Infinitely many new sequences of surfaces of general type with maximal Picard number converging to the Severi line

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Abstract

Examples of algebraic surfaces of general type with maximal Picard number are not abundant in the literature. Moreover, most known examples either possess low invariants, lie near the Noether line $K^2 = 2\chi - 6$ or are somewhat scattered. A notable exception is Persson's sequence of double covers of the projective plane with maximal Picard number, whose invariants converge to the Severi line $K^2 = 4\chi$. This note is devoted to the construction of infinitely many new sequences of surfaces of general type with maximal Picard number whose invariants converge to the Severi line.

1 Introduction

The self-intersection of the canonical class K_X^2 and the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ of an algebraic surface X are two of its main numerical invariants. If X is a minimal surface of general type defined over the complex numbers $\mathbb{C}$, it is well-known that $(K_X^2, \chi(\mathcal{O}_X))$ is a pair of strictly positive integers satisfying both Noether's inequality $K_X^2 \geq 2\chi(\mathcal{O}_X) - 6$ and the Bogomolov-Miyaoka-Yau inequality $K_X^2 \leq 9\chi(\mathcal{O}_X)$ (cf. [2, Chapter VII]). More challenging is the inverse problem of determining whether given an admissible pair, i.e. a pair of strictly positive integers (K^2, χ) satisfying both Noether's inequality and the Bogomolov-Miyaoka-Yau inequality, there exists a minimal surface of general type X such that $K_X^2 = K^2$ and $\chi(\mathcal{O}_X) = \chi$. The first systematic approach to this problem, known as the geographical question, was taken by Persson [21], who filled in most of the region $2\chi - 6 \leq K^2 \leq 8\chi$. Chen [10], [11] then filled in some of the remaining gaps, extending coverage to a substantial portion of the region $8\chi < K^2 < 9\chi$. Despite the presence of some missing examples, the current consensus is that these instances primarily stem from technical factors rather than indicating any inherent issues.

Given the prevailing agreement that the original geographical question can be given an affirmative answer, numerous authors directed their attention towards exploring the geography of surfaces of general type with special features such as genus 2 fibrations [20], simply-connectedness [21], 2-divisibility of the canonical class [23], global 1-forms [15], $\mathbb{Z}_2^2$-actions [16], etc. Specially intriguing is the case of the geography of surfaces of general type with maximal Picard number.

The Picard number $\rho(X)$ of a smooth projective surface X is the rank of its Neron-Severi group $NS(X)$ which, defined as the group of divisors of X modulo numerical equivalence, is finitely generated. The Picard number of X is bounded above by the Hodge number $h^{1,1}(X) = \dim H^1(X, \Omega_X^1)$ and X is said to have maximal Picard number if $\rho(X) = h^{1,1}(X)$. Although there is no a priori reason to believe the geography of minimal surfaces of general type should not be highly populated by surfaces with maximal Picard number, the examples of such surfaces are scarce in the literature (see [5] for an overview on the subject). In summary:

- Surfaces X with geometric genus $p_g(X) = 0$ have maximal Picard number.

- The first non-trivial examples of algebraic surfaces of general type with maximal Picard number were published by Persson [22]. On the one hand, if we denote by $\mathfrak{M}_{K^2,\chi}$ Gieseker's moduli space [14] of canonical models of surfaces of general type with self-intersection of the canonical class K^2 and holomorphic Euler characteristic χ, Persson [22, Theorem 1] proved that given an admissible pair (K^2, χ) such that $K^2 = 2\chi - 6$ and $\chi \not\equiv 0$ mod 6, then all the connected components of $\mathfrak{M}_{K^2,\chi}$ contain canonical models whose minimal resolution has maximal Picard number. On the other hand, Persson [22, Theorem 3] proved that for every integer $n \geq 4$, if we denote $K^2 = 2(n-3)^2$ and $\chi = \frac{1}{2}(n-1)(n-2) + 1$, then $\mathfrak{M}_{K^2,\chi}$ contains canonical models whose minimal resolution has maximal Picard number.

- Examples of surfaces of general type with maximal Picard number and low geometric genus can be found in [5, 6.2, 6.3], [3], [7].

- Some scattered examples can be found in [5, 6.5], [25].

- Further examples can be found in [1], [26].

- The authors [6] proved that $\mathfrak{M}_{K^2,\chi}$ contains canonical models whose minimal resolution has maximal Picard number for every admissible pair such that $K^2 \leq \frac{5}{2}\chi - 11$. In addition, some scattered examples one above the Noether line were provided.

The main result of [6] draws inspiration from [22, Theorem 1]. Whereas Persson obtained the examples of [22, Theorem 1] as double covers of rational surfaces whose branch locus has a very particular configuration, the authors [6] obtained theirs as bidouble covers of rational surfaces. Similarly, this paper introduces new families of surfaces of general type with maximal Picard number drawing inspiration from [22, Theorem 3], where surfaces of general type with maximal Picard number were obtained as double covers of the projective plane $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2])$ whose branch locus derives from the curve:

$$C = [(X_0^n + X_1^n + X_2^n)^2 - 4 \cdot ((X_0 X_1)^n + (X_0 X_2)^n + (X_1 X_2)^n) = 0].$$

Considering bidouble covers of the projective plane whose branch locus derives from the curve C, one can get the following:

Theorem 1. *Given $n \in \mathbb{Z}_{\geq 2}$ let us define*

$$K^2 = 4n^2 - 12n + 9 \qquad and \qquad \chi = n^2 - n + 1.$$

Then $\mathfrak{M}_{K^2,\chi}$ contains canonical models X_n whose minimal resolution has maximal Picard number.

Blowing-up $\mathbb{P}^2$ at a point, taking cyclic covers branched along two fibers of the blown-up surface and then considering a bidouble cover whose branch locus derives from C, one can get the following:

Theorem 2. *Given $m \in \mathbb{Z}_{\geq 3}$ and $n \in 2 \cdot \mathbb{Z}_{\geq 1}$ let us define*

$$K^2 = 4mn^2 - 4(m+2)n + 8 \qquad and \qquad \chi = mn^2 - n + 1.$$

Then $\mathfrak{M}_{K^2,\chi}$ contains canonical models $X_{m,n}$ whose minimal resolution has maximal Picard number.

Finally, blowing-up $\mathbb{P}^2$ at a point, taking cyclic covers branched along two fibers of the blown-up surface and then considering a double cover whose branch locus derives from C, one can get the following:

Theorem 3. *Given $m \in \mathbb{Z}_{\geq 2}$ and $n \in 2 \cdot \mathbb{Z}_{\geq 2}$ let us define*

$$K^2 = 2mn^2 - 4(m+1)n + 8 \qquad and \qquad \chi = \frac{1}{2}mn(n-1) + 1.$$

Then $\mathfrak{M}_{K^2,\chi}$ contains canonical models $X_{m,n}$ whose minimal resolution has maximal Picard number.

The region of the plane (K^2, χ) covered by the pairs described in Theorem 1, Theorem 2 and Theorem 3 may seem unstructured at a first glance, but paying attention to the slopes $\mu = K^2/\chi$ one can infer the following:
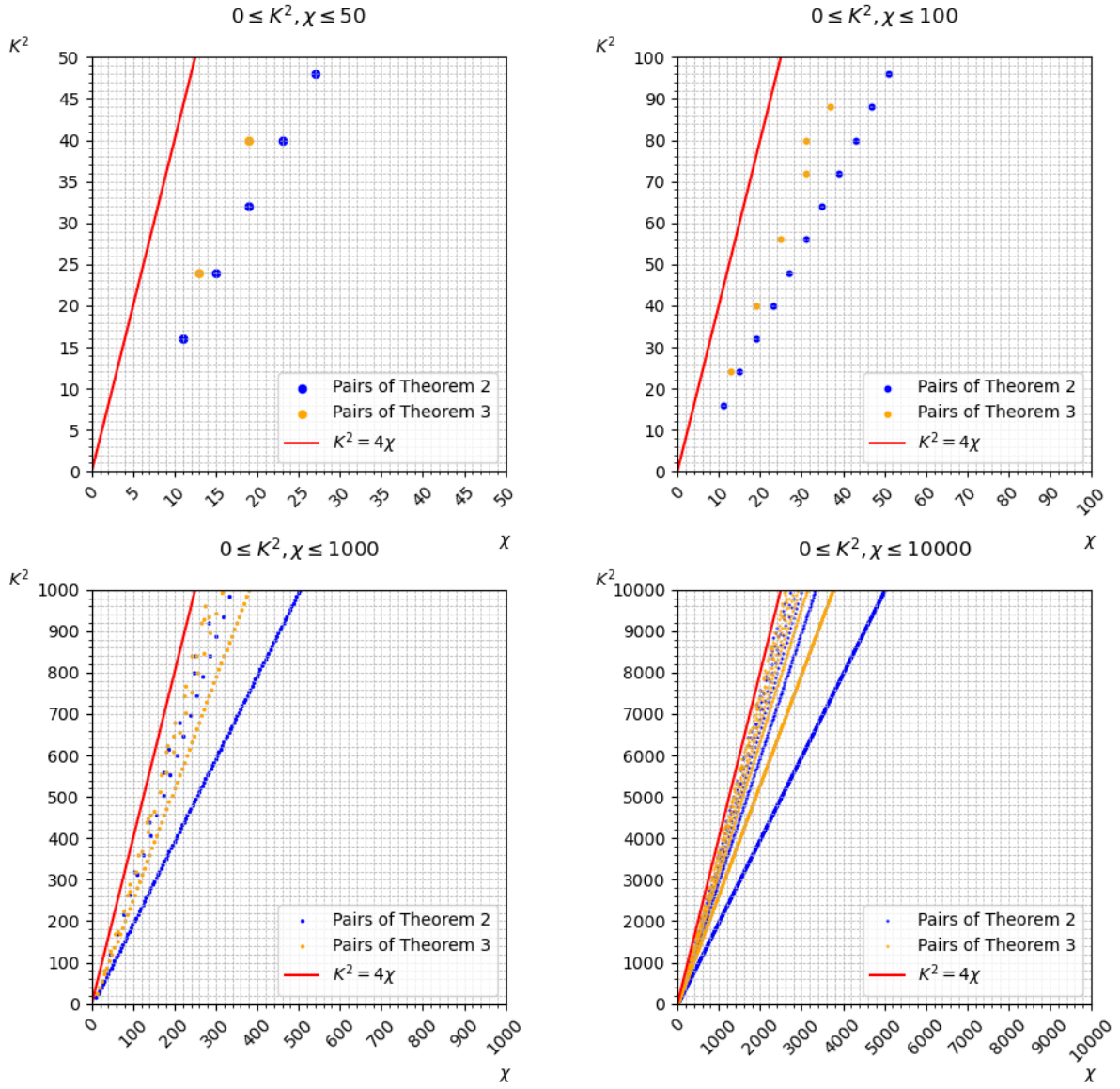
Figure 1: Multi-scale representation of the plane (K^2, χ) highlighting the pairs described in Theorem 2, the pairs described in Theorem 3 and the Severi line.

Corollary 1. *Denoting by $\mu(S)$ the slope $K_S^2/\chi(\mathcal{O}_S)$ of an algebraic surface S:*

i) *For each $n \in 2 \cdot \mathbb{Z}_{\geq 1}$ there exists an unbounded sequence $\{X_m^n\}_{m \in \mathbb{Z}_{\geq 3}}$ of surfaces of general type with maximal Picard number such that:*

$$\lim_{m \to \infty} \mu(X_m^n) = 4 - \frac{4}{n}.$$

ii) *For each $m \in \mathbb{Z}_{\geq 3}$ there exists a distinct unbounded sequence $\{Y_n^m\}_{n \in 2 \cdot \mathbb{Z}_{\geq 1}}$ of surfaces of general type with maximal Picard number such that:*

$$\lim_{n \to \infty} \mu(Y_n^m) = 4.$$

The inequality $K^2 \geq 4\chi$ is a necessary condition for a minimal smooth complex projective surface to have maximal Albanese dimension. Although this fact was not fully proven until the 2000s by Pardini [18], $K^2 \geq 4\chi$ (resp. $K^2 = 4\chi$) is known as the Severi inequality (resp. line) because Francesco Severi [13] stated and gave an incorrect proof of the aforementioned result in the 1930s.

It follows from Corollary 1, *ii*) that there exist countably many sequences of surfaces of general type with maximal Picard number that converge to the Severi line. While Persson's constructions [22, Theorem 3] also give rise to a sequence $\{S_n\}$ of surfaces of general type with maximal Picard number that converges to the Severi line, our sequences are not only disjoint from $\{S_n\}$ (see Remark 4 below) but there are infinitely many such sequences.

The remainder of this paper is structured as follows. Section 2 contains the basics on abelian covers that will be needed throughout this article. Section 3 provides an overview of key facts regarding the Picard number of algebraic surfaces that are relevant to our discussion. Finally, in Section 4 we construct families of surfaces of general type with maximal Picard number that will allow to prove Theorem 1, Theorem 2 and Theorem 3.

Notation and conventions. Throughout this paper, we assume the ground field to be the field of complex numbers $\mathbb{C}$ and all varieties to be algebraic and projective. We will use $\cong$ to indicate linear equivalence of line bundles or divisors, while $\equiv$ will represent congruence modulo a given integer. The Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ will be referred to as $\mathbb{F}_e$. For simplicity, the negative section Δ_0 and a general fiber F of $\mathbb{F}_e$ will just mean two intersecting fibers if $e = 0$. Finally, given an integer n, we will denote by $\mathbb{Z}_{\geq n}$ the set of integers bigger than or equal to n. The rest of the notation follows conventional usage in algebraic geometry.

2   Abelian covers

Given a finite abelian group G, a finite map $f \colon X \to Y$ in conjunction with a faithful action of the group G on X such that f allows Y to be expressed as the quotient X/G, is known as a G-cover. The general scenario was first explored by Pardini [17], but here we will limit our discussion to $\mathbb{Z}_2$-covers and $\mathbb{Z}_2^2$-covers, also known as double covers and bidouble covers respectively. A key result about cyclic covers (which are G-covers where G is a cyclic group and therefore include $\mathbb{Z}_2$-covers) of Hirzebruch surfaces will also be presented.

Comessatti [12] had already studied cyclic covers of surfaces, but [19] is a good reference for the specific case $G = \mathbb{Z}_2$. The structure theorem for smooth $\mathbb{Z}_2^2$-covers was established by Catanese [8].

2.1   Double covers

According to [19, Section II] or [17, Theorem 2.1], the building data $\{L, B\}$ suffices to define a normal $\mathbb{Z}_2$-cover $X \to Y$ of a smooth and irreducible projective variety Y, where:

- The branch locus B is a reduced and effective divisor on Y.

- L is a non-trivial line bundle on Y such that $L \cong 2B$.

Proposition 1 ([19, Section II] or [17, Proposition 4.2]). *Let Y be a smooth surface and $f\colon X \to Y$ a smooth $\mathbb{Z}_2$-cover with building data $\{L, B\}$. Then:*

$$K_X \cong f^*(K_Y + B),$$
$$K_X^2 = 2(K_Y + B)^2,$$
$$p_g(X) = p_g(Y) + h^0(K_Y + L),$$
$$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_Y) + \frac{1}{2}L(L + K_Y).$$

We observe that Proposition 1 applies when X has ADE singularities (see [2] for details on ADE singularities on curves and surfaces). For a broader statement, the reader is referred to Bauer and Pignatelli's work [4, Section 2].

2.2   Bidouble covers

According to [8, Section 2] or [17, Theorem 2.1], the building data $\{L_i, B_j\}_{i,j\in\{1,2,3\}}$ suffices to define a normal $\mathbb{Z}_2^2$-cover $X \to Y$ of a smooth and irreducible projective variety Y, where:

- B_1, B_2, B_3 are effective divisors on Y such that the branch locus $B = B_1 + B_2 + B_3$ is reduced.

- L_1, L_2, L_3 are non-trivial line bundles on Y satisfying $2L_1 \cong B_2 + B_3$, $2L_2 \cong B_1 + B_3$ and $L_3 \cong L_1 + L_2 - B_3$.

Proposition 2 ([8, Section 2] or [17, Proposition 4.2]). *Let Y be a smooth surface and $f\colon X \to Y$ a smooth $\mathbb{Z}_2^2$-cover with building data $\{L_i, B_j\}_{i,j}$. Then:*

$$2K_X \cong f^*(2K_Y + B_1 + B_2 + B_3),$$
$$K_X^2 = (2K_Y + B_1 + B_2 + B_3)^2,$$
$$p_g(X) = p_g(Y) + \sum_{i=1}^{3} h^0(K_Y + L_i),$$
$$\chi(\mathcal{O}_X) = 4\chi(\mathcal{O}_Y) + \frac{1}{2}\sum_{i=1}^{3} L_i(L_i + K_Y).$$

We observe that Proposition 2 also applies when X has ADE singularities. Again, the reader is referred to [4, Section 2].

Corollary 2 ([6, Corollary 1]). *Let $f\colon X \to Y$ be a $\mathbb{Z}_2^2$-cover of a smooth surface Y with building data $\{L_i, B_j\}_{i,j}$ and let $Q = f(P)$ be an intersection point of B_1 and B_2 that is not contained in B_3.*

- *i) If both B_1 and B_2 are smooth at Q but they intersect in such a way that $B_1 + B_2$ has a singularity of type $A_{2n+1}, n \geq 1$ at Q, then X has a singularity of type A_n at P.*

- *ii) If B_1 has a singularity of type $A_n, n \geq 1$ at Q, B_2 is smooth at Q and they intersect in such a way that $B_1 + B_2$ has a singularity of type D_{n+3} at Q, then X has a singularity of type A_{2n+1} at P.*

Remark 1. *It is worth noting that:*

- *i) If $n = 0$ in Corollary 2, i) then X is smooth at P.*

- *ii) An ADE singularity of $B_i, i \in \{1, 2, 3\}$ that is disjoint from the two other divisors of the branch locus induces two singularities of the same type on X.*

For more information about the singularities arising from bidouble covers the reader is addressed to [9] or [17].

2.3 Cyclic covers of Hirzebruch surfaces

The following result is a restatement of [22, Lemma 4.1] that will be essential to prove Theorem 2 and Theorem 3.

Lemma 1 ([22, Lemma 4.1] or [6, Lemma 2]). *Given a Hirzebruch surface $\mathbb{F}_e$ with two disjoint fibers F_1, F_2 there is for each integer d a unique $\mathbb{Z}_d$-cover $\pi_d \colon \mathbb{F}_{de} \to \mathbb{F}_e$ branched at F_1 and F_2. Furthermore:*

a) *If $C \cong \mathcal{O}_{\mathbb{F}_e}(a\Delta_0 + bF)$ is an effective divisor not having F_1 and F_2 as components then $C_d := \pi_d^* C \cong \mathcal{O}_{\mathbb{F}_{de}}(a\Delta_0 + dbF)$.*

b) *If C has at most ADE singularities, then C_d has at most ADE singularities if and only if the only singularities of C lying on F_1 or F_2 are of type A_n and they are transversal to these fibers.*

c) *Given $n \in 2 \cdot \mathbb{Z}_{\geq 1}$ a singularity of type A_{n-1} of C on $F_i, i \in \{1, 2\}$ that is transversal to this fiber gives rise to a singularity of type A_{dn-1} of C_d.*

d) *A singularity of C not lying on $F_1 \cup F_2$ gives rise to d singularities of the same type on C_d.*

Remark 2. *The authors suspect that Lemma 1, c) remains valid when $n \in \mathbb{Z}_{\geq 2}$ is odd, despite this case not being addressed in Persson's original statement [22, Lemma 4.1, c)]. To support this, they explicitly wrote π_d using the description of Hirzebruch surfaces provided in [24, Chapter 2]. Then, they studied whether the pullback of a singularity of type A_{n-1} of C on $F_i, i \in \{1, 2\}$ that is transversal to this fiber gives rise to a singularity of type A_{dn-1} of C_d. If confirmed, this claim could broaden the scope of Theorem 2 and Theorem 3. However, the authors have chosen not to include it as a precautionary measure.*

3 On the Picard Number of an algebraic surface

Determining the Picard number of a smooth projective surface can be challenging in general. However, there is a relatively straightforward method to demonstrate that a smooth projective surface X has maximal Picard number.

Indeed, on the one hand, the number of algebraically independent divisor classes in $NS(X)$ that one is able to identify is a lower bound on $\rho(X)$. On the other hand, $h^{1,1}(X) = 10\chi(\mathcal{O}_X) - K_X^2 - 2q(X)$ is an upper bound on $\rho(X)$.

Hence, if we are able to find $h^{1,1}(X)$ divisors on X whose intersection matrix has rank $h^{1,1}(X)$, then X has maximal Picard number. See [25, Section 2] for other approaches.

From the discussion above one can easily infer the following:

Lemma 2 ([6, Lemma 1]). *Let X be a canonical model with minimal resolution $\hat{X} \to X$ and whose singular set consists of:*

- *α_i singularities of type $A_i, i \in \mathbb{Z}_{\geq 1}$,*

- *β_j singularities of type $D_j, j \in \mathbb{Z}_{\geq 4}$,*

- *γ_k singularities of type $E_k, k \in \{6, 7, 8\}$.*

Let us suppose that X admits a surjective morphism $\pi : X \to Y$ onto a smooth and projective surface Y and there exist numerically independent divisors $C_1, \ldots, C_n$ on Y. Then:

$$\rho(\hat{X}) \geq \sum_i i \cdot \alpha_i + \sum_j j \cdot \beta_j + \sum_k k \cdot \gamma_k + n.$$

Remark 3. *Lemma 2 will just be applied in the following scenarios:*

- *When $\pi \colon X \to \mathbb{P}^2$ is a $\mathbb{Z}_2^2$-cover of the projective plane $\mathbb{P}^2$, $n = 1$ and C_1 is a line.*

- *When $\pi \colon X \to \mathbb{F}_e$ is either a $\mathbb{Z}_2$-cover or a $\mathbb{Z}_2^2$-cover of the Hirzebruch surface $\mathbb{F}_e$ for some integer $e \geq 0$, $n = 2$ and the divisors C_1, C_2 are the negative section and a fiber.*

4   Surfaces with maximal Picard number

This section is devoted to construct families of surfaces of general type with maximal Picard number from which Theorem 1, Theorem 2 and Theorem 3 will follow. Given that all these surfaces are obtained as abelian covers whose branch locus derives from the curve

$$C = [(X_0^n + X_1^n + X_2^n)^2 - 4 \cdot ((X_0 X_1)^n + (X_0 X_2)^n + (X_1 X_2)^n) = 0]$$

on $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2])$, we will begin by establishing the common setup and notation.
Denoting $l_1 = [X_0 = 0]$, $l_2 = [X_1 = 0]$ and $l_3 = [X_2 = 0]$, the curve C has n singularities of type A_{n-1} on l_1, n singularities of type A_{n-1} on l_2 and n singularities of type A_{n-1} on l_3 by [22, Lemma 7.5].
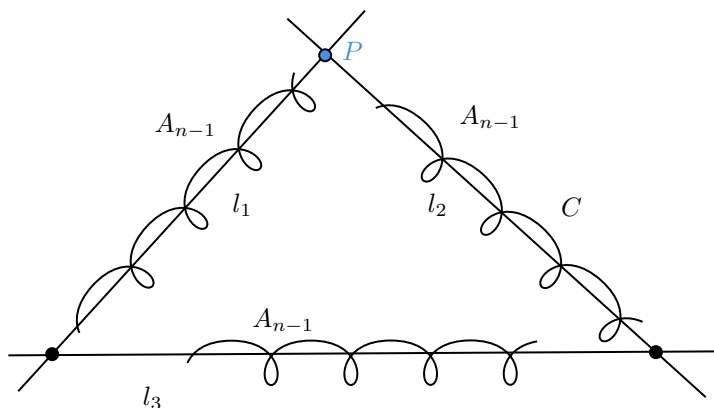

Figure 2: Schematic depiction of curve C.

Let $\pi \colon \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at P where P is the intersection of l_1 and l_2. We denote by:

$$\bar{l}_3, \overline{C}: \quad \text{the pull-back via } \pi \text{ of } l_3, C \text{ respectively,}$$
$$l_1', l_2': \quad \text{the strict transforms via } \pi \text{ of } l_1, l_2 \text{ respectively.}$$

Note that $l_1' \cong l_2' \cong \mathcal{O}_{\mathbb{F}_1}(F)$, $\bar{l}_3 \cong \mathcal{O}_{\mathbb{F}_1}(\Delta_0 + F)$, and $\overline{C} \cong \mathcal{O}_{\mathbb{F}_1}(2n\Delta_0 + 2nF)$.

Let $\psi \colon \mathbb{F}_m \to \mathbb{F}_1$ be the $\mathbb{Z}_m$-cover branched at $l_1' + l_2'$ (see Lemma 1).

Firstly, since $\bar{l}_3 \cong \mathcal{O}_{\mathbb{F}_1}(\Delta_0 + F)$ and $\overline{C} \cong \mathcal{O}_{\mathbb{F}_1}(2n\Delta_0 + 2nF)$ are effective divisors not having l_1' and l_2' as components, then $\tilde{l}_3 := \psi^*\left(\bar{l}_3\right) \in |\mathcal{O}_{\mathbb{F}_m}(\Delta_0 + mF)|$ and $\tilde{C} := \psi^*\left(\overline{C}\right) \in |\mathcal{O}_{\mathbb{F}_m}(2n\Delta_0 + 2nmF)|$ by Lemma 1, where ψ^* is the pull-back of ψ.

Secondly, being ψ branched at $l_1' + l_2'$, we obtain that

$$\psi^*\left(l_1'\right) = m\tilde{l}_1,$$
$$\psi^*\left(l_2'\right) = m\tilde{l}_2,$$

for some fibers $\tilde{l}_1, \tilde{l}_2 \in |\mathcal{O}_{\mathbb{F}_m}(F)|$.

Proof of Theorem 1. We consider the following divisors on $\mathbb{P}^2$:

$$B_1 := l_1,$$
$$B_2 := l_2,$$
$$B_3 := l_3 + C.$$

Note that $B_1, B_2 \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ and $B_3 \in |\mathcal{O}_{\mathbb{P}^2}(2n+1))|$.

We also consider the following line bundles on $\mathbb{P}^2$:

$$L_1 := \mathcal{O}_{\mathbb{P}^2}(n+1),$$
$$L_2 := \mathcal{O}_{\mathbb{P}^2}(n+1),$$
$$L_3 := \mathcal{O}_{\mathbb{P}^2}(1).$$

The building data $\{L_i, B_j\}_{i,j}$ defines a $\mathbb{Z}_2^2$-cover $\varphi \colon X \to \mathbb{P}^2$. On the one hand, taking into account [22, Lemma 7.5], we get:

- The singular set of B_3 restricted to $B_3 \setminus (B_1 \cup B_2)$ consists of n singularities of type D_{n+2}.

- The singular set of $B_1 + B_3$ restricted to $B_1 \cap B_3$ consists of n singularities of type D_{n+2}.

- The singular set of $B_2 + B_3$ restricted to $B_2 \cap B_3$ consists of n singularities of type D_{n+2}.

Thus, the singular set of X consists of:

- $2n$ singularities of type D_{n+2} by Remark 1.

- n singularities of type A_{2n-1} by Corollary 2,ii).

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On the other hand, by Proposition 2 we have that K_X is ample because $2K_X$ is the pull-back via φ of the ample divisor

$$\mathcal{O}_{\mathbb{P}^2}(2n-3).$$

Moreover:

$$K_X^2 = 4n^2 - 12n + 9,$$
$$\chi(\mathcal{O}_X) = n^2 - n + 1,$$
$$p_g(X) = n^2 - n,$$
$$q(X) = 0,$$
$$h^{1,1}(X) = 6n^2 + 2n + 1.$$

We conclude that $X \in \mathfrak{M}_{K^2, \chi}$ is a canonical model whose minimal resolution has maximal Picard number $h^{1,1}(X) = 6n^2 + 2n + 1$ by Lemma 2.

$\square$

Proof of Theorem 2. We consider the following divisors on $\mathbb{F}_m$:

$$B_1 := 0,$$
$$B_2 := \tilde{l}_1 + \tilde{l}_2 \in |\mathcal{O}_{\mathbb{F}_m}(2F)|,$$
$$B_3 := \Delta_0 + \tilde{l}_3 + \tilde{C} \in |\mathcal{O}_{\mathbb{F}_m}((2n+2)\Delta_0 + (2n+1)mF)|,$$

if m is even and:

$$B_1 := \tilde{l}_1 \in |\mathcal{O}_{\mathbb{F}_m}(F)|,$$
$$B_2 := \tilde{l}_2 \in |\mathcal{O}_{\mathbb{F}_m}(F)|,$$
$$B_3 := \Delta_0 + \tilde{l}_3 + \tilde{C} \in |\mathcal{O}_{\mathbb{F}_m}((2n+2)\Delta_0 + (2n+1)mF)|,$$

if m is odd.

We also consider the following line bundles on $\mathbb{F}_m$:

$$L_1 := \mathcal{O}_{\mathbb{F}_m}\left((n+1)\Delta_0 + \left(mn + \frac{1}{2}n + 1\right)F\right),$$

$$L_2 := \mathcal{O}_{\mathbb{F}_m}\left((n+1)\Delta_0 + \left(mn + \frac{1}{2}n\right)F\right),$$

$$L_3 := \mathcal{O}_{\mathbb{F}_m}(F),$$

if m is even and:

$$L_1 := \mathcal{O}_{\mathbb{F}_m}\left((n+1)\Delta_0 + \left(mn + \frac{1}{2}(n-1) + 1\right)F\right),$$

$$L_2 := \mathcal{O}_{\mathbb{F}_m}\left((n+1)\Delta_0 + \left(mn + \frac{1}{2}(n-1) + 1\right)F\right),$$

$$L_3 := \mathcal{O}_{\mathbb{F}_m}(F),$$

if m is odd.

The building data $\{L_i, B_j\}_{i,j}$ defines a $\mathbb{Z}_2^2$-cover $\varphi\colon X \to \mathbb{F}_m$. On the one hand, taking into account [22, Lemma 7.5], we get:

- The singular set of B_3 restricted to $B_3 \setminus (B_1 \cup B_2)$ consists of mn singularities of type D_{n+2}.

- The singular set of $B_1 + B_2 + B_3$ restricted to $(B_1 \cap B_3) \cup (B_2 \cap B_3)$ consists of $2n$ singularities of type D_{mn+2}.

Thus, the singular set of X consists of:

- $2mn$ singularities of type D_{n+2} by Remark 1.

- $2n$ singularities of type A_{2mn-1} by Corollary 2,ii).

On the other hand, by Proposition 2 we have that K_X is ample because $2K_X$ is the pull-back via φ of the ample divisor

$$\mathcal{O}_{\mathbb{F}_m}\left((2n-2)\Delta_0 + (2mn - m - 2)F\right).$$

Moreover:

$$K_X^2 = \left((2n-2)\Delta_0 + (2mn - m - 2)F\right)^2 = 4mn^2 - 4(m+2)n + 8,$$
$$\chi(\mathcal{O}_X) = mn^2 - n + 1,$$
$$p_g(X) = mn^2 - n,$$
$$q(X) = 0,$$
$$h^{1,1}(X) = 6mn^2 + (4m-2)n + 2.$$

We conclude that $X \in \mathfrak{M}_{K^2,\chi}$ is a canonical model whose minimal resolution has maximal Picard number $h^{1,1}(X) = 6mn^2 + (4m-2)n + 2$ by Lemma 2. $\qquad\square$

Proof of Theorem 3. We consider following the divisor on $\mathbb{F}_m$:

$$B = \tilde{C} + \tilde{l}_1 + \tilde{l}_2 \in |\mathcal{O}_{\mathbb{F}_m}(2n\Delta_0 + (2mn + 2)F)|$$

The divisor B defines a $\mathbb{Z}_2$-cover $\varphi\colon X \to \mathbb{F}_m$. On the one hand, taking into account [22, Lemma 7.5], we get that the singular set of B consists of $2n$ singularities of type D_{mn+2} and mn singularities of type A_{n-1}.

Thus, the singular set of X consists of $2n$ singularities of type D_{mn+2} and mn singularities of type A_{n-1}.

On the other hand, we have that K_X is ample because K_X is the pull-back via φ of the ample divisor

$$\mathcal{O}_{\mathbb{F}_m}\left((n-2)\,\Delta_0 + (mn - m - 1)\,F\right).$$

Moreover:

$$K_X^2 = 2\left((n-2)\,\Delta_0 + (mn - m - 1)\,F\right)^2 = 2mn^2 - 4\,(m+1)\,n + 8,$$
$$\chi\left(\mathcal{O}_X\right) = \frac{1}{2}mn(n-1) + 1,$$
$$p_g\left(X\right) = \frac{1}{2}mn(n-1),$$
$$q\left(X\right) = 0,$$
$$h^{1,1}\left(X\right) = 3mn^2 + (4-m)\,n + 2.$$

We conclude that $X \in \mathfrak{M}_{K^2,\chi}$ is a canonical model whose minimal resolution has maximal Picard number $h^{1,1}\left(X\right) = 3mn^2 + (4-m)\,n + 2$ by Lemma 2. $\qquad\square$

Proof of Corollary 1. Given $m \in \mathbb{Z}_{\geq 3}$ and $n \in 2 \cdot \mathbb{Z}_{\geq 1}$, let us consider the surface $X_{m,n}$ described in Theorem 2. Then:

$$\mu(X_{m,n}) = \frac{K_{X_{m,n}}^2}{\chi(\mathcal{O}_{X_{m,n}})} = 4 + \frac{4(1 - n - mn)}{1 - n + mn^2}. \tag{1}$$

Defining $X_m^n = Y_n^m = X_{m,n}$, the result easily follows from equation (1). $\qquad\square$

Remark 4. *Let us consider the following subsets of $\mathbb{Z} \times \mathbb{Z}$:*

$$A_1 := \left\{\left(4n^2 - 12n + 9, n^2 - n + 1\right) : n \in \mathbb{Z}_{\geq 2}\right\},$$
$$A_2 := \left\{\left(4mn^2 - 4(m+2)n + 8, mn^2 - n + 1\right) : m \in \mathbb{Z}_{\geq 3}, n \in 2 \cdot \mathbb{Z}_{\geq 1}\right\},$$
$$A_3 := \left\{\left(2mn^2 - 4(m+1)n + 8, \frac{1}{2}mn(n-1) + 1\right) : m \in \mathbb{Z}_{\geq 2}, n \in 2 \cdot \mathbb{Z}_{\geq 2}\right\},$$
$$B := \left\{\left(2(n-3)^2, \frac{1}{2}(n-1)(n-2) + 1\right) : n \in \mathbb{Z}_{\geq 4}\right\}.$$

These sets contain the invariants of the surfaces of Theorem 1, Theorem 2, Theorem 3 and [22, Theorem 3] respectively. We claim that:

i) $A_1 \cap B = A_2 \cap B = A_3 \cap B = A_1 \cap A_2 = A_1 \cap A_3 = \varnothing$.

ii) Both $A_2 \setminus A_3$ and $A_3 \setminus A_2$ contain an infinite amount of pairs.

iii) $A_2 \cap A_3$ contains an infinite amount of pairs.

The equalities $A_1 \cap B = A_2 \cap B = A_3 \cap B = \varnothing$ follow noticing that $\frac{1}{2}K^2$ is a perfect square if $(K^2, \chi) \in B$, which is not the case for pairs $(K^2, \chi) \in A_1 \cup A_2 \cup A_3$.
The equalities $A_1 \cap A_2 = A_1 \cap A_3 = \varnothing$ follow noticing that K^2 is odd for every pair $(K^2, \chi) \in A_1$ whereas K^2 is even for every pair $(K^2, \chi) \in A_2 \cup A_3$.
The fact that $A_3 \setminus A_2$ contains an infinite amount of pairs follows noticing that χ is odd for every pair $(K^2, \chi) \in A_2$ but it is easy to construct an infinite sequence of pairs $(K^2, \chi) \in A_3$ such that χ is even.
In order to prove that $A_2 \setminus A_3$ contains an infinite amount of pairs it suffices to notice that the intersection of A_2 with the Noether line is

$$\{(8m - 8, 4m - 1) : m \in \mathbb{Z}_{\geq 3}\}$$

and the intersection of A_3 with the Noether line is empty.
To prove that $A_2 \cap A_3$ contains an infinite amount of pairs it suffices to show that $A_2 \cap A_3$ contains the set

$$T := \left\{\left(2t(t-1)(t-4) + 8, \frac{1}{2}t(t-1)(t-3) + 1\right) : t \in 2 \cdot \mathbb{Z}_{\geq 3}\right\}.$$

Now, setting $m = \frac{t}{2} - 1$ and $n = t - 1$, we get $T \subseteq A_2$, and setting $m = t - 3$ and $n = t$, we get $T \subseteq A_3$.

Remark 5. *Note that, while some of our examples lie in the region $K^2 \leq \frac{5}{2}\chi - 11$ covered by [6, Theorem 1], we have constructed infinitely many surfaces of general type with maximal Picard number that do not. This claim follows from Corollary 1 taking into account that $K^2/\chi \leq \frac{5}{2}$ if $K^2 \leq \frac{5}{2}\chi - 11$.*

Remark 6. *As Figure 1 suggests, the examples of Theorem 2 and Theorem 3 are arranged in lines. On the one hand, given $n \in 2 \cdot \mathbb{Z}_{\geq 1}$, it is easy to see that the surface $X_{m,n}$ of Theorem 2 lies on the line*

$$K^2 = 4\frac{n-1}{n}\chi - 4\frac{(n+1)(n-1)}{n}$$

for every integer $m \geq 3$.
On the other hand, given an integer $n \in 2 \cdot \mathbb{Z}_{\geq 2}$, it is easy to see that the surface $X_{m,n}$ of Theorem 3 lies on the line

$$K^2 = 4\frac{n-2}{n-1}\chi - 4\frac{n(n-2)}{n-1}$$

for every integer $m \geq 2$.

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