Convergence rate of Smoluchowski–Kramers approximation with stable Lévy noise *

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Abstract The small mass limit of the Langevin equation perturbed by β-stable Lévy noise is considered by rewriting it in the form of slow-fast system, and spliting the fast component into three parts. By exploring the three parts respectively, the limit equation and the convergence rate are derived.

Keywords Langevin equation, Smoluchowski–Kramers approximation, Averaging, Singular pertubation

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1 Introduction

Smoluchowski–Kramers (SK for short) approximation is initially proposed by Smoluchowski [15] and Kramers [11] to derive an effective approximation to a Langevin equation which describes the motion of a particle with small mass. Roughly speaking, the equation

$$\epsilon\ddot{u}^\epsilon + \dot{u}^\epsilon = b(u^\epsilon) + \sigma(u^\epsilon)\dot{W}$$

is approximated, as $\epsilon \to 0$, in some sense by the equation

$$\dot{u} = b(u) + \sigma(u)\dot{W}.$$

Formally, the limit equation is obtained by dropping the term $\epsilon\ddot{u}$.

There is fruitful work on SK approximation for Langevin equations with Gaussian white noise [4–6, 12, 14, 16, 17, e.g.]. The case that W is an infinite dimensional Brownian motion is firstly studied by Cerrai and Freidlin [2, 3]. There is also some work concerned with SK approximation with colored noise which is highly oscillating in time [7, 8] or with Lévy noise [20, 21]. In this paper, we consider the following Langevin equation driven by a stable Lévy process

$$\begin{cases} \epsilon\ddot{u}^\epsilon(t) + \dot{u}^\epsilon(t) = f(u^\epsilon(t)) + \epsilon^\alpha \dot{L}(t), \\ u^\epsilon(0) = u_0, \quad \dot{u}^\epsilon(0) = v_0. \end{cases} \tag{1.1}$$

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Here, $0 \leq \alpha < 1$ is a constant, and L is a β-stable process whose properties are detailed in Section 2. System (1.1) describes the motion of a particle with mass ϵ in an athermal fluctuation environment. Equation (1.1) can also be seen as a singularly perturbed differential equation with a random noise, which has attracted many researchers' interest [18, 19, e.g.]. β-stable process is an important class of Lévy processes due to its self-similarity and scaling property [9, 10]. One of the difficulty lies on the fact that, unlike Brownian motions or Lévy processes without big jump, β-stable processes do not have finite second order moment. Another difficulty is that the Lévy measure of a β-stable process is infinite.

Formally, the effective approximation model of (1.1) can also be obtained by dropping the $\epsilon \ddot{u}^\epsilon$ term, that is,

$$\dot{\bar{u}}^\epsilon(t) = f(\bar{u}^\epsilon(t)) + \epsilon^\alpha \dot{L}(t), \quad \bar{u}^\epsilon(0) = u_0. \tag{1.2}$$

Obviously, the statement above reduces to the classical SK approximation in the case $\alpha = 0$. Here we introduce a splitting technique of the solution [19] to show the approximation rigorously. Moreover, we also obtain the convergence rate.

Rewrite the equation (1.1) as

$$\begin{cases} \dot{u}^\epsilon(t) = v^\epsilon(t), \\ \dot{v}^\epsilon(t) = \epsilon^{-1}[-v^\epsilon(t) + f(u^\epsilon(t))] + \epsilon^{\alpha-1}\dot{L}(t), \\ u^\epsilon(0) = u_0, \quad v^\epsilon(0) = v_0. \end{cases} \tag{1.3}$$

Equation (1.3) has a form of "slow-fast system". Inspired by a splitting technique introduced by Lv et al. [19], we make the following important decomposition, which makes the analysis to (1.3) considerably more clear

$$\begin{cases} \dot{\bar{v}}_1^\epsilon(t) = -\epsilon^{-1}\bar{v}_1^\epsilon(t), \\ \dot{\bar{v}}_2^\epsilon(t) = -\epsilon^{-1}[\bar{v}_2^\epsilon(t) - f(u^\epsilon(t))], \\ \dot{\bar{v}}_3^\epsilon(t) = -\epsilon^{-1}\bar{v}_3^\epsilon(t) + \epsilon^{-\frac{1}{\beta}}\dot{L}(t), \\ \bar{v}_1^\epsilon(0) = \epsilon v_0, \quad \bar{v}_2^\epsilon(0) = 0, \quad \bar{v}_3^\epsilon(0) = 0. \end{cases} \tag{1.4}$$

Direct calculation yields

$$v^\epsilon = \epsilon^{-1}\bar{v}_1^\epsilon + \bar{v}_2^\epsilon + \epsilon^{\alpha+\frac{1}{\beta}-1}\bar{v}_3^\epsilon. \tag{1.5}$$

Then we consider the three parts of v^ϵ respectively to pass the limit $\epsilon \to 0$.

The paper is organized as follows. In Section 2, we impose some assumptions and state the main result. In Section 3, we give several technical lemmas with proofs in details. After these preparation, we prove the main result in Section 4.

2 Preliminary and Main Result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, on which there is a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual condition, where $0 < T < \infty$ is fixed throughout the paper. Let L be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. In the rest of the paper, $|x| := \sqrt{\sum_{i=1}^d x_i^2}$ for each $x = (x_1, ..., x_d) \in \mathbb{R}^d$, and $||A|| := \sup_{x\in\mathbb{R}^d, |x|=1} |Ax|$ for each matrix $A \in \mathbb{R}^{d\times d}$.

We make the following assumptions.

$(\mathbf{A_1})$ L is a pure jump isotropic β-stable Lévy process on $\mathbb{R}^d$ with $1 \le \beta < 2$, and for each $c > 0$, the Lévy–Itô decomposition for L is

$$L(t) = \int_0^t \int_{|x|<c} x\tilde{N}(dsdx) + \int_0^t \int_{|x|\ge c} xN(dsdx)\,, \tag{2.1}$$

with the Lévy measure $\nu(dx) = \frac{1}{|x|^{\beta+d}}dx$.

$(\mathbf{A_2})$ $f : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz, that is, there exists a constant $L_f > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \le L_f|x - y|.$$

Remark 2.1. *From $(\mathbf{A_1})$ we notice that $\int_{|y|\ge 1} |y|^p \nu(dy) < \infty$ if and only if $p < \beta$. This fact is used frequently in the following part.*

We establish several moment estimates. In the following part, C denotes constant whose value may change from line to line. Unless otherwise stated, the value of C may depend on T and the Lévy measure ν, but it never depends on ϵ. We use the notation $x \lesssim y$ to indicate that there exists a constant C such that $x \le Cy$.

Our main result is the following theorem.

Theorem 2.1. *(i) Let $0 \le \alpha < 1$. Under assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$,*

$$\mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| \lesssim \epsilon^\alpha. \tag{2.2}$$

(ii) Let $\alpha = 0$. Under assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$,

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|u^\epsilon(t) - \bar{u}^\epsilon(t)| = 0. \tag{2.3}$$

Remark 2.2. *Obviously, part (i) of Theorem 2.1 does not give a convergence result in the case $\alpha = 0$. The reason that causes the difference between cases $\alpha = 0$ and $\alpha > 0$ is presented at the end of Section 4.*

3 Several Technical Lemmas

In this section, we establish several moment estimates, which are used in Section 4. We always assume that $0 \le \alpha < 1$ and **(A1)** and **(A2)** hold true.

Lemma 3.1. *For each $0 < \epsilon \le 1$, the equation (1.3) admits a unique strong solution.*

Proof. Let $\psi^\epsilon := \begin{pmatrix} u^\epsilon \\ v^\epsilon \end{pmatrix}$. The equation (1.3) can be rewritten as

$$\dot{\psi}^\epsilon = \mathcal{A}^\epsilon \psi^\epsilon + \mathcal{F}^\epsilon(\psi^\epsilon) + \dot{\mathcal{L}}^\epsilon, \tag{3.1}$$

where $\mathcal{A}^\epsilon := \begin{pmatrix} 0 & Id \\ 0 & -\epsilon^{-1}Id \end{pmatrix}$, $\mathcal{F}^\epsilon \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \epsilon^{-1}f(u) \end{pmatrix}$ and $\mathcal{L}^\epsilon := \begin{pmatrix} 0 \\ \epsilon^{\alpha-1}L \end{pmatrix}$. Since f is Lipschitz, $\mathcal{A}^\epsilon + \mathcal{F}^\epsilon$ is also Lipschitz which leads to the existence and uniqueness [1]. $\square$

Lemma 3.2. *(i) For $\alpha \in [0, \frac{1}{\beta})$ and $p \in [1, \beta)$,*

$$\mathbb{E} \sup_{0 \le t \le T} |u^\epsilon(t)|^p \lesssim 1 + \frac{1}{\epsilon^{(1-\frac{1}{\beta})p}}.$$

(ii) For $\alpha \in [\frac{1}{\beta}, 1)$ and $p \in [1, \beta)$,

$$\sup_{0<\epsilon\le 1} \mathbb{E} \sup_{0 \le t \le T} |u^\epsilon(t)|^p \lesssim 1.$$

Remark 3.1. *Obviously the bound in (i) is not uniform in ϵ, but the estimate is enough for our following discussion.*

Proof. Let $\tilde{v}^\epsilon := \epsilon^{1-\frac{1}{\beta}} v^\epsilon$, then

$$\begin{cases} \dot{u}^\epsilon = \epsilon^{\frac{1}{\beta}-1} \tilde{v}^\epsilon, \\ \dot{\tilde{v}}^\epsilon = \epsilon^{-1}[-\tilde{v}^\epsilon + \epsilon^{1-\frac{1}{\beta}} f(u)] + \epsilon^{\alpha - \frac{1}{\beta}} \dot{L}, \\ u^\epsilon(0) = u_0, \quad \tilde{v}^\epsilon(0) = \epsilon^{1-\frac{1}{\beta}} v_0. \end{cases} \tag{3.2}$$

Consider the following linear SDE

$$\dot{\eta}^\epsilon = -\epsilon^{-1}\eta^\epsilon + \epsilon^{\alpha-\frac{1}{\beta}} \dot{L}, \quad \eta^\epsilon(0) = \epsilon^{1-\frac{1}{\beta}} v_0. \tag{3.3}$$

Let $\phi(x) := (|x|^2 + 1)^{p/2}$, then there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$,

$$|D\phi(x)| \le C|x|^{p-1}, \tag{3.4}$$

and

$$\|D^2\phi(x)\| \le C. \tag{3.5}$$

We take $c = \epsilon^{\frac{1}{\beta}}$ in the Lévy–Itô decomposition, and by Itô's formula,

$$\begin{aligned} & \phi(\eta^\epsilon(s)) \\ = \;& \phi(\epsilon^{1-\frac{1}{\beta}} v_0) + \int_0^t (D\phi(\eta^\epsilon(s)), -\epsilon^{-1}\eta^\epsilon(s)) ds \\ & + \int_0^t \int_{|x|<\epsilon^{\frac{1}{\beta}}} \phi(\eta^\epsilon(s-) + \epsilon^{\alpha-\frac{1}{\beta}} x) - \phi(\eta^\epsilon(s-)) \tilde{N}(dsdx) \\ & + \int_0^t \int_{|x|\ge\epsilon^{\frac{1}{\beta}}} \phi(\eta^\epsilon(s-) + \epsilon^{\alpha-\frac{1}{\beta}} x) - \phi(\eta^\epsilon(s-)) N(dsdx) \\ & + \int_0^t \int_{|x|<\epsilon^{\frac{1}{\beta}}} \phi(\eta^\epsilon(s-) + \epsilon^{\alpha-\frac{1}{\beta}} x) - \phi(\eta^\epsilon(s-)) - (D\phi(\eta^\epsilon(s)), \epsilon^{\alpha-\frac{1}{\beta}} x) \nu(dx) ds \\ =: \;& \phi(\epsilon^{1-\frac{1}{\beta}} v_0) + \sum_{k=1}^4 J_k^\epsilon(t). \end{aligned} \tag{3.6}$$

Now we deal with the five terms on the right hand side of the equation above. Note that

$$\phi(\epsilon^{1-\frac{1}{\beta}} v_0) \le C, \tag{3.7}$$

a consequence of the fact that $\beta \geq 1$. For J_1^ϵ, since $D\phi(y) = \frac{py}{(|y|^2+1)^{1-p/2}}$,

$$
\begin{aligned}
& J_1^\epsilon(t) \\
= & \int_0^t \frac{p}{\epsilon} \frac{-|\eta^\epsilon(s)|^2}{(|\eta^\epsilon(s)|^2+1)^{1-p/2}} ds \\
\leq & \int_0^t -\frac{p}{2\epsilon}(|\eta^\epsilon(s)|^2+1)^{p/2} + \frac{p}{\epsilon} ds \\
= & \int_0^t -\frac{p}{2\epsilon}\phi(\eta^\epsilon(s))ds + \frac{p}{\epsilon}t,
\end{aligned}
\tag{3.8}
$$

where the last inequality comes from the elementary inequality

$$
\frac{-|x|^2}{(|x|^2+1)^{1-p/2}} \leq -\frac{1}{2}(|x|^2+1)^{p/2} + 1.
$$

Taking expectation and using Fubini theorem,

$$
\mathbb{E}J_1^\epsilon(t) \leq \int_0^t -\frac{p}{2\epsilon}\mathbb{E}\phi(\eta^\epsilon(s))ds + \frac{p}{\epsilon}t. \tag{3.9}
$$

The martingale property leads to

$$
\mathbb{E}J_2^\epsilon(t) = 0\,. \tag{3.10}
$$

We next establish the estimate for J_3^ϵ. By making a change-of-variable $y = \epsilon^{-\frac{1}{\beta}}x$, we have $\nu(dx) = \frac{1}{\epsilon}\nu(dy)$ due to the fact that $\nu(dx) = \frac{1}{|x|^{\beta+d}}dx$. By the isometry property of Poisson integral, Taylor's formula, Cauchy–Schwarz inequality and (3.4)

$$
\begin{aligned}
& \mathbb{E}J_3^\epsilon(t) \\
= & \mathbb{E}\int_0^t \int_{|y|\geq 1} [\phi(\eta^\epsilon(s-) + \epsilon^\alpha y) - \phi(\eta^\epsilon(s-))]\frac{1}{\epsilon}\nu(dy)ds \\
= & \epsilon^{-1}\mathbb{E}\int_0^t \int_{y\geq 1} (D\phi(\eta^\epsilon(s-) + \theta\epsilon^\alpha y), \epsilon^\alpha y)\nu(dy)ds \\
\leq & \epsilon^{-1}\mathbb{E}\int_0^t \int_{|y|\geq 1} |D\phi(\eta^\epsilon(s-) + \theta\epsilon^\alpha y)||\epsilon^\alpha y|\nu(dy)ds \\
= & \epsilon^{\alpha-1}\mathbb{E}\int_0^t \int_{|y|\geq 1} |D\phi(\eta^\epsilon(s-) + \theta\epsilon^\alpha y)||y|\nu(dy)ds \\
\leq & C\epsilon^{\alpha-1}\mathbb{E}\int_0^t \int_{|y|\geq 1} |\eta^\epsilon(s-) + \theta\epsilon^\alpha y|^{p-1}|y|\nu(dy)ds \\
\leq & C\epsilon^{\alpha-1}\mathbb{E}\int_0^t \int_{|y|\geq 1} |\eta^\epsilon(s-)|^{p-1}|y|\nu(dy)ds + C\epsilon^{\alpha-1}\mathbb{E}\int_0^t \int_{|y|\geq 1} |\epsilon^\alpha y|^{p-1}|y|\nu(dy)ds \\
= & C\epsilon^{\alpha-1}\int_0^t |y|\nu(dy)\mathbb{E}\int_0^t |\eta^\epsilon(s)|^{p-1}ds + C\epsilon^{\alpha p-1}\int_{|y|\geq 1} |y|^p\nu(dy)t \\
\leq & \epsilon^{\alpha-1}\mathbb{E}\int_0^t \frac{p}{4}|\eta^\epsilon(s)|^p + Cds + C\epsilon^{\alpha p-1}t \\
\leq & \frac{p\epsilon^{\alpha-1}}{4}\int_0^t \mathbb{E}\phi(\eta^\epsilon(s))ds + (C + C\epsilon^{\alpha p-1})t \\
\leq & \frac{p}{4\epsilon}\int_0^t \mathbb{E}\phi(\eta^\epsilon(s))ds + Ct + C\epsilon^{-1}t.
\end{aligned}
\tag{3.11}
$$

For J_4^ϵ, the change-of-variable $y = \epsilon^{-\frac{1}{\beta}}x$, Taylor's formula and (3.5) yield

$$
\begin{aligned}
& J_4^\epsilon(t) \\
= & \int_0^t \int_{|y|<1} [\phi(\eta^\epsilon(s-) + \epsilon^\alpha y) - \phi(\eta^\epsilon(s-)) - (D\phi(s), \epsilon^\alpha y)] \frac{1}{\epsilon} \nu(dy) ds \\
= & \int_0^t \int_{|y|<1} \epsilon^{2\alpha-1} D^2(\phi(\eta^\epsilon(s-) + \theta\epsilon^\alpha y)(y \otimes y) \nu(dy) ds \\
\leq & \int_0^t \int_{|y|<1} \epsilon^{2\alpha-1} C|y|^2 \nu(dy) ds \\
\leq & C\epsilon^{2\alpha-1} t.
\end{aligned}
\tag{3.12}
$$

Taking expectation,

$$
\mathbb{E} J_4^\epsilon(t) \leq C\epsilon^{2\alpha-1} t \leq C\epsilon^{-1} t. \tag{3.13}
$$

Taking expectation on both sides of (3.6), and combining estimates (3.9)–(3.13),

$$
\mathbb{E}\phi(\eta^\epsilon(t)) \leq -\frac{p}{4\epsilon} \int_0^t \mathbb{E}\phi(\eta^\epsilon(s)) ds + Ct + \frac{Ct}{\epsilon}.
$$

Differentiating on both sides and using the comparison principle, we derive that

$$
\sup_{0<\epsilon\leq 1} \sup_{0\leq t\leq T} \mathbb{E}\phi(\eta^\epsilon(t)) \leq C. \tag{3.14}
$$

Let $\xi^\epsilon = \tilde{v}^\epsilon - \eta^\epsilon$. By (3.2) and (3.3),

$$
\xi^\epsilon(t) = \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} f(u^\epsilon(s)) ds.
$$

By the Lipschitz continuity of f, equation (3.2), Fubini theorem and the fact that $\beta \geq 1$,

$$
\begin{aligned}
& |\xi^\epsilon(t)| \\
\lesssim & \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} |u^\epsilon(s)| ds + \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} ds \\
\leq & \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} |u^\epsilon(s)| ds + \epsilon^{1-\frac{1}{\beta}} \\
\leq & \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} |u_0 + \epsilon^{\frac{1}{\beta}-1} \int_0^s \tilde{v}^\epsilon(r) dr| ds + \epsilon^{1-\frac{1}{\beta}} \\
\leq & \epsilon^{-\frac{1}{\beta}} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} |u_0| ds + \epsilon^{-1} e^{-\epsilon^{-1}t} \int_0^t \int_0^s e^{\epsilon^{-1}s} |\tilde{v}^\epsilon(r)| dr ds + \epsilon^{1-\frac{1}{\beta}} \\
\leq & \epsilon^{1-\frac{1}{\beta}} |u_0| + \epsilon^{-1} e^{-\epsilon^{-1}t} \int_0^t |\tilde{v}^\epsilon(r)| \int_r^t e^{\epsilon^{-1}s} ds dr + \epsilon^{1-\frac{1}{\beta}} \\
\leq & \epsilon^{1-\frac{1}{\beta}} |u_0| + \int_0^t |\tilde{v}^\epsilon(r)| dr + \epsilon^{1-\frac{1}{\beta}} \\
\leq & |u_0| + \int_0^t |\tilde{v}^\epsilon(r)| dr + 1 \\
\leq & |u_0| + \int_0^t |\xi^\epsilon(r)| dr + \int_0^t |\eta^\epsilon(r)| dr + 1 \\
\lesssim & \int_0^t |\xi^\epsilon(r)| dr + \int_0^t |\eta^\epsilon(r)| dr + 1.
\end{aligned}
\tag{3.15}
$$

Then by Hölder's inequality, taking expectation and (3.14)

$$\mathbb{E}|\xi^\epsilon(t)|^p \lesssim \int_0^t \mathbb{E}|\xi^\epsilon(r)|^p dr + 1\,,$$

which yields, by Gronwall's inequality,

$$\mathbb{E}|\xi^\epsilon(t)|^p \le C. \tag{3.16}$$

As a consequence of (3.14) and (3.16), we conclude that

$$\mathbb{E}|\tilde{v}^\epsilon(t)|^p \le C\,. \tag{3.17}$$

Now for each $0 \le t \le T$, by the fact that $\dot{u}^\epsilon = \epsilon^{\frac{1}{\beta}-1}\tilde{v}^\epsilon$,

$$|u(t)|^p \lesssim |u_0|^p + \epsilon^{(\frac{1}{\beta}-1)p}\int_0^T |\tilde{v}^\epsilon(t)|^p dt\,,$$

which yields

$$\mathbb{E}\sup_{0\le t\le T}|u^\epsilon(t)|^p \lesssim 1 + \epsilon^{(\frac{1}{\beta}-1)p}\,,$$

by Gronwall's inequality, taking expectation and (3.17). The proof of (i) is complete. In order to prove (ii), one should work with equations (1.3) instead of (3.2). The result follows from a similar argument, which is easier since the noise behaves less singularity. We omit the detail. □

Next we treat the velocity part.

Lemma 3.3. $\mathbb{E}\sup_{0\le t\le T}|\epsilon^{-1}\int_0^t \bar{v}_1^\epsilon(s)ds| \lesssim \epsilon$.

Proof. From (1.4)

$$\bar{v}_1^\epsilon(t) = \epsilon v_0 e^{-\epsilon^{-1}t}. \tag{3.18}$$

As a consequence, for all $0 \le t \le T$,

$$\begin{aligned} \left|\epsilon^{-1}\int_0^t \bar{v}_1^\epsilon(s)ds\right| &= \left|\int_0^t v_0 e^{-\epsilon^{-1}s}ds\right| \\ &\le |v_0|\int_0^T e^{-\epsilon^{-1}s}ds \\ &\le \epsilon|v_0|. \end{aligned} \tag{3.19}$$

Taking supremum and expectation yields the result. □

Lemma 3.4. *(i) For $\alpha \in [0, \frac{1}{\beta})$,*

$$\epsilon\mathbb{E}\sup_{0\le t\le T}|\bar{v}_2^\epsilon(t)| \lesssim \epsilon + \epsilon^{\frac{1}{\beta}}\,.$$

(ii) For $\alpha \in [\frac{1}{\beta}, 1)$,

$$\epsilon\mathbb{E}\sup_{0\le t\le T}|\bar{v}_2^\epsilon(t)| \lesssim \epsilon.$$

Proof. From (1.4),

$$\bar{v}_2^\epsilon(t) = \epsilon^{-1} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} f(u^\epsilon(s))ds. \tag{3.20}$$

Thanks to the Lipschitz continuity of f, for all $0 \le t \le T$

$$\begin{aligned} & |\bar{v}_2^\epsilon(t)| \\ \lesssim\ & \epsilon^{-1} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} |u^\epsilon(s)| ds + \epsilon^{-1} e^{-\epsilon^{-1}t} \int_0^t e^{\epsilon^{-1}s} ds \\ \le\ & \epsilon^{-1} e^{-\epsilon^{-1}t} \sup_{0\le t\le T} |u^\epsilon(t)| \int_0^t e^{\epsilon^{-1}s} ds + 1 \\ \le\ & \sup_{0\le t\le T} |u^\epsilon(t)| + 1, \end{aligned} \tag{3.21}$$

from which we obtain

$$\mathbb{E} \sup_{0\le t\le T} |\bar{v}_2^\epsilon(t)| \lesssim \mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t)| + 1.$$

Multiplying both sides by ϵ,

$$\epsilon \mathbb{E} \sup_{0\le t\le T} |\bar{v}_2^\epsilon(t)| \lesssim \epsilon \mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t)| + \epsilon.$$

Now we are in a position to prove both cases of this lemma right after using the Lemma 3.2. □

Lemma 3.5. $\sup_{0<\epsilon\le 1} \left[\epsilon^{\frac{1}{\beta}} \mathbb{E} \sup_{0\le t\le T} |\bar{v}_3^\epsilon(t)| \right] < \infty.$

Proof. Let $\bar{v}_4^\epsilon(t) := \epsilon^{\frac{1}{\beta}} \bar{v}_3^\epsilon(t)$, then

$$\dot{\bar{v}}_4^\epsilon(t) = -\epsilon^{-1} \bar{v}_4^\epsilon(t) + \dot{L}(t).$$

Next we show the following more general statement

$$\sup_{0<\epsilon\le 1} \mathbb{E} \sup_{0\le t\le T} |\bar{v}_4^\epsilon(t)|^p < \infty\,, \quad 1 \le p < \beta\,.$$

Similar to the proof of Lemma 3.2, by applying Itô's formula for $\phi(x) = (|x|^2 + 1)^{p/2}$ and choosing $c = 1$ in the Lévy–Itô decomposition (2.1),

$$\begin{aligned} & \phi(\bar{v}_4^\epsilon(t)) \\ =\ & 1 + \int_0^t (D\phi(\bar{v}_4^\epsilon(s)), -\epsilon^{-1} \bar{v}_4^\epsilon(s)) ds \\ & + \int_0^t \int_{|x|<1} \phi(\bar{v}_4^\epsilon(s-) + x) - \phi(\bar{v}_4^\epsilon(s-)) \tilde{N}(dsdx) \\ & + \int_0^t \int_{|x|\ge 1} \phi(\bar{v}_4^\epsilon(s-) + x) - \phi(\bar{v}_4^\epsilon(s-)) N(dsdx) \\ & + \int_0^t \int_{|x|<1} \phi(\bar{v}_4^\epsilon(s) + x) - \phi(\bar{v}_4^\epsilon(s)) - (D\phi(\bar{v}_4^\epsilon(s)), x) \nu(dx) ds \\ =:\ & 1 + \sum_{k=1}^4 H_k^\epsilon(t). \end{aligned} \tag{3.22}$$

Since

$$D\phi(y) = \frac{py}{(|y|^2+1)^{1-p/2}},$$

one immediately obtain

$$H_1^\epsilon(t) = \int_0^t \frac{-p|\bar{v}_4^\epsilon(s)|^2}{\epsilon(|\bar{v}_4^\epsilon(s)|^2+1)^{1-p/2}} ds \le 0,$$

and

$$\mathbb{E} \sup_{0\le t\le T} H_1^\epsilon(t) \le 0. \tag{3.23}$$

By Burkholder–Davis–Gundy inequality [13], Jensen's inequality, Taylor's formula and (3.4),

$$\begin{aligned}
& \mathbb{E} \sup_{0\le t\le T} H_2^\epsilon(t) \\
\lesssim\ & \mathbb{E}\sqrt{\int_0^T \int_{|x|<1} |\phi(\bar{v}_4^\epsilon(s-)+x) - \phi(\bar{v}_4^\epsilon(s-))|^2 N(dsdx)} \\
\le\ & \sqrt{\mathbb{E}\int_0^T \int_{|x|<1} |\phi(\bar{v}_4^\epsilon(s-)+x) - \phi(\bar{v}_4^\epsilon(s-))|^2 N(dsdx)} \\
=\ & \sqrt{\mathbb{E}\int_0^T \int_{|x|<1} |\phi(\bar{v}_4^\epsilon(s)+x) - \phi(\bar{v}_4^\epsilon(s))|^2 \nu(dx)ds} \\
=\ & \sqrt{\mathbb{E}\int_0^T \int_{|x|<1} |(D\phi(\bar{v}_4^\epsilon(s)+\theta x), x)|^2 \nu(dx)ds} \\
\lesssim\ & \sqrt{\mathbb{E}\int_0^T \int_{|x|<1} |\bar{v}_4^\epsilon(s)+\theta x|^{2p-2}|x|^2 \nu(dx)ds} \\
\le\ & \sqrt{\mathbb{E}\int_0^T \int_{|x|<1} |\bar{v}_4^\epsilon(s)|^{2p-2}|x|^2 \nu(dx)ds + \int_0^T \int_{|x|<1} |x|^{2p}\nu(dx)ds} \\
\le\ & \mathbb{E}\int_0^T \int_{|x|<1} |\bar{v}_4^\epsilon(s)|^{2p-2}|x|^2 \nu(dx)ds + \int_0^T \int_{|x|<1} |x|^{2p}\nu(dx)ds + 1 \\
\lesssim\ & \int_0^T \mathbb{E}|\bar{v}_4^\epsilon(s)|^{2p-2} ds + 1 \\
\le\ & \int_0^T \mathbb{E} \sup_{0\le s\le t} \phi(\bar{v}_4^\epsilon(s))dt + 1
\end{aligned} \tag{3.24}$$

where we have used $\sqrt{z} \le z+1$ and $2p-2 \le p$.

For H_3^ϵ, we have

$$\begin{aligned}
& \mathbb{E} \sup_{0\le t\le T} H_3^\epsilon(t) \\
\le\ & \mathbb{E}\int_0^T \int_{|x|\ge 1} |\phi(\bar{v}_4^\epsilon(s-)+x) - \phi(\bar{v}_4^\epsilon(s-))| N(dsdx) \\
=\ & \mathbb{E}\int_0^T \int_{|x|\ge 1} |\phi(\bar{v}_4^\epsilon(s)+x) - \phi(\bar{v}_4^\epsilon(s))| \nu(dx)ds.
\end{aligned} \tag{3.25}$$

Then by the same procedure that derives (3.11), we further have

$$\mathbb{E}\sup_{0\le t\le T} H_3^\epsilon(t) \lesssim \int_0^T \mathbb{E}\phi(\bar{v}_4^\epsilon(s))ds + 1 \le \int_0^T \mathbb{E}\sup_{0\le s\le t}\phi(\bar{v}_4^\epsilon(s))dt + 1. \tag{3.26}$$

Lastly we turn to H_4^ϵ. By Taylor's formula and (3.5),

$$\begin{aligned}
&\sup_{0\le t\le T} H_4^\epsilon(t)\\
\le\;& \int_0^T\int_{|x|<1} |\phi(\bar{v}_4^\epsilon(s)+x)-\phi(\bar{v}_4^\epsilon(s))-(D\phi(\bar{v}_4^\epsilon(s)),x)|\nu(dx)ds\\
\le\;& \int_0^T\int_{|x|<1} |D^2\phi(\bar{v}_4^\epsilon(s)+\theta x)(x\otimes x)|\nu(dx)ds\\
\lesssim\;& \int_0^T\int_{|x|<1} |x|^2\nu(dx)ds\\
\lesssim\;& 1,
\end{aligned} \tag{3.27}$$

which means

$$\mathbb{E}\sup_{0\le t\le T} H_4^\epsilon(t) \lesssim 1. \tag{3.28}$$

Taking supremum and expectation in (3.22) and using (3.23)–(3.28),

$$\mathbb{E}\sup_{0\le t\le T}\phi(\bar{v}_4^\epsilon(t)) \lesssim \int_0^T \mathbb{E}\sup_{0\le s\le t}\phi(\bar{v}_4^\epsilon(s))dt + 1,$$

and we end our proof by Gronwall's inequality. □

Lemma 3.6. $\sup_{0<\epsilon\le 1}\sup_{0\le t\le T}\mathbb{E}|\bar{v}_3^\epsilon(t)| < \infty.$

Proof. Applying Itô's formula to $\phi(x) = (|x|^2+1)^{p/2}$ and choosing $c = \epsilon^{\frac{1}{\beta}}$ in the Lévy–Itô decomposition (2.1) ,

$$\begin{aligned}
&\phi(\bar{v}_3^\epsilon(t))\\
=\;& 1 + \int_0^t (D\phi(\bar{v}_3^\epsilon(s)), -\epsilon^{-1}\bar{v}_3^\epsilon(s))ds\\
&+ \int_0^t\int_{|x|<\epsilon^{\frac{1}{\beta}}} \phi(\bar{v}_3^\epsilon(s-)+\epsilon^{-\frac{1}{\beta}}x)-\phi(\bar{v}_3^\epsilon(s-))\tilde{N}(dsdx)\\
&+ \int_0^t\int_{|x|\ge\epsilon^{\frac{1}{\beta}}} \phi(\bar{v}_3^\epsilon(s-)+\epsilon^{-\frac{1}{\beta}}x)-\phi(\bar{v}_3^\epsilon(s-))N(dsdx)\\
&+ \int_0^t\int_{|x|<\epsilon^{\frac{1}{\beta}}} \phi(\bar{v}_3^\epsilon(s)+\epsilon^{-\frac{1}{\beta}}x)-\phi(\bar{v}_3^\epsilon(s))-(D\phi(\bar{v}_3^\epsilon(s)),\epsilon^{-\frac{1}{\beta}}x)\nu(dx)ds\\
=:\;& 1+\sum_{k=1}^4 M_k^\epsilon(t).
\end{aligned} \tag{3.29}$$

By the same way which leads to (3.8), we derive that

$$\mathbb{E}M_1^\epsilon(t) \le -\frac{p}{2\epsilon}\int_0^t \phi(\bar{v}_3^\epsilon(s))ds + \frac{p}{\epsilon}t. \tag{3.30}$$

$$\mathbb{E}M_2^\epsilon(t)=0, \tag{3.31}$$

due to the martingale property. Similar to the argument which derives (3.11) and (3.13),

$$\mathbb{E}M_3^\epsilon(t)\le \frac{p}{4\epsilon}\int_0^t \mathbb{E}\phi(\bar v_3^\epsilon(s))ds+\frac{Ct}{\epsilon}, \tag{3.32}$$

and

$$\mathbb{E}M_4^\epsilon(t)\le \frac{Ct}{\epsilon}. \tag{3.33}$$

Combining (3.30)–(3.33),

$$\mathbb{E}\phi(\bar v_3^\epsilon(t))\le -\frac{p}{4\epsilon}\int_0^t \mathbb{E}\phi(\bar v_3^\epsilon(s))ds+\frac{Ct}{\epsilon}.$$

Differentiating on both sides and using the comparison principle, we obtain the desired result. □

4 Proof of the Main Result

With the preparation made above, we are in a position to prove our main result. From (1.3) and (1.5) we have

$$u^\epsilon(t)=u_0+\epsilon^{-1}\int_0^t \bar v_1^\epsilon(s)ds+\int_0^t \bar v_2^\epsilon(s)ds+\epsilon^{\alpha+\frac{1}{\beta}-1}\int_0^t \bar v_3^\epsilon(s)ds.$$

By (1.4),

$$\bar v_2^\epsilon(t)=\int_0^t -\epsilon^{-1}[\bar v_2^\epsilon(s)-f(u^\epsilon(s))]ds.$$

Combining the two equations above,

$$u^\epsilon(t)=u_0+\epsilon^{-1}\int_0^t \bar v_1^\epsilon(s)ds+\int_0^t f(u^\epsilon(s))ds-\epsilon\bar v_2^\epsilon(t)+\epsilon^{\alpha+\frac{1}{\beta}-1}\int_0^t \bar v_3^\epsilon(s). \tag{4.1}$$

From (1.2) and (4.1) we deduce that

$$\begin{aligned}
&|u^\epsilon(t)-\bar u^\epsilon(t)|\\
\le\ & \left|\epsilon^{-1}\int_0^t \bar v_1^\epsilon(s)ds\right|+\left|\int_0^t f(u^\epsilon(s))-f(\bar u^\epsilon(s))ds\right|+\epsilon|\bar v_2^\epsilon(t)|+\left|\epsilon^{\alpha+\frac{1}{\beta}-1}\int_0^t \bar v_3^\epsilon(s)-\epsilon^\alpha L(t)\right|\\
=:\ & \sum_{k=1}^4 I_k^\epsilon(t).
\end{aligned} \tag{4.2}$$

As a consequence of Lemma 3.3,

$$\mathbb{E}\sup_{0\le t\le T} I_1^\epsilon(t)\lesssim \epsilon. \tag{4.3}$$

Due to the fact that f is Lipschitz, for all $0\le t\le T$,

$$\begin{aligned}
&\left|\int_0^t f(u^\epsilon(s))-f(\bar u^\epsilon(s))ds\right|\\
\le\ & \int_0^t |f(u^\epsilon(s))-f(\bar u^\epsilon(s))|ds\\
\le\ & \int_0^T \sup_{0\le r\le s}|f(u^\epsilon(r))-f(\bar u^\epsilon(r))|ds\\
\lesssim\ & \int_0^T \sup_{0\le r\le s}|u^\epsilon(r)-\bar u^\epsilon(r)|ds.
\end{aligned}$$

Taking supremum and expectation and using Fubini theorem,

$$\mathbb{E}\sup_{0\le t\le T} I_2^\epsilon(t) \lesssim \int_0^T \mathbb{E}\sup_{0\le s\le t}|u^\epsilon(s)-\bar{u}^\epsilon(s)|dt. \tag{4.4}$$

Now we deal with I_4^ϵ. From (1.4),

$$\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t) = L(t) - \epsilon^{\frac{1}{\beta}-1}\int_0^t \bar{v}_3^\epsilon(s)ds, \tag{4.5}$$

so as a consequence Lemma 3.5, there exists a constant $C>0$ such that for all $0<\epsilon\le 1$,

$$\mathbb{E}\sup_{0\le t\le T}|L(t)-\epsilon^{\frac{1}{\beta}-1}\int_0^t \bar{v}_3^\epsilon(s)ds| \le C.$$

Multiplying both sides by ϵ^α,

$$\mathbb{E}\sup_{0\le t\le T} I_4^\epsilon(t) = \mathbb{E}\sup_{0\le t\le T}|\epsilon^\alpha L(t) - \epsilon^{\alpha+\frac{1}{\beta}-1}\int_0^t \bar{v}_3^\epsilon(s)ds| \le C\epsilon^\alpha. \tag{4.6}$$

For I_3^ϵ, in the case $0\le\alpha<\frac{1}{\beta}$, we have by Lemma 3.4 (i),

$$\mathbb{E}\sup_{0\le t\le T} I_3^\epsilon(t) \lesssim \epsilon + \epsilon^{\frac{1}{\beta}}. \tag{4.7}$$

Taking supremum and expectation on both sides of (4.2) and combining (4.3)–(4.7),

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T}|u^\epsilon(t)-\bar{u}^\epsilon(t)| \\ \lesssim\ & \int_0^T \mathbb{E}\sup_{0\le s\le t}|u^\epsilon(s)-\bar{u}^\epsilon(s)|dt + \epsilon + \epsilon^\alpha + \epsilon^{\frac{1}{\beta}} \\ \le\ & \int_0^T \mathbb{E}\sup_{0\le s\le t}|u^\epsilon(s)-\bar{u}^\epsilon(s)|dt + \epsilon^\alpha, \end{aligned} \tag{4.8}$$

where the last inequality follows from the fact that $0\le\alpha<\frac{1}{\beta}\le 1$. By Gronwall's inequality,

$$\mathbb{E}\sup_{0\le t\le T}|u^\epsilon(t)-\bar{u}^\epsilon(t)| \lesssim \epsilon^\alpha.$$

In the case $\frac{1}{\beta}\le\alpha<1$, from Lemma 3.4 (ii),

$$\mathbb{E}\sup_{0\le t\le T} I_3^\epsilon(t) \lesssim \epsilon. \tag{4.9}$$

Taking supremum and expectation on both sides of (4.2) and combining (4.3)–(4.6) and (4.9),

$$\begin{aligned} &\mathbb{E}\sup_{0\le t\le T}|u^\epsilon(t)-\bar{u}^\epsilon(t)| \\ \lesssim\ & \int_0^T \mathbb{E}\sup_{0\le s\le t}|u^\epsilon(s)-\bar{u}^\epsilon(s)|dt + \epsilon + \epsilon^\alpha \\ \le\ & \int_0^T \mathbb{E}\sup_{0\le s\le t}|u^\epsilon(s)-\bar{u}^\epsilon(s)|dt + \epsilon^\alpha, \end{aligned} \tag{4.10}$$

where the last inequality follows from the fact that $0 \le \alpha < 1$. By Gronwall's inequality,

$$\mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| \lesssim \epsilon^\alpha.$$

Combining the two cases above, we finish the proof for part (i) of Theorem 2.1.

Let us turn to the proof of part (ii) of Theorem 2.1, which is more subtle. In this case, from (4.2)–(4.3), (4.5), (4.7) and the Lipschitz property of f,

$$\mathbb{E}|u^\epsilon(t) - \bar{u}^\epsilon(t)| \lesssim \int_0^t \mathbb{E}|u^\epsilon(s) - \bar{u}^\epsilon(s)|ds + \epsilon + \epsilon^{\frac{1}{\beta}} + \mathbb{E}|\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)|,$$

which means

$$\begin{aligned} & \sup_{0\le t\le T} \mathbb{E}|u^\epsilon(t) - \bar{u}^\epsilon(t)| \\ \lesssim\ & \int_0^T \mathbb{E}|u^\epsilon(t) - \bar{u}^\epsilon(t)|dt + \epsilon + \epsilon^{\frac{1}{\beta}} + \sup_{0\le t\le T} \mathbb{E}|\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)| \\ \le\ & \int_0^T \sup_{0\le s\le t} \mathbb{E}|u^\epsilon(s) - \bar{u}^\epsilon(s)|dt + \epsilon + \epsilon^{\frac{1}{\beta}} + \sup_{0\le t\le T} \mathbb{E}|\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)|. \end{aligned} \tag{4.11}$$

Then we have

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|u^\epsilon(t) - \bar{u}^\epsilon(t)| = 0$$

provided that

$$\lim_{\epsilon\to 0} \sup_{0\le t\le T} \mathbb{E}|\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)| = 0, \tag{4.12}$$

which is an immediate consequence of Lemma 3.6. The proof is complete. □

We point out that, it is not clear for us whether the stronger convergence

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| = 0,$$

is true, but we can give a necessary and sufficient condition for it. From (4.2)–(4.5) and (4.7),

$$\mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| \lesssim \int_0^T \mathbb{E} \sup_{0\le s\le t} |u^\epsilon(s) - \bar{u}^\epsilon(s)|dt + \epsilon + \epsilon^{\frac{1}{\beta}} + \mathbb{E} \sup_{0\le t\le T} |\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)|.$$

Then we have

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| = 0$$

provided that

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |\epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t)| = 0. \tag{4.13}$$

Conversely, from (1.2), (4.1) and (4.5),

$$\begin{aligned} & u^\epsilon(t) - \bar{u}^\epsilon(t) \\ =\ & \epsilon^{-1}\int_0^t \bar{v}_1^\epsilon(s)ds + \int_0^t f(u^\epsilon(s)) - f(\bar{u}^\epsilon(s))ds - \epsilon\bar{v}_2^\epsilon(t) + \epsilon^{\frac{1}{\beta}-1}\int_0^t \bar{v}_3^\epsilon(s) - L(t) \\ =\ & \epsilon^{-1}\int_0^t \bar{v}_1^\epsilon(s)ds + \int_0^t f(u^\epsilon(s)) - f(\bar{u}^\epsilon(s))ds - \epsilon\bar{v}_2^\epsilon(t) + \epsilon^{\frac{1}{\beta}}\bar{v}_3^\epsilon(t). \end{aligned}$$

Now we take $v_0 = 0$ and $f = 0$. From (3.18) and (3.20),

$$u^\epsilon(t) - \bar{u}^\epsilon(t) = \epsilon^{\frac{1}{\beta}} \bar{v}_3^\epsilon(t).$$

The equation above, together with our previous analysis, implies that

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{0 \le t \le T} |u^\epsilon(t) - \bar{u}^\epsilon(t)| = 0$$

if and only if (4.13) holds.

However, to our best knowledge, it is unclear whether (4.13) holds true.

Remark 4.1. *We can also consider the situation when assumption* $(\mathbf{A_1})$ *is replaced by*

$(\mathbf{A_3})$ L *is a pure jump Lévy process without big jump, that is, the Lévy–Itô decomposition reads*

$$L(t) = \int_0^t \int_{|x|<c} x\tilde{N}(dsdx), \tag{4.14}$$

for some constant $c > 0$.

Theorem 2.1 is true under $(\mathbf{A_2})$ *and* $(\mathbf{A_3})$. *To prove the theorem in this case, one can take* $\beta = 2$ *in (1.4) and (1.5), write Lemma 3.1–3.6 in a similar manner, and obtain the convergence rate by repeating the procedure in this section. Again, the case* $\alpha = 0$ *needs a finer analysis.*

References

[1] D. Applebaum. Levy Processes and Stochastic Calculus. *Cambridge Studies in Advanced Mathematics*, 116, 2009.

[2] S. Cerrai and M. Freidlin. On the Smoluchowski–Kramers approximation for a system with an infinite number of degrees of freedom. *Probability Theory and Related Fields*, 135:363–394, 2006.

[3] S. Cerrai and M. Freidlin. Smoluchowski–Kramers approximation for a general class of SPDEs. *Journal of Evolution Equations*, 6(4):657–689, 2006.

[4] S. Cerrai and G. Xi. A Smoluchowski–Kramers approximation for an infinite dimensional system with state-dependent damping. *The Annals of Probability*, 50(3):874–904, 2022.

[5] M. Freidlin and W. Hu. Smoluchowski–Kramers approximation in the case of variable friction. *Journal of Mathematical Sciences*, 179(1):184–207, 2011.

[6] M. Freidlin, W. Hu, and A. Wentzell. Small mass asymptotic for the motion with vanishing friction. *Stochastic Processes and their Applications*, 123(1):45–75, 2013.

[7] S. Hottovy. *The Smoluchowski–Kramers approximation for stochastic differential equations with arbitrary state dependent friction*. PhD thesis, The University of Arizona, 2013.

[8] S. Hottovy, A. McDaniel, G. Volpe, and J. Wehr. The Smoluchowski–Kramers limit of stochastic differential equations with arbitrary state-dependent friction. *Communications in Mathematical Physics*, 336:1259–1283, 2015.

[9] O. Kallenberg. *Foundations of Modern Probability*, volume 2. Springer, 1997.

[10] S. Ken-Iti. *Lévy Processes and Infinitely Divisible Distributions*, volume 68. Cambridge University Press, 1999.

[11] H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7(4):284–304, 1940.

[12] X. Liu, Q. Jiang, and W. Wang. The Smoluchowski–Kramers approximation for a system with arbitrary friction depending on both state and distribution. *arXiv preprint arXiv:2406.18056*, 2024.

[13] S. Peszat and J. Zabczyk. *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach*, volume 113. Cambridge University Press, 2007.

[14] C. Shi and W. Wang. Small mass limit and diffusion approximation for a generalized Langevin equation with infinite number degrees of freedom. *Journal of Differential Equations*, 286:645–675, 2021.

[15] M. v. Smoluchowski. Drei vortrage uber diffusion, brownsche bewegung und koagulation von kolloidteilchen. *Zeitschrift fur Physik*, 17:557–585, 1916.

[16] D. Su and W. Wang. Approximation for a generalized Langevin equation with high oscillation in time and space. *Stochastics and Dynamics*, 22(08):2240030, 2022.

[17] M. Wang, D. Su, and W. Wang. Averaging on macroscopic scales with application to Smoluchowski–Kramers approximation. *Journal of Statistical Physics*, 191(2):22, 2024.

[18] W. Wang and Y. Lv. Limit behavior of nonlinear stochastic wave equations with singular perturbation. *Discrete and Continuous Dynamical Systems-B*, 13(1):175–193, 2009.

[19] W. Wang, Y. Lv, and A. Roberts. On the approximation for singularly perturbed stochastic wave equations. *arXiv preprint arXiv:1109.3000*, 2011.

[20] Z. Wang, L. Lv, Y. Zhang, J. Duan, and W. Wang. Small mass limit for stochastic interacting particle systems with Lévy noise and linear alignment force. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 34(2), 2024.

[21] S. Zhang. *Smoluchowski–Kramers approximation for stochastic equations with Lévy-noise*. PhD thesis, Purdue University, 2008.