FUNCTIONAL CENTRAL LIMIT THEOREM FOR TOPOLOGICAL FUNCTIONALS OF GAUSSIAN CRITICAL POINTS

CHRISTIAN HIRSCH AND RAPHAËL LACHIÈZE-REY

Abstract We consider Betti numbers of the excursion of a smooth Euclidean Gaussian field restricted to a rectangular window, in the asymptotics where the window grows to $\mathbb{R}^d$. With motivations coming from Topological Data Analysis, we derive a functional Central Limit Theorem where the varying argument is the thresholding parameter, under assumptions of regularity and covariance decay for the field and its derivatives. We also show fixed-level CLTs coming from martingale based techniques inspired from the theory of geometric stabilisation, and limiting non-degenerate variance.

Keywords: Gaussian fields, geometric excursions, Betti numbers, functional CLT, stabilisation, topological data analysis.

1. Introduction

Stationary Gaussian fields are a dominant model to represent spatially homogeneous continuous data in several dimensions. In particular, the *excursion sets*, or *upper level sets*, obtained by thresholding at a given level, are the topic of intensive research across several fields, and under many angles: image analysis, percolation, materials science, neurology, cosmology. Their geometric characteristics, such as the volume, perimeter, or topological indexes, have been the subject of several works on limit theorems. See for instance [1, 3, 12] for theoretical tools, results, and case studies for smooth Gaussian fields.

Recently, a large variety of disciplines have adopted methods from the mathematical domain of topological data analysis (TDA) in order to analyze data exhibiting complex topological features. In this area, the (persistent) Betti numbers are the key tool, which, loosely speaking, describe the number of holes of a fixed dimension contained in a considered data set. In order to put TDA on a rigorous statistical foundation and to allow for the derivation of hypothesis tests, it is crucial to develop the theory of normal approximation for the Betti numbers. This has been successfully achieved for topology models based on the Poisson point process [14, 34]. While point process models are interesting, when looking into applications, random-field models are of central importance [27, 28, 32]. However, when considering the asymptotics of persistent Betti numbers, then much less is known. One exception is a work on discretely-indexed Gaussian excursions, which is however restricted to the sparse regime [31].

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Central limit theorems (CLTs) for geometric functionals have been a topic of intensive research in the last 20 years. Here, the most thoroughly studied model is that of a Poisson point process in Euclidean space. For instance, we refer the reader to [16, 17, 18, 26] and the references therein. Motivated by the emerging domain of TDA, we focus in this work on topological indexes of the fields. While there has been a lot of progress in the normal approximation of geometric functionals on Poisson processes, the situation is much less clear for the case of Gaussian fields. The first such work is the CLT for the Euler characteristic established by Estrade & Léon [10]. However, their arguments critically rely on the local nature of this functional and therefore do not generalize to the more globally defined Betti numbers. Much more recently, [4] applied a general martingale technique to derive a CLT for the component count of Gaussian random fields under suitable moment assumptions, in the spirit of the theory of Geometric stabilisation [26]. Later, McAuley [19] established a similar CLT for geometric functionals of the unbounded connected component.

The results mentioned above in [34, 14] concern a CLT at a fixed level. However, in applications, it is typically not at all clear what is the appropriate scale. Also, TDA is concerned with the evolution of the topology when a real parameter is varying, this role is played here by the level u. Therefore, it is essential to have a functional CLT that allows to vary the level. In our main result, we derive such a result under suitable moment conditions where the level is allowed to vary in sub-critical and super-critical regime of random-field percolation. We note that even in the point-process case such a process-level result is only known to hold for quasi-1D domains or under truncation of the Betti number [15, 6]. Here, the particular challenge in functional CLTs is the question of tightness for topological functionals. One important ingredient in our arguments is the derivation of exponential tails for the diameter of bounded components coming from continuous percolation.

Besides the tightness, we derive CLTs at fixed levels for a large class of non-local topological functionals. We also discuss the question of positivity of the limiting variance. While we can reuse some parts of the strategies from the component-case considered in [4], the general Betti numbers require a more careful geometric analysis.

The rest of the manuscript is organized as follows. In Section 2 we introduce the Gaussian fields and recall some useful properties about them. In Section 2.1, we introduce the topological functionals and their fundamental properties. In Section 2.2 we state our main results. In Section 3, we discuss key results on Gaussian random fields such as the white-noise decomposition. Section 4 contains important topological preliminaries that will be used in the proofs. In Section 5, we prove the fixed-level CLT, and show the positivity of the limiting variance. Finally, the proof of the functional CLT is given in Section 6.

2. Results

A stationary Gaussian field is a random function $F : \mathbb{R}^d \to \mathbb{R}$ with Gaussian finite-dimensional marginals $(F(t_1), \dots, F(t_d))$, and which is distributionally invariant under translations, i.e. $F(t + \cdot) \overset{(d)}{=} F$ for $t \in \mathbb{R}^d$. We assume furthermore that the field is centered with unit variance (i.e. $F(x) \sim \mathcal{N}(0, 1)$ for $x \in \mathbb{R}^d$), in which case the law of F is characterised by the covariance function

$$C(x) = \mathbf{E}(F(0)F(x)), x \in \mathbb{R}^d.$$

Let us state some assumptions ensuring asymptotic independence and regularity of the field.

Assumption 1 (Gaussian fields assumptions)**.**

- *Regularity: C should be of class $\mathcal{C}^{2q_0+3}$ for some $q_0 > 2^{13}$.*
- *Covariance decay: C and its derivatives $\partial^\alpha C$ for a multi-index α with $|\alpha| \leqslant 3$ should decay at ∞ as $\|x\|^{-\eta}$ for some $\eta > 55d^2$.*

By [2, Section 1], this assumption implies in particular that the Gaussian field a.s. has sample paths of class $\mathcal{C}^{q_0+1}$. In practice, only class $\mathcal{C}^3$ is necessary to properly assess quantitatively topological properties of the field, but by [11] this regularity yields that the number of critical points has locally finite moments of order q_0 (Theorem 12), which is useful for using Hölder's inequality in several parts of the proof.

We study the topological properties of the excursion sets

$$\mathsf{E}(u) = \mathsf{E}(u; F) = \{x : F(x) \geqslant u\}, u \in \mathbb{R},$$

intersected through a rectangular window $W \subset \mathbb{R}^d$. We consider topological functionals applied to the connected components of $\mathsf{E}(u) \cap W$. It is more convenient to work with components which do not touch ∂W, and we shall assume that these components have a controllable size. For $A \subset \mathbb{R}^d$, denote by $\mathscr{C}(A)$ the set of bounded connected components of A, for bounded $Q \subset A$, let $\mathscr{C}(A; Q)$ the set of $C \in \mathscr{C}(A)$ with $C \cap Q \neq \varnothing$, and let $\mathcal{C}(A; Q)$ the union of the $C \in \mathscr{C}(A; Q)$. The functional CLT established in this paper works on an interval I away from the critical regime. To make this precise, we henceforth let $u_c^s \geqslant 0$ be the threshold of sharp phase transition: for $u > u_c^s$, for any $a > 0$ there is $c_a < \infty$ with

$$\mathbf{P}(\mathrm{diam}(\mathcal{C}(\{F \geqslant u\}, \{0\})) \geqslant r) \leqslant c_a r^{-a}.$$

Remark that it is not automatic that $u_c^s < \infty$, this condition is discussed below. Before that, we first state our assumption on the interval I.

Assumption 2. *$I \subset (u_c^s, \infty) \cup (-\infty, -u_c^s)$.*

Sharp phase transition is traditionally defined with exponential decay, but we require a weaker assumption here. Most available results actually give exponential decay. This assumption is discussed at Section 2.4.

2.1. **Topological functionals.** For simplicity we assume here that $W = W_n$ is a *stratified window*, i.e., a union of rectangles parallel to the axes and in general position (all pairs of rectangles have an intersection that is either empty or full dimensional). Our aim is to treat Betti numbers of Gaussian excursions, such as number of connected components, number of cavities, etc... We introduce a more general class of topological functionals adapted to Morse excursions.

We call $\mathscr{E}^k$ the class of k-dimensional compact manifolds of $\mathbb{R}^d$ (with boundary) that can be written as the level set $\{f \geqslant u\}$ of a $\mathcal{C}^k$ Morse function $f : \mathbb{R}^d \to \mathbb{R}$, such that for two critical points x, y of f, $f(x) \neq f(y)$. The latter requirement could be avoided, but it is not restrictive for Gaussian fields and more convenient this way. Such a representation of A is called a *Morse representation*.

Definition 3. Say that a function $\beta : \mathscr{E}^k \to \mathbb{R}$ is *topologically additive* if it admits the representation on bounded connected components C

$$\beta(A) = \sum_{C \in \mathscr{C}(A)} \beta(C)$$

and

- $\beta(C)$ only depends on the isotopy class of C, where two sets C, C' are in the same isotopy class if there is a continuous function $\gamma : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d, t \in [0,1]$ such that $\gamma(0,x) = x$ and $\gamma(1,C) = C'$, and for each t, $\gamma(t,\cdot)$ is a homeomorphism.
- $|\beta(C)|$ is bounded by the number of critical points of f over C above level u for any Morse representation $C = \{f \geqslant u\}$.

Our main example will be the Betti numbers, which are such additive topological functionals. Here, we only give a loose description of this quantity. The precise definition relies on the concept of homology, whose mathematical definition is beyond the scope of the present work. We refer the reader to [20, 13].

Definition 4 (Betti numbers)**.** Betti numbers are such additive topological functionals $\beta_k, k \leqslant d$ where $\beta_k(A)$ counts the number of equivalence classes of k-dimensional cycles of A.

For instance, a 0-dimensional cycle is in the same class as a point, hence $\beta_0(A)$ counts the number of classes of points which are not topologically equivalent in A, hence $\beta_0(A)$ is the number of bounded connected components of A.

The first property from Definition 3 holds for Betti numbers since they are invariant by homotopy, see [13, Theorem 2.10]. For the second property, note that if $u \leqslant u'$ are two levels of critical points of f such that there is no other critical point with level in $[u, u']$, then the relative homology groups $H_k\big(\{f \geqslant u\}, \{f \geqslant u'\}\big)$ are of rank 1 for at most one value of k and 0 otherwise. Therefore, the second property follows from the long exact sequence in homology. We refer the reader to [20, Section I.5] for details.

We will be looking at here a $\mathcal{C}^k$ Gaussian field F which level sets are not compact, but we only observe the connected components in the interior of a window W_n, so that we have indeed

$$\bigcup_{C \in \mathscr{C}(\mathsf{E}(u), W_n, \partial W_n)} C = \{f \geqslant u\},$$

for some $\mathcal{C}^k$ function f on W_n which coincides with F on a neighbourhood of the components. It is well known that such restricted excursions of smooth Gaussian fields a.s. correspond to this framework, use Lemma 17 for a rigourous proof. We set

$$\beta_n(u) := \beta_n(u; F) := \sum_{C \in \mathscr{C}(\mathsf{E}(u), W_n, \partial W_n)} \beta(C) \tag{1}$$

hence we consider topologically additive functionals as defined above.

2.2. **Fixed-level CLT and variance lower bounds.** We first state the fixed-level CLTs for Betti numbers, then the FCLT in the variable u, where we also assume that $\#\{i : Q_i \neq \varnothing\} \sim \mathrm{Vol}(W_n) \sim n$. In this section, we state a CLT for the Betti numbers at a fixed level. To ensure positivity of the limiting variance, we need an additional condition. To state this precisely, let

$$\mu(u) := \lim_{n\to\infty} |W_n|^{-1}\mathbf{E}\big(\beta_n(u)\big), \tag{2}$$

supposing that this limit exists. Henceforth, we set $\gamma := \eta/d - 1/2$, which satisfies $\gamma > 54d$ under Assumption 1. Define

$$\tilde{\beta}_n(u) = n^{-1/2}(\beta_n(u) - \mathbf{E}(\beta_n(u))).$$

Theorem 5 (Fixed-level CLT)**.** *Consider a Gaussian random field satisfying Assumption 1. Furthermore, assume that u is as in Assumption 2. Then,*

$$\widetilde{\beta_n}(u) \Rightarrow \mathcal{N}(0, \sigma(u)^2),$$

where $\mathcal{N}(0, \sigma(u)^2)$ is a normal distribution with mean 0 and some variance $\sigma(u)^2 \geqslant 0$. If $\mu(u)$ exists and is not 0, then $\sigma(u) > 0$.

Remark 6. Our bound for q_0 is likely very conservative. We believe it can be improved substantially by optimizing further within proofs.

Theorem 5 is proven in Section 5 by invoking a general CLT for stabilizing functionals on Gaussian random fields from [4, Theorem 1.2]. The latter result is modeled after a classical CLT for stabilizing functionals of a Poisson point process [26, Theorem 3.1]. Very recently, there also has been a general CLT implying the asymptotic normality of the volume, surface area and Euler characteristic of an unbounded component in the excursion set [19].

Concerning the positivity of variance, we then proceed along the lines of [4, Theorem 1.3]. However, a crucial ingredient in that proof is [4, Lemma 3.13(1)], which contains a delicate argument based on properties of the spectral measure to show that with positive probability at least some connected component of the excursion set intersects $[0, 1]^d$.

2.3. **Functional central limit theorem.** After having established the CLT at a fixed level in Theorem 5, the next step is to prove a CLT where the functional is considered as a stochastic process in the level. To this end, we must ensure that the process $u \mapsto \beta_q(u)$ is in the CADLAG space:

Lemma 7. *If $|Y_n([u_-, u_+])| = 0$, $\beta_q(u_-) = \beta_q(u_+)$.*

Since, by Lemma 17, the critical values over a compact window are almost surely locally finite, the process β_q a.s. jumps finitely many times on each compact by Lemma 17 below. Therefore, we henceforth consider $\beta_q(u)$ as an element in the space of CADLAG functions equipped with the standard Skorokhod topology, see [5, Section 12].

Theorem 8 (FCLT for Betti numbers)**.** *Consider a Gaussian random field satisfying Assumption 1. Let I be an interval satisfying Assumption 2. Then, as $n \to \infty$, as a process, in the Skorokhod topology,*

$$\widetilde{\beta_n}(\cdot) \Rightarrow Z,$$

where Z is a centered Gaussian process on the interval I.

2.4. **Discussion of the percolation assumption.** This question is highly technical and outside the scope of the present paper, many recent works deal with the percolation properties of Gaussian excursions, and in particular give conditions under which the percolation regime undergoes a *sharp phase transition*, under which this assumption is satisfied. We briefly discuss what is known about the validity of Assumption 2 on the exponential decay of the cluster diameters. A more elementary question concerns the existence of a level above which with probability 1, there exists an unbounded connected component. That is, we define

$$u_c := \sup\Big\{u \in \mathbb{R} : \mathbf{P}\Big(\mathrm{diam}(\mathcal{C}(\mathsf{E}(u); [0,1]^d)) = \infty\Big) > 0\Big\}$$

as the standard critical level of percolation. The fact that for sufficiently low u there is an unbounded connected component under mild assumptions goes back to Stepanov and Molchanov [21]. Here are some more precise statements. We mainly give results in the supercritical regime, as it is in general easier to give bounds on the size of bounded components of the subcritical regime.

(1) Consider the planar case, i.e., $d = 2$. Here, $u_c = u_c^s = 0$ by self-duality under very mild assumptions. Then, [24, Theorem 1.7] shows the sharpness of the phase transition under some correlation decay assumptions. In particular, Assumption 2 is implied by Assumption 1 in the planar case, meaning it holds for every interval I not containing $\{0\}$.
(2) For general dimensions $d \geqslant 3$, the situation is more delicate. First, the typical situation is to have percolation of both phases at level 0, hence $u_c > 0$; it has been proved to hold in [9] under stronger assumptions, and in general the value of u_c is not known, as in most non-symmetric percolation models. [30, Theorem 1.2] shows the sharpness of the phase transition under mild hypotheses, meaning that Assumption 2 holds under Assumption 1 for $I \subset [-u_c, u_c]^c$. However, the arguments need positive association (a.k.a. the FKG inequality) and therefore only apply for nonnegative covariance kernels. That is, we must additionally assume $\inf_x C(x) \geqslant 0$.
(3) Without positive association, for the components in the subcritical regime, we can apply [22, Theorem 3.7]: if the covariance decays monotonically at a rate faster than any polynomial, then we have the threshold

$$u_c' = \inf\{u \in \mathbb{R} : \liminf_{R\to\infty} \sup_{x\in\mathbb{R}^d} \mathbf{P}(\mathcal{C}(\{F \geqslant u\}) \cap \partial B(x, 2R), B(x,R)) \neq \varnothing) = 0\}$$

where $B(x,R)$ is the ball centred in x with radius R, that satisfies $u_c \leqslant u_c^s < \infty$, and for $u > u_c^s$,

$$\limsup_{r\to\infty} (\log r)^{-1} \log \mathbf{P}(\ \mathrm{diam}(\mathcal{C}(\{F \geqslant u\}, \{x\})) > r) = -\infty.$$

It is expected that $u_c^s = u_c$, but proved only in the planar case.
(4) Finally, [23, Theorem 1.2] concerns again the sharpness of the phase transition for fields not satisfying the FKG condition in dimension $d \geqslant 3$. The arguments here rely on a suitable finite-range decomposition of the considered random field. In particular, it is proved in particular that exponential decay occurs for some (non necessarily positive) covariances decaying polynomially with an arbitrary negative exponent.

To summarise:

Theorem 9 (Muirhead,Rivera,Severo)**.** *Assume Assumption 1 holds. Then Assumption 2 holds in the following cases:*
- $d = 2$ *and* $0 \notin \bar{I}$.
- $d \geqslant 3$ *and* $C(x) \geqslant 0$ *and* $I \subset (-\infty, -u_c^s) \cup (u_c^s, \infty)$.
- $d \geqslant 3$ *and* $I \subset (-\infty, -u_c^s)$ *for some* $u_c^s > u_c$ *if* $C(x)$ *decays sufficiently fast.*
- $d \geqslant 3$ *and* $I \subset (u_c^s, \infty)$ *for some classes of covariances of "finite range" (it also probably works for* $I \subset (-\infty, -u_c^s)$*).*

3. Properties of Gaussian fields

3.1. **White noise convolution.** Many assumptions are more conveniently stated through the spectral measure, defined as the unique probability measure μ on $\mathbb{R}^d$ such that

$$C(x) = \int e^{ixu}\mu(du), \qquad x \in \mathbb{R}^d.$$

Let $\mathcal{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$. A Gaussian white noise is a random signed measure seen as a random field $\mathcal{W} : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ such that
- $\mathcal{W}(A) \sim \mathcal{N}(0, |A|)$ for A Borel
- $\mathcal{W}(A \cup B) = \mathcal{W}(A) + \mathcal{W}(B)$ a.s. for every disjoint Borel sets $A, B \subset \mathbb{R}^d$
- $\mathcal{W}(A)$ and $\mathcal{W}(B)$ are independent for every disjoint Borel sets $A, B \subset \mathbb{R}^d$.

See [1, Section 1.4.3] for an explicit construction. It satisfies in particular for f, g square integrable

$$\mathsf{Cov}(\int f d\mathcal{W}, \int g d\mathcal{W}) = \int fg. \tag{3}$$

Assumption 1 implies that the spectral measure μ has a smooth L^2 density, denoted by ρ. Hence $C = \hat{\rho} = q \star q$ with $q = \widehat{\sqrt{\rho}}$. Let $\mathcal{W}$ be a centred stationary Gaussian white noise on $\mathbb{R}^d$. Writing $\star$ for the classical convolution operator, F admits the representation

$$F(x) \overset{(d)}{=} F(x; \mathcal{W}) := q \star \mathcal{W}(x) \tag{4}$$

because

$$C(x) = q \star q(x) = \int q(y)q(y + x)dy = \mathbf{E}(F(0)F(x))$$

and the covariance uniquely determines the law of the Gaussian field.

Not all Gaussian covariances can be written in this way, for instance the random planar wave model cannot as its spectral measure is singular. The renormalisation assumption means that

$$\mathsf{Var}(F(x)) = \|q\|_{L^2} = 1.$$

3.2. **Non-degeneracy.** Let us recall the formulae linking derivatives of the field and the covariances [1, (5.5.4)-(5.5.5)]: if a covariance function C is $\mathcal{C}^{2k+}$, F is a.s. of class $\mathcal{C}^k$ and for natural integers $\alpha, \eta, \gamma, \delta$ such that $\alpha + \eta \leqslant k, \gamma + \delta \leqslant k$, coordinates $1 \leqslant i, j \leqslant d$,

$$\mathbf{E}\left(\partial_i^\alpha \partial_j^\eta F(t) \cdot \partial_i^\gamma \partial_j^\delta F(s)\right) = \frac{\partial^{\alpha+\eta+\gamma+\delta}}{\partial t_i{}^\alpha \partial t_j{}^\eta \partial s_i{}^\gamma \partial s_j{}^\delta} C(t - s), s, t \in \mathbb{R}^2. \tag{5}$$

For instance, by symmetry,

$$\nabla C(0) = 0$$
$$\mathsf{Cov}(\nabla F(t)) = -\operatorname{Hess}_C(0) = -\begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_d \end{pmatrix}$$

where $\mathsf{Cov}(M)$ denotes the covariance matrix of a random vector M, and $\lambda_i = \mathsf{Var}(\partial_i F(0))$. We have in particular the expansion

$$C(x) = 1 - \frac{1}{2}(\sum_i \lambda_i x_i^2) + o(x^2), x \in \mathbb{R}^d.$$

The following standard result states that the field's derivatives are not degenerate at disjoint locations under Assumption 1. Let

$$\mathcal{I} = \{(1),(2),(1,1),(2,2),(1,2),(1,1,1),(1,1,2),(1,2,2),(2,2,2)\}$$

the set of orderer multi-indexes of length $\leqslant 3$ in $\{1,2\}$.

Proposition 10. *Assume there is an open set in the support of the spectral density ρ. Let the Gaussian vector $V(x) = (\partial_\alpha F(x); \alpha \in \mathcal{I}) \subset \mathbb{R}^{\#\mathcal{I}}, x \in \mathbb{R}^d$ and for $x, y \in \mathbb{R}^d$, $V(x,y) \in \mathbb{R}^{2\#\mathcal{I}}$ the random vector obtained by concatenating $V(x)$ and $V(y)$. Then for all $x \in \mathbb{R}^d$, the derivatives $\partial^\alpha F(x), \alpha \in \mathcal{I}$ form a non-degenerate Gaussian vector, i.e.* $\det(\mathsf{Cov}(V(x)) > 0$. *Also, for $\delta > 0$,*

$$\inf_{|x-y|>\delta} |\det(\mathsf{Cov}\,(V(x,y)))| > 0.$$

Proof. The proof is based on the fact that for a Gaussian vector $V = (V_1, \dots, V_m)$, $\det(\mathsf{Cov}(V)) = 0$ iff there is a non-trivial linear relation

$$\sum_{i=1}^m a_i V_i = 0 \text{ a.s..}$$

Let $(a_\alpha)_{\alpha \in I}, (b_{\alpha'})_{\alpha' \in I}$ finite collections of complex numbers indexed by multi-indices, and let $x \in \mathbb{R}^d$. By recalling that $C = \hat{\rho}$, we have by (5), for some polynomials $P_1, P_2, Q : \mathbb{C}^d \to \mathbb{C}$, for $x \in \mathbb{R}^d \backslash \{0\}$,

$$\operatorname{Var}(\sum_\alpha a_\alpha \partial^\alpha F(0) + \sum_{\alpha'} b_{\alpha'} \partial^{\alpha'} F(x)) = \int_{\mathbb{C}^d} [\underbrace{P_1(\lambda) + P_2(\lambda) + e^{i\lambda x} Q(\lambda)}_{R(\lambda)}] \rho(\lambda) d\lambda,$$

where resp. P_1, P_2 are obtained when $(b_{\alpha'}) \equiv 0$, resp. $(a_\alpha) \equiv 0$, and Q is the cross term. This formula is valid for any L^2 spectral density ρ and corresponding stationary Gaussian field F, hence the right hand side is nonnegative for any ρ. It implies that $R(\lambda) \in \mathbb{R}_+$, and similarly P_1, P_2 are nonnegative.

Assume now that for some ρ having a nonempty open set O in its support, this quantity vanishes, which equivalently means that there exist deterministic complex (a_α), $(b_{\alpha'})$ such that almost surely,

$$\sum_\alpha a_\alpha F(0) + \sum_{\alpha'} b_{\alpha'} \partial^{\alpha'} F(x) = 0$$

It means that $R(\lambda) = 0$ over O, and as an analytic function, it means it vanishes on $\mathbb{C}^d$. Hence, $e^{i\lambda x} Q(\lambda)$ should have a finite expansion with $x \neq 0$, which means that $Q = 0$, and then $P_1 + P_2 = 0$ as well. Since they are nonnegative, all coefficients of P_1, P_2 are zero, which easily implies with (5) that that the coefficients a_α and $b_{\alpha'}$

are all 0. We hence proved by contradiction that $\det(\mathsf{Cov}(V(x,y))) > 0$ for $x \neq y$. Looking at the first $|\mathcal{I}|$ coordinates of $V(x,y)$, it means that $V(x)$ is non-degenerate, which proves the first statement.

For the second statement, note that the map $(x,y) \mapsto \det(\mathsf{Cov}(V(x,y)))$ is continuous and does not vanish, hence for $0 < \delta < K < \infty$,

$$\inf_{\delta < |x-y| < K} |\det(\mathsf{Cov}(V(x,y)))| > 0.$$

Let us finally prove that the infimum over distant x, y is non-zero as well. If it is zero, it means by stationarity that for some sequence $x_n \to \infty$,

$$\det \underbrace{\mathsf{Cov}(V(0,x_n))}_{=:\Gamma_{x_n}} \xrightarrow[x_n\to\infty]{} 0.$$

Denoting by U_{x_n} a unit vector associated to the smallest eigenvalue of Γ_{x_n}, we have $\Gamma_{x_n} U_{x_n} \to 0$ in $\mathbb{R}^d$. By compactness of $\mathbb{S}^{d-1}$, it means we can find coefficients $a_\alpha, b_{\alpha'}$ which constitute the limit of a subsequence of U_{x_n} in $\mathbb{S}^{d-1}$ and such that

$$\sum_\alpha a_\alpha \partial^\alpha F(0) + \sum_{\alpha'} b_{\alpha'} \partial^{\alpha'} F(x_n) \to 0.$$

Since by stationarity both terms of the left hand side have a constant positive variance, it means their correlation goes to 1, which is in contradiction with the assumption that for each $\alpha, \alpha' \in I$, the partial derivative of the covariance function $\partial^\alpha \partial^{\alpha'} C(x_n) \to 0$ as $x_n \to \infty$.

□

3.3. **Concentration.** Finally, it is standard a result in Gaussian processes that the field and its derivatives concentrate well. Nevertheless, to make the presentation self-contained, we provide the proof.

Proposition 11. *There is $c_d, c'_d < \infty$ such that the following holds: given any continuous centered Gaussian field G on some domain $A \subset \mathbb{R}^d$, with $\sigma_A = \sup_A \sqrt{\mathsf{Var}(G(x))}$, for all $t \geqslant 0$,*

$$\mathbf{P}(\|G\|_A \geqslant t) \leqslant c_d(1 + \mathsf{diam}(A)^d) \exp(-c'_d(t/\sigma_A)^2). \tag{6}$$

Proof. Assume first $\mathsf{Var}(G(x)) \leqslant 1$. Decomposing $A = \cup_{i=1}^{[\mathsf{diam}(A)]^d+1} A_i$ where the A_i have diameter at most 1,

$$\mathbf{P}(\|G\|_A \geqslant t) \leqslant \sum_i \mathbf{P}(\|G\|_{A_i} \geqslant t) \leqslant c_d(1 + \mathsf{diam}(A)^d)\mathbf{P}(S > t)$$

by stationarity, where $S = \|G\|_{B(0,1/2)}$. Then Borell-TIS inequality yields

$$\mathbf{P}(|S| > \mathbf{E}(|S|) + s) \leqslant 2\exp(-s^2)$$

and $\mathbf{E}(|S|) \leqslant c''_d$ with for instance [1, Th.1.3.3]. In the general case apply the previous reasoning to the field $G(x)\sigma_A^{-1}$.

□

4. Topological analysis

4.1. **Morse representation.** Following the theory of Morse functions [20], the central objects of investigation in our work are the random critical points in the compact sampling window $W_n \subset \mathbb{R}^d$ of volume n. More precisely, for a bounded Borel set $W \subset \mathbb{R}^d$, given $F : W \to \mathbb{R}$ smooth, we let

$$Y(W \times I; F) := Y(W \times I) := \{(x, F(x)) \in W \times I : \nabla F(x) = 0\}$$

be the marked point process of critical points of the field F.

The index n indicates the restriction to some large rectangular window $W_n \subset \mathbb{R}^d$

$$Y_n = Y(W_n \times \cdot).$$

Furthermore, let Q_i be the intersection of W_n with the half-open straight cube centred in i with sidelength 1. Nonempty intersection of W_n with affine subspaces that are maximal with respect to inclusion are called facets, their dimension is the minimal possible dimension of the intersecting affine subspace. In this context, call *stratified critical point* any x belonging to a facet f of W such that $F(x) = 0, \nabla_{\mathsf{f}} F(x) = 0$, where $\nabla_{\mathsf{f}} F(x) = (\partial_{u_i} F(x))_i$, for some basis u_i spanning the hyperplane containing f. Denote by $Y^{\partial}(W \times I; F) = Y^{\partial}(W \times I)$ the corresponding process of stratified critical points, note that it contains $Y(W \times I)$.

The following result ensures that we have finiteness of the moments of the measure Y_n:

Theorem 12 (Gass, Stecconi [11])**.** *Assume F is of class $\mathcal{C}^{p+1}$ on some bounded Borel set $A \subset \mathbb{R}^d$. Then*

$$\mathbf{E}((\#Y(A \times \mathbb{R}))^p) < \infty.$$

The following lemma ensures some sort of topological stability of such a manifold $A = \{f \geqslant u\}$ perturbed by a $\mathcal{C}^k$ real function $\Delta : B \to \mathbb{R}$ where B is a neighbourhood of A. Call *quasi critical point* of (f, Δ) at level u a couple $t \in [0,1], x \in \mathbb{R}^d$ such that $(f + t\Delta)(x) = u, \nabla(f + t\Delta)(x) = 0$. In a stratified window W, call more generally *stratified quasi critical point* a couple (t, x) with $x \in \mathsf{f}$ for some facet f of W such that $(f + t\Delta)(x) = u$ and $\nabla_{\mathsf{f}}(f + t\Delta)(x) = 0$. Denote by $\mathscr{C}(A, Q, B)$ the set of $C \in \mathscr{C}(A, Q)$ with $C \cap B = \varnothing$. We typically consider the components $\mathscr{C}(\{F \geqslant u\}, W, \partial W)$ of the excursion set interior to W.

Lemma 13 (Fundamental lemma of stratified Morse theory, Lemma B.1 of [4])**.** *Assume (f, Δ) does not have stratified quasi critical point at level u on some stratified window W. Then there is a one-to-one mapping ζ between $\mathscr{C}(\{f \geqslant u\}, W, \partial W)$ and $\mathscr{C}(\{f + \Delta \geqslant u\}, W, \partial W)$, and for any $C \in \mathscr{C}(\{f \geqslant u\}, W, \partial W)$, C and $\zeta(C)$ are isotopic.*

This lemma implies that contributions in a topological functional over the manifold are determined by the (stratified) critical points of the function.

Let us give an alternative representation of $\beta_n(u, F)$ obtained by scanning the levels and account carefully for critical points on ∂W_n:

Proposition 14. *For $x \in \mathsf{E}(u)$, let $C_x = \mathcal{C}(\{F \geqslant u\}, \{x\})$ be the bounded connected component containing x, or the empty set. We have with Definition 3*

$$\beta_n(u; F) = \sum_{(x,v) \in Y^{\partial}(W_n \times [u, \infty))} \delta(x, v, C_x)$$

where $\delta(x, v, C_x)$ satisfies

$$|\delta(x, v, C_x)| \leqslant \#Y^\partial(C_x \times [u, \infty))$$

and $\delta(x, v, C_x)$ only depends on the isotropy class of C_x (hence is not modified upon a Δ-perturbation without stratified quasi critical point by Lemma 13). We then define the positive and negative parts:

$$\beta_n^+(u; F) = \sum_{(x,v)} [\delta(x, v, C_x)]_+$$

$$\beta_n^-(u; F) = \sum_{(x,v)} [\delta(x, v, C_x)]_-$$

Proof. Recall that

$$\beta_n(u; F) = \sum_C \beta(C)$$

and, exploiting the fact that there are a.s. finitely many critical points in any compact set (Lemma 17), all with disjoint values, define for $(x, v) \in Y^\partial(C \times [u, \infty))$

$$\delta(x, v, C) = \beta_n(v^+; F) - \beta_n(v^-; F)$$

where exponents $+$ and $-$ denote respectively lower and upper limits in v. □

This representation of β_n as the difference between two non-decreasing functionals will be useful in Section 6 when assessing the uniform tightness of the functional $u \mapsto \beta_n(u; F)$. Let us enumerate some situations for a triple (x, v, C) where $\delta(x, v, C) \neq 0$. It is important to remember that this value is allocated in the scanning process and only depends on levels $w \geqslant v$, and the value will not change when crossing critical points below.

(1) x only involves internal components, such as when it is the birth point of a component or the death point of a hole, or the merging of two internal components $C, C' \in \mathscr{C}(\{F \geqslant v\}; W_n, \partial W_n)$, in the latter case $\delta(x, v, C) = \beta(C \cup C') - \beta(C) - \beta(C')$.
(2) x is a stratified critical point where a component C touches ∂W_n at level v, hence $\delta(x, v, C) = -\beta(C)$,
(3) x is an internal critical point which merges an internal component C with an external one, in which case $\delta(x, v, C) = -\beta(C)$.

Here is an alternative strategy of assigning weights to critical points.

Definition 15 (Reference point)**.** For a compact connected component C of $\mathscr{C}(A; W; \partial W)$, call $x(C; F) = x(C) \in W \backslash \partial W$ the critical point of C lowest with respect to the lexicographic order, called *reference point of* C, and define

$$\delta^{ref}(x, v) = \beta(C)\mathbf{1}[x = x(C)]$$

for $(x, v) \in Y(C \times [u, \infty))$. We have indeed

$$\beta_n(u, F) = \sum_{(x,v) \in Y^\partial(W_n \times [u, \infty))} \delta^{ref}(x, v, C_x).$$

It will be apparent in the proof of Lemma 18 that this representation is easier to handle when one tries to evaluate the probability that the topology is not modified upon the perturbation by some field Δ, i.e. $\beta_n(u; F) = \beta_n(u; F + \Delta)$. The reason is that it is easier to bound the probability that two critical points exchange lexicographic order during the perturbation than to bound the probability that they exchange value, i.e. that one becomes lower than the other.

4.2. **Topological perturbation.** For $B \subset \mathbb{R}^d$, define by $\mathcal{W}^{(B)}$ an independent resampling of the Gaussian white noise $\mathcal{W}$ in B, i.e.

$$\mathcal{W}^{(B)}(A) = \mathcal{W}(A \backslash B) + \mathcal{W}'(A \cap B)$$

where $\mathcal{W}'$ is a white noise independent of $\mathcal{W}$ with the same law. We will only consider countably many such resamplings, so we can assume all $\mathcal{W}^{(B)}$ are independent. Define as in (4)

$$F^{(B)}(x) := q \star \mathcal{W}^{(B)}(x)$$
$$\Delta_B := F - F^{(B)}$$

Remark that $F^{(B)}$ has the same distribution as F because $\mathcal{W}$ and $\mathcal{W}^{(B)}$ have the same law, and Δ_B should be small far away from B.

Proposition 16. *Let $|\alpha| \leqslant 3$. If for some $\eta > d, c_q < \infty$, $|\partial_\alpha q(x)| < c_q(1+|x|)^{-\eta}$, there is finite $c > 0$ such that for $A \subset \mathbb{R}^d$*

$$\mathbf{P}(\|\partial_\alpha \Delta_B(x)\|_A > t) \leqslant c_d(1 + \text{ diam } (A)^2) \exp\left(-c\frac{t^2}{(1+d(A,B))^{-2\eta+d}}\right).$$

Proof. Let $r = d(A,B)$. By (3)

$$\begin{aligned}\mathsf{Var}(\Delta_B(x)) = \mathsf{Var}(q \star (1_B(\mathcal{W} - \mathcal{W}'))) =& 2\int_B |q(x-y)|^2 dy \\ &\leqslant c' \int_{B(0,r)^c} (1 + \|x-y\|)^{-2\eta} dy \leqslant c'' r^{-2\eta+d}.\end{aligned}$$

Hence, with Proposition 11, the maximum over some A satisfies the conclusion. Then replace Δ_B with $\partial_\alpha \Delta_B$ for $\alpha \neq 0$ to arrive at the same conclusion. □

Henceforth, we only consider $B := H_{i,j}$ the open half-space of points closer from some $j \in \mathbb{Z}^d$ than from some $i \in \mathbb{Z}^d$, in which case use the shorthand notation

$$F^{(i,j)} = F^{(H_{i,j})}; \ \Delta_{i,j} = \Delta_{H_{i,j}}$$

and remark that $\Delta_{i,j}$ is independent from $\Delta_{j,i}$ because $H_{i,j} \cap H_{j,i}$ has negligible intersection.

In the remainder of this section and the entire paper, it will be essential to be able to bound the expected number of critical points whose value is contained in a certain interval I. This will be done with the Kac-Rice formula.

Lemma 17 (Kac-Rice)**.** *let $m \geqslant 1$, $Q \subset \mathbb{R}^m$ compact and $F : Q \to \mathbb{R}$ a $\mathcal{C}^3$ smooth centred Gaussian field such that for each $x \in Q$,*

(7) $\quad (F(x), \nabla F(x); \partial_{i,j} F(x); i \leqslant j)$ *is a non-degenerate Gaussian vector.*

Then

$$\mathbf{E}(\#Y(Q \times I)) \leqslant c|Q||I|$$

where c depends on the law of F. More precisely, in general, the constant c can depend on Q. However, if F is stationary, then there is no such dependence.

Proof. Assumption (7) yields by Corollary 11.2.2 in [1] that

$$\mathbf{E}(\#Y(Q\times I)) = \int_Q \mathbf{E}(|\det H_F(x)|\mathbf{1}_{\{F(x)\in I\}}|\nabla F(x)=0)dx.$$

Let $W_x = (\nabla F(x), \partial_{i,j}F(x), i\leqslant j)$. Denote by Λ_v the (Gaussian) conditional distribution of $F(x)$ given $W_x = v$. By basic results on conditional Gaussian vectors, we have $\Lambda_v \sim \mathcal{N}(m(v,x),\sigma_x)$ where σ_x is a deterministic value not depending on v. Hence

$$\begin{aligned}\mathbf{E}(|\det H_F(x)|\mathbf{1}_{\{F(x)\in I\}}|\nabla F(x)=0) =&\mathbf{E}(\mathbf{P}(F(x)\in I|W_x)|\det H_F(x)||\nabla F(x)=0)\\ =&\mathbf{E}(\mathbf{P}(\mathcal{N}(m(W_x,\sigma_x)\in|I|))|\det H_F(x)||\nabla F(x)=0)\\ \leqslant& c\frac{|I|}{\sigma_x}\mathbf{E}(|\det H_F(x)||\nabla F(x)=0).\end{aligned}$$

The minimal and maximal eigenvalues of the covariance matrix of W_x are bounded from above and below as its covariance matrix is continuous on a compact and its determinant does not vanish; hence it yields the desired bound. □

4.3. **Topological lemma.** Fix some stratified window W_n. Let Q_i the unit cube centered at some $i\in\mathbb{Z}^d$ with faces parallel to the axes, and let $Q_i := \mathsf{Q}_i\cap W_n$. Let β a topologically additive functional as in Definition 3. We have the decomposition

$$\beta(u;W_n) = \sum_{i:Q_i\neq\varnothing}\beta_{[i]}(u;W_n)$$

where

$$\begin{aligned}\beta_{[i]}(u;W_n) := \beta_{[i]} :=& \sum_{(x,v)\in Y(Q_i,[u,\infty))}\delta^{ref}(x,v,C_x)\\ =& \sum_{C\in\mathscr{C}(\mathsf{E}(u),Q_i,\partial W_n)}\beta(C)\mathbf{1}_{\{x(C;F)\in Q_i\}}.\end{aligned}$$

We use implicitly that a.s. no critical point is on the boundary of a Q_i, formally proved with Lemma 17 applied with the Lebesgue-zero set $\cup_i\partial Q_i$.

In this section, we consider the effect above Q_i of a perturbation applied to the field. More precisely, the white noise is resampled far away in $H_{i,j}$ (points closer from j than i), and we denote by $\tilde\beta_{[i],j}$ the value of $\beta_{[i]}$ after perturbation, i.e. for the field $F+\Delta_{i,j}$:

$$\tilde\beta_{[i],j}(u;W_n) := \tilde\beta^j_{[i]} : = \sum_{C\in\mathscr{C}(\{F+\Delta_{i,j}\geqslant u\},Q_i,\partial W_n)}\beta(C)\mathbf{1}_{\{x(C,F+\Delta_{i,j})\in Q_i\}}.$$

We extend these definitions to the interval $I=[u_-,u_+]$:

$$\beta_{[i]}(I) = \beta_{[i]}(u_+;W_n) - \beta_{[i]}(u_-;W_n)$$

and similarly for $\tilde\beta^j_{[i]}(I)$. The content of the following lemma is to show that both values are equal with high probability.

Lemma 18. *Let* $\delta_j = 1+\frac{1}{3}\|i-j\|, \varepsilon>0$. *Then,*

$$\mathbf{P}(\exists n :\beta_{[i]}(I;W_n)\neq\tilde\beta_{[i],j}(I;W_n)) \leqslant c_\varepsilon\min(\delta_j^{d/2-\beta+\varepsilon}\ ,\ |I|). \tag{8}$$

We prove separately that the LHS is bounded by each of the terms in the minimum in the RHS. First, we have the trivial bound, exploiting the fact that $\beta_{[i]}$ and $\tilde{\beta}_{[i],j}$ have the same law,

$$\begin{aligned}\mathbf{P}(\exists n : \beta_{[i]} \neq \tilde{\beta}_{[i],j}) \leqslant & \mathbf{P}(\exists n : \beta_{[i]} \neq 0 \text{ or } \tilde{\beta}_{[i],j} \neq 0) \\ \leqslant & 2\mathbf{P}(\exists n : \beta_{[i]} \neq 0) \\ \leqslant & 2\mathbf{P}(Y(Q_i \times I) \neq \emptyset) \leqslant 2\mathbf{E}(\# Y(Q_i \times I)).\end{aligned}$$

The bound by $|I|$ hence follows from Lemma 17, invoking also Proposition 10.

Bounding by the first term in the minimum in Lemma 18 is much more tricky. We see the resampling as a continuous temporal evolution that leads from $\beta_{[i]}$ to $\tilde{\beta}^j_{[i]}$. We interpolate between the original field $F = F_0 := F(\cdot; \mathcal{W})$ and the resampled field $F_1 := F(\cdot; \mathcal{W}^{(H_{i,j})})$. The resampling around j is done continuously through the evolution $F_t = F + t\Delta, t \in [0,1]$, where we recall that $\Delta := \Delta_{i,j}$ is small around i. The proof of Lemma 18 resides in the idea that the topology of $\{F \geqslant u\} \cap Q_i$ is the same as $\{F + \Delta \geqslant u\} \cap Q_i$ if no "topological event" occurs during the evolution $t \to F_t$. It is formalised by the "deterministic" Lemma 19 below.

We must first control the size of the connected component of Q_i. Let $m = \delta_j^{\varepsilon}$. Let Q_i^m the cube with faces parallel to axes centred in i with sidelength m. By Assumption 2, there is $\xi > 0$ such that

$$\mathbf{P}(\mathsf{diam}(\mathcal{C}(\{F \geqslant u - \xi\})) \geqslant m) \leqslant c\delta_j^{-\eta + d/2}.$$

Introduce the events

$$\begin{aligned}\Omega_1 :=& \{\mathcal{C}(\{F \geqslant u - \xi\}, Q_i) \subset Q_i^m\} \\ \Omega_2 :=& \{\sup_{x \in Q_i^m} \|\Delta(x)\| < \xi\}.\end{aligned}$$

If Ω_1, Ω_2 are satisfied, indeed $F + t\Delta \geqslant F - \xi$ on Q_i^m, hence for $t \in [0,1]$,

$$(\{F + t\Delta > u\} \cap Q_i^m) \subset (\{F > u - \xi\} \cap Q_i^m)$$

and

$$\mathcal{C}(\{F + t\Delta \geqslant u\}, Q_i) \subset \mathcal{C}(\{F \geqslant u - \xi\}, Q_i) \subset Q_i^m.$$

These events are indeed dominant using also Proposition 11:

$$\mathbf{P}(\Omega_1^c) + \mathbf{P}(\Omega_2^c) \leqslant c\delta_j^{-\eta + d/2}$$

where the constant depends on the law of F and u, ξ, ε.

Lemma 19. *Recall $I = [u_-, u_+]$. Assume that Ω_1, Ω_2 and the following hold:*

1. *There is no quasi critical point $(t, x) \in [0,1] \times Q_i^m$ with $F_t(x) \in \{u_-, u_+\}$*
2. *There is no stratified quasi critical point $(t, x) \in [0,1] \times \mathsf{f}$ for some facet f of Q_i or Q_i^m or $\partial W_n \cap Q_i^m$, i.e. such that $F_t(x) \in \{u_-, u_+\}, \nabla_{\mathsf{f}} F_t(x) = 0$.*
3. *There is no $(t, x) \in [0,1] \times \partial Q_i$ such that $\nabla F_t(x) = 0$.*
4. *There are no two critical points $x \neq y \in Q_i^m$ of F_t such that $x_1 = y_1$.*
5. *For $(x, t) \in Q_i^m \times [0,1], \det H_{F_t}(x) \neq 0$ if $\nabla F_t(x) = 0$.*

Then,

$$\beta_i(I; W_n) = \tilde{\beta}_{[i],j}(I; W_n).$$

Proof. Denote by $y_1(t), \ldots, y_{p(t)}(t)$ the critical points of $\mathcal{C}(\{F_t \geqslant u_-\}, Q_i) \subset Q_i^m$ at time t such that $F_t(y_k(t)) \in I$. Then,

(i) $y_k(t)$ satisfies the equation $\nabla F_t(y_k(t)) = 0$,
(ii) the Jacobian matrix of $y \mapsto \nabla F_t$ is $J_t(y) = (\partial_j \partial_i F_t(y)) = H_{F_t}(y)$,
(iii) and by (4), $\det J_t(y_k(t)) \neq 0$.

Hence, by the Inverse Function Theorem, $t \mapsto y_k(t)$ can be $\mathcal{C}^1$ extended on a neighbourhood of t, which means the $y_k(t)$ are $\mathcal{C}^1$ trajectories $[0,1] \to Q_i^m$.

The number of critical points could in principle depend on t, but since the component is contained in Q_i^m during the evolution process they cannot escape Q_i^m, and due to the previous lemma, points stay all along the evolution and $p(t) = p = const$. The corresponding values $F_t(y_k(s))$ remain in I thanks to point (1) and by continuity in t.

According to Lemma 13, points (1) and (2) yield an isotopy Γ between connected components of $\{F_t \geqslant u\}$ in Q_i^m at times $t = 0$ and $t \in [0,1]$. By point (2), those of these components that touch Q_i at time 0 are the same than at time t, because there is no stratified quasi critical point on the boundary ∂Q_i. Let us label such components at time $t = 0$ by $C_1, \dots, C_\ell$, and $C_1' = \Gamma(C_1), \dots, C_\ell' = \Gamma(C_\ell)$ the components of $\{F_t \geqslant u\}$ touching Q_i at some time $t \in [0,1]$. We used that since Ω_1, Ω_2 are satisfied, all these components are contained in Q_i^m. In particular, the isotopy yields that $\beta(C_k) = \beta(C_k')$ for $1 \leqslant k \leqslant \ell$. By point (3), two critical points cannot exchange order in the lexicographic order, hence we can write that the reference points are $y_{i_1}(t), \dots, y_{i_\ell}(t)$ for some fixed indexes $i_1, \dots, i_\ell \in \{1, \dots, p\}$. Hence,

$$\begin{aligned}\beta_i(u_-; W_n) - \tilde{\beta}^j_{[i]}(u_-; W_n) &= \sum_{k=1}^{\ell} \beta(C_k)\mathbf{1}_{\{y_{i_k(0)} \in Q_i, C_k \subset W_n\}} - \beta(C_k')\mathbf{1}_{\{y_{i_k(1)} \in Q_i, C_k' \subset W_n\}} \\ &= \sum_{k=1}^{\ell} \beta(C_k)(\mathbf{1}_{\{y_{i_k}(0) \in Q_i, C_k \subset W_n\}} - \mathbf{1}_{\{y_{i_k}(1) \in Q_i, C_k' \subset W_n\}})\end{aligned}$$

(and a similar representation holds for $\beta_{[i]}(u_+) - \tilde{\beta}^j_{[i]}(u_+)$). Since (2) is satisfied, $y_{i_k}(t)$ stays at a positive distance from ∂Q_i, hence $1_{\{y_{i_k}(0) \in Q_i\}} = 1_{\{y_{i_k}(t) \in Q_i\}}$.

Finally, since there is no tangency point of some component C_k, C_k' with the boundary of W_n by (2), the status $C_k \subset W_n$ or $C_k' \subset W_n$ cannot change, and the previous sum vanishes.

□

To complete the proof of Lemma 18, we hence have

$$\begin{aligned}\mathbf{P}(\exists n : \beta_{[i]} \neq \tilde{\beta}^j_{[i]}) \leqslant& \mathbf{P}(\Omega_1^c) + \mathbf{P}(\Omega_2^c) + \sum_k \underbrace{\mathbf{P}((k) \text{ is not satisfied })}_{=:\mathsf{b}_k} \\ \leqslant& c\delta_j^{\eta - d/2} + \mathsf{b}_1 + \mathsf{b}_2 + \mathsf{b}_3 + \mathsf{b}_4 + \mathsf{b}_5.\end{aligned}$$

Let us estimate the b_k's. First note that point (1) is a particular case of (2), with $\mathsf{f} = Q_i^m$. Point (2) is implied by

$$|F(x) - u| > \|\Delta F(x)\| \text{ or } \|\nabla_{\mathsf{f}} F(x)\| > \|\nabla_{\mathsf{f}} \Delta(x)\|, x \in \partial W_n \cap Q_i^m$$

and similarly for v. Point (3) is implied by

$$\|\nabla F(x)\| > \|\nabla \Delta(x)\|, x \in \partial Q_i$$

$$\begin{aligned}\mathsf{b}_2 \leqslant &\mathbf{P}(\exists x \in \mathsf{f} : |F(x) - u_-| < |\Delta(x)|, \|\nabla_{\mathsf{f}} F(x)\| < \|\nabla_{\mathsf{f}}\Delta(x)\|)\\ \mathsf{b}_3 \leqslant &\mathbf{P}(\exists x \in \partial Q_i : |\nabla F(x)| < \|\nabla\Delta(x)\|)\\ \mathsf{b}_4 = &\mathbf{P}(\exists t \in [0,1], \exists x \neq y \in Q_i^m \text{ such that } x_1 = y_1 \text{ and } \nabla F_t(x) = \nabla F_t(y) = 0)\\ \mathsf{b}_5 = &\mathbf{P}(\exists t, x \in [0,1] \times Q_i^m : \nabla F_t(x) = 0, \det H_{F_t}(x) = 0)\end{aligned}$$

All the terms are dealt with a lemma bounding from above the probability that the minimum of a Gaussian field on a compact is small, it is a quantitative version of Bulinskaya's lemma [7] in the spirit of the Nazarov and Sodin's version [25], applicable also to some non-Gaussian fields such as $\det H_{F_t}(x)$. This lemma is proved later in this section.

Lemma 20 (Bulinskaya)**.** *Let $p' < p$ integers, A a bounded Borel set of $\mathbb{R}^{p'}$. Let $g_1, \dots, g_p$ be smooth centred Gaussian fields $A \to \mathbb{R}^{d_i}$ resp. for some $d_i \in \mathbb{N}^*$, and such that*

$$G(x) := (g_i(x)) \in \mathbb{R}^{p^*}$$

has a uniformly bounded density, where $p^* = \sum_i d_i$. Let $h_i : \mathbb{R}^{d_i} \to \mathbb{R}$ be subpolynomial functions, and $f_i(x) := h_i(g_i(x)), 1 \leqslant i \leqslant p$. Assume also that each $h_i(X)$ has a density bounded on $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ and $X \sim \mathcal{N}(0, I_{d_i})$. Let

$$\psi(x) = (f_i(x))_{i=1}^p \in \mathbb{R}^p,$$

Let $\alpha < 1$. There is c_α such that for $\tau > 0$,

$$\mathbf{P}(\inf_{x \in A} \|\psi(x)\| \leqslant \tau) \leqslant c_\alpha(|A| \vee 1)\tau^\alpha.$$

Furthermore for a centred Gaussian field $\varphi : A \to \mathbb{R}^p$, let $\sigma_A := \max_{x \in A} \mathsf{Var}(\varphi(x))$, we have

$$\mathbf{P}(\exists x \in A, \|\psi(x)\| < \|\varphi(x)\|) \leqslant c'_\alpha(|A| \vee 1)\sigma_A^\alpha.$$

In all cases except b_5, we will apply the lemma with $h_i(s) = s$ or $h_i(s) = s - u$ or $h_i(s) = s - v$, which obviously have a bounded density on Gaussian input. Recall that $\gamma = \eta/d - 1/2$

(b_2) Let $p' = \mathsf{dim}(\mathsf{f}) \leqslant d$. Lemma 20 can be applied on $A = \mathsf{f} \subset \mathbb{R}^{p'}$ with $p = p' + 1, \psi(x) = (\partial_1 F(x), \dots, \partial_{p'} F(x), F(x) - u_-), h_i(s) = s$ for $i \leqslant p', h_p(s) = s - u_-$. The Gaussian field $G(x) = (\nabla F(x), F(x))$ is non-degenerate for each x with Proposition 10, hence since A is compact and $x \to \det \mathsf{Cov}\,(G(x))$ is continuous, $G(x)$ has a uniformly bounded density.

Let $\varphi(x) = (\nabla\Delta(x), \Delta(x))$. We compute in the proof of Proposition 16 that with $\partial_0\Delta := \Delta$,

$$\max_{k \leqslant d} \mathrm{Var}(\partial_k \Delta(x)) \leqslant cm^d \delta_j^{-\gamma d}, 0 \leqslant k \leqslant d$$

hence Bulinskaya's lemma yields

$$\begin{aligned}\mathsf{b}_2 \leqslant &\mathbf{P}(\|\psi(x)\| \leqslant \|\varphi(x)\|, x \in Q_i^m)\\ \leqslant &c_\alpha |Q_i^m|^\alpha \max_{x \in Q_i^m, 0 \leqslant k \leqslant d} \mathrm{Var}(\partial_k \Delta(x)^\alpha)\\ \leqslant &cm^{d\alpha} \delta_j^{-\gamma d\alpha}\end{aligned}$$

for $1 - \alpha$ sufficiently small. This is of the right order for Lemma 18.

(b_3) For b_3, reason similarly with $p' = d-1, p = d$, $\psi(x) = \nabla F(x), h_i(s) = s, \varphi(x) = \nabla\Delta(x)$, to show

$$\mathsf{b}_2 \leqslant c\delta_j^{(-\eta+d/2)\alpha}.$$

Once again $\psi(x) = G(x)$ is non-degenerate by Proposition 10.

(b_4) For $(x,\hat{y}) = (x,(y_2,\dots,y_d)) \in \mathbb{R}^{2d-1}$, denote by $\tilde{y} = (x_1, y_2, \dots, y_d)$. Let $p' = 2d-1$ and

$$A = \{(x,\hat{y}) \in Q_i^m \times \mathbb{R}^{d-1} : \tilde{y} \in Q_i^m, x \neq \tilde{y}\}.$$

Define the operator $D^2F(x,\hat{y}) = \|x-\tilde{y}\|^{-1}(\nabla F(x) - \nabla F(\tilde{y}))$ for $x \neq \tilde{y}$. We want to estimate the probability that there is some $t \in [0,1]$ such that

$$\nabla F(x) + t\nabla\Delta(x) = \nabla F(\tilde{y}) + t\nabla\Delta(\tilde{y}) = 0.$$

It implies (technique of the divided differences)

$$\|\nabla F(x)\| \leqslant \|\nabla\Delta(x)\|, \|D^2F(x,\hat{y})\| \leqslant \|D^2\Delta(x,\hat{y})\|.$$

We wish to apply Bulinskaya's lemma with $p' = 2d-1, p = 2d$ to the fields

$$\psi(x,\hat{y}) = (\nabla F(x), D^2F(x,\hat{y})) \in \mathbb{R}^{2d},$$
$$\varphi(x,\hat{y}) = (\nabla\Delta(x), D^2\Delta(x,\hat{y}))$$

on A, with $h_i(s) = s$. It would yield with the same mechanism

$$\mathsf{b}_4 \leqslant \mathbf{P}(\exists (x,\hat{y}) \in A : \|\psi(x,\hat{y})\| \leqslant \|\varphi(x,\hat{y})\|) \leqslant c''_\alpha m^d \delta_j^{-\gamma d\alpha}.$$

It remains to check the non-degeneracy condition. Since A is not closed, the non-degeneracy of each $G(x,\hat{y}) = \psi(x,\hat{y})$ is not enough. Reason by contradiction: assume that for some sequence $(x_n,\hat{y}_n)_n$ in A the max density of $\psi(x_n,\hat{y}_n)$ grows to ∞. By compactness, up to taking a sub-sequence, $(x_n,\tilde{y}_n) \to (x_\infty,\tilde{y}_\infty) \in (Q_i^m)^2$.

- First possibility: $x_\infty \neq \tilde{y}_\infty$, hence the law of $(\nabla F(x_\infty), \nabla F(\tilde{y}_\infty))$ is degenerate, contradicts Proposition 10.
- Second possibility: $x_\infty = \tilde{y}_\infty$. Up to taking a sub-subsequence, by compactness of $\mathbb{S}^{d-1}$, there is a unit vector z such that $(x_n - \tilde{y}_n)\|x_n - \tilde{y}_n\|^{-1} \to z$. Therefore $D^2F(x_n,\hat{y}_n) \to \partial_z \nabla F(x_\infty)$. It follows that $(\nabla F(x_\infty), \partial_z\nabla F(x_\infty))$ is degenerate, which again contradicts Proposition 10.

(b_5) Denote $d\det$ the differentiable application of the determinant on the space of $d\times d$ matrices, $\|H\|$ the norm of a matrix. Remark that for any matrices H, D

$$\begin{aligned}|\det(H+D) - \det(H)| &\leqslant \|D\| \sup_{t\in[0,1]} \|d\det_{H+tD}\| \\ &\leqslant \|D\| \sup_t \sum_i |\operatorname{com}(H+tD)_{i,i}| \\ &\leqslant c\|D\|(\|H\| + \|D\|)^{d-1},\end{aligned}$$

where $\operatorname{com}(H)$ denotes the comatrix of H. Denote by $H_f(x)$ the Hessian matrix at x of some $\mathcal{C}^2$ field $f : A \to \mathbb{R}$, and $\|H_f\|_{Q_i^m} = \sup_{x\in Q_i^m} \|H_f(x)\|$.

The latter inequality yields that $\det H_{F_t}(x) = \det(H_F(x) + tH_\Delta(x))$ can only vanish for some $t \in [0,1]$ if

$$|\det(H_F(x))| < c\|H_\Delta\|_{Q_i^m}(\|H_F\|_{Q_i^m} + \|H_\Delta\|_{Q_i^m})^{d-1}.$$

Hence, b_5 is upper bounded by

$$\mathbf{P}\left(\exists x \in Q_i^m : \nabla F(x) \leqslant \|\nabla\Delta(x)\|, \det H_F(x)| < c\|H_\Delta\|_{Q_i^m}(\|H_F\|_{Q_i^m} + \|H_\Delta\|_{Q_i^m})^{d-1}\right).$$

We still invoke Bulinskaya's lemma with $p' = d, p = d+1, A = Q_i^m$,

$$\begin{aligned}(g_1(x), \dots, g_d(x)) =& \nabla F_t(x),\\ g_{d+1}(x) =& (\partial_{i,j}F(x), i \leqslant j) \in \mathbb{R}^{d(d+1)/2},\\ h_i(s) =& s, s \in \mathbb{R}, i \leqslant d,\\ h_{d+1}((a_{i,j})_{i\leqslant j}) =& \det((a_{i,j}))\end{aligned}$$

hence

$$\psi(x) = (\nabla F(x), \det H_F(x)).$$

By Proposition 10, $G(x) := (g_i(x))_{1\leqslant i\leqslant d+1}$ has a non-degenerate distribution, and $A = Q_i^m$ is compact. The fact that $h_{d+1}(A_{i,j}) = \det((A_{i,j}))$ has a bounded density for i.i.d. Gaussian entries $A_{i,j}, i \leqslant j$ follows from Lemma 21 below. We hence have for $\alpha < 1$ some c such that, by Lemma 20,

$$p_\tau := \mathbf{P}(\exists x \in Q_i^m : \|\nabla F(x)\| < \tau, |\det H_F(x)| < \tau) \leqslant c\tau^\alpha, \tau > 0.$$

Let $\tau = c\delta_j^{-\gamma d\alpha}$. We then have for $\varepsilon > 0$ sufficiently small, by Proposition 16

$$\begin{aligned}\mathsf{b}_5 \leqslant& p_\tau + \mathbf{P}(\|\nabla\Delta\| > \tau) + \mathbf{P}(\|H_\Delta\|(\|H_F\| + \|H_\Delta\|)^{d-1} > \tau)\\ \leqslant& c\tau^\alpha + \mathbf{P}(\|\nabla\Delta\| > \delta_j^{\eta-d/2-\varepsilon}) + \mathbf{P}(\|H_\Delta\| > \tau^{1+\varepsilon}) + \mathbf{P}(\|H_F\| > \delta_j^{\varepsilon'})\\ \leqslant& c\delta_j^{(-d/2+\eta)(1-\varepsilon'')} + c\exp(-c\delta_j^{-\varepsilon})\end{aligned}$$

where $\varepsilon', \varepsilon'' > 0$ are as small as we want.

Finally, all terms have been dealt with and give a contribution whose magnitude is not larger than the RHS of (8), thereby concluding the proof.

Proof of Bulinskaya's lemma. If $\inf_{x_0\in A}\max_{i\leqslant p}|f_i(x_0)| < \tau$, then,

$$\max_{y\in B(x_0,\tau)}|f_i(y)| \leqslant \tau(1 + \|\nabla f_i\|_{B(x_0,\tau)}\|).$$

Let $\gamma \in (0,1)$, and

$$X = \int_A \frac{1}{\prod_{i=1}^p |f_i(x)|^\gamma}dx.$$

Let $\Omega = \{\inf_{x_0\in A}\|\psi(x_0)\| \leqslant \tau\}$. If Ω is realised,

$$\sup_{x\in B(x_0,\tau)}|f(x)| \leqslant \tau + \tau \sup_{y\in B(x_0,\tau)}\|\nabla f(y)\|,$$

then for some $c > 0$ deterministic,

$$X \geqslant \int_{B(x_0,\tau)} \frac{1}{\prod_{i=1}^p |f_i(x)|^\gamma}dx \geqslant c\tau^{p'}\prod_i \frac{1}{\tau^\gamma(1 + \|\nabla f\|_{B(x_0,\tau)})^\gamma} =: c\tau^{p'-p\gamma}\frac{1}{B}.$$

Remark that for $q > 1$, $\mathbf{E}(B^q) < C$ where C depends on A, q and $\sup_{x,i} \mathsf{Var}(f_i(x))$. Let $q' = (1 - q^{-1})^{-1}$. Hence, with Hölder's inequality

$$\begin{aligned}\mathbf{P}(\Omega) \leqslant & \tau^{p\gamma - p'} \mathbf{E}(BX) = \tau^{p\gamma-p'}\mathbf{E}(B^q)^{\frac{1}{q}} \left[\mathbf{E}\left(\int_A \prod_i |f_i(x)|^{-\gamma} dx \right)^{q'} \right]^{1/q'} \\ \leqslant & c\tau^{p\gamma-p'} \left[|A|^{q'} \mathbf{E}\left(\int_A \prod_i |f_i(x)|^{-\gamma q'} \frac{dx}{|A|} \right) \right]^{1/q'} \\ \leqslant & c\tau^{p\gamma-p'} |A| (\sup_{x\in A} \mathbf{E}(\prod_i |f_i(x)|^{-\gamma q'}))^{1/q'}\end{aligned}$$

Since $p \geqslant p' + 1$, we can choose γ such that $p\gamma - p' = \alpha$ and then q such that $\gamma q' < 1$.

By assumption, the covariance matrix of $G(x) = (g_i(x))_{i=1}^m \in \mathbb{R}^{p^*}$ is not degenerate for each $x \in A \subset \mathbb{R}^{m'}$, it is also a continuous function of x because the field is smooth, hence its determinant is a non-vanishing continuous function, hence uniformly (in x) bounded from below, and the density of $G(x)$ is uniformly bounded from above: there is C such that for $X_1, \ldots, X_p$ independent Gaussian vectors with $X_i \sim \mathcal{N}(0, I_{p_i})$, for all $\varphi \geqslant 0$, for all x,

$$\mathbf{E}(\varphi(G(x))) = \int_{\mathbb{R}^{m*}} \varphi(y) d\mathbf{P}_{G(x)}(y) \leqslant C\mathbf{E}(\varphi(X_1, \ldots, X_p))$$

In particular,

$$\begin{aligned}\sup_{x\in A} \mathbf{E}(\prod_{i=1}^p |f_i(x)|^{-\gamma q}) = & \sup_{x\in A} \mathbf{E}(\prod_{i=1}^p |h_i(g_i(x))|^{-\gamma q'}) \\ \leqslant & C\mathbf{E}(\prod_{i=1}^p |h_i(X_i)|^{-\gamma q'}) \\ = & C \prod_{i=1}^p \mathbf{E}(|h_i(X_i)|^{-\gamma q'}) \\ \leqslant & C(\varepsilon^{-\gamma q} + \int_{[-\varepsilon,\varepsilon]} |u|^{-\gamma q'} du)^p < \infty.\end{aligned}$$

For the second statement of the lemma, assume wlog $\sigma_A \leqslant 1$. Let $\alpha' = \sqrt{\alpha} < 1$. Let $\tau = \sigma_A^{\alpha'} \geqslant \sigma_A$, with Proposition 11

$$\begin{aligned}\mathbf{P}(\exists x : \|\psi(x)\| < \|\varphi(x)\|) \leqslant & \mathbf{P}(\exists x, \|\psi(x)\| < \tau) + \mathbf{P}(\sup_x \|\varphi(x)\| > \tau) \\ \leqslant & c_{\alpha'}\tau^{\alpha'} + c_d(|A| \vee 1) \exp(-c'_d \tau^2/\sigma_A^2) \\ \leqslant & c_\alpha \sigma_A^\alpha + c_d(|A| \vee 1) \exp(-c'_d \sigma_A^{2\alpha'-2})\end{aligned}$$

□

Lemma 21. *Let a random symmetric matrix M with centred Gaussian entries in the upper diagonal forming a non-degenerate Gaussian vector. Then* $\det(M)$ *has a bounded density in* 0.

Proof. Let $G = (M_{i,j}, i \leqslant j)$ be the centred Gaussian vector forming the entries, and let $G' = (M'_{i,j})$ the GOE model, i.e. the $M'_{i,j}$ are independent with $\mathsf{Var}(M'_{ii}) =$

$2, \mathsf{Var}(M'_{i,j}) = 1, i < j$. The non-degeneracy condition yields that the density f_G is uniformly bounded by $cf_{G'}(c'\cdot)$ for some constants c, c', hence it is enough to prove the result for the GOE model.

We prove by induction that there is $C_d < \infty$ such that, if M_d is the d-th order GOE, $P(|\det(M_d)| < a) < C_d a$ for $a > 0$. We use the classical result of random matrix theory that the eigenvalues $(l_1, \dots, l_d)$ of M_d have the explicit joint density $c_d \prod_{i<j} |l_i - l_j| f(l_1) \dots f(l_j)$ for some Gaussian density f. Hence, we can explicitly write

$$\begin{aligned}
&P(|\det(M_{d+1})| < a) \\
&\quad = c_d \int_{\mathbb{R}^{n+1}} 1_{|l_1 \dots l_{d+1}| < a} \prod_{i<j} |l_i - l_j| f(l_1) \dots f(l_{d+1}) dl_1 \dots dl_{d+1} \\
&\quad = 2c_d \int_0^\infty \left(\int_{\mathbb{R}^n} 1_{|l_1 \dots l_{d+1}| < a} \prod_{i<j} |l_i - l_j| f(l_1) \dots f(l_d) dl_1 \dots dl_d \right) f(l_{d+1}) dl_{d+1}
\end{aligned}$$

using the symmetry of f.

Up to a combinatorial term, we can reduce the multiple integral to $(d+1)$-tuples such that $l_{d+1} > |l_{d+1}| > \cdots > |l_1| > 0$, and we have the crude bound $|l_{d+1} - l_i| < 2l_{d+1}$ for $i < d+1$. We then have, using the induction hypothesis,

$$\begin{aligned}
P(|\det(M_{d+1})| < a) \leqslant & c'_d \int_0^\infty l_{d+1}^d \int 1_{|l_1 \dots l_d| < a/l_{d+1}} \prod_{i<j\leqslant d} |l_i - l_j| f(l_1) \dots f(l_{d+1}) dl_1 \dots dl_{d+1} \\
= & c'_d \int_0^\infty l_{d+1}^d P(|\det(M_d)| < a/l_{d+1}) f(l_{d+1}) dl_{d+1} \\
\leqslant & c'_d \int_0^\infty l_{d+1}^d C_d \frac{a}{l_{d+1}} f(l_{d+1}) dl_{d+1} \\
\leqslant & c''_d a \int_0^\infty l^d \frac{f(l)}{l} dl \\
\leqslant & C_{d+1} a.
\end{aligned}$$

□

5. Proof of the fixed-level CLT, Theorem 5

In this section, we prove the fixed-level CLT from Theorem 5. We proceed in two steps. First, in Section 5.1, we prove the CLT asserted in Theorem 5 with a possibly vanishing variance. Second, in Section 5.2, we show the positivity of the limiting variance.

5.1. **Fixed-level CLT.** Now, we establish the asymptotic normality of the Betti numbers of the excursion set $\mathsf{E}(u)$.

Essentially, the proof idea for asymptotic normality at a fixed level relies on the stabilisation and moment arguments from [26, Theorem 3.1]. This technique was designed for dealing with functionals from a Poisson point process in a bounded domain, which is inconvenient in the current setting. Indeed, the white noise $\mathcal{W}$ is defined in all of $\mathbb{R}^d$, and modifications at large distances still have a small but non-vanishing effect on the number of critical points in a given domain. This problem also appears in the investigation of the component counts of excursion sets in [4]. To provide a suitable analog of [26, Theorem 3.1] for the setting where the white noise is defined in the entire Euclidean space, [4, Theorem 3.2] is established.

One of the key proof steps of [4, Theorem 1.2] is that the proof of the CLT is reduced to a stabilisation and a moment condition [4, Lemmas 3.6, 3.7]. To state these conditions precisely, we first need to introduce the costs associated with resampling. Moreover, since Assumption 2 guarantees that we are in the subcritical percolation regime, we may assume that all bounded connected components of the excursion set $\mathsf{E}(u)$ that intersect W_n are contained in W_{2n}. We write $\beta_n(u;F)$ as in (1). We set the global variation on Q_j resampling

$$B^{(n)}_{\Delta,j} := \beta_n\big(u;F\big) - \beta_n\big(u;F^{(Q_j)}\big) = \sum_{i\in W_n} B_{\Delta,i,j} \tag{9}$$

where $B_{\Delta,i,j} = \beta_{[i]}(u,W_n) - \tilde\beta_{[i],j}(u,W_n)$ denotes the contribution to $B^{(n)}_{\Delta,j}$ coming from critical points in the cube Q_i. $B^{(n)}_{\Delta,j}$ represents the total variation when Q_j is resampled. Recall that $\gamma := \eta/d - 1/2$.

Proposition 22 (Stabilisation condition; Lemma 3.7 of [4])**.** *Assume that $\gamma > 1$. Then, the sequence $\{B^{(n)}_{\Delta,o}\}_n$ converges almost surely to some almost surely finite random variable $B^{(\infty)}_{\Delta,o}$.*

Proof. We want to show that $B_{\Delta,i,o}(u,W_n) = 0$ for all $i \in \mathbb{Z}^d$ with $|i| \geqslant R$ and $n \geqslant 1$, where R is an almost surely finite random variable. This will be achieved via the Borel-Cantelli lemma. More precisely, first, Lemma 18 implies that

$$\mathbf{P}(\cup_{n\geqslant 1}\{B_{\Delta,i,o}(u,W_n) \neq 0\}) \in O\big(|i|^{-\gamma d+\varepsilon}\big). \tag{10}$$

In particular, since $\gamma > 1$, the products $|i|^{d-1}\mathbf{P}(\cup_{n\geqslant 1}\{B_{\Delta,i,o}(u,W_n) \neq 0\})$ are summable for $i \in \mathbb{Z}^d$. Hence, applying the Borel-Cantelli lemma shows that the existence of the asserted random $R < \infty$ such that $B_{\Delta,i,o}(u,W_n) = 0$ for all $i \in \mathbb{Z}^d$ with $|i| \geqslant R$ and $n \geqslant 1$. □

Now, we write $d_{j,n} := \mathsf{dist}(j,W_{2n})+1$ for the distance of a site j to the window W_{2n}. To ease notation, we assume henceforth that $\mathbf{E}\big[Y(Q_1\times\mathbb{R})^{q_0}\big] < \infty$ for some $q_0 > 2$ which results from $F \in \mathcal{C}^{q_0}$ by Theorem 12. Moreover, we write $q = q(\varepsilon) = 2+\varepsilon$ in the rest of this section. For the proof of Proposition 23 below, we assume that $q_0 > 4$ and that $\gamma > 12$. Here, we also note that these assumptions imply that $\gamma > \frac{9q_0}{q_0+6}$.

Proposition 23 (Moment conditions; Lemma 3.6 of [4])**.** *Assume that $q_0 > 4$ and $\gamma > 12$. Then, for any sufficiently small $\varepsilon > 0$, it holds that*

(1) $\sup_{n\geqslant 1}\sup_{j\in\mathbb{Z}^d}\mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)}) < \infty$;
(2)

$$\sup_{n,k\geqslant 1}\frac{\sum_{j\in\mathbb{Z}^d:\, d_{j,n}>k}\mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)})}{|W_n|^3k^{-\eta/3}(k^d+k|W_n|^{(d-1)/d})} < \infty;$$

(3) $\sup_{n\geqslant 1}|W_n|^{-1}\sum_{j\in\mathbb{Z}^d}\mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)}) < \infty$.

The key step in the proof of Proposition 23 are the following moment bounds on $B^{(n)}_{\Delta,j}$. Henceforth, we set $|(x_1,\dots,x_d)| := \max_{i\leqslant d}|x_i|$ for the ℓ_∞ norm in $\mathbb{R}^d$.

Lemma 24 (Moment bound on $B^{(n)}_{\Delta,j}$)**.** *Let $q_0 > 2$, $\gamma > 1$ and $2 < m < q_0\gamma/(q_0+\gamma)$. Then, for every $\varepsilon > 0$,*

$$\mathbf{E}(|B^{(n)}_{\Delta,j}|^m) \in O\big(d_{j,n}^{-\gamma d(q_0-m)/q_0+\varepsilon}(|W_n| \wedge d^d_{j,n})^m\big).$$

Henceforth, we let

$$g_{j,n}(k) := \big|\{i \in \bar W_n : |i-j| = k\}\big| \in O\big(k^{d-1} \wedge n^{1-1/d}\big).$$

denote the number of elements of $\bar W_n := \{i \in \mathbb{Z}^d : W_n \cap Q_i \ne \varnothing\}$ at distance $k \geqslant 0$ from $j \in \mathbb{Z}^d$. To ease notation, we henceforth often write $B_{\Delta,i,j}$ instead of the more verbose $B_{\Delta,i,j}(u, W_n)$ when the value of n is clear from the context.

Proof. First, by the Jensen inequality, $\mathbf{E}(|B^{(n)}_{\Delta,j}|^m) \leqslant \Big(\sum_{i\in\bar W_n}\mathbf{E}(|B_{\Delta,i,j}|^m)^{1/m}\Big)^m$. Now, invoking (10) and the Hölder inequality with $q' = q_0/m$ and $p' = q_0/(q_0-m)$ gives that

$$\mathbf{E}(B^m_{\Delta,i,j}) \leqslant \big(\mathbf{E}(B^{q_0}_{\Delta,i,j})\big)^{1/q'}\mathbf{P}(B_{\Delta,i,j} \ne 0)^{1/p'} \in O\big(|i-j|^{-\gamma d/p'+\varepsilon}\big) \tag{11}$$

Note that by our moment bound, we have $\mathbf{E}(B^{q_0}_{\Delta,i,j}) < \infty$. Now,

$$\sum_{i\in\bar W_n}|i-j|^{-\gamma d/(p'm)} \leqslant C\sum_{k\geqslant d_{j,n}} g_{j,n}(k)k^{-\gamma d/(p'm)+\varepsilon},$$

where the right-hand side is in $O\big((|W_n| \wedge d^d_{j,n})d_{j,n}^{-\gamma d/(p'm)+\varepsilon}\big)$, as asserted. □

Finally, we complete the proof of Proposition 23.

Proof of Proposition 23. Note that our assumptions on q_0 and γ imply that $3 < q_0\gamma/(q_0+\gamma)$. In particular, part (1) follows from Lemma 24 , and we concentrate on parts (2) and (3). In both parts, we write

$$g^{(n)}_k := |\{i \in \mathbb{Z}^d : \mathsf{dist}(i, W_{2n}) = k\}|$$

for the number of sites that are at distance $k \geqslant 1$ from W_n. Moreover, putting $\rho(\varepsilon) := 1 - q(\varepsilon)/q_0$, we also note that for $k \geqslant n^{1/d}$, Lemma 24 and Jensen give that

$$\sum_{j:\ d_{j,n}\geqslant k}\mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)}) \leqslant C\sum_{\ell\geqslant k} g^{(n)}_\ell |W_n|^{q(\varepsilon)}\ell^{-\gamma d\rho(\varepsilon)}. \tag{12}$$

Since $g^{(n)}_\ell \in O\big(\ell^{d-1}\big)$, the right-hand side is of order $O\big(|W_n|^{q(\varepsilon)}k^{d-\gamma d\rho(\varepsilon)}\big)$.

Part (3). Since the number of $j \in \mathbb{Z}^d$ with $d_{j,n} \leqslant n^{1/d}$ is of order $O(|W_n|)$, it suffices to deal with the case, where $d_{j,n} \geqslant n^{1/d}$. Then, by (12) with $k = \lfloor n^{1/d}\rfloor$,

$$|W_n|^{-1}\sum_{j:\ d_{j,n}\geqslant n^{1/d}}\mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)}) \in O\big(|W_n|^{q(\varepsilon)-\gamma\rho(\varepsilon))}\big).$$

Hence, invoking the assumption $\gamma > 3q_0/(q_0-2)$ concludes the proof of part (3).

Part (2). First, (12) implies that for $k \geqslant n^{1/d}$,

$$\sum_{j:\ d_{j,n}\geqslant k}\mathbf{E}(|B^{(n)}_{\Delta,j}|^q) \in O\big(|W_n|^{q(\varepsilon)}k^{d-\gamma d\rho(\varepsilon))}\big).$$

In particular, noting that $\eta = \gamma d + d/2$, $q_0 > 4$ and $\gamma > 6$, for $k \geqslant n^{1/d}$, the ratio

$$\frac{|W_n|^{q(\varepsilon)}k^{d-\gamma\rho(\varepsilon)d}}{|W_n|^3k^{d-\eta/3}} = |W_n|^{q(\varepsilon)-3}k^{(1/6-\gamma(\rho(\varepsilon)-1/3))d}$$

remains bounded.

Second, we consider the case $d_{j,n} \leqslant n^{1/d}$. Then, by Lemma 24,

$$\sum_{j\colon k \leqslant d_{j,n} \leqslant n} \mathbf{E}(|B^{(n)}_{\Delta,j}|^{q(\varepsilon)}) \leqslant C \sum_{k \leqslant \ell \leqslant n} g^{(n)}_\ell |W_n|^{q(\varepsilon)} \ell^{(1-\gamma\rho(\varepsilon))d}.$$

Since $\max_{\ell \leqslant n} g^{(n)}_\ell \in O\big(n^{1-1/d}\big)$, this is of order $O\big(n^{3d-1+d\varepsilon} k^{1+(1-\gamma\rho(\varepsilon))d}\big)$. Again, since $\gamma > 12$, the ratio

$$\frac{|W_n|^{3+\varepsilon-1/d} k^{1+(1-\gamma\rho(\varepsilon))d}}{|W_n|^{4-1/d} k^{d-\eta/3}} = |W_n|^{-1+\varepsilon} k^{1+d/6-\gamma(\rho(\varepsilon)-1/3)d}$$

remains bounded, thereby concluding the part of part (2). □

5.2. **Positivity of variance.**

In this section, we establish the positivity of the limiting variance $\sigma(u)^2$ from Theorem 5. The strategy is to proceed similarly as in the case of the component count considered in [4, Theorem 1.3]. The idea is that in the proof the martingale CLT from [4, Theorem 1.3], the authors also derive a non-trivial and highly useful representation of the limiting variance. First, we recall from Proposition 22 that $B^{(\infty)}_{\Delta,o}$ is the difference of the Betti numbers before and after the resampling of the white noise in Q_1. Now, we have

$$\sigma^2 := \sigma(u)^2 := \mathbf{E}\big[\mathbf{E}(B^{(\infty)}_{\Delta,o} \,|\, \mathcal{F}_0)^2\big], \tag{13}$$

where

$$\mathcal{F}_0 := \sigma\big(\mathcal{W} \cap Q_i \colon Q_i \prec Q_1\big)$$

is the σ-algebra generated by the white noise $\mathcal{W}$ in all cubes of the form Q_i which center i precedes 1 in the lexicographic order. This representation provides a starting point for the positivity proof.

We now proceed along the lines of [4, Theorem 1.3] to show that $\sigma^2 > 0$. In order to avoid redundancy, we sometimes sketch the general argument and concentrate on the steps that are markedly different. We note that some of our steps are in fact simpler because we work in the subcritical percolation regime. We now briefly comment on the assumption that $\mu(u) \neq 0$.

Remark 25. Recall that $\mu(u) = \lim_{n\uparrow\infty} |W_n|^{-1}\mathbf{E}(\beta(W_n \cap \mathsf{E}(u)))$ for the normalised expected functional of the excursion set $\mathsf{E}(u)$. A key step in the positivity proof of [4, Theorem 1.3] is the positivity of $\mu(u)$. While for the considered case of connected components this can be enforced by assuming that the support of the spectral measure has a support containing an open set, the situation is more complicated and we plan to proceed as in [33, Theorem 1.2c] and [29, Proposition 5.2].

Nevertheless, in the case of actual Betti numbers, the positivity of $\mu(u)$ can be verified along the lines of the [29, 33]. More precisely [29, 33] deal with Betti numbers of level sets whereas we need excursion sets and general topological functionals. Although this change does not cause major differences, we briefly recall the argument from [29, Lemma 5.5] to make our presentation self-contained. We first consider any smooth function $h\colon \mathbb{R}^d \to \mathbb{R}$ with support in some compact set $D \subseteq \mathbb{R}^d$ such that the excursion set at level u has a positive Betti number. Now, as in Lemma [29, Lemma 5.5], since the field F is smooth it allows for a series representation in terms of eigenfunctions, and we can conclude that with positive probability, h is at most ε away from F in $\mathcal{C}^1(D)$-distance. Hence, we conclude from

Morse theory in the form of Lemma 13 that with positive probability the excursion sets of F and of h are isotopic. Hence, the excursion set $A(u; F)$ has a positive Betti number with positive probability.

Recall that $\beta_n(F; u)$ is the functional of the union of components of the excursion set $\mathsf{E}(u)$ in the interior of the window W_n. Similarly to [4], a key step is that the expected functional of the excursion set becomes smaller after a suitable perturbation of the underlying random field.

Lemma 26 (Reduction of expected functional by perturbation)**.** *Assume that* $q_0 > 2 \cdot 64^2$ *and that* $\gamma > 3$*. Then, there exists* $m \geqslant 1$ *and a nonempty open set* $S \subseteq \mathbb{R}$ *such that if* $\mu(u) > 0$*, then*

$$\sup_{s \in S} \lim_{n \uparrow \infty} \big(\mathbf{E}(\beta_n(F + s(q \star \mathbb{1}_{mQ_0}), u)) - \mathbf{E}(\beta_n(F, u))\big) < 0.$$

and if $\mu(u) < 0$, then

$$\sup_{s \in S} \lim_{n \uparrow \infty} \big(\mathbf{E}(\beta_n(F + s(q \star \mathbb{1}_{mQ_0}), u)) - \mathbf{E}(\beta_n(F, u))\big) > 0.$$

We first explain how Lemma 26 implies $\sigma^2 > 0$ if $\mu(u) > 0$, the case $\mu(u) < 0$ is symmetric. After that, we prove Lemma 26. Since the proof is parallel to that of [4, Lemma 3.12], we only give the main idea. The positivity proof relies on the representation (13). In fact, it will be convenient to generalise the definition of $B^{(\infty)}_{\Delta,o}$ so as to compare the excursion functional after resampling to those obtained by a deterministic perturbation of the considered random field. More precisely, for $w \in \mathcal{C}^4$ satisfying $\lim_{|x| \uparrow \infty} |w(x)| = 0$, we set

$$D(w) := \lim_{n \uparrow \infty} \big(\beta_n(F + w; u) - \beta_n\big(F^{(Q_1)}; u\big)\big),$$

where the right-hand side converges almost surely by our assumption that the level u is in the subcritical regime. Note that we cannot directly use Proposition 22 because this only deals with $w = 0$. More precisely, $D(0) = B^{(\infty)}_{\Delta,o}$.

Proof of positivity of the limiting variance. In the proof, we rely on a variant $B^{(\infty)}_{\Delta,0;m}$ of $B^{(\infty)}_{\Delta,o}$, where instead of a partition into side length 1 boxes, we use boxes of side length $m \geqslant 1$. To avoid confusion, we stress that there is no clash in notation in the sense that $B^{(\infty)}_{\Delta,0;m}$ is different from $B_{\Delta,i,j}$.

Let $Z_0 := \mathcal{W}(mQ_0)$, which is normal random variable with variance m^d. Then, the $\mathbf{E}(B^{(\infty)}_{\Delta,0;m} \,|\, Z_0) = G(Z_0)$ for some measurable function $G : \mathbb{R} \to \mathbb{R}$. Now, the white noise $\mathcal{W}$ on mQ_0 decomposes into $Z_0 \mathbb{1}_{mQ_0}(\cdot)$ and an orthogonal part $\mathcal{W}_1$. Hence, we can represent the random field F

$$F = q \star (\mathcal{W}|_{(mQ_0)^c} + Z_0 \mathbb{1}_{mQ_0}(\cdot) + \mathcal{W}_1),$$

and also, for any fixed $s \in \mathbb{R}$,

$$q \star (\mathcal{W}|_{(mQ_0)^c} + (Z_0 + s)\mathbb{1}_{mQ_0}(\cdot) + \mathcal{W}_1) = F + w,$$

where $w = s(q \star \mathbb{1}_{mQ_0})$. In particular, for any fixed $s \in \mathbb{R}$, we have $\mathbf{E}(G(Z_0 + s)) = \mathbf{E}(D(s(q \star \mathbb{1}_{mQ_0})))$, so that $\mathbf{E}(G(Z_0)) = 0$. Moreover, Lemma 26 implies that $\mathbf{E}(G(Z_0 + s)) < 0$ for s contained in an open set. Then, the formula for the conditional variance gives the asserted $\sigma^2 \geqslant \mathsf{Var}(G(Z_0)) > 0$, where σ^2 is the variance defined in (13), which inside this proof might still depend on m. □

It remains to prove the perturbation property asserted in Lemma 26. Before starting the proof, we recall that we assume that for the critical points, we have moments up to order $q(\varepsilon) = 2 + \varepsilon$.

Proof of Lemma 26. In the proof, we may assume that $\int_{\mathbb{R}^d} q(x)\mathrm{d}x = 1$. Otherwise, one repeats the proof below after s by $s/\int_{\mathbb{R}^d} q(x)\mathrm{d}x$. Moreover, we also set

$$\beta\big(A, F\big) := \beta\big(A \cap \mathsf{E}(u; F)\big),$$

and the same for $\beta\big(A, F + w\big)$.

Here, we include in this definition all bounded components at level u whose reference point is contained in A. Note that since $\mu(u)$ is defined as an average over growing windows, the contributions coming from components intersecting the boundary are negligible in the limit. The key observation is to realise that $\lim_{u\uparrow\infty} \mu(u) = 0$. Indeed, we first note that $\mu(u)$ is bounded above by the expected number of critical points above level u per unit volume. Since the expected number of such critical points is finite so that the dominated convergence theorem yields that $\lim_{u\uparrow\infty} \mu(u) = 0$. Now, we use that $\mu(u) > 0$ for every $u \in \mathbb{R}$. To justify this note that Theorem 1.2c in (Wigman, 2021) shows the positivity $\mu(u)$ in the setting of level sets, which also extends to excursion sets. Hence, we conclude that there exists $\zeta > 0$ and an open $S \subseteq \mathbb{R}$ such that $\sup_{s\in S} \mu(u - s) - \mu(u) < -7\zeta$. Invoking the definition of μ therefore gives for $k > k_0$ sufficiently large

$$\mathbf{E}\big[\beta\big(B(k), F + s\big)\big] - \mathbf{E}\big[\beta\big(B(k), F\big)\big] < -6\zeta k^d, \tag{14}$$

where we write $B(k) := B(0, k)$. Then, as in [4, Theorem 1.3], we proceed in the following four steps where which are valid for $n > m^d$, provided that m is chosen sufficiently large. While it would be possible to extract the specific lower bound for m from the proof, it is not needed for our aims. For such m, we set $k := m - \sqrt{m}$, $s \in S$ and where we set $w := w_{s,m} := sq \star \mathbb{1}_{mQ_0}$:

$$\mathbf{E}\big[\big|\beta\big(W_n, F\big) - \beta\big(B(k), F\big) - \beta\big(W_n \backslash B(k), F\big)\big|\big] \leqslant \zeta m^d \tag{15}$$

$$\mathbf{E}\big[\big|\beta\big(W_n, F + w\big) - \beta\big(B(k), F + w\big) - \beta\big(W_n \backslash B(k), F + w\big)\big|\big] \leqslant \zeta m^d \tag{16}$$

$$\mathbf{E}\big[\big|\beta\big(W_n \backslash B(k), F + w\big) - \beta\big(W_n \backslash B(k), F\big)\big|\big] \leqslant \zeta m^d \tag{17}$$

$$\mathbf{E}\big[\big|\beta\big(B(k), F + w\big) - \beta\big(B(k), F + s\big)\big|\big] \leqslant \zeta m^d. \tag{18}$$

Hence, as soon as (15)–(17) are satisfied, we obtain the desired

$$\sup_{s\in S} \lim_{n\uparrow\infty} \big(\mathbf{E}(\beta_n(F + s(q \star \mathbb{1}_{mQ_0}))) - \mathbf{E}(\beta_n(F))\big) \leqslant -\zeta m^d,$$

We now verify the individual claims separately.

Bounds (15) **&** (16)**.** We only deal with (15) since the arguments for (16) are analogous. The key observation is that the Betti number is additive in the connected components. Hence, the Betti numbers of components contained in $B(k)$ are taken into account in $\beta(W_n, F)$. Similarly also the Betti numbers of components contained in $W_n \backslash B(k)$ are accounted for in $\beta(W_n, F)$. Hence, the deviations in (15) come from components intersecting $\partial B(k)$.

Hence, it therefore suffices to show that $\sup_{i\in\mathbb{Z}^d} \mathbf{E}\big[\beta\big(\mathcal{C}(Q_i, \mathsf{E}(u))\big)\big] < \infty$. To prove this claim, we let $\Omega_{i,m}$ denote the event that $m \geqslant 1$ is the smallest integer such that all connected components hitting Q_i are contained in Q_i^m. Henceforth, we write $Y = Y^\partial$ for the set critical and stratified critical points of the field F. Then, by the Cauchy-Schwarz inequality,

$$\sup_{i \in W_n} \mathbf{E}\big[\beta\big(\mathcal{C}(Q_i, \mathsf{E}(u))\big)\big] \leqslant \sum_{m \geqslant 1} \mathbf{E}\big[Y(B(m))\mathbb{1}\{\Omega_{i,m}\}\big]$$
$$\leqslant \sup_{i \in W_n} \sqrt{\mathbf{E}\big[Y(Q_i)^2\big]} \sum_{m \geqslant 1} |B(m)| \sqrt{\mathbf{P}(\Omega_{i,m})} < \infty,$$

where the last step follows since the excursion set is in the subcritical regime.

It remains to deal with (17) and (18), where the general idea is to apply Lemmas 18 and 19. First, we note that that $\beta(B(k), F + w)$ decomposes into the contributions coming from the cubes Q_j. That is,

$$\beta(B(k), F + w) =: \sum_{j \in B(k) \cap \mathbb{Z}^d} B_j(F + w),$$

and similarly for $\beta(B(k), F+s)$. The key step is now to obtain bounds on $\mathbf{E}(|B_j(F) - B_j(F + w)|)$, i.e., on the contribution to the perturbation from each cube Q_j.

To this end, we first need to show that the field perturbation w is small. Indeed, we have that

$$\sup_{x \in \mathbb{R}^d} |w(x)| \leqslant |s| \sup_{x \in \mathbb{R}^d} \int_{B(m)} |g(x - y)| \mathrm{d}y \leqslant |s| \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |g(x - y)| \mathrm{d}y < \infty, \tag{19}$$

and a similar computation shows the boundedness of derivatives up to order 3.

Next, we recall that by Theorem 12 on the finiteness of moments of the number of critical points, we have $\mathbf{E}(|B_j(F)|^q) < \infty$. Moreover, in Remark 1.3, it is argued that this moment bound is uniform over all fields with an upper bound on the absolute value of their derivatives. Hence, we deduce that we also have $\mathbf{E}(|B_j(F + w)|^q) < \infty$ uniformly over all considered perturbations w.

We will use this observation as follows. The Hölder inequality with $q' = q = q(\varepsilon)$ and $q_{\mathsf{M}} = q/(q-1)$ shows that for every $j \in \mathbb{Z}^d$ the expression $\mathbf{E}(|B_j(F) - B_j(F+w)|)$ is at most

$$\mathbf{P}\big(B_j(F) \neq B_j(F + w)\big)^{1/q_{\mathsf{M}}} \Big(\mathbf{E}(|B_j(F)|^q)^{1/q} + \mathbf{E}(|B_j(F + w)|^q)^{1/q}\Big),$$

where as argued above, the second factor is of constant order. Therefore,

$$\mathbf{E}(|B_j(F) - B_j(F + w)|) \leqslant C\mathbf{P}\big(B_j(F) \neq B_j(F + w)\big)^{1/q_{\mathsf{M}}}. \tag{20}$$

Relying on this observation, we now conclude the proofs of (17) and (18).

Bound (17)**.** First,

$$\mathbf{E}\big(\big|\beta(W_n \backslash B(k), F + w) - \beta(W_n \backslash B(k), F)\big|\big)$$
$$\leqslant \sum_{j:\ k < |j| \leqslant m} \mathbf{E}(|B_j(F + w) - B_j(F)|) + \sum_{j:\ |j| \geqslant m + \sqrt{m}} \mathbf{E}(|B_j(F + w) - B_j(F)|)$$

Note that the number of summands in the first sum is of order $O(m^{d-1/2})$. Hence, according to (20), it suffices to show that

$$\max_{j:\ k < |j| \leqslant m} \mathbf{P}(B_j(F + w) \neq B_j(F)) \in o(1), \tag{21}$$

and

$$\sum_{j\colon |j|\geqslant m+\sqrt{m}} \mathbf{P}(B_j(F+w)\neq B_j(F))^{1/q_{\mathsf{M}}} \in o(m^d). \tag{22}$$

In both cases, we rely on Lemmas 18 and 19. We apply these results to the family of perturbations $F^{(t)} := F + tw$. Then, for $x \in B(m)^c$,

$$|w(x)| = \Big|\int_{B(m)} g(x-u)\mathrm{d}u\Big| \leqslant \int_{|u|>\mathsf{dist}(x,B(m))} |g(u)|\mathrm{d}u \in O(\mathsf{dist}(x,B(m))^{d-\beta}).$$

In particular, Lemmas 18 and 19 give that $\mathbf{P}(B_j(F+w)\neq B_j(F)) \in O(m^{d-\beta})$, thereby implying (21). Moreover, since $\eta/d > q_{\mathsf{M}} + 1$, we can bound (22) by

$$c_2 \sum_{j\colon |j|\geqslant m+\sqrt{m}} \mathsf{dist}(j,B(m))^{-(\eta-d)/q_{\mathsf{M}}} \leqslant c_3 \sum_{i\geqslant m+\sqrt{m}} i^{d-1}(i-m)^{-(\eta-d)/q_{\mathsf{M}}} \in O(m^{d-1/2}),$$

thereby concluding the proof of (22).

Bound (18)**.** First, arguing as in (17), it suffices to show that

$$\max_{j\colon |j|\leqslant k} \mathbf{P}\big(B_j(F+s)\neq B_j(F+w)\big) \in o(1).$$

In both cases, we rely again on Lemmas 18 and 19. We apply this result to the family of perturbations $F^{(t)} := F + s + t(w-s)$. The arguments are now very similar to the proof of (18) but to make the presentation self-contained, we give some details. Indeed, we have that

$$\sup_{x\in B(k)} |s-w(x)| \leqslant |s|\int_{\mathbb{R}^d\setminus B(\sqrt{m})} g(x) \in O\big(m^{-(\eta-d)/2}\big).$$

Therefore,

$$\mathbf{P}\big(B_j(F+s)\neq B_j(F+w)\big) \in O(m^{-(\eta-d)/2}).$$

Hence, noting that $\eta > d$ concludes the proof of the (18). □

6. Proof of the FCLT, Theorem 8

In this section, we prove the functional CLT from Theorem 8. After having established the fixed-level CLT in Theorem 5, we now need to prove tightness. Recall that we assume that the critical points have moments of order at least 32. The percolation Assumption 2 allows us to restrict our attention to components contained in W_{2n}.

Henceforth, we write $Y = Y^\partial$ for the set critical and stratified critical points above level u of the field F, where the stratification is with respect to W_n, see Section 2.1. For deriving the functional CLT, it will be essential to ensure that the decomposed Betti numbers β_n^+ and β_n^- are both non-increasing in the level u. To ensure this, we recall from Proposition 14 $\beta_n = \beta_n^+ - \beta_n^-$ with

$$\beta_n^\pm(u;F) = \sum_{(x,v,C)} [\delta(x,v,C)]_\pm$$

Since the sum of tight processes is tight and symmetry considerations it suffices to prove the tightness statement when replacing β_n by β_n^+.

Set $\bar\beta_n^+(I) := \beta_n^+(I) - \mathbf{E}(\beta_n^+(I))$. We prove tightness by verifying the Chentsov-condition from [5, Theorem 15.6],

$$\mathbf{E}\big[\bar\beta_n^+(I)^4\big] \leqslant c|W_n|^2|I|^{5/4}, \tag{23}$$

for suitable $c > 0$, where for $I = [u_-, u_+]$, we set $\bar\beta_n^+(I) := \bar\beta_n^+(u_+) - \bar\beta_n^+(u_-)$. The crucial tool is the cumulant expansion

$$\mathbf{E}\big[\bar\beta_n^+(I)^4\big] = 3\mathsf{Var}\big(\bar\beta_n^+(I)\big)^2 + c_4\big(\bar\beta_n^+(I)\big). \tag{24}$$

In Section 6.1 below, we rely on a trick from [8, Corollary 2] reducing the verification of (23) to *n-big* intervals I, i.e., to intervals with $|I| \geqslant |W_n|^{-2/3}$. Hence, to establish condition (23), we derive refined bounds on variances and cumulants.

Proposition 27. *Let $q_0 > 16$ and $\gamma > 54d/(1 - 8/q_0)$. Then,*

$$\sup_{n\geqslant 1}\sup_{I\text{ is }n\text{-big}} |W_n|^{-1}|I|^{-5/8}\mathsf{Var}\big(\beta_n^+(I)\big) + |W_n|^{-7/6}c_4\big(\beta_n^+(I)\big) < \infty.$$

The variance and cumulant bounds in Proposition 27 give the tightness.

Proof of (23)*; n-big intervals.* First, by combining Proposition 27 and the identity

$$\mathbf{E}\big[\bar\beta_n^+(I)^4\big] \leqslant 3c|W_n|^2|I|^{5/4} + c|W_n|^{7/6}.$$

Since $|I|$ is n-big, we deduce that $|W_n|^{7/6} \leqslant |W_n|^2|I|^{5/4}$, as asserted. It remains to prove the reduction step, see below. □

After the reduction step, we prove Proposition 27 in Section 6.2. To achieve this goal, the key task is bound both the moments for the critical points inside percolation sets, and also to to refine the moment condition so as to reflect the level. To allow for clear reference, we state the result as a separate lemma.

Let $\mathcal{C}_i = \mathcal{C}(\{F \geqslant u_c\}; Q_i)$ denote the union of all connected components of $\{F \geqslant u_c\}$ intersecting the cube Q_i.

Lemma 28 (Moment bounds)**.** *Let $q_0 \geqslant 2^{13}$. Assume that* $\sup_{i\in\mathbb{Z}^d}\mathbf{E}\big[Y^\partial(B(i,1)\times\mathbb{R})^{q_0}\big] < \infty$. *Then,*

(1) $\sup_{i\in\mathbb{Z}^d}\mathbf{E}\big[Y^\partial(\mathcal{C}_i\times\mathbb{R})^{q_0/2}\big] < \infty$;

(2) $\sup_{I\subseteq I_{\mathsf{b}}}\sup_{i\in\mathbb{Z}^d}|I|^{-31/32}\mathbf{E}\big[Y^\partial(\mathcal{C}_i\times I)^{\sqrt{q_0/2}}\big] < \infty$ *for every compact* $I_{\mathsf{b}} \subseteq \mathbb{R}$.

Proof. We prove the two parts separately.

Part (1). Let $K_i \geqslant 1$ be the smallest integer such that $\mathcal{C}_i \subseteq B(i, K_i)$. Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\sup_{i\in\mathbb{Z}^d}\mathbf{E}\big[Y^\partial(\mathcal{C}_i\times\mathbb{R})^{q_0/2}\big] &\leqslant \sup_{i\in\mathbb{Z}^d}\sum_{k\geqslant 1}\mathbf{E}\big[Y^\partial(B(i,k)\times\mathbb{R})^{q_0/2}\mathbb{1}\{K_i = k\}\big]\\ &\leqslant \sup_{i\in\mathbb{Z}^d}\sum_{k\geqslant 1}\sqrt{\mathbf{E}\big[Y^\partial(B(i,k)\times\mathbb{R})^{q_0}\big]}\sqrt{\mathbf{P}(K_i = k)}\\ &\leqslant \sup_{i\in\mathbb{Z}^d}\sqrt{\mathbf{E}\big[Y^\partial(B(i,1)\times\mathbb{R})^{q_0}\big]}\sum_{k\geqslant 1}(2k)^{1+q_0/2}\sqrt{\mathbf{P}(K_i = k)}.\end{aligned}$$

Finally, the sum converges because we assume that we are in the subcritical regime of percolation, see Assumption 2. Note that the first term is finite by applying Theorem 12 to all faces of $B(i, 1)$.

Part (2). Set $M := \sqrt{q_0/2}$. We proceed similarly as in the proof of part (1). The difference is that instead of the Cauchy-Schwarz inequality, we use the Hölder inequality with $q = M$ and $p = M/(M-1)$. Then,

$$\begin{aligned}\mathbf{E}\big[Y^\partial(\mathcal{C}_i \times I)^M\big] &\leqslant \sum_{k\geqslant 1} \mathbf{E}\big[Y^\partial(B(i,k) \times I)^M \mathbb{1}\{K_i = k\}\big]\\ &\leqslant \sum_{k\geqslant 1} \big(\mathbf{E}\big[Y^\partial(B(i,k) \times I)^{pM}\big]\big)^{1/p}\mathbf{P}(K_i = k)^{1/q}\\ &\leqslant \big(\mathbf{E}\big[Y^\partial(B(i,1) \times I)^{pM}\big]\big)^{1/p} \sum_{k\geqslant 1}(2k)^{M+1}\mathbf{P}(K_i = k)^{1/q}.\end{aligned}$$

As in part (1), the convergence of the sum follows from the assumption of subcritical percolation. To ease notation, write Y^∂ short for $Y^\partial(B(i,1) \times I)$. Note that $(Y^\partial)^{pM} \leqslant (Y^\partial)^{pM+p}$. Hence, another application of the Hölder inequality gives that

$$\mathbf{E}\big[Y^\partial(B(i,1) \times I)^{pM}\big] \leqslant \mathbf{E}(Y^\partial(B(i,1) \times \mathbb{R})^{pM^2})^{1/M}\mathbf{E}(Y^\partial(B(i,1) \times I))^{1/p}$$

Since $2M^2 \leqslant q_0$, we deduce that the first factor is finite, whereas the second is of order $O\big(|I|^{1/p}\big)$ by the Kac-Rice Lemma 17 applied to each facet of $B(i,1)$. Since, $M \geqslant 64$, we conclude that $1/p^2 \geqslant 31/32$, thereby concluding the proof. □

6.1. **Reduction to n-big intervals.** In this section, we explain how to reduce the verification of the moment bound to n-big intervals. The key idea is to consider the approach from [8, Corollary 2]. In order to be able to apply this result, we recall from Section 3 that $\beta_n^+(u)$ is decreasing in u.

Now, [8, Corollary 2] allows to carry out the reduction to n-big intervals provided that $\mathbf{E}(\beta_n^+(I)) \in o\big(\sqrt{|W_n|}\big)$ holds for all n-small intervals $I \subseteq I_{\mathsf{b}}$. Now, we show that $\mathbf{E}(\beta_n^+(I)) \in o\big(\sqrt{|W_n|}\big)$. We recall from Section 3 that $|\delta(x,v,C)| \leqslant c\#Y^\partial \cap (C \times [u,\infty))$ for $v \geqslant u \geqslant u_c$. Therefore, with $Y_j := Y^\partial(Q_j \times I)$,

$$\beta_n^+(I) \leqslant c \sum_{j\in\bar{W}_n} Y^\partial\big(\mathcal{C}_j \times [u_c,\infty)\big)\mathbb{1}\{Y_j \neq 0\}.$$

Hence, taking expectations and using Lemma 28, we arrive at

$$\mathbf{E}\big[\beta_n^+(I)\big] \leqslant c \sum_{j\in\bar{W}_n} \mathbf{E}\big[Y^\partial\big(\mathcal{C}_j \times [u_c,\infty)\big)\mathbb{1}\{Y_j \neq 0\}\big]\big] \tag{25}$$

Now, we bound the second summand in (25). First, by the Hölder inequality,

$$\mathbf{E}\big[Y^\partial\big(\mathcal{C}_j \times [u_c,\infty)\big)\mathbb{1}\{Y_j \neq 0\}\big]\big] \leqslant \big(\mathbf{E}\big[Y^\partial\big(\mathcal{C}_j \times [u_c,\infty)\big)^{32}\big]\big)^{\frac{1}{32}}\mathbf{P}\big(Y_j \neq 0\big).^{\frac{31}{32}}$$

The first factor is bounded by part (1) of Lemma 28. Finally, the second factor is of order $O(|I|^{31/32})$ by Lemma 17. Now, referring again to the smallness of I concludes the proof.

6.2. **Proof of Proposition 27.** To prove Proposition 27, we rely on the martingale technique that was already implemented in the setting of cylindrical networks [15]. To make the presentation self-contained, we recollect here the basic set-up.

We let $\mathcal{G}_j$ be the σ-algebra generated by the restriction of $\mathcal{W}$ to boxes of the form Q_i with $i \leqslant_{\mathsf{lex}} j$. Then, setting $B_{+,j}^{(n)}(I) := \beta_n^+(I;F) - \beta_n^+(I;F^{(Q_j)})$ as in (9), we decompose the centered increment $\bar{\beta}_n^+(I)$ as

$$\beta_n^+(I) - \mathbf{E}(\beta_n^+(I)) = \sum_{j\in\mathbb{Z}^d} \mathbf{E}(B_{+,j}^{(n)}(I) \,|\, \mathcal{G}_j).$$

We now prove the variance bounds in Proposition 27.

Proof of Proposition 27 – variance. The key observation from [26] is that $\{B^{(n)}_{+,i}\}_{i\in\mathbb{Z}^d}$ is a martingale-difference sequence because $\mathbf{E}(\beta_n^+(I;F^{(Q_j)})\,|\,\mathcal{G}_j) = \mathbf{E}(\beta_n^+(I;F)\,|\,\mathcal{G}_{j-1})$. Hence, the Cauchy-Schwarz inequality implies that

$$\mathsf{Var}(\beta_n^+(I)) = \sum_{j\in\mathbb{Z}^d}\mathsf{Var}\big(\mathbf{E}(B^{(n)}_{+,j}(I)\,|\,\mathcal{G}_j)\big) \leqslant \sum_{j\in\mathbb{Z}^d}\mathbf{E}\big[B^{(n)}_{+,j}(I)^2\big].$$

We now let $B_{+,i,j}$ denote the positive contribution to $B_{+,j}$ coming from critical points in the cube Q_i. Then, we have a decomposition

$$B^{(n)}_{+,j}(I) = \sum_{i\in W_{2n}} B_{+,i,j}(I). \tag{26}$$

Now, let $M = 16$ and $p' = 16/15$. Then, by Cauchy-Schwarz and Hölder,

$$\mathbf{E}\big[B^{(n)}_{+,j}(I)^2\big]^{1/2} \leqslant \sum_{i\in\bar W_n}\sqrt{\mathbf{E}\big[B_{+,i,j}(I)^2\big]} \leqslant \sum_{i\in\bar W_n}\mathbf{E}\big[B_{+,i,j}(I)^{2M}\big]^{(2M)^{-1}}\mathbf{P}(B_{+,i,j}(I)\neq 0)^{(2p')^{-1}}$$

We claim that the first factor is of constant order. Indeed, we first note that F and $F^{(Q_j)}$ have the same distribution. Moreover, by the weak Morse inequality, the Betti number of a component is bounded by the number of critical points. Finally, Lemma 24 implies that

$$\mathbf{E}\big[Y^\partial(\mathcal{C}_0\times\mathbb{R})^{2M}\big] < \infty,$$

thus showing the asserted finiteness of the first factor.

For the second factor, an application of (10) shows that

$$\mathbf{P}(B_{+,i,j}(I)\neq 0)^{1/5}\mathbf{P}(B_{+,i,j}(I)\neq 0)^{4/5} \in O\big(|i-j|^{-\gamma d/5}|I|^{4/5}\big)$$

Therefore, $\mathbf{E}\big[B^{(n)}_{+,j}(I)^2\big]^{1/2} \leqslant O\big(\sum_{i\in\bar W_{2n}}|i-j|^{-\gamma d/8}|I|^{3/8}\big)$. Hence, we conclude that if $d_{j,n} := \mathsf{dist}(j, W_{2n}) \leqslant n^{1/d}$, then $\mathbf{E}\big[B^{(n)}_{+,j}(I)^2\big] \in O(|I|^{3/4})$. Moreover, if $d_{j,n} \geqslant n^{1/d}$, then $\mathbf{E}\big[B^{(n)}_{+,j}(I)^2\big] \in O(d_{j,n}^{2(1-\gamma/8)d}|I|^{3/4})$. Finally,

$$\sum_{j:\,d_{j,n}\geqslant n} d_{j,n}^{2(1-\gamma/8)d} \in |W_n|^{3-\gamma/4}.$$

Thus, recalling the assumption $\gamma > 8$ concludes the proof.

□

Hence, it remains to establish the cumulant bounds in Proposition 27. To achieve this goal, we need to have a suitable control on the correlation structure. To make this precise, we let $\mathfrak{d}(\boldsymbol{z}) := \max_{\{S,T\}\prec\{i,j,k,\ell\}}\mathfrak{d}_{S,T}(\boldsymbol{z})$, where the maximum is taken over all possible partitions of $\boldsymbol{z} = \{i,j,k,\ell\}\subset\mathbb{Z}^d$ into two non-empty groups, and where $\mathfrak{d}_{S,T}(\boldsymbol{z}) := \mathsf{dist}\big(\{s\}_{s\in S},\{t\}_{t\in T}\big)$ is the inter-partition distance, when partitioning $\boldsymbol{z}$ into two groups indexed by S and T, respectively. We set $\boldsymbol{\mathcal{D}}(I) := \big\{B^{(n)}_{+,i}(I), B^{(n)}_{+,j}(I), B^{(n)}_{+,k}(I), B^{(n)}_{+,\ell}(I)\big\}$. Again, we state the decorrelation property now, and postpone the proof.

Lemma 29 (Spatial decorrelation)**.** *Let $q_0 > 2^{13}$ and $\gamma > 1$. Assume that $\sup_{i\in\mathbb{Z}^d}\mathbf{E}\big[Y^\partial(B(i,1)\times\mathbb{R})^{q_0}\big] < \infty$. Let $\boldsymbol{z}\subseteq\mathbb{Z}^d$ with $|\boldsymbol{z}|\leqslant 4$. Then, for every $\varepsilon > 0$,*

$$\big|c_4\big(B^{(n)}_{+,i}(I), B^{(n)}_{+,j}(I), B^{(n)}_{+,k}(I), B^{(n)}_{+,\ell}(I)\big)\big| \in O\big(\mathfrak{d}(\boldsymbol{z})^{-\gamma d(1-8/q_0)}\big).$$

Relying on Lemma 29, we now complete the proof of the cumulant bound. To ease notation, we henceforth drop the dependence on I in the quantity $B^{(n)}_{+,i}(I)$.

Proof of Proposition 27 – cumulant bounds. First, by multilinearity of cumulants,

$$c_4\big(\bar\beta^+_n(I)\big) = \sum_{i,j,k,\ell\in\mathbb{Z}^d} a_{i,j,k,\ell}\, c_4(B^{(n)}_{+,i}, B^{(n)}_{+,j}, B^{(n)}_{+,k}, B^{(n)}_{+,\ell}), \tag{27}$$

where the $a_{i,j,k,\ell} \geqslant 1$ are suitable combinatorial constants. They only depend on which indices i, j, k, ℓ are equal.

We now decompose (27) into contributions with indices in the set $\mathcal{I} := \{i : d_{i,n} \leqslant n^{1/d}\}$ and those with some indices outside $\mathcal{I}$. We bound the contributions from $\mathcal{I}^c$ and $\mathcal{I}$ separately. In both cases, we use that each summand in (27) can be bounded as

$$|c_4(B^{(n)}_{+,i}, B^{(n)}_{+,j}, B^{(n)}_{+,k}, B^{(n)}_{+,\ell})| \in O\Big(\prod_{m\in\{i,j,k,\ell\}} \mathbf{E}((B^{(n)}_{+,m})^4)^{1/4}\Big). \tag{28}$$

Now, by Lemma 24, we have $\mathbf{E}(|B^{(n)}_{+,i}|^4)^{1/4} \in O\big(|W_n|^{5/4} d_{i,n}^{-\gamma d/5}\big)$. In particular,

$$\sum_{i:\ d_{i,n}>n^{1/d}} \mathbf{E}(|B^{(n)}_{+,i}|^4)^{1/4} \in O\big(|W_n|^{1+5/4-\gamma/5}\big),$$

so that

$$\sum_i \mathbf{E}(|B^{(n)}_{+,i}|^4)^{1/4} = \sum_{i:d_{i,n}\leqslant n^{1/d}} \mathbf{E}(|B^{(n)}_{+,i}|^4)^{1/4} + \sum_{i:d_{i,n}>n^{1/d}} \mathbf{E}(|B^{(n)}_{+,i}|^4)^{1/4} \in O(|W_n|)$$

and

$$\begin{aligned}\sum_{\mathcal{I}^c} |c_4(B^{(n)}_{+,i}, B^{(n)}_{+,j}, B^{(n)}_{+,k}, B^{(n)}_{+,\ell})| &\leqslant C \sum_{i:\ d_{i,n}>n^{1/d}} \mathbf{E}(|B^{(n)}_{+,i}|^4)^{\frac14} \Big(\sum_{j\in\mathbb{Z}^d} \mathbf{E}(|B^{(n)}_{+,j}|^4)^{\frac14}\Big)^3 \\ &\in O\big(|W_n|^{4+5/4-\gamma/5}\big).\end{aligned}$$

Hence, noting that $\gamma > 22$ shows that the last line is in $o(|W_n|)$.

Therefore, it remains to deal with indices in $\mathcal{I}$. Here, we partition the sum over $\mathcal{I}$ again into two parts $\Sigma_1 + \Sigma_2$, where Σ_1 and Σ_2 are determined as follows. The part Σ_1 contains all summands corresponding to indices $(i, j, k, \ell) \in \mathcal{I}$ such that $\mathfrak{d}(\{i, j, k, \ell\}) \leqslant |W_n|^{\varepsilon/d}$ with $\varepsilon = 1/18$, and Σ_2 contains the remaining summands.

The sum Σ_1 consists of $O(|W_n|^{1+3\varepsilon})$ summands each of which is of constant order, because of the bound (28). Thus, $\Sigma_1 \in O\big(|W_n|^{7/6}\big)$. Second, we note that Σ_2 consists of $O(|W_n|^4)$ summands. Moreover, by Lemma 29 each of them is in $O\big(|W_n|^{-\gamma\varepsilon(1-4/\sqrt{q_0})/d}\big)$. Thus, since $\gamma\varepsilon(1 - 8/q_0) \geqslant 3d$, we concludes the proof. □

It remains to prove the spatial decorrelation asserted in Lemma 29. To that end, we will proceed along the blueprint provided in [15, Lemma 2].

Proof of Lemma 29. We may assume that $k_0 := \mathfrak{d}(\boldsymbol{z}) := \mathsf{dist}\big(\{i, k\}, \{j, \ell\}\big)$. The other cases are similar but easier. Moreover, by the cluster-decomposition of the cumulant from [26, Lemma 5.1], it suffices to show the claim when replacing the cumulant by

$$\mathsf{Cov}\big(B^{(n)}_{+,i}B^{(n)}_{+,j}, B^{(n)}_{+,k}B^{(n)}_{+,\ell}\big).$$

Now, letting $\boldsymbol{z'} = (i', j', k', \ell') \in \mathbb{Z}^{4d}$, this covariance decomposes as

$$\sum_{\boldsymbol{z'}} \mathsf{Cov}\big(B'_{+,i',i}B'_{+,j',j}, B'_{+,k',k}B'_{+,\ell',\ell}\big).$$

Now, we may use the resampling representation in order to conclude the proof. More precisely, to construct $\tilde{B}'_{+,i',i}$ and $\tilde{B}'_{+,j',j}$, we resample the white noise in the half-space of consisting of all points that are closer to i than to k. Similarly, we define $\tilde{B}'_{+,k',k}$ and $\tilde{B}'_{+,\ell',\ell}$ by resampling the white noise in the half-space of consisting of all points that are closer to k than to i. Therefore,

$$\begin{aligned}&\Big|\mathsf{Cov}\big(B'_{+,i',i}B'_{+,j',j}, B'_{+,k',k}B'_{+,\ell',\ell}\big)\Big|\\ &= \Big|\mathbf{E}\Big[B'_{+,i',i}B'_{+,j',j}B'_{+,k',k}B'_{+,\ell',\ell} - \tilde{B}'_{+,i',i}\tilde{B}'_{+,j',j}\tilde{B}'_{+,k',k}\tilde{B}'_{+,\ell',\ell}\Big]\Big|.\end{aligned}$$

Then, by the Hölder inequality with $q' = M$ and $p' = M/(M-1)$ where $M := q_0/8$, we have that

$$\begin{aligned}&\Big|\mathsf{Cov}\big(B'_{+,i',i}B'_{+,k',k}, B'_{+,j',j}B'_{+,\ell',\ell}\big)\Big|\\ &\leqslant 2\mathbf{P}\big(B'_{+,i',i}B'_{+,k',k}B'_{+,j',j}B'_{+,\ell',\ell} \neq \tilde{B}'_{+,i',i}\tilde{B}'_{+,k',k}\tilde{B}'_{+,j',j}\tilde{B}'_{+,\ell',\ell}\big)^{1/p'}\\ &\quad\times \sup_{m,m'\in\mathbb{Z}^d} \mathbf{E}\Big[\big|B'_{+,m,m'}\big|^{4M}\Big]^{1/(4M)}.\end{aligned}$$

Lemma 18 shows that $\mathbf{P}\big(B'_{+,i',i} \neq \tilde{B}'_{+,i',i}\big) \in O(\mathfrak{d}(z)^{-\gamma d+\varepsilon})$, and the arguments for j, k, ℓ are analogous. By the moment bound in Lemma 28, the second factor remains bounded uniformly over all indices. This concludes the proof.

□

Acknowledgements

The authors thank S. Muirhead, F. Severo and I. Wigman for very helpful suggestions and references to literature.

This research was conducted while the second author was in Paris, supported by the *Programme d'invitations internationales scientifiques, campagne 2025* from Université Paris Cité.

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