$\mu_k \geq C \left(\frac{k}{|\Omega|} \right)^{\frac{2}{n}}, \quad (14)$$

where $C = ((2\pi)^n / (2\omega_n))^{n/2}$ and in particular depends only on n . However, by scaling,

$$\mu_k = \lambda_1^D(B_k) = \left(\frac{|\mathbb{B}|}{|B_k|} \right)^{\frac{2}{n}} \lambda_1^D(\mathbb{B}) = C |B_k|^{-\frac{2}{n}}.$$

Plugging this equation into (14) gives the desired conclusion. \square

Now we fix $\epsilon \in (0, 1)$ and $\delta > 0$ sufficiently small. (This ‘‘sufficiently small’’ may depend only on the measure of Ω and on n . See the Appendix 31.) We supplement our fixed ϵ and δ by fixing a set Ω_δ and constants θ_0 and θ_1 which satisfy Proposition 3.4 and Lemma 3.2. We may also choose θ_1 so that $\theta_1 \leq \frac{1}{2}$ and $\theta_1 \leq \epsilon_2$, where ϵ_2 is defined below (note ϵ_2 does not depend on θ_1 , so that the argument is not circular).

Finally, we fix $\epsilon_1, \epsilon_2 \in (0, 1)$ sufficiently small and independent of both ϵ and δ – see below. For each k satisfying

$$k \geq \max \left\{ k_0, \frac{2}{\theta_0} C |\Omega| \right\}, \quad (15)$$

with C as in Lemma 3.5, we categorise the nodal domains D_j of the eigenfunction u_k into four disjoint sets (type-I, type-II, type-III, and type- $\tilde{\text{III}}$) as follows.

- I. $|D_j| > (1 + \epsilon_1)|B_k|$;
- II. $|D_j| \leq (1 + \epsilon_1)|B_k|$ and $|D_j \cap \Omega_\delta| > \theta_1 |D_j|$;
- III. $|D_j| \leq (1 + \epsilon_1)|B_k|$, $|D_j \cap \Omega_\delta| \leq \theta_1 |D_j|$, and $\tilde{A}(D_j) \leq \epsilon_2$;
- $\tilde{\text{III}}$. $|D_j| \leq (1 + \epsilon_1)|B_k|$, $|D_j \cap \Omega_\delta| \leq \theta_1 |D_j|$, and $\tilde{A}(D_j) > \epsilon_2$.

Let N_{I} , N_{II} , N_{III} , and $N_{\tilde{\text{III}}}$ denote the number of nodal domains in each family. Note that $\mathcal{N}_\Omega^h(k) = N_{\text{I}} + N_{\text{II}} + N_{\text{III}} + N_{\tilde{\text{III}}}$. We denote by $\Omega_\#$ the union of all nodal domains of type $\#$. Note also that our condition (15) on k guarantees that

$$|B_k| \leq \min \left\{ C \frac{|\Omega|}{k}, \frac{\theta_0}{2} \right\}.$$

Remark 3.6. We can now be specific about our choice of ϵ_1 and ϵ_2 . They are chosen independent of (sufficiently small) ϵ , and in a way such that type-III domains can only pack a $\tilde{\rho}(n) < 1$ fraction of Ω_δ^c ; we show in our proof of Lemma 3.7 below that this is possible.

Our first step is to show that type-III domains do not have density 1 – that is, that the fraction of nodal domains which are *not* type-III is bounded away from zero. This is a sphere-packing argument.

Lemma 3.7. *There exist $k_1 = k_1(\delta, \epsilon, \epsilon_1, \epsilon_2, \bar{\Omega}) \in \mathbb{N}$ and $\tilde{\rho} = \tilde{\rho}(n) < 1$ such that for all $k \geq k_1$,*

$$\frac{|\Omega_{\text{III}}|}{|\Omega|} \leq \tilde{\rho}.$$

To prove this lemma, we begin with an auxiliary result.

Lemma 3.8. *Suppose that D is a type-III nodal domain. Then*

$$A(D \cap \Omega_\delta^c) \leq \frac{\epsilon_2}{1 - \theta_1} + \frac{2\theta_1}{1 - \theta_1} \leq 6\epsilon_2. \quad (16)$$

Proof. We claim a slightly more general result: if S and T are domains with $A(S) < \epsilon_2$, $T \subseteq S$, and $|T|/|S| \geq 1 - \theta_1$, then

$$A(T) \leq \frac{\epsilon_2}{1 - \theta_1} + \frac{2\theta_1}{1 - \theta_1}. \quad (17)$$

Assuming this claim for the moment, we know because D is a type-III nodal domain that there exists a U with $\Omega_\delta^c \subseteq U \subseteq \Omega$ for which $A(D \cap U) < \epsilon_2$. We then apply (17) with $S = D \cap U$ and $T = D \cap \Omega_\delta^c$, immediately proving the first inequality in (16). The last inequality in (16) follows immediately from the inequalities $\theta_1 \leq \frac{1}{2}$ and $\theta_1 \leq \epsilon_2$.

It remains to prove (17). By definition of the Fraenkel asymmetry, there is a ball B such that $|B| = |S|$ and $|B \Delta S| < \epsilon_2 |B|$. Let \tilde{B} be a ball of volume $|T|$ with $\tilde{B} \subseteq B$. Then

$$|\tilde{B} \Delta T| = |\tilde{B} \setminus T| + |T \setminus \tilde{B}| \leq |B \setminus T| + |S \setminus \tilde{B}| \leq (|B \setminus S| + |S \setminus T|) + (|S \setminus B| + |B \setminus \tilde{B}|).$$

Now $|S \setminus T| \leq \theta_1 |S|$ and the same is true for $|B \setminus \tilde{B}|$. So

$$|\tilde{B} \Delta T| \leq |B \Delta S| + 2\theta_1 |S|.$$

Dividing by $|\tilde{B}| = |T|$ gives

$$A(T) \leq \frac{|\tilde{B} \Delta T|}{|\tilde{B}|} \leq \frac{1}{|S|} \frac{|S|}{|T|} (|B \Delta S| + 2\theta_1 |S|),$$

and now (17) follows from the assumptions on S and T . \square

Proof of Lemma 3.7. Let D be a type-III nodal domain associated with μ_k . By Lemma 3.2 part b), we have

$$|D|^{2/n} \lambda_1(D) \geq (1 - \epsilon) \lambda_1^D(\mathbb{B}) |\mathbb{B}|^{2/n} = (1 - \epsilon) \lambda_1^D(B_k) |B_k|^{2/n} = (1 - \epsilon) \mu_k |B_k|^{2/n}.$$

By inequality (8), there exists \tilde{k}_1 depending only on Ω and ϵ for which $k \geq \tilde{k}_1$ implies

$$(1 - \epsilon) \mu_k |B_k|^{2/n} \leq \lambda_1(D) |D|^{2/n} \leq (1 + \epsilon) \mu_k |D|^{2/n}.$$

Therefore, also using the fact that D is a type-III nodal domain,

$$\left(\frac{1 - \epsilon}{1 + \epsilon} \right)^{n/2} |B_k| \leq |D| \leq (1 + \epsilon_1) |B_k|. \quad (18)$$

Now, suppose that D and \tilde{D} are any two nodal domains associated to μ_k . Let B and \tilde{B} be balls such that

$$|B| = |D \cap \Omega_\delta^c|, \quad |\tilde{B}| = |\tilde{D} \cap \Omega_\delta^c|.$$

By Lemma 3.8, the centers of B and \tilde{B} can be chosen so that

$$|B \Delta (D \cap \Omega_\delta^c)| \leq 6\epsilon_2 |B|, \quad |\tilde{B} \Delta (\tilde{D} \cap \Omega_\delta^c)| \leq 6\epsilon_2 |\tilde{B}|.$$

We claim that B and \tilde{B} have comparable volume and small overlap.

To show that B and \tilde{B} have comparable volume, first observe that by the definition of a type-III nodal domain, then using (18) for both D and \tilde{D} , as well as the fact that $\theta_1 \leq \epsilon_2$,

$$\frac{|B|}{|\tilde{B}|} = \frac{|D \cap \Omega_\delta^c|}{|\tilde{D} \cap \Omega_\delta^c|} \leq \left(\frac{1 + \theta_1}{1 - \theta_1} \right) \frac{|D|}{|\tilde{D}|} \leq \left(\frac{1 + \epsilon_2}{1 - \epsilon_2} \right) (1 + \epsilon_1) \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{n/2}. \quad (19)$$

We can assume $\epsilon \leq \epsilon_2 \leq \epsilon_1 \leq \frac{1}{2}$. Thus,

$$\frac{|B|}{|\tilde{B}|} \leq (1 + \epsilon_1) \left(1 + \frac{2\epsilon_1}{1 - \epsilon_1} \right)^{(n+2)/2} \leq 1 + C\epsilon_1. \quad (20)$$

The same upper bound holds for $|\tilde{B}|/|B|$.

To show that B and \tilde{B} have small overlap, observe that since D and \tilde{D} are disjoint,

$$|B \cap \tilde{B}| \leq |B \Delta (D \cap \Omega_\delta^c)| + |\tilde{B} \Delta (\tilde{D} \cap \Omega_\delta^c)| \leq 6\epsilon_2(|B| + |\tilde{B}|),$$

and so using (20),

$$\frac{|B \cap \tilde{B}|}{|B|} \leq 6\epsilon_2 \left(1 + \frac{|\tilde{B}|}{|B|} \right) \leq 6\epsilon_2(2 + C\epsilon_1) \leq C\epsilon_2, \quad (21)$$

where again by abuse of notation, C in the left-hand side of the last inequality is equal to $6(2 + C)$.

Now consider such a ball for *each* type-III nodal domain of μ_k . Note that the volume of each of these balls is less than $\frac{C|\Omega|}{k}$. Let $\rho_o = \rho_o(n) < 1$ represent the sphere packing density of \mathbb{R}^n . The sphere packing density of Ω is bounded above by ρ_o . Therefore, there exists a threshold $\zeta > 0$ for which the packing density of Ω with balls of any fixed size less than ζ is bounded above by $(\rho_o + 1)/2$. We choose k sufficiently large so that $\frac{C|\Omega|}{k} < \zeta$. By choosing ϵ_1 to be sufficiently small, the balls will have almost the same radius. We then shrink the balls by a controlled factor, which depends only on ϵ_1 and n , so that they all have the same radius. Next, we select ϵ_2 to be small enough so that their overlap is very small. By shrinking the balls again by a controlled factor, depending only on ϵ_2 , we can make them disjoint, thereby obtaining a sphere packing of Ω_δ^c – indeed, a sphere packing of Ω itself. Therefore,

$$(1 - c(n, \epsilon_1, \epsilon_2)) \sum_{j \in \text{III}} |D_j \cap \Omega_\delta^c| < \frac{1 + \rho_o}{2} |\Omega|,$$

where $c(n, \epsilon_1, \epsilon_2)$ is a positive function and goes to zero when ϵ_1 and ϵ_2 tend to zero. Hence, we can choose ϵ_1 and ϵ_2 small enough such that $\frac{(1 + \rho_o)/2}{1 - c(n, \epsilon_1, \epsilon_2)} < (1 - \epsilon_2)\tilde{\rho}$, where $\tilde{\rho}$ is a fixed constant in the interval $(\rho, 1)$. Therefore since we have type-III nodal domains,

$$|\Omega_{\text{III}}| = \sum_{j \in \text{III}} |D_j| \leq \sum_{j \in \text{III}} \frac{1}{1 - \theta_1} |D_j \cap \Omega_\delta^c| \leq \frac{1}{1 - \epsilon_2} \sum_{j \in \text{III}} |D_j \cap \Omega_\delta^c| \leq \tilde{\rho} |\Omega|.$$

This completes the proof of Lemma 3.7. \square

We now proceed to bound, for each $\sharp \in \{\text{I}, \text{II}, \text{III}, \tilde{\text{III}}\}$, the quantity $\frac{N_\sharp}{k}$. From adding the bounds we obtain and taking the limsup as $k \rightarrow \infty$, we will be able to read off our sharpened Pleijel theorem.

Type-I nodal domains. Suppose that D_j is a nodal domain of type I. Then it is immediate from the definition of type I that

$$\lambda_1^D(B_k)^{n/2} |B_k| (1 + \epsilon_1) = \mu_k^{n/2} |B_k| (1 + \epsilon_1) \leq \mu_k^{n/2} |D_j|.$$

Summing this equation over all nodal domains of type I yields

$$N_I \lambda_1^D(B_k)^{n/2} |B_k| (1 + \epsilon_1) \leq \mu_k^{n/2} |\Omega_I|,$$

and thus

$$\frac{N_I}{k} \leq \frac{\mu_k^{n/2} |\Omega|}{\lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}| k} \cdot \frac{|\Omega_I|}{|\Omega| (1 + \epsilon_1)}. \quad (22)$$

Type-II nodal domains. We do the same treatment as for type-II nodal domains in the proof of Theorem 3.1 to obtain

$$\frac{N_{II}}{k} \leq \frac{(\mu_k + o(\mu_k))^{n/2} \delta}{\theta_1 c_2(n, \Omega)^{n/2} k}, \quad (23)$$

where $c_2(n, \Omega)$ is the constant in Lemma (3.2) (a).

Type-III nodal domains. Type-III nodal domains satisfy the assumption of Lemma 3.2 (b). Thus, we have

$$\lambda_1(D_j) |D_j|^{2/n} \geq (1 - \epsilon) \lambda_1^D(\mathbb{B}) |\mathbb{B}|^{2/n}.$$

By inequality (8),

$$(\mu_k + o(\mu_k))^{n/2} |\Omega_{\mathbb{III}}| \geq \sum_{j \in \mathbb{III}} \lambda_1(D_j)^{n/2} |D_j| \geq (1 - \epsilon)^{n/2} \lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}| N_{\mathbb{III}}.$$

Solving for $N_{\mathbb{III}}$ and dividing by k , we see as in the proof for Type I nodal domains that

$$\frac{N_{\mathbb{III}}}{k} \leq \frac{(\mu_k + o(\mu_k))^{n/2} |\Omega|}{\lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}| k} \cdot \frac{|\Omega_{\mathbb{III}}|}{(1 - \epsilon)^{n/2} |\Omega|}. \quad (24)$$

Type-III~ nodal domains. The analysis of type-III~ nodal domains is very similar to the analysis of type-III nodal domains, but now we have a lower bound on $\tilde{A}(D_j)$. Moreover, by Lemma 3.5 and inequality (8), for $k \geq k_0$, using also the trivial bound $\epsilon_1 \leq 1$,

$$\lambda_1(D_j) |D_j|^{2/n} \leq C \left(\frac{|\Omega|}{k} \right)^{2/n} (\mu_k + o(\mu_k)),$$

which by Weyl's law is bounded by C . Hence Proposition 3.4 applies taking $C_0 = C$, and therefore

$$\lambda_1(D_j) \geq (1 - \epsilon + C\epsilon_2^4) \lambda_1^D(D_j^*).$$

Taking ϵ small enough such that $2\epsilon \leq C\epsilon_2^4$ and applying the same logic as for type-III, we obtain

$$\frac{N_{\mathbb{III}^{\sim}}}{k} \leq \frac{(\mu_k + o(\mu_k))^{n/2} |\Omega|}{\lambda_1^D(\mathbb{B})^{n/2} |\mathbb{B}| k} \cdot \frac{|\Omega_{IV}|}{(1 + C\epsilon_2^4)^{n/2} |\Omega|}. \quad (25)$$

Note that in (25), by abuse of notation $C = \frac{C}{2}$.

Completing the proof. Combining (22), (23), (24), and (25) yields

$$\frac{N_k}{k} \leq \frac{\delta}{|\Omega| \theta_1 C} \frac{(1 + o(1)) \mu_k^{n/2} |\Omega|}{k} + \frac{(1 + o(1)) \mu_k^{n/2} |\Omega|}{k |\mathbb{B}| \lambda_1^D(\mathbb{B})^{n/2}} \left(\frac{1}{1 + \epsilon_1} \frac{|\Omega_I|}{|\Omega|} + \frac{1}{(1 - \epsilon)^{n/2}} \frac{|\Omega_{\mathbb{III}}|}{|\Omega|} + \frac{1}{(1 + C\epsilon_2^4)^{n/2}} \frac{|\Omega_{\mathbb{III}^{\sim}}|}{|\Omega|} \right). \quad (26)$$

Now take the limsup. Using Weyl's law yields

$$\limsup_{k \rightarrow \infty} \frac{N_k}{k} \leq \frac{C\delta}{\theta_1|\Omega|} + \gamma(n) \limsup_{k \rightarrow \infty} \left(\frac{1}{1 + \epsilon_1} \frac{|\Omega_I|}{|\Omega|} + \frac{1}{(1 - \epsilon)^{n/2}} \frac{|\Omega_{\mathbb{I}}|}{|\Omega|} + \frac{1}{(1 + C\epsilon_2^4)^{n/2}} \frac{|\Omega_{\tilde{\mathbb{I}}}|}{|\Omega|} \right). \quad (27)$$

Pulling out the $(1 - \epsilon)^{-n/2}$ and choosing $\epsilon_3 > 0$, again independent of ϵ and δ , so that

$$(1 - \epsilon_3) \geq \max \left\{ \frac{1}{1 + \epsilon_1}, \frac{1}{(1 + C\epsilon_2^4)^{n/2}} \right\}$$

gives

$$\limsup_{k \rightarrow \infty} \frac{N_k}{k} \leq \frac{C\delta}{\theta_1|\Omega|} + \frac{\gamma(n)}{(1 - \epsilon)^{n/2}} \limsup_{k \rightarrow \infty} \left((1 - \epsilon_3) \frac{|\Omega_I| + |\Omega_{\tilde{\mathbb{I}}}|}{|\Omega|} + \frac{|\Omega_{\mathbb{I}}|}{|\Omega|} \right), \quad (28)$$

which in turn means

$$\limsup_{k \rightarrow \infty} \frac{N_k}{k} \leq \frac{C\delta}{\theta_1|\Omega|} + \frac{\gamma(n)}{(1 - \epsilon)^{n/2}} \limsup_{k \rightarrow \infty} \left((1 - \epsilon_3) \left(1 - \frac{|\Omega_{\mathbb{I}}|}{|\Omega|} \right) + \frac{|\Omega_{\mathbb{I}}|}{|\Omega|} \right). \quad (29)$$

By Lemma 3.7, for $k \geq k_1$, $|\Omega_{\mathbb{I}}|/|\Omega| \leq \tilde{\rho}$. Therefore

$$\limsup_{k \rightarrow \infty} \frac{N_k}{k} \leq \frac{C\delta}{\theta_1|\Omega|} + \frac{\gamma(n)}{(1 - \epsilon)^{n/2}} \left((1 - \epsilon_3)(1 - \tilde{\rho}) + \tilde{\rho} \right). \quad (30)$$

Now let $\delta \rightarrow 0$, using the fact that θ_1 , ϵ_3 , and $\tilde{\rho}$ are independent of δ , we see

$$\limsup_{k \rightarrow \infty} \frac{N_k}{k} \leq \frac{\gamma(n)}{(1 - \epsilon)^{n/2}} \left((1 - \epsilon_3)(1 - \tilde{\rho}) + \tilde{\rho} \right).$$

Finally, we let $\epsilon \rightarrow 0$, using that ϵ_3 and $\tilde{\rho}$ are independent of ϵ , and we observe that $(1 - \epsilon_3)(1 - \tilde{\rho}) + \tilde{\rho}$ is strictly less than 1 and depends only on n . This completes the proof. \square

4. GEOMETRIC UPPER BOUNDS FOR THE NUMBER OF COURANT-SHARP ROBIN EIGENVALUES

The goal of this section is to obtain an upper bound for the number of Courant-sharp Robin eigenvalues of an open, bounded, connected, convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with C^2 boundary, that is explicit in terms of the geometric quantities of Ω and the Robin parameter.

To do this, we first derive a comparison between the Robin and Neumann counting functions for open, bounded, connected, Lipschitz sets in \mathbb{R}^n . We then derive an upper bound for the largest positive Courant-sharp Robin eigenvalue of an open, bounded, connected set $\Omega \subset \mathbb{R}^n$ with C^2 boundary. By combining these two results with the result from [GL20, Appendix A] which holds for convex sets, we obtain an upper bound for the number of Courant-sharp Robin eigenvalues of Ω . We consider the case where Ω has C^2 boundary as this setting allows us to glean explicit geometric control in the geometric bounds that follow.

4.1. Comparison between Neumann and Robin counting functions. In order to use previous work estimating the Neumann eigenvalues and the Neumann counting function, we prove, for an arbitrary Lipschitz domain, a comparison result between the Neumann and the Robin spectra. This can be stated either as a lower bound for the Robin eigenvalues or an upper bound for the Robin counting function. For $\mu > 0$, we define the Robin counting function as

$$N_{\Omega}^h(\mu) := \#\{k \in \mathbb{N} : \mu_k(\Omega, h) < \mu\}.$$

We denote the Neumann counting function by $N_{\Omega}^N(\mu) = N_{\Omega}^0(\mu)$.

We recall that for any Lipschitz domain Ω with outward-pointing vector field F and for all $u \in H^1(\Omega)$, we have

$$\int_{\partial\Omega} u^2 ds \leq \Gamma_1(\Omega, F) \int_{\Omega} u^2 dx + \Gamma_2(\Omega, F) \left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}, \quad (31)$$

with $\Gamma_1(\Omega, F)$ and $\Gamma_2(\Omega, F)$ the constants defined in Proposition 2.4.

This can be interpreted as a kind of trace inequality, and it is the starting point of our analysis.

Proposition 4.1. *Let Ω be a Lipschitz domain and let $h \in L^\infty(\partial\Omega)$ be a real-valued function defined on its boundary $\partial\Omega$. Let F be an outward-pointing vector field relative to Ω , and let $H := \|h^-\|_{L^\infty(\partial\Omega)}$ where $h^- = \max\{0, -h\}$. Then, setting*

$$\begin{aligned} K_1(\Omega, F) &:= \Gamma_1(\Omega, F); \\ K_2(\Omega, F) &:= \frac{\Gamma_2(\Omega, F)^2}{4}; \end{aligned}$$

(with $\Gamma_1(\Omega, F)$, $\Gamma_2(\Omega, F)$ defined in (4), (5) respectively), for any $\eta \in (0, 1)$ and any $k \geq 1$, we have

$$\mu_k(\Omega, h) \geq (1 - \eta)\mu_k^N(\Omega) - \left(K_1(\Omega, F)H + K_2(\Omega, F)\frac{H^2}{\eta} \right). \quad (32)$$

Equivalently, we have

$$N_\Omega^h(\mu) \leq N_\Omega^N \left(\frac{1}{1 - \eta} \left(\mu + K_1(\Omega, F)H + K_2(\Omega, F)\frac{H^2}{\eta} \right) \right) \quad (33)$$

for all $\mu \in \mathbb{R}$.

Proof. Without loss of generality we can assume that $H > 0$. It follows immediately from Inequality (31) that, for all $u \in H^1(\Omega)$ and for any parameter $A > 0$,

$$\int_{\partial\Omega} u^2 ds \leq \Gamma_1(\Omega, F) \int_{\Omega} u^2 dx + \frac{\Gamma_2(\Omega, F)A}{2} \int_{\Omega} u^2 dx + \frac{\Gamma_2(\Omega, F)}{2A} \int_{\Omega} |\nabla u|^2 dx.$$

Choosing $A = (2\eta)^{-1}\Gamma_2(\Omega, F)H$, we obtain

$$\int_{\partial\Omega} u^2 ds \leq \frac{\eta}{H} \int_{\Omega} |\nabla u|^2 dx + \left(K_1(\Omega, F) + K_2(\Omega, F)\frac{H}{\eta} \right) \int_{\Omega} u^2 dx. \quad (34)$$

Inequality (34) implies

$$\begin{aligned} \int_{\partial\Omega} h u^2 ds &\geq -H \int_{\partial\Omega} u^2 ds \\ &\geq -\eta \int_{\Omega} |\nabla u|^2 dx - \left(K_1(\Omega, F)H + K_2(\Omega, F)\frac{H^2}{\eta} \right) \int_{\Omega} u^2 dx. \end{aligned} \quad (35)$$

Now let q_h denote the quadratic form associated to the Robin problem with boundary function h :

$$q_h[u] := \int_{\Omega} |\nabla u|^2 dx + \int_{\partial\Omega} h u^2 ds. \quad (36)$$

It is closed with domain $H^1(\Omega)$. From the min-max principle, we have

$$\mu_k(\Omega, h) = \min_{V_k} \max_{u \in V_k \setminus \{0\}} \frac{q_h(u)}{\|u\|_{L^2(\Omega)}^2}, \quad (37)$$

where V_k runs over all k -dimensional subspaces of $H^1(\Omega)$. In particular, Equation (37) also holds for $h = 0$, that is, for the Neumann eigenvalue problem.

It follows immediately from (35) that, for all $u \in H^1(\Omega)$,

$$q_h[u] \geq (1 - \eta)q_0(u) - \left(K_1(\Omega, F)H + K_2(\Omega, F)\frac{H^2}{\eta} \right) \|u\|_{L^2(\Omega)}^2. \quad (38)$$

From the min-max principle, we get, for all $k \geq 1$,

$$\mu_k(\Omega, h) \geq (1 - \eta)\mu_k^N(\Omega) - \left(K_1(\Omega, F)H + K_2(\Omega, F)\frac{H^2}{\eta} \right).$$

This is Inequality (32). Inequality (33) is straightforwardly equivalent. \square

We see from the previous result that to obtain an upper bound for the Robin counting function, it is sufficient to find an upper bound for a corresponding Neumann counting function. This has been done in [GL20], under the additional assumption that Ω is convex, to derive explicit upper bounds for the number of Courant-sharp Neumann eigenvalues. The proof of the following result is given in [GL20, Appendix A].

Proposition 4.2. *For any convex domain Ω and any $\mu > 0$,*

$$N_\Omega^N(\mu) \leq \frac{n^{\frac{n}{2}}}{\pi^n} \mu^{\frac{n}{2}} \left| \Omega + \frac{\pi}{\sqrt{\mu}} \mathbb{B} \right|. \quad (39)$$

where the last factor in the right-hand side is the volume (i.e. the Lebesgue measure) of the Minkowski sum of the two convex sets.

From Inequality (39) we see that an upper bound for the volume $|\Omega + \delta \mathbb{B}|$ translates into an upper bound for the Neumann counting function. The volume can either be expressed with the help of the so-called *quermassintegrals* (see for instance [Sch14, Chap. 4]) or estimated from above using the maximal scalar curvature of $\partial\Omega$ in the case where Ω is a C^2 domain.

Corollary 4.3. *For any convex domain Ω and any $\mu > 0$,*

$$N_\Omega^N(\mu) \leq \frac{n^{\frac{n}{2}}}{\pi^n} |\Omega| \mu^{\frac{n}{2}} + \frac{n^{\frac{n}{2}}}{\pi^{n-1}} |\partial\Omega| \mu^{\frac{n-1}{2}} + n^{\frac{n}{2}} \sum_{j=2}^n \binom{n}{j} W_j(\Omega) \left(\frac{\mu}{\pi^2} \right)^{\frac{n-j}{2}}, \quad (40)$$

where $W_j(\Omega)$ stands for the j -th quermassintegral of Ω .

If, in addition, Ω is of class C^2 , then for any $\mu > 0$,

$$N_\Omega^N(\mu) \leq \frac{n^{\frac{n}{2}}}{\pi^n} |\Omega| \mu^{\frac{n}{2}} + n^{\frac{n}{2}} |\partial\Omega| \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} \kappa_{\max}^j \left(\frac{\mu}{\pi^2} \right)^{\frac{n-j-1}{2}}, \quad (41)$$

where κ_{\max} is the maximum over $\partial\Omega$ of the largest principal curvature.

Proof. By definition of the quermassintegrals, for all $\delta > 0$,

$$|\Omega + \delta \mathbb{B}| = \sum_{j=0}^n \binom{n}{j} W_j(\Omega) \delta^j.$$

Taking $\delta = \frac{\pi}{\sqrt{\mu}}$, using Inequality (39) and recalling that $W_0(\Omega) = |\Omega|$ and $W_1(\Omega) = \frac{1}{n} |\partial\Omega|$, we get (40).

In the same way, (41) follows from the inequality

$$|\Omega + \delta \mathbb{B}| \leq |\Omega| + |\partial\Omega| \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} (\kappa_{\max})^j \delta^{j+1},$$

which is proved in [GL20, Appendix A]. \square

Thus, in order to obtain an upper bound for the number of Courant-sharp Robin eigenvalues of Ω , it is sufficient to obtain an upper bound for the largest Courant-sharp Robin eigenvalue and substitute it into the bounds for the counting functions using Proposition 4.1 and Corollary 4.3.

4.2. Geometric upper bound for Courant-sharp Robin eigenvalues. In this section we take $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to be an open, bounded, connected set with C^2 boundary. We will see that the assumption that the boundary is C^2 allows us to obtain explicit geometric control in the desired bounds.

Throughout we use the following notation

- $V := |\Omega|$ is the Lebesgue measure,
- $S := |\partial\Omega|$ is the surface measure,
- $\rho := S/V^{1-\frac{1}{n}}$ is the isoperimetric ratio,
- t_+ is the minimal radius of curvature (i.e., t_+^{-1} is the supremum of the maximum modulus of the principal curvatures $\{\kappa_1, \dots, \kappa_{n-1}\}$ of $\partial\Omega$),
- δ_0 , is the minimum between t_+ and the cut-distance with respect to the interior of Ω (see, e.g., [GL20, Section 3] for a precise definition of the cut-distance),
- δ_1 is the minimum between t_+ and the cut-distance with respect to the entire complement of $\partial\Omega$ (i.e., the interior and exterior).
- $H := \max\{-h(x) : x \in \partial\Omega\}$, where h is the real-valued function appearing in the Robin boundary condition.

Our main result is the following.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, connected set with C^2 boundary. There exists a constant C , depending only on n , such that any Courant-sharp Robin eigenvalue μ satisfies*

$$\mu \leq C \left(\frac{V^{\frac{2}{n}}}{\delta_1^4} + \frac{\rho^4}{V^{\frac{2}{n}}} + V^{\frac{2}{n}} H^4 \right).$$

From the proof of Theorem 4.4, we will see that the constant C may be very large.

We first need to estimate the constants $\Gamma_1(\Omega, F)$, $\Gamma_2(\Omega, F)$ from Proposition 2.4 in terms of n and of some of the geometric quantities of Ω . To do this we will construct an outward-pointing function and use it to obtain an outward-pointing vector field. We have the following.

Lemma 4.5. *Let $n \geq 2$. There exists a constant $C > 0$, depending only on n , such that, for any set $\Omega \subset \mathbb{R}^n$, open, bounded, connected with C^2 boundary, we can find an outward-pointing vector field F for which*

$$\Gamma_1(\Omega, F) \leq \frac{C}{\delta_0}, \quad \Gamma_2(\Omega, F) \leq C.$$

Proof. In order to find such an F and to estimate $\Gamma_1(\Omega, F)$, $\Gamma_2(\Omega, F)$, we first construct an outward-pointing function. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function such that

- (i) $\varphi(t) = t$ for $t \leq \frac{1}{2}$,
- (ii) $\varphi(t) = c \in (\frac{1}{2}, \frac{3}{4})$ for $t \geq \frac{3}{4}$.

We define

$$g(x) := \varphi \left(\sigma(x) \frac{d(x, \partial\Omega)}{\delta_0} \right),$$

where

$$\sigma(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega, \\ -1 & \text{if } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

If $x \in \Omega$ and $d(x, \partial\Omega) \geq \frac{3}{4}\delta_0$, $g(x) = c$. On the other-hand, it follows from the definition of δ_0 that the function $x \mapsto d(x, \partial\Omega)$ is of class C^2 in the open inner tubular neighborhood of the boundary

$$\partial\Omega_{\delta_0}^+ := \{x \in \Omega : d(x, \partial\Omega) < \delta_0\}.$$

(See, for example, [GT01, Lemma 14.16]). It follows that g is C^2 in Ω , and in fact is C^2 in an open-neighborhood¹ of $\bar{\Omega}$.

For x in the inner tubular neighborhood $\partial\Omega_{\delta_0}^+$, the chain rule implies

$$\nabla g(x) = \frac{1}{\delta_0} \varphi' \left(\frac{d(x, \partial\Omega)}{\delta_0} \right) \nabla d(x, \partial\Omega). \quad (42)$$

It is well-known that

$$|\nabla d(x, \partial\Omega)| = 1$$

and it is clear that there exists some absolute constant C_1 such that

$$|\varphi'(t)| \leq C_1$$

for all $t \in \mathbb{R}$. It follows that

$$|\nabla g(x)| \leq \frac{C_1}{\delta_0}$$

for all $x \in \partial\Omega_{\delta_0}^+$. We note that Formula (42) extends by continuity to $x \in \partial\Omega$. Since $\varphi'(0) = 1$, this implies, for all $x \in \partial\Omega$,

$$\nabla g(x) \cdot \nu(x) = \frac{1}{\delta_0}. \quad (43)$$

In particular, g is an outward-pointing function with constant $\gamma_g = \frac{1}{\delta_0}$.

Furthermore, from Property (ii) of φ , $\nabla g(x) = 0$ for all $x \in \Omega$ such that $d(x, \partial\Omega) \geq \frac{3}{4}\delta_0$. It follows that

$$|\nabla g(x)| \leq \frac{C_1}{\delta_0} \quad (44)$$

for all $x \in \bar{\Omega}$.

Differentiating Formula (42) once more, we find, for $x \in \partial\Omega_{\delta_0}^+$, that

$$\begin{aligned} \Delta g(x) &= \frac{1}{\delta_0^2} \varphi'' \left(\frac{d(x, \partial\Omega)}{\delta_0} \right) |\nabla d(x, \partial\Omega)|^2 + \frac{1}{\delta_0} \varphi' \left(\frac{d(x, \partial\Omega)}{\delta_0} \right) \Delta d(x, \partial\Omega) \\ &= \frac{1}{\delta_0^2} \varphi'' \left(\frac{d(x, \partial\Omega)}{\delta_0} \right) + \frac{1}{\delta_0} \varphi' \left(\frac{d(x, \partial\Omega)}{\delta_0} \right) \Delta d(x, \partial\Omega). \end{aligned} \quad (45)$$

It is clear that there exists some absolute constant C_2 such that

$$|\varphi''(t)| \leq C_2$$

for all $t \in \mathbb{R}$. Property (ii) of φ implies that $\Delta g(x) = 0$ for all $x \in \Omega$ such that $d(x, \partial\Omega) \geq \frac{3}{4}\delta_0$. It remains to estimate $|\Delta d(x, \partial\Omega)|$. Since $\partial\Omega$ is C^2 , we have, for all $x \in \partial\Omega_{\delta_0}^+$,

$$\Delta d(x, \partial\Omega) = \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d(x, \partial\Omega)},$$

where the $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures at y , the unique point in $\partial\Omega$ such that $|x-y| = d(x, \partial\Omega)$. (See, for example, [GT01, Appendix 14.6]). Let us recall that, by definition, $\delta_0 \leq t_+$ and $\kappa_i \leq t_+^{-1}$ for all $1 \leq i \leq n-1$. Then,

$$|\kappa_i d(x, \partial\Omega)| \leq |\kappa_i| \cdot \frac{3\delta_0}{4} \leq |\kappa_i| \cdot \frac{3t_+}{4} \leq \frac{3}{4}.$$

¹The distance from $\partial\Omega$ up to which g is C^2 in the complement of Ω will depend on Ω , through the geometric parameter δ_1 . However, since we only consider the values of g and its derivatives in $\bar{\Omega}$, we just need g to be C^2 in some open neighborhood of $\bar{\Omega}$, even very small, and only the parameter δ_0 will play a role.

Therefore, if $x \in \partial\Omega_{3\delta_0/4}^+$,

$$|\Delta d(x, \partial\Omega)| \leq \sum_{i=1}^{n-1} 4|\kappa_i| \leq \frac{4(n-1)}{t_+} \leq \frac{4(n-1)}{\delta_0}.$$

This finally gives us

$$|\Delta g(x)| \leq \frac{C_2}{\delta_0^2} + \frac{4(n-1)C_1}{\delta_0^2}. \quad (46)$$

Now we set

$$C := C_2 + 4(n-1)C_1$$

(and we note that $C \geq 2C_1$). Since g is an outward-pointing function with constant γ_g , $F := \nabla g$ is an outward-pointing vector field, with constant

$$\gamma_F = \gamma_g = \frac{1}{\delta_0}.$$

Using the formulas for $\Gamma_1(\Omega, F)$ and $\Gamma_2(\Omega, F)$ from Proposition 2.4, we obtain

$$\begin{aligned} \Gamma_1(\Omega, F) &= \frac{1}{\gamma_F} \sup_{\Omega} |\operatorname{div}(F)| = \delta_0 \sup_{\Omega} |\Delta g| \leq \delta_0 \frac{C}{\delta_0^2} = \frac{C}{\delta_0}, \\ \Gamma_2(\Omega, F) &= \frac{2}{\gamma_F} \sup_{\Omega} |F| = 2\delta_0 \sup_{\Omega} |\nabla g| \leq 2\delta_0 \frac{C_1}{\delta_0} \leq 2C_1 \leq C. \end{aligned}$$

□

In order to prove Theorem 4.4, we make use of the following standard inequalities.

Remark 4.6. Let a and b be non-negative numbers, $0 \leq \theta \leq 1$ and $p \geq 1$.

- (i) $(a+b)^\theta \leq a^\theta + b^\theta$. If $\theta < 1$ and $a, b > 0$, the inequality is strict. A useful special case is $\theta = \frac{1}{2}$: $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.
- (ii) $ab^{p-1} \leq C(a^p + b^p)$, with $C = C(p)$.²
- (iii) If $a \leq b$, then $b^p - a^p \leq p b^{p-1}(b-a)$.
- (iv) $(a+b)^p \leq C(a^p + b^p)$, with $C = C(p)$.
- (v) For all $\varepsilon > 0$, $(a+b)^p \leq (1+\varepsilon)a^p + C b^p$, with $C = C(p, \varepsilon)$.

By using these inequalities in the proof of Theorem 4.4, we obtain the existence of a constant $C(n)$, depending only on n , as claimed. At each step of the proof, we take the maximum of various constants depending only on n and thus any closed-form expression arising would be very complicated and far from optimal. Hence we use $C = C(n)$ throughout to denote a constant depending on n which may change from line to line.

Proof of Theorem 4.4. We define

$$\hat{\mu}_H^{\frac{1}{2}} := \left(\mu + C \frac{H}{\delta_0} \right)^{\frac{1}{2}} + CH, \quad (47)$$

where C is a constant depending only on n , defined by Lemma 4.5. The idea of the proof is to apply similar arguments to those used in [GL20] to $\hat{\mu}_H$ instead of to μ .

We assume $\mu \geq 0$. Then, according to Remark 4.6 part (i),

$$\hat{\mu}_H^{\frac{1}{2}} \leq \mu^{\frac{1}{2}} + \Delta_H,$$

²By this, we mean that for a given p , there exists a constant C , depending only on p , such that Inequality (ii) is satisfied for all $a, b \geq 0$. The meaning in the other statements in Remark 4.6 is similar.

with

$$\Delta_H := \left(C \frac{H}{\delta_0} \right)^{\frac{1}{2}} + CH.$$

We fix $\hat{\varepsilon}(n) \in (0, 1)$ (to be specified later). We consider a spectral pair (eigenvalue-eigenfunction) (μ, u) for the Robin eigenvalue problem. As in [Lén19, GL20, HS24], we consider ν_0 and ν_1 , respectively the number of bulk and boundary nodal domains (defined using $\varepsilon_0 = \hat{\varepsilon}(n)$), and $\hat{\mu}_H$, associated with μ via Equation (47).

If

$$\left(\frac{V^{\frac{2}{n}}}{\hat{\mu}_H} \right)^{\frac{1}{4}} \leq \delta_1, \quad (48)$$

that is, if

$$\hat{\mu}_H \geq \frac{V^{\frac{2}{n}}}{\delta_1^4}, \quad (49)$$

then, by following the same arguments as in [GL20], we have

$$\nu_0 \leq \frac{V}{\Lambda(n)^{\frac{n}{2}}} \left((1 + \varepsilon(n)) \hat{\mu}_H + C \frac{\hat{\mu}_H^{\frac{1}{2}}}{V^{\frac{1}{n}}} \right)^{\frac{n}{2}}, \quad (50)$$

where $\varepsilon(n) > 0$ is defined by

$$1 + \varepsilon(n) = \frac{1 + \hat{\varepsilon}(n)}{1 - \hat{\varepsilon}(n)}$$

and can be chosen arbitrarily small by taking $\hat{\varepsilon}(n)$ small enough, at the cost of a larger C . In what follows, we express the inequalities in terms of $\varepsilon(n)$, for which we will specify some properties later (this correspondingly specifies $\hat{\varepsilon}(n)$).

Still under Condition (49), we have³

$$\nu_1 \leq C(n) S V^{\frac{1}{2n}} \hat{\mu}_H^{-\frac{1}{4}} \left(\hat{\mu}_H + \frac{\hat{\mu}_H^{\frac{1}{2}}}{V^{\frac{1}{n}}} \right)^{\frac{n}{2}}. \quad (51)$$

Now, as in [GL20, Section 2.1], let $(\lambda_k(\Omega))_{k \geq 1}$ denote the Dirichlet eigenvalues of the Laplacian on Ω and, for $\mu > 0$, define the Dirichlet counting function as:

$$N_{\Omega}^D(\mu) := \#\{k \in \mathbb{N} : \lambda_k(\Omega) < \mu\},$$

and the corresponding remainder $R_{\Omega}^D(\mu)$ such that

$$N_{\Omega}^D(\mu) = \frac{\omega_n |\Omega|}{(2\pi)^n} \mu^{n/2} - R_{\Omega}^D(\mu), \quad (52)$$

where the first term in the right-hand side of Equation (52) corresponds to Weyl's law.

By monotonicity of the Robin eigenvalues, we have

$$N_{\Omega}^h(\mu) \geq N_{\Omega}^D(\mu) = w_n V \mu^{\frac{n}{2}} - R_{\Omega}^D(\mu)$$

³Inequalities (50) and (51) essentially correspond to the upper bounds on page 23 of [GL20]. However, there is a gap in the proof of the upper bound for ν_1 in that reference. It can be fixed at the cost of changing the values of the constants and putting a stronger condition on μ . Namely, one must impose that Inequality (48) is satisfied, rather than Inequality (34) in [GL20], meaning that δ_0 must be replaced with δ_1 . That way, one obtains (51).

with $w_n := \omega_n / (2\pi)^n$. From the known bounds on the Dirichlet counting function (see [GL20, Section 9.2] and [vdBG16, Section 2]), we find that under a condition of the form

$$\mu \geq C \frac{V^2}{S^2 \delta_0^4}, \quad (53)$$

we have

$$R_\Omega^D(\mu) \leq C (SV)^{\frac{1}{2}} \mu^{\frac{n}{2} - \frac{1}{4}} \leq C (SV)^{\frac{1}{2}} (\hat{\mu}_H)^{\frac{n}{2} - \frac{1}{4}}.$$

It follows that

$$N_\Omega^h(\mu) \geq w_n V \hat{\mu}_H^{\frac{n}{2}} - C (SV)^{\frac{1}{2}} \hat{\mu}_H^{\frac{n}{2} - \frac{1}{4}} - w_n V \left(\hat{\mu}_H^{\frac{n}{2}} - \mu^{\frac{n}{2}} \right).$$

Using Remark 4.6 part (iii) with $a = \mu^{\frac{1}{2}}$, $b = \hat{\mu}_H^{\frac{1}{2}}$ and $p = n$, we get

$$\hat{\mu}_H^{\frac{n}{2}} - \mu^{\frac{n}{2}} \leq n \hat{\mu}_H^{\frac{n}{2} - \frac{1}{2}} \left(\hat{\mu}_H^{\frac{1}{2}} - \mu^{\frac{1}{2}} \right) \leq n \hat{\mu}_H^{\frac{n}{2} - \frac{1}{2}} \Delta_H.$$

Hence we obtain

$$N_\Omega^h(\mu) \geq w_n V \hat{\mu}_H^{\frac{n}{2}} - C (SV)^{\frac{1}{2}} \hat{\mu}_H^{\frac{n}{2} - \frac{1}{4}} - C V \hat{\mu}_H^{\frac{n}{2} - \frac{1}{2}} \Delta_H. \quad (54)$$

A necessary condition for (μ, u) to be a Courant-sharp pair is

$$N_\Omega^h(\mu) - \nu_0 - \nu_1 < 0.$$

If $\mu \geq 0$ and Conditions (49) and (53) are satisfied, we have

$$\begin{aligned} N_\Omega^h(\mu) - \nu_0 - \nu_1 &\geq w_n V \hat{\mu}_H^{\frac{n}{2}} - C (SV)^{\frac{1}{2}} \hat{\mu}_H^{\frac{n}{2} - \frac{1}{4}} - C V \hat{\mu}_H^{\frac{n}{2} - \frac{1}{2}} \Delta_H \\ &\quad - \frac{V}{\Lambda(n)^{\frac{n}{2}}} \left((1 + \varepsilon(n)) \hat{\mu}_H + C \frac{\hat{\mu}_H^{\frac{1}{2}}}{V^{\frac{1}{n}}} \right)^{\frac{n}{2}} \\ &\quad - C S V^{\frac{1}{2n}} \hat{\mu}_H^{-\frac{1}{4}} \left(\hat{\mu}_H + \frac{\hat{\mu}_H^{\frac{1}{2}}}{V^{\frac{1}{n}}} \right)^{\frac{n}{2}}. \end{aligned} \quad (55)$$

To simplify the analysis, we define the dimensionless (i.e. scaling invariant) quantities

$$\begin{aligned} \xi &:= V^{\frac{1}{n}} \mu^{\frac{1}{2}}, \\ \hat{\xi} &:= V^{\frac{1}{n}} \hat{\mu}_H^{\frac{1}{2}}. \end{aligned}$$

Inequality (55) then becomes

$$\begin{aligned} N_\Omega^h(\mu) - \nu_0 - \nu_1 &\geq w_n \hat{\xi}^n - C \rho^{\frac{1}{2}} \hat{\xi}^{n - \frac{1}{2}} - C V^{\frac{1}{n}} \Delta_H \hat{\xi}^{n-1} \\ &\quad - \frac{1}{\Lambda(n)^{\frac{n}{2}}} \left((1 + \varepsilon(n)) \hat{\xi}^2 + C \hat{\xi} \right)^{\frac{n}{2}} \\ &\quad - C \rho \hat{\xi}^{-\frac{1}{2}} \left(\hat{\xi}^2 + \hat{\xi} \right)^{\frac{n}{2}}. \end{aligned}$$

If we apply Remark 4.6 part (v) to the term on the second line (with $p = \frac{n}{2}$ and $\varepsilon = \varepsilon(n)$) and rearrange the right-hand side, we find

$$\begin{aligned} N_\Omega^h(\mu) - \nu_0 - \nu_1 &\geq \left(w_n - \frac{(1 + \varepsilon(n))^{\frac{n}{2} + 1}}{\Lambda(n)^{\frac{n}{2}}} \right) \hat{\xi}^n - C \rho^{\frac{1}{2}} \hat{\xi}^{n - \frac{1}{2}} - C V^{\frac{1}{n}} \Delta_H \hat{\xi}^{n-1} \\ &\quad - C \hat{\xi}^{\frac{n}{2}} - C \rho \hat{\xi}^{-\frac{1}{2}} \left(\hat{\xi}^2 + \hat{\xi} \right)^{\frac{n}{2}}. \end{aligned} \quad (56)$$

Since

$$w_n > \frac{1}{\Lambda(n)^{\frac{n}{2}}},$$

we can choose $\varepsilon(n) > 0$ such that

$$\ell(n) := w_n - \frac{(1 + \varepsilon(n))^{\frac{n}{2}+1}}{\Lambda(n)^{\frac{n}{2}}} > 0.$$

Furthermore, we have, from the isoperimetric inequality, $\rho \geq n\omega_n^{\frac{1}{n}}$. It follows that for any $0 \leq \alpha < \beta$,

$$\rho^\alpha \leq C\rho^\beta,$$

with $C = C(n, \alpha, \beta) := (n\omega_n^{1/n})^{\alpha-\beta}$. In addition, if $\hat{\xi} \geq \rho$, we have, with the same C ,

$$\hat{\xi}^\alpha \leq C\hat{\xi}^\beta.$$

Using these remarks in Inequality (56), we obtain, if $\hat{\xi} \geq \rho$,

$$N_\Omega(\mu) - \nu_0 - \nu_1 \geq \ell(n)\hat{\xi}^n - C\left(\rho + V^{\frac{1}{n}}\Delta_H\right)\hat{\xi}^{n-\frac{1}{2}}.$$

Now, using Remark 4.6 part (ii) with $a = \delta_0^{-\frac{1}{2}}$, $b = H^{\frac{1}{2}}$ and $p = 2$, we find

$$\Delta_H \leq C\left(\frac{1}{\delta_0} + H\right).$$

Therefore, if $\hat{\xi} \geq \rho$,

$$N_\Omega(\mu) - \nu_0 - \nu_1 \geq \ell(n)\hat{\xi}^n - C\left(\rho + \frac{V^{\frac{1}{n}}}{\delta_0} + V^{\frac{1}{n}}H\right)\hat{\xi}^{n-\frac{1}{2}}. \quad (57)$$

If $\hat{\xi} \geq \hat{\xi}^*$, with

$$\hat{\xi}^* := \frac{C^2}{\ell(n)^2} \left(\rho + \frac{V^{\frac{1}{n}}}{\delta_0} + V^{\frac{1}{n}}H\right)^2,$$

then the right-hand side of Inequality (57) is non-negative. Note that by applying Remark 4.6 part (iv) with $p = 2$ repeatedly, we find that there exists C large enough so that

$$\hat{\xi}^* \leq C\left(\rho^2 + \frac{V^{\frac{2}{n}}}{\delta_0^2} + V^{\frac{2}{n}}H^2\right).$$

Putting everything together, we conclude that there exists a constant C (large enough) so that under conditions (49) and (53), if

$$\hat{\xi} \geq C\left(\rho^2 + \frac{V^{\frac{2}{n}}}{\delta_0^2} + V^{\frac{2}{n}}H^2\right),$$

then (u, μ) cannot be Courant-sharp (we note that C can be chosen large enough that the above inequality implies $\hat{\xi} \geq \rho$).

Conditions (49) and (53) are not independent. Since $\delta_1 \leq \delta_0$ by definition and $\rho \geq n\omega_n^{1/n}$, we have

$$\frac{V^2}{S^2\delta_0^4} \leq \frac{V^{\frac{2}{n}}}{\rho^2\delta_1^4} \leq C\frac{V^{\frac{2}{n}}}{\delta_1^4},$$

with $C = (n\omega_n)^{-2}$. Under the condition $\mu \geq 0$, we have $\hat{\mu}_H \geq \mu$. Therefore, we can choose C (large enough) so that

$$\mu \geq C \frac{V_n^{\frac{2}{n}}}{\delta_1^4}$$

implies both conditions (49) and (53).

It follows that, if (μ, u) is Courant-sharp,

$$\xi \leq \max \left\{ C \frac{V_n^{\frac{2}{n}}}{\delta_1^2}, C \left(\rho^2 + \frac{V_n^{\frac{2}{n}}}{\delta_0^2} + V_n^{\frac{2}{n}} H^2 \right) \right\},$$

which, since $\delta_1 \leq \delta_0$, implies

$$\xi \leq C \left(\frac{V_n^{\frac{2}{n}}}{\delta_1^2} + \rho^2 + V_n^{\frac{2}{n}} H^2 \right). \quad (58)$$

From $\mu = \xi^2 / V_n^{\frac{2}{n}}$ and from Remark 4.6 part (iv), applied repeatedly with $p = 2$, we obtain that, for (μ, u) Courant-sharp,

$$\mu \leq C \left(\frac{V_n^{\frac{2}{n}}}{\delta_1^4} + \frac{\rho^4}{V_n^{\frac{2}{n}}} + V_n^{\frac{2}{n}} H^4 \right).$$

□

4.3. Geometric upper bound for the number of Courant-sharp Robin eigenvalues of a convex, C^2 domain. In this section we take $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to be an open, bounded, connected, convex set with C^2 boundary. We use the same notation as was introduced at the beginning of Section 4.2. We note that the convexity assumption on Ω allows us to employ Corollary 4.3.

Our main result is the following.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, connected, convex set with C^2 boundary. There exists a constant C , depending only on n , such that the number of Courant-sharp Robin eigenvalues of Ω is at most*

$$C \left(\frac{V^2}{t_+^{2n}} + \rho^{2n} + V^2 H^{2n} \right).$$

Roughly speaking, the strategy of the proof is to substitute the result of Theorem 4.4 into that of Corollary 4.3. As the constant given in the statement of Theorem 4.4 may be very large, the constant C given in the statement of Theorem 4.7 may be very large too.

Proof. By Lemma 4.5, we see that there exists a constant $C > 0$, depending only on n and a vector field $F : \Omega \rightarrow \mathbb{R}^n$, outward-pointing relative to Ω , such that Proposition 4.1 holds with

$$K_1(\Omega, F) \leq \frac{C}{\delta_0}, \quad K_2(\Omega, F) \leq C.$$

Thus, there exists a constant K , depending only on n , such that, for all $\eta \in (0, 1)$ and all $x \geq 0$,

$$N_\Omega^h(x) \leq N_\Omega^N \left(\frac{1}{1-\eta} \left(x + K \left(\frac{H}{\delta_0} + \frac{H^2}{\eta} \right) \right) \right). \quad (59)$$

Since Ω is convex, inequality (41) gives an upper bound for the Neumann counting function, which we can put into a simplified form: there exists a constant C , depending only on n , such that for all $x \geq 0$,

$$N_\Omega^N(x) \leq C x^{\frac{n}{2}} \left(V + S x^{-\frac{1}{2}} \left(1 + \frac{1}{t_+ \sqrt{x}} \right)^{n-1} \right). \quad (60)$$

We begin by deducing a weaker inequality from Inequality (60) which is easier to use. Setting $y = V^{\frac{1}{n}} x^{\frac{1}{2}}$ (dimensionless variable), we have

$$N_{\Omega}^N(x) \leq C y^n \left(1 + \rho y^{-1} \left(1 + \frac{V^{\frac{1}{n}}}{t_+ y} \right)^{n-1} \right) = C \left(y^n + \rho \left(y + \frac{V^{\frac{1}{n}}}{t_+} \right)^{n-1} \right).$$

Using Remark 4.6 part (iv) and part (ii) (twice), we get

$$N_{\Omega}^N(x) \leq C \left(y^n + \rho y^{n-1} + \rho \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^{n-1} \right) \leq C \left(y^n + \rho^n + y^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right).$$

We obtain, for $x \geq 0$,

$$N_{\Omega}^N(x) \leq C \left(y^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right). \quad (61)$$

From Inequality (59) (fixing, for instance, $\eta = \frac{1}{2}$), and Inequality (61), we get, for all x ,

$$N_{\Omega}^h(x) \leq C \left(V \left(C \left(x_+ + \frac{H}{t_+} + H^2 \right) \right)^{\frac{n}{2}} + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right),$$

where $x_+ = \max\{0, x\}$ ⁴. Setting $y_+ := V^{\frac{1}{n}} x_+^{\frac{1}{2}}$, using repeatedly Remark 4.6 part (iv) with $p = \frac{n}{2}$ and part (ii) with $p = 2$, we get

$$\begin{aligned} N_{\Omega}^h(x) &\leq C \left(y_+^n + \left(\frac{V^{\frac{2}{n}} H}{t_+} \right)^{\frac{n}{2}} + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right) \\ &\leq C \left(y_+^n + \left(\left(\frac{V^{\frac{1}{n}}}{t_+} \right)^2 + (V^{\frac{1}{n}} H)^2 \right)^{\frac{n}{2}} + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right) \\ &\leq C \left(y_+^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n + (V^{\frac{1}{n}} H)^n + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right). \end{aligned}$$

Finally, we obtain

$$N_{\Omega}^h(x) \leq C \left(y_+^n + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right). \quad (62)$$

We now denote the largest Courant-sharp Robin eigenvalue by $\mu = \mu_k(\Omega, h)$. Then, the number of Courant-sharp Robin eigenvalues (counted with multiplicities) is at most k . For $\varepsilon > 0$ (using the notation $\mu_+ = \max\{\mu, 0\}$ and $\xi = V^{\frac{1}{n}} \mu_+^{\frac{1}{2}}$), we have

$$k \leq N_{\Omega}^h(\mu_+ + \varepsilon) \leq C \left((\mu_+ + \varepsilon)^{\frac{n}{2}} V + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right),$$

so that, taking $\varepsilon \rightarrow 0^+$, we get

$$k \leq C \left(\xi^n + (V^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V^{\frac{1}{n}}}{t_+} \right)^n \right). \quad (63)$$

⁴We give this definition to deal with the case where $x < 0$, which is relevant since the Robin Laplacian has negative eigenvalues in general.

We now use Inequality (58) to obtain

$$\xi \leq C \left(\frac{V_n^{\frac{2}{n}}}{t_+^2} + \rho^2 + V_n^{\frac{2}{n}} H^2 \right), \quad (64)$$

where we used the fact that $\delta_1 = \delta_0 = t_+$ since Ω is convex. Substituting Inequality (64) for ξ in Inequality (63) and using Remark 4.6 part (iv) with $p = n$ repeatedly, we get

$$k \leq C \left(\left(\frac{V_n^{\frac{1}{n}}}{t_+} \right)^{2n} + \rho^{2n} + (V_n^{\frac{1}{n}} H)^{2n} + (V_n^{\frac{1}{n}} H)^n + \rho^n + \left(\frac{V_n^{\frac{1}{n}}}{t_+} \right)^n \right). \quad (65)$$

To complete the proof, we note that $\rho \geq n\omega_n^{\frac{1}{n}}$ (from the isoperimetric inequality), so that

$$\begin{aligned} \rho^n &\leq C\rho^{2n}, \\ (V_n^{\frac{1}{n}} H)^n &\leq C\rho^n (V_n^{\frac{1}{n}} H)^n \leq C\left(\rho^{2n} + (V_n^{\frac{1}{n}} H)^{2n}\right), \\ \left(\frac{V_n^{\frac{1}{n}}}{t_+}\right)^n &\leq C\rho^n \left(\frac{V_n^{\frac{1}{n}}}{t_+}\right)^n \leq C\left(\rho^{2n} + \left(\frac{V_n^{\frac{1}{n}}}{t_+}\right)^{2n}\right). \end{aligned}$$

Substituting these bounds for the n -th powers in Inequality (65), we conclude that

$$k \leq C \left(\left(\frac{V_n^{\frac{1}{n}}}{t_+} \right)^{2n} + \rho^{2n} + (V_n^{\frac{1}{n}} H)^{2n} \right).$$

□

Remark 4.8. We note that the bounds derived in the proof of Theorem 4.4 implicitly contain a bound on the number of negative Robin eigenvalues of Ω . Indeed, for all $x < 0$, $x_+ = 0$ and therefore $y_+ = 0$. This means that for negative values of x , taking $y_+ = 0$ in the right-hand side of (62) gives an upper bound for the number of negative Robin eigenvalues, $N_\Omega^h(x)$, which is explicit in terms of H and some of the geometric quantities of Ω .

APPENDIX A. PROOF OF PROPOSITION 3.4

The idea behind the quantitative version of the Faber-Krahn inequality for the first Dirichlet eigenvalue (see [FMP09] and the references therein) employs the classical method of comparing the Dirichlet energy of the eigenfunction with that of its radially symmetric, monotone rearrangement on a Euclidean ball, known as the Pólya-Szegő inequality, together with a quantitative version of the Euclidean isoperimetric inequality. The proof of Proposition 3.4 follows a similar line of reasoning. We first establish a quantitative version of the isoperimetric inequality in terms of the perimeter of the interior boundary of ‘small’ domains in Ω and a modified Fraenkel asymmetry (see Proposition A.1). We follow the proof of the (non-quantitative) isoperimetric inequality in [DFV24], incorporating the quantitative isoperimetric inequality and applying it to part of the domain distant from the boundary. It in turn gives an improvement in the Pólya-Szegő inequality. Then we follow the proof in [DFV24] while adapting the approach in [FMP09] to obtain a quantitative version of the Faber-Krahn inequality in the setting of Proposition 3.4. While the method of the proof is based on the same ideas as in [FMP09, DFV24], the presence of a modified Fraenkel asymmetry and a modification of the eigenfunction in the Pólya-Szegő Inequality pose some challenges and require adaptation at each step of the proof.

We first start with some definitions and notation. Throughout this section, we use $C_i = C_i(n)$ to denote constants depending only on n , and for the rest of the constants we mainly use $c_i = c_i(\cdot)$.

For any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ open, we define

$$|\mathbf{D}f|(U) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_U |\nabla f_n| : f_n \in \text{Lip}_{\text{loc}}(U), f_n \rightarrow f \text{ in } L^1_{\text{loc}}(U) \right\}.$$

For any Borel set $A \subseteq \mathbb{R}^n$, we set

$$|\mathbf{D}f|(A) := \inf \{ |\mathbf{D}f|(U) : A \subset U, \text{ and } U \text{ is open in } \mathbb{R}^n \}.$$

Subsequently for any two Borel sets A and E we define $\text{Per}(E, A) = |\mathbf{D}\chi_E|(A)$. We denote $\text{Per}(E, \mathbb{R}^n)$ by $\text{Per}(E)$ or $|\partial E|$. We say that E is of finite perimeter when $\text{Per}(E) < \infty$.

Let Ω be an open Lipschitz domain in \mathbb{R}^n . Note that $\text{Per}(E, \Omega)$ gives the interior perimeter of E in Ω . For simplicity, we use the following notation:

$$\text{Per}(E, \Omega) = |\partial E \cap \Omega|.$$

De Ponti, Farinelli, and Violo [DFV24, Theorem 4.1] proved an almost sharp isoperimetric inequality for any Borel set E in Ω with ‘small’ volume having its interior boundary $\text{Per}(E, \Omega)$ instead of $\text{Per}(E)$ in the inequality. We shall see that with some modification of their proof, we get a quantitative version. It is an interesting question whether the quantitative version we obtain for the Lipschitz domains can be generalised to the general setting of the PI spaces considered in [DFV24].

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, Lipschitz domain. For any $\epsilon, \delta \in (0, 1)$ there exist a neighbourhood Ω_δ of $\partial\Omega$ with $|\Omega_\delta| < \delta$ and positive constants $C_1 = C_1(n)$, $\alpha = \alpha(n, \epsilon, \delta)$ and $\beta = \beta(\epsilon)$ such that for every Borel set $E \subset \Omega$ with*

$$0 < |E| \leq \alpha, \quad \frac{|E \cap \Omega_\delta|}{|E|} \leq \beta, \quad (66)$$

we have

$$|\partial E \cap \Omega| \geq (1 - \epsilon) \left(1 + C_1 \tilde{A}(E)^2 \right) n \omega_n^{1/n} |E|^{\frac{n-1}{n}} \quad (67)$$

where $\tilde{A}(E) := \inf_U A(E \cap U)$, where the infimum is taken over all open sets $U \subseteq \Omega$ such that $\Omega \setminus \Omega_\delta \subset U$.

Proof. Let $U_t = \{x \in \overline{\Omega} : d(x, \partial\Omega) > t\}$. Take $t_0 > 0$ such that $\Omega_\delta := \Omega \setminus U_{t_0}$ has volume less than δ . We first show that for any Borel set $E \subset \overline{\Omega}$, there exists $t_1 \in (0, t_0)$ such that

$$|\partial(E \cap U_{t_1})| \leq |\partial E \cap \Omega| + c_1 |E|,$$

where $c_1 = c_1(\Omega, \delta)$ is a positive constant.

Note that the function $d(\cdot, \partial\Omega) \in \text{Lip}(\overline{\Omega})$. By the co-area formula [Mir03, Theorem 4.2], we have

$$\int_{t_0/2}^{t_0} \text{Per}(U_t, E) dt \leq \int_0^\infty \text{Per}(U_t, E) dt = \int_E |\nabla d| \leq |E|.$$

By Markov’s inequality,

$$\left| \left\{ t \in \left(\frac{t_0}{2}, t_0 \right) : \text{Per}(U_t, E) \geq \frac{3|E|}{t_0} \right\} \right| \leq \frac{t_0}{3|E|} \int_{t_0/2}^{t_0} \text{Per}(U_t, E) dt \leq \frac{t_0}{3}.$$

Hence, the set

$$T := \left\{ t \in \left(\frac{t_0}{2}, t_0 \right) : \text{Per}(U_t, E) < \frac{3|E|}{t_0} \right\}$$

has non-zero measure. Therefore, there exists $t_1 \in T$ such that $\text{Per}(E, \partial U_{t_1}) = 0$ following the same argument as in [APP22, Corollary 2.6]. Here, t_0 and t_1 depend only on Ω and δ . Moreover, since $\text{Per}(E, \partial U_{t_1}) = 0$, we can apply [APPV23, Proposition 2.6] to get

$$\begin{aligned} |\partial(E \cap U_{t_1})| &= \text{Per}(E \cap U_{t_1}) \leq \text{Per}(E, U_{t_1}) + \text{Per}(U_{t_1}, E^\circ) \\ &\leq \text{Per}(E, \Omega) + \frac{3|E|}{t_0}. \end{aligned}$$

Since $E \cap U_{t_1} \subset \Omega$, we can use the quantitative version of the isoperimetric inequality [FMP08]:

$$|\partial(E \cap U_{t_1})| \geq (1 + C_1 A(E \cap U_{t_1})^2) n \omega_n^{1/n} |E \cap U_{t_1}|^{\frac{n-1}{n}}.$$

Combining the above inequalities, we obtain

$$\begin{aligned} |\partial E \cap \Omega| = \text{Per}(E, \Omega) &\geq (1 + C_1 A(E \cap U_{t_1})^2) n \omega_n^{1/n} |E \cap U_{t_1}|^{\frac{n-1}{n}} - \frac{3|E|}{t_0} \\ &\geq (1 + C_1 A(E \cap U_{t_1})^2) n \omega_n^{1/n} |E \setminus \Omega_\delta|^{\frac{n-1}{n}} - \frac{3|E|}{t_0} \\ &\geq (1 + C_1 A(E \cap U_{t_1})^2) n \omega_n^{1/n} |E|^{\frac{n-1}{n}} \left(1 - \left(\frac{|E \cap \Omega_\delta|}{|E|} \right)^{\frac{n-1}{n}} - \frac{3|E|^{1/n}}{t_0 n \omega_n^{1/n}} \right). \end{aligned}$$

Hence, it is enough to have

$$\left(\frac{|E \cap \Omega_\delta|}{|E|} \right)^{\frac{n-1}{n}} \leq \frac{\epsilon}{2}, \quad \frac{3|E|^{1/n}}{t_0 n \omega_n^{1/n}} \leq \frac{\epsilon}{2}.$$

We conclude by taking $\beta = \left(\frac{\epsilon}{2}\right)^{\frac{n}{n-1}}$ and $\alpha = \frac{t_0^n}{6^n} n^n \omega_n \epsilon^n$. □

Another main ingredient of the proof is a generalization of the Pólya-Szegő inequality. The classical Pólya-Szegő inequality is stated for functions in $H_0^1(U)$ where U is a bounded open set in \mathbb{R}^n . Recently the Pólya-Szegő inequality has been extended to the setting of metric-measure spaces [MS20, NV22, DFV24]. Its main difference with the classical version is that we can view $\overline{\Omega}$ as a complete metric-measure space and compactly supported functions in $\overline{\Omega}$ can be nonzero on the boundary of Ω . Without any extra condition, the Pólya-Szegő inequality fails in this general setting. However, it holds true on subdomains of $\overline{\Omega}$ with small volume. In [DFV24, Theorem 2.22], it is stated that it holds for open sets in a bounded *PI space*. We do not need to know the definition of a PI space. We use [DFV24, Theorem 3.9], where it is shown that a bounded Lipschitz domain $\overline{\Omega}$ is a bounded PI space. Hence, we state Lemma A.2 below, which is a version of [DFV24, Theorem 2.22], only for Lipschitz domains. Let us first introduce some notation.

Let Ω be an open, bounded, Lipschitz domain in \mathbb{R}^n . Let $U \subsetneq \overline{\Omega}$ be an open set in $\overline{\Omega}$. For a non-negative Borel function u on U we define

$$\mu(t) = |\{u > t\}|.$$

We denote by U^* the ball centred at the origin having the same volume as U , and $u^* : U^* \rightarrow [0, \infty)$ the symmetric decreasing rearrangement of u . By abuse of notation, we use $\text{Lip}_c(U)$ to denote the space of Lipschitz functions compactly supported in U with induced topology from $\overline{\Omega}$.

Lemma A.2 (Pólya-Szegő inequality). *Let Ω be an open, bounded, Lipschitz domain in \mathbb{R}^n . There exists a constant $c = c(\Omega)$ such that for any open set $U \subsetneq \overline{\Omega}$ in $\overline{\Omega}$ with $|U| \leq c$, any*

$0 \neq u \in \text{Lip}_c(U)$ non-negative with $|\nabla u| \neq 0$ a.e. in $\{u > 0\}$, and any $0 < s \leq T := \sup_U u$ we have $u^* \in \text{Lip}_c(U^*)$ and

$$\int_{\{u \leq s\}} |\nabla u|^2 - \int_{\{u^* \leq s\}} |\nabla u^*|^2 \geq \int_0^s \frac{\text{Per}(\{u > t\}, \Omega)^2 - \text{Per}(\{u^* > t\})^2}{|\mu'(t)|}.$$

Proof. By [DFV24, Lemma 3.7], $\bar{\Omega}$ satisfies the assumption of [DFV24, Theorem 2.18]. Thus there exist constants $c_0 = c_0(\Omega) > 0$ and $c_1 = c_1(\Omega, n)$ such that for any U with $|U| \leq c_0$ and any Borel set $E \subset U$ we have

$$\text{Per}(E, \Omega) = |\partial E \cap \Omega| \geq c_1 |E|^{\frac{n-1}{n}}.$$

Hence, the assumption of [DFV24, Theorem 2.22] is met. Therefore, $u^* \in \text{Lip}_c(U^*)$. Moreover, for any $s \in (0, T)$, we have (see e.g. [NV22, Lemma 2.25])

$$-\mu'(t) = \int_{\{u^*=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u^*|} = \int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}, \quad a.e.$$

and as a result (see [DFV24, Inequality (2.29)])

$$\int_{\{u \leq s\}} |\nabla u|^2 \geq \int_0^s \frac{\text{Per}(\{u > t\}, \Omega)^2}{-\mu'(t)} dt. \quad (68)$$

Since $|\nabla u^*|$ is constant on $\{u^* = t\}$, we have

$$\int_{\{u^* \leq s\}} |\nabla u^*|^2 = \int_0^s dt \int_{\{u^*=t\}} |\nabla u^*| d\mathcal{H}^{n-1} = \int_0^s \frac{\text{Per}(\{u^* > t\})^2}{-\mu'(t)} dt. \quad (69)$$

We obtain the result by taking the difference of (68) and (69). \square

We now use Proposition A.1, and Lemma A.2 to prove Proposition 3.4.

Proof of Proposition 3.4. We choose $\theta_0 = \theta_0(\Omega, \epsilon, \delta)$, $\theta_1 = \theta_1(\Omega, \epsilon) > 0$ small enough such that θ_0 satisfies inequality (74) below, and θ_1 satisfies a set of inequalities (70), (73), (77), (81), (86), (90) below. Let $f : D \rightarrow \mathbb{R}$ be a $\lambda_1(D)$ eigenfunction, where D satisfies the assumption of the proposition. We know that f is strictly positive on D .

The strategy is to use Proposition A.1 to estimate the right-hand side of the Polya-Szegő inequality in Lemma A.2. Hence, we need to ensure that the condition of Proposition (A.1) is met for the super-level sets $\{f > 0\}$. For this reason, we restrict the range of t . We consider

$$\tilde{t} = \sup\{t : |\{f > t\}| \geq 2\sqrt{\theta_1}|D|\}$$

and define

$$\tilde{f} = (f - \tilde{t})^+, \quad \hat{f} = \min\{f, \tilde{t}\} =: f \wedge \tilde{t}.$$

Note that $|\{f > \tilde{t}\}| \leq 2\sqrt{\theta_1}|D|$ and for any $t \in (0, \tilde{t})$, we have $|\{f > t\}| \geq 2\sqrt{\theta_1}|D|$. We break down the proof into several steps.

Step 1. We start by giving a lower bound for the Rayleigh quotient of \tilde{f} . From [DFV24, Corollary 5.2], we know that there exists a constant $c_1 = c_1(\Omega)$ such that for any $u \in W^{1,2}(\Omega)$ with $|\text{supp}(u)| \leq c_1$ we have

$$\frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2} \geq \frac{c_1}{|\text{supp}(u)|^{2/n}}.$$

We assume θ_1 is chosen such that

$$2\sqrt{\theta_1}|D| < c_1. \quad (70)$$

Then for $\tilde{f} = (f - \tilde{t})^+$ we have

$$\frac{\int_{\Omega} |\nabla \tilde{f}|^2}{\int_{\Omega} \tilde{f}^2} \geq \frac{c_1}{(2\sqrt{\theta_1}|D|)^{2/n}}. \quad (71)$$

Step 2. We now estimate the Dirichlet energy of \hat{f} using the Pólya-Szegő inequality. Up to scaling, we can assume $\int \hat{f}^2 = 1$.

Note that for any $t < \tilde{t}$, we have $|\{f > t\}| \geq 2\sqrt{\theta_1}|D|$. Hence,

$$\frac{|\{f > t\} \cap \Omega_{\delta}|}{|\{f > t\}|} \leq \frac{|D \cap \Omega_{\delta}|}{2\sqrt{\theta_1}|D|} \leq \frac{\theta_1|D|}{2\sqrt{\theta_1}|D|} = \sqrt{\theta_1}/2. \quad (72)$$

We can assume θ_1 is chosen so that it satisfies

$$\sqrt{\theta_1} < 2\beta, \quad (73)$$

where β is as in Proposition A.1 with $\epsilon = \epsilon/2$. By assumption we also have $|\{f > t\}| \leq |D| \leq \theta_0$. We can assume that

$$\theta_0 \leq \min\{\alpha, c(\Omega)\} \quad (74)$$

where α is as in Proposition A.1, again with $\epsilon = \epsilon/2$, and $c = c(\Omega)$ is the constant in Lemma A.2. Hence, by Proposition A.1, for a.e. $t < \tilde{t}$, we have

$$\text{Per}(\{f > t\}, \Omega) = |\partial\{f > t\} \cap \Omega| \geq (1 - \frac{\epsilon}{2}) \left(1 + C_1 \tilde{A}(\{f > t\})^2\right) n\omega_n^{1/n} |\{f > t\}|^{\frac{n-1}{n}}.$$

We can rewrite the above inequality as

$$\text{Per}(\{f > t\}, \Omega)^2 - (1 - \epsilon) \text{Per}(\{f^* > t\})^2 \geq C_2(1 - \epsilon) \tilde{A}(\{f > t\})^2 \mu(t)^{\frac{2(n-1)}{n}}, \quad (75)$$

where $C_2 = 2C_1$. We now use Lemma A.2, which applies since $\theta_0 \leq c$, and inequality (75) to obtain

$$\int_{\{f \leq \tilde{t}\}} |\nabla f|^2 - (1 - \epsilon) \int_{\{f^* \leq \tilde{t}\}} |\nabla f^*|^2 \geq C_2(1 - \epsilon) \int_0^{\tilde{t}} \frac{\tilde{A}(\{f > t\})^2 \mu(t)^{\frac{2(n-1)}{n}}}{|\mu'(t)|} dt. \quad (76)$$

Here, $f^* : D^* \rightarrow \mathbb{R}$ is the monotone rearrangement of f , where D^* is the Euclidean ball centred at the origin with the same volume as D .

Step 3. Let us define $\eta(D) := \frac{\lambda_1(D)}{\lambda_1^D(D^*)} - 1$. In this step, we relate the left-hand side of (76) to $\eta(D)$. Observe that

$$\begin{aligned} \int_{\{f^* \leq \tilde{t}\}} |\nabla f^*|^2 &= \int_{D^*} |\nabla(f^* \wedge \tilde{t})|^2 \\ &\geq \lambda_1^D(D^*) \int_{D^*} (f^* \wedge \tilde{t})^2 \\ &= \lambda_1^D(D^*) \int_D \hat{f}^2 \\ &= \lambda_1^D(D^*). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \lambda_1(D) &= \frac{\int_D |\nabla f|^2}{\int_D f^2} \\ &\geq \frac{\int_D |\nabla \hat{f}|^2 + \int_D |\nabla \tilde{f}|^2}{1 + \int_D \tilde{f}^2 + 2 \left(\int_D \tilde{f}^2\right)^{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned}
\text{by inequality (71)} &\geq \frac{\int_D |\nabla \hat{f}|^2 + c_1(2\sqrt{\theta_1}|D|)^{-\frac{2}{n}} \int_D \tilde{f}^2}{1 + \int_D \tilde{f}^2 + 2 \left(\int_D \tilde{f}^2 \right)^{\frac{1}{2}}} \\
&\geq \frac{\int_D |\nabla \hat{f}|^2}{1} + c_1^{-1}(2\sqrt{\theta_1}|D|)^{\frac{2}{n}} \int_D |\nabla \hat{f}|^2,
\end{aligned}$$

where in the last inequality, we minimize over possible values of $\int_D \tilde{f}^2$ as in [DFV24, Proof of Theorem 5.3]. By rearranging, we get

$$\left(1 - c_1^{-1}(2\sqrt{\theta_1}|D|)^{\frac{2}{n}} \lambda_1(D)\right) \int_D |\nabla \hat{f}|^2 \leq \lambda_1(D).$$

By the assumption, $\lambda_1(D)|D|^{2/n} \leq C_0$. Hence, we can assume θ_1 is small enough such that

$$C_0 c_1^{-1} (2\sqrt{\theta_1})^{\frac{2}{n}} \leq \epsilon. \quad (77)$$

Therefore,

$$\int_D |\nabla \hat{f}|^2 \leq \frac{\lambda_1(D)}{1 - \epsilon}.$$

In summary, we get

$$C_2(1 - 2\epsilon) \int_0^{\tilde{t}} \frac{\tilde{A}(\{f > t\})^2 \mu(t)^{\frac{2(n-1)}{n}}}{|\mu'(t)|} \leq \lambda_1^D(D^*) (\eta(D) + 2\epsilon). \quad (78)$$

Note that we use the trivial inequality $(1 - \epsilon)^2 \geq 1 - 2\epsilon$.

The remaining part of the proof follows the argument in [FMP09] for the proof of a quantitative version of the Faber-Krahn inequality. The goal is to obtain a lower bound for $\eta(D)$ in terms of $\tilde{A}(D)$.

Step 4. In this step, we establish a relation between $\tilde{A}(D)$ and $\tilde{A}(\{f > t\})$ as in [FMP09]. More precisely, we show that there is a positive constant $C_3 = C_3(n)$ such that for any $t \in (0, \tilde{t})$ we have

$$\tilde{A}(D) \leq C_3(t\sqrt{|D|} + \eta(D) + \tilde{A}(\{f > t\}) + \epsilon). \quad (79)$$

Since $\tilde{A}(D) \leq 2$, inequality (79) holds for $t > \frac{1}{4}|D|^{-1/2}$ and for any constant $C_3 \geq 8$, see [FMP09]. Thus we assume $t \leq \frac{1}{4}|D|^{-1/2}$.

Let $B \subset \mathbb{R}^n$ be a ball centered at the origin with $|B| = |D \cap U|$. W.l.o.g., we can also assume $|D \setminus U| \leq |D \cap U|$. Thus

$$|B| \leq |D| \leq 2|B|. \quad (80)$$

For any $x_0 \in \mathbb{R}^n$, following the same line of argument as on [FMP09, Page 59-60], we have

$$\begin{aligned}
|B|A(D \cap U) &\leq |(D \cap U)\Delta(x_0 + B)| = 2|(x_0 + B) \setminus (D \cap U)| \\
&\leq 2|(x_0 + B) \setminus (\{f > t\} \cap U)| \\
&\leq 2|(x_0 + B \cap \{f^* \leq s\}) \setminus (\{f > t\} \cap U)| + 2|x_0 + (\{f^* > s\}) \setminus (\{f > t\} \cap U)| \\
&\leq 2(|B \cap \{f^* \leq s\}| + |x_0 + \{f^* > s\}| \setminus (\{f > t\} \cap U)|) \\
&\leq 2(|B \cap \{f^* \leq s\}| + |x_0 + \{f^* > s\}|\Delta(\{f > t\} \cap U)|)
\end{aligned}$$

where $s \geq t$ is chosen such that $|\{f^* > s\}| = |\{f > t\} \cap U|$. Thus, optimizing over x_0 and then taking the infimum over all such U , we get

$$\tilde{A}(D) \leq 2 \left(\frac{|B \cap \{f^* \leq s\}|}{|B|} + \tilde{A}(\{f > t\}) \right).$$

We first establish an upper bound for s in terms of t .

$$\begin{aligned} \mu(t) \geq \mu(s) &= |\{f^* > s\}| = |\{f > t\} \cap U| \\ &\geq |\{f > t\} \cap \Omega_\delta^c| \\ &= |\{f > t\}| - |\{f > t\} \cap \Omega_\delta| \\ \text{by (72)} \quad &\geq \left(1 - \frac{\sqrt{\theta_1}}{2}\right) |\{f > t\}| \\ &= \left(1 - \frac{\sqrt{\theta_1}}{2}\right) \mu(t). \end{aligned}$$

Let us consider the generalised inverse μ^{-1} of μ defined by

$$\mu^{-1}(m) = \begin{cases} \sup f & m = 0 \\ \inf\{t : \mu(t) < m\} & m > 0 \end{cases}$$

It is non-increasing and left-continuous ([DFV24, Definition 2.20]). Let $m_0 := \mu(t)$ and $m = \mu(s)$. We have $m \in \left(\left(1 - \frac{\sqrt{\theta_1}}{2}\right) m_0, m_0\right)$. There exists a positive constant $c_2 = c_2(\Omega, \epsilon)$ such that for

$$\theta_1 \leq c_2 \tag{81}$$

we have

$$\mu^{-1}(m) - \mu^{-1}(m_0) = s - t \leq \epsilon.$$

Thus, to obtain (79), it reduces to show that

$$\frac{|B \cap \{f^* \leq s\}|}{|B|} \leq C_4(s\sqrt{|D|} + \eta(D) + \epsilon) \leq C_4(t\sqrt{|D|} + \eta(D) + 2\epsilon) \tag{82}$$

for every $s \in [t, t + \epsilon)$ and $t \leq \min\{\tilde{t}, \frac{1}{4}|D|^{-1/2}\}$. We can assume $\epsilon \leq \frac{1}{4}|\Omega|^{-1/2} \leq \frac{1}{4}|D|^{-1/2}$. Thus

$$s \leq \frac{1}{2}|D|^{-1/2}. \tag{83}$$

We consider the function $f_s(x) = (f^*(x) - s)^+ \in H_0^1((1-r)B)$, where $r > 0$ is chosen such that $\{f_s > 0\} = \{f^* > s\} = (1-r)B$. Note that

$$|B \cap \{f^* \leq s\}| = |B \setminus \{f_s > 0\}| = (1 - (1-r)^n)|B| \leq nr|B|. \tag{84}$$

Hence, we need to estimate r . We can use $f_s(x)$ as a test function for $\lambda_1^D((1-r)B)$.

$$\frac{\lambda_1^D(B)}{(1-r)^2} = \lambda_1^D((1-r)B) \leq \frac{\int_{(1-r)B} |\nabla f_s|^2}{\int_{(1-r)B} f_s^2} \leq \frac{\int_B |\nabla f^*|^2}{\int_{(1-r)B} f_s^2}$$

$$\begin{aligned} \text{By Lemma A.2} &\leq \frac{\int_D |\nabla f|^2}{\int_{(1-r)B} f_s^2} \\ &\leq \frac{\lambda_1^D(D^*)(1 + \eta(D)) \int_D f^2}{\int_{(1-r)B} f_s^2} \\ &\leq \frac{\lambda_1^D(B)(1 + \eta(D)) \int_D f^2}{\int_{(1-r)B} f_s^2} \end{aligned}$$

Taking the square root of both sides and using the fact that $\|f\|_{L^2(D)} \leq 1 + \|\tilde{f}\|_{L^2(D)}$, we get:

$$\frac{1}{(1-r)} \leq \frac{(1 + \eta(D))^{1/2} (1 + (\int_D \tilde{f}^2)^{1/2})}{(\int_{(1-r)B} f_s^2)^{1/2}}. \quad (85)$$

We now estimate $\|\tilde{f}\|_{L^2(D)}$. Using inequality (71), we obtain

$$\begin{aligned} \|\tilde{f}\|_{L^2(D)} &\leq c_1^{-1/2} (2\sqrt{\theta_1}|D|)^{1/n} \left(\int_D |\nabla \tilde{f}|^2 \right)^{1/2} \\ &\leq c_1^{-1/2} (2\sqrt{\theta_1}|D|)^{1/n} \left(\int_D |\nabla f|^2 \right)^{1/2} \\ &\leq c_1^{-1/2} (2\sqrt{\theta_1})^{1/n} \left(|D|^{\frac{2}{n}} \lambda_1(D) \right)^{1/2} \left(1 + \|\tilde{f}\|_{L^2(D)} \right) \\ &\leq c_1^{-1/2} (2\sqrt{\theta_1})^{1/n} C_0^{1/2} \left(1 + \|\tilde{f}\|_{L^2(D)} \right) \end{aligned}$$

where in the last inequality we use the assumption $\lambda_1(D)|D|^{2/n} \leq C_0$. Let

$$A := c_1^{-1/2} (2\sqrt{\theta_1})^{1/n} C_0^{1/2}.$$

Thus, for θ_1 small enough such that

$$1 - A > 0, \quad \text{and} \quad \frac{A}{1-A} < \epsilon, \quad (86)$$

we have

$$\|\tilde{f}\|_{L^2(D)} \leq \frac{A}{1-A} \leq \epsilon. \quad (87)$$

Therefore, after rearranging inequality (85), we get

$$\begin{aligned} \left(\int_{(1-r)B} f_s^2 \right)^{1/2} &\leq (1-r)(1 + \eta(D))^{1/2} (1 + \epsilon) \\ &\leq (1-r) \left(1 + \frac{\eta(D)}{2} \right) (1 + \epsilon) \\ &\leq (1 + \epsilon) \left(1 - r + \frac{\eta(D)}{2} \right) \\ &\leq (1 + \epsilon)(1-r) + \eta(D). \end{aligned} \quad (88)$$

On the other hand, we have

$$\begin{aligned} \left(\int_{(1-r)B} f_s^2 \right)^{1/2} &= \left(\int_{(1-r)B} (f^* - s)_+^2 \right)^{1/2} \\ &= \left(\int_{(1-r)B} (f^* - s)^2 \right)^{1/2} \\ \text{by the triangle inequality} &\geq \left(\int_B (f^*)^2 - \int_{B \setminus (1-r)B} (f^*)^2 \right)^{1/2} - s((1-r)^n |B|)^{1/2} \\ &\geq \left(\int_B (f^*)^2 - \int_{B \setminus (1-r)B} (f^*)^2 \right)^{1/2} - s|B|^{1/2} \\ &\geq \left(\int_{D^*} (f^*)^2 - \left(\int_{D^* \setminus B} + \int_{B \setminus (1-r)B} \right) (f^*)^2 \right)^{1/2} - s|B|^{1/2} \end{aligned}$$

$$\begin{aligned}
 \text{using } \|f\|_{L^2(D)} \geq \|\hat{f}\|_{L^2(D)} = 1 &\geq \left(1 - \left(\int_{D^* \setminus B} + \int_{B \setminus (1-r)B}\right) (f^*)^2\right)^{1/2} - s|B|^{1/2} \\
 &\geq (1 - s^2(|D^* \setminus (1-r)B|))^{1/2} - s|B|^{1/2} \\
 &\geq (1 - s^2|D^*|)^{1/2} - s|B|^{1/2} \\
 &\geq 1 - s|D^*|^{1/2} - s|B|^{1/2} \\
 &\geq 1 - s \left(\frac{|B|}{1 - \theta_1}\right)^{1/2} - s|B|^{1/2}.
 \end{aligned} \tag{89}$$

The inequality, (80) and the bound on s in (83) imply that $1 - s^2|D^*| \geq \frac{1}{2} > 0$, so (89) and the two inequalities above are valid. The last inequality is the consequence of the following. We use the assumption $|D \cap \Omega_\delta| \leq \theta_1|D|$ to get $|D \cap U^c| \leq |D \cap \Omega_\delta| \leq \theta_1|D|$. Thus,

$$|B| = |D \cap U| = |D| - |D \cap U^c| \geq |D|(1 - \theta_1).$$

Assuming

$$1 - \theta_1 \geq \frac{1}{4}, \tag{90}$$

we have

$$\left(\int_{(1-r)B} f_s^2\right)^{1/2} \geq 1 - 3s|B|^{1/2}. \tag{91}$$

Combining and rearranging (88) and (91), we get

$$r \leq \epsilon + \eta(D) + 3s\sqrt{|B|} \leq \epsilon + \eta(D) + 3s\sqrt{|D|}.$$

Putting this into (84), we obtain inequality (79) with $C_3 = \max\{8, 6n\}$.

Step 5. For the final step, we use (78) to show that there exists $t \in (0, \tilde{t})$ such that $\tilde{A}(\{f > t\})$ is bounded above in terms of $\eta(D)$. Then we use it in inequality (79) to conclude.

Using the identity $\lambda_1^D(D^*)|D^*|^{\frac{2}{n}} = \lambda_1^D(\mathbb{B})|\mathbb{B}|^{\frac{2}{n}}$, we can rewrite inequality (78) as

$$\int_0^{\tilde{t}} \frac{\tilde{A}(\{f > t\})^2 \mu(t)^{\frac{2(n-1)}{n}}}{|\mu'(t)|} \leq C_5 |D|^{-\frac{2}{n}} (\eta(D) + 2\epsilon). \tag{92}$$

W.l.o.g, we assume $\eta(D) \leq \delta_3$ where $\delta_3 < 1$ is small enough (depending only on n) to be chosen later. We now use (82) to obtain a lower bound for $\mu(t)$. Note that in (82), $|B| = |D \cap U|$ where $U \subseteq \Omega$ is any open set containing $\Omega \setminus \Omega_\delta$. We take $U = \Omega$, and thus $|B| = |D \cap \Omega| = |D|$. Hence, for every $t \leq \min\{\frac{1}{4}|D|^{-1/2}, \tilde{t}\}$, we have

$$\begin{aligned}
 \mu(t) &= |\{f > t\}| \\
 &\geq |D| - |D \cap \{f \leq t\}| \\
 &\geq |B| - |B \cap \{f^* \leq t\}| \\
 &\geq |B| - |B|C_4(t\sqrt{|D|} + \eta(D) + \epsilon).
 \end{aligned}$$

Let $t_1 = \min\left\{\frac{1}{6C_4\sqrt{|D|}}, \frac{1}{4\sqrt{|D|}}, \tilde{t}\right\}$, $\delta_3 \leq \frac{1}{6C_4}$ and $\epsilon \leq \frac{1}{6C_4}$. Then for any $t \in (0, t_1)$, using (80),

$$\mu(t) \geq \frac{|B|}{2} = \frac{|D|}{2}$$

for any $t \in (0, t_1)$. As a result we obtain

$$\int_0^{t_1} \frac{|D|^2 \tilde{A}(\{f > t\})^2}{|\mu'(t)|} \leq C_6 (\eta(D) + 2\epsilon). \quad (93)$$

We now follow the same line of argument as in [FMP09] and introduce two positive parameters θ and σ (that are fixed below). Let us estimate the 1-dimensional Hausdorff measure of the following two sets.

$$I_1 := \left\{ t \in (0, t_1) : \frac{\tilde{A}(\{f > t\})^2}{|\mu'(t)|} \geq \theta \right\}, \quad I_2 := \{ t \in (0, t_1) : |\mu'(t)| \geq |D|\theta^{-\sigma} \}.$$

From inequality (93), we have

$$|I_1| \leq \frac{C_6(\eta(D) + 2\epsilon)}{|D|^2\theta}, \quad |I_2| \leq \int_{I_2} |\mu'(t)| |D|^{-1} \theta^\sigma \leq \theta^\sigma,$$

and therefore

$$|I_1 \cup I_2| \leq \left(\frac{C_6(\eta(D) + 2\epsilon)}{|D|^2\theta} + \theta^\sigma \right).$$

Taking $\theta = \left(\frac{\eta(D) + 2\epsilon}{|D|^2} \right)^{\frac{1}{1+\sigma}}$ and $\sigma = \frac{1}{3}$, we get:

$$|I_1 \cup I_2| \leq C_7 \left(\frac{\eta(D) + 2\epsilon}{|D|^2} \right)^{\frac{\sigma}{1+\sigma}} = \frac{C_7(\eta(D) + 2\epsilon)^{\frac{1}{4}}}{\sqrt{|D|}}.$$

Now note that

$$\tilde{t}^2 |D| \geq \int_{\{f \leq \tilde{t}\}} f^2 + \int_{\{f > \tilde{t}\}} \tilde{t}^2 = \int_D \hat{f}^2 = 1.$$

Thus, $\tilde{t} \geq \frac{1}{\sqrt{|D|}}$. As a result $t_1 \geq \frac{C_8}{\sqrt{|D|}}$, where $C_8 = \min\{\frac{1}{6C_4}, \frac{1}{4}\}$. Assuming δ_3 and ϵ are sufficiently small so that $\epsilon \leq \frac{\delta_3}{2}$ and $2C_7(2\delta_3)^{1/4} \leq C_8$ implies that the set

$$\begin{aligned} \mathcal{B} &= (0, t_1) \cap \left(0, \frac{2C_7(\eta(D) + 2\epsilon)^{1/4}}{\sqrt{|D|}} \right) \setminus (I_1 \cup I_2) \\ &= \left(0, \frac{2C_7(\eta(D) + 2\epsilon)^{1/4}}{\sqrt{|D|}} \right) \setminus (I_1 \cup I_2). \end{aligned}$$

has nonzero measure. In particular for any $t \in \mathcal{B}$, we have

$$\tilde{A}(\{f > t\})^2 \leq |\mu'(t)|\theta \leq |D|\theta^{1-\sigma} = |D| \left(\frac{\eta(D) + 2\epsilon}{|D|^2} \right)^{\frac{1-\sigma}{1+\sigma}} = (\eta(D) + 2\epsilon)^{\frac{1}{2}}.$$

Plugging into (79) for such t , we get

$$\begin{aligned} \tilde{A}(D) &\leq C \left((\eta(D) + 2\epsilon)^{\frac{\sigma}{1+\sigma}} + \eta(D) + \epsilon + (\eta(D) + 2\epsilon)^{\frac{1-\sigma}{2(1+\sigma)}} \right) \\ &\leq C(\eta(D) + 2\epsilon)^{\frac{1}{4}}. \end{aligned} \quad (94)$$

Note that the choice of $\sigma = \frac{1}{3}$ is also made so that the power of the first and the last term in the right-hand side of (94) are equal. We conclude that

$$\lambda_1(D) \geq \left(1 - 2\epsilon + C\tilde{A}(D)^4 \right) \lambda_1^D(D^*).$$

Note that we only get a quantitative version when the term $C\tilde{A}(D)^4 - 2\epsilon$ is positive. \square

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DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, MATHEMATICAL SCIENCES AND COMPUTER SCIENCE BUILDING MCS2030, UPPER MOUNTJOY CAMPUS, STOCKTON ROAD, DURHAM DH1 3LE, UNITED KINGDOM.

Email address: `katie.gittins@durham.ac.uk`

UNIVERSITY OF BRISTOL, SCHOOL OF MATHEMATICS, FRY BUILDING, WOODLAND ROAD, BRISTOL, BS8 1UG, U.K.

Email address: `asma.hassannezhad@bristol.ac.uk`

UNIVERSITÀ DEGLI STUDI DI PADOVA, DIPARTIMENTO DI TECNICA E GESTIONE DEI SISTEMI INDUSTRIALI (DTG), STRADELLA S. NICOLA 3, 36100 VICENZA, ITALY

UNIVERSITÀ DEGLI STUDI DI PADOVA, DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”, VIA TRIESTE 63, 35121 PADOVA, ITALY

Email address: `corentin.lena@unipd.it`

DEPAUL UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, 2320 N. KENMORE AVE., CHICAGO IL 60614, USA

Email address: `dsher@depaul.edu`