# LIOUVILLE THEOREMS FOR HARMONIC METRICS ON GRADIENT RICCI SOLITONS

CHENGHONG HE, DI WU, AND XI ZHANG

Abstract. In this paper, we prove two Liouville theorems for harmonic metrics on complex flat line bundles on gradient steady Ricci solitons and gradient shrinking Kähler-Ricci solitons, which imply that they arise from fundamental group representations into  $S^1$ .

### 1. INTRODUCTION

As is well-known, the classical Liouville theorem for harmonic functions states that there is no nontrivial positive harmonic function on  $\mathbb{R}^n$ . It is Yau [\[Ya1](#page-9-0)] who successfully extended this theorem to the context of complete Riemannian manifolds with nonnegative Ricci curvatures nearly fifty years ago. For  $p > 1$ , Yau [\[Ya2](#page-9-1)] also proved that any nonnegative  $L^p$  sub-harmonic function on a complete manifold must be a constant. In particular, any harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant if the energy is finite. Since then, the subject of Liouville properties for harmonic functions on complete Riemannian manifolds has witnessed a tremendous growth.

It is natural and interesting to study various Liouville type properties for other partial differential equations and find its applications. This paper initiates the study of Liouville theorems for harmonic metrics on vector bundles. Let  $(E, \nabla)$  be a vector bundle on a Riemannian manifold  $(M, g)$ , where  $\nabla$  is a connection on E. Throughout this paper, unless indicated explicitly otherwise, vector bundles could be real or complex, whose metrics are Riemannian or Hermitian respectively. Motivated by calculating the Euler characteristic number via the Gauss-Bonnet-Chern formula and its ramifications, one may ask: How to find metric compat*ible connections on*  $E$ ? We may write for a metric H on E such that

$$
\nabla = \nabla_H + \psi_H,
$$

where  $\nabla_H$  is a connection preserving H and  $\psi_H$  is an End(E)-valued 1-form. We define

(1.2) 
$$
\mathcal{E}_{\nabla}(H) = \frac{1}{2} \int_M |\psi_H|^2 \, \mathrm{d} \mathrm{vol}_g,
$$

and the point is to minimize  $\mathcal{E}_{\nabla}(H)$  when H varies.

**Definition 1.1.** The critical point H of  $\mathcal{E}_{\nabla}$  is called a harmonic metric and it satisfies

$$
\nabla_H^* \psi_H = 0,
$$

where  $\nabla_H^*$  is the formal adjoint operator of  $\nabla_H$ .

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In general, detecting harmonic metrics on vector bundles is a nonlinear system generalization of solving the Laplace equation and obstruction emerges in a natural way. A basic issue of harmonic metrics is to investigate its existence theory, there is a Riemannian Kobayashi-Hitchin correspondence for harmonic metrics analogous to the celebrated Kobayashi-Hitchin correspondence [\[Do1](#page-8-0), [Do2](#page-8-1), [UY](#page-9-2)] in complex geometry, which was well established by Corlette [\[Co](#page-8-2)] and Donaldson [\[Do3](#page-8-3)] for flat vector bundles in the late 1980s, by Wu-Zhang [\[WZ2](#page-9-3)] in full generality for arbitrary vector bundles until very recently.

For a compact Riemannian manifold, by the Riemannian Kobayashi-Hitchin correspondence and Bochner formula for harmonic metrics, we can conclude that any semi-simple flat vector bundle must arise from a unitary representation of the fundamental group provided the base space admits a metric with nonnegative Ricci curvature, see [\[WZ1\]](#page-9-4) which also considered the noncompact case. On the other hand, recall the famous Chern conjecture predicts that any compact affine manifold has vanishing Euler characteristic. In particular, the Chern conjecture holds for a compact affine manifold whose tangent bundle is semi-simple and which admits a Riemannian metric with nonnegative Ricci curvature.

Motivated by Liouville theorems for harmonic functions on gradient Ricci solitons studied by Munteanu-Sešum [\[MS\]](#page-8-4), the present paper mainly focus on harmonic metrics on gradient Ricci solitons. A noncompact complete gradient Ricci soliton  $(M, g, f)$  consists of a noncompact complete Riemannian manifold  $(M, g)$ , a smooth function f (called the potential function), and a constant  $\rho$  such that

(1.4) 
$$
\operatorname{Ric}_g + \nabla df = \rho g,
$$

where we also denote by  $\nabla$  the Levi-Civita connection on TM. After rescaling the metric we may assume that  $\rho \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$  $\frac{1}{2}$ . Depending on the behavior of the Ricci flow on solitons, they are called shrinking if  $\rho = \frac{1}{2}$  $\frac{1}{2}$ , steady if  $\rho = 0$  and expanding if  $\rho = -\frac{1}{2}$  $\frac{1}{2}$ . Gradient Ricci solitons arise often as singularity models of the Ricci flow and that is why underlying them is an important question in the field.

It is known that Liouville theorems for harmonic functions can be used to investigate the geometry of gradient Ricci solitons at infinity. In this paper, we observe that Liouville theorems for harmonic metrics on complex flat line bundles on gradient Ricci solitons can be used to distinguish line bundles that arise from  $S^1$ -representations of fundamental groups.

<span id="page-1-0"></span>Our main results are stated as follows.

**Theorem 1.1.** Let  $(L, \nabla)$  be a complex flat line bundle on a noncompact complete gradient steady Ricci solition  $(M, g, f)$  and H be a harmonic metric with finite energy, then  $\psi_H = 0$ . In particular,  $\nabla$  is unitary and L arises from a representation of  $\pi_1(M)$  into  $S^1$ .

<span id="page-1-1"></span>**Theorem 1.2.** Let  $(L, \nabla)$  be a complex flat line bundle on a noncompact complete gradient shrinking Kähler-Ricci solition  $(M, g, f)$  and H be a harmonic metric with finite energy, then  $\psi_H = 0$ . In particular,  $\nabla$  is unitary and L arises from a representation of  $\pi_1(M)$  into  $S^1$ .

Similar results also hold for real flat line bundles, even though the above two theorems are stated for complex flat line bundles. Along the way of the proof, among other things, an important notion of pluri-harmonicit should be introduced as follows.

**Definition 1.2.** Let E be a flat vector bundle on a Kähler manifold  $(M, q)$  and  $\nabla$  be a connection on E, a metric H on E is called pluri-harmonic if

(1.5) 
$$
G_H \triangleq (D_H^{0,1} + \psi_H^{1,0})^2 = 0,
$$

where  $\bullet^{p,q}$  denotes the  $(p,q)$ -component of  $\bullet$ .

On a compact Kähler manifold, it is easy to see that  $H$  being a harmonic metric if and only if  $\Lambda G_H = 0$ , where  $\Lambda$  is obtained as the formal adjoint of the multiplication of the associated fundamental form. Moreover, any harmonic metric is automatically pluri-harmonic due to Corlette, Deligne, Sampson, Simpson, Siu, see [\[Co](#page-8-2), [Sa,](#page-9-5) [Sim2,](#page-9-6) [Siu\]](#page-9-7). Crucially, we have the pluri-harmonicity of harmonic metrics on noncompact manifolds in the following sense.

**Proposition 1.1** (See Proposition [2.1\)](#page-6-0). Let  $(E, \nabla)$  be a flat vector bundle on a noncompact complete Kähler manifold  $(M, g)$  and H be a harmonic metric with  $\int_{B_p(R)} |\psi_H|^2 dvol_g = o(R^2)$ as R goes infinity, then H must be pluri-harmonic.

This rigidity phenomenon appears as a nonlinear system analogue of Li's classical result [\[Li\]](#page-8-5) for harmonic functions. It may be mentioned that the pluri-harmonicity of harmonic metrics turns out to be a core property in studying the so-called nonabelian Hodge theory [\[Co](#page-8-2), [Do3,](#page-8-3) [Hi](#page-8-6), [Sim1,](#page-9-8) Sim2 on compact Kähler manifolds and this property can be also investigated in various noncompact contexts by some other methods that presented in [\[JZ](#page-8-7), [Sim1,](#page-9-8) [WZ1\]](#page-9-4).

### 2. Proofs

2.1. Gradient Ricci solitons. Let  $(M, g, f)$  be a complete noncompact gradient Ricci soliton, the drifted Laplacian and Bakry-Emery Ricci curvature are given by ´

(2.1) 
$$
\Delta_{g,f} = \Delta_g - \langle \nabla f, \bullet \rangle, \text{ Ric}_{g,f} = \text{Ric}_g + \nabla df.
$$

In [\[Ha](#page-8-8)], Hamilton showed that the scalar curvature  $Sc_{q}$  satisfies

(2.2) 
$$
\operatorname{Sca}_g + |\nabla f|^2 - 2\rho f = \lambda,
$$

for a constant  $\lambda$ . If  $\rho = 0$  (that is Ric<sub>g</sub> =  $\nabla df$ ), since every steady soliton is in particular an ancient solution to the Ricci flow, we have (see [\[Ca,](#page-8-9) [Ch\]](#page-8-10))

<span id="page-2-0"></span>(2.3) 
$$
\operatorname{Sca}_g \ge 0, \ \lambda \ge 0, \ |\nabla f| \le \sqrt{\lambda}.
$$

On the other hand, Munteanu-Sešum [\[MS\]](#page-8-4) proved the volume growth

<span id="page-2-1"></span>(2.4) 
$$
\frac{r}{c_1} \leq \text{vol}_g(B_p(R)) \leq c_2 e^{\sqrt{R}}, R > R_0,
$$

for three uniform positive constants  $c_1, c_2, R_0$ . Next let us consider the case  $\rho = \frac{1}{2}$  $rac{1}{2}$  (that is  $\text{Ric}_g = \frac{1}{2}\nabla df$ , after normalizing we have

(2.5) 
$$
Scag + |\nabla f|^2 = \lambda.
$$

It is known from [\[Ca](#page-8-9), [Ch](#page-8-10)] that  $Scag \geq 0$ . On the other hand, Cao-Zhou [\[CZ](#page-8-11)] proved that for any fixed  $p \in M$ , we have

<span id="page-2-3"></span>(2.6) 
$$
\frac{1}{4}(\text{dist}(\bullet, p) - c_3)^2 \le f \le \frac{1}{4}(\text{dist}(\bullet, p) + c_3)^2,
$$

for a uniform constant  $c_3$ . In particular, it holds

<span id="page-2-2"></span>(2.7) 
$$
|\nabla f| \leq \frac{1}{2}(\text{dist}(\bullet, p) + c_3).
$$

2.2. Connections on vector bundles. We assume  $(E, \nabla)$  is a vector bundle on a Riemannian manifold  $(M, g)$  and H is a metric on E. We set  $A^p(M, E) = \Gamma(\Lambda^p T^* M \otimes E)$ , the space of p-forms on M with values in E. Associated to  $\nabla$ , the exterior differential operator

(2.8) 
$$
D: A^p(M, E) \to A^{p+1}(M, E)
$$

is defined by requiring  $D(\alpha \otimes u) = d\alpha \otimes u + (-1)^p \alpha \wedge \nabla u$  for any  $\alpha \in A^p(M)$  and  $u \in \Gamma(E)$ . Then the curvature is given by  $F_{\nabla} = D \circ \nabla \in A^2(M, \text{End}(E))$ . Using the Levi-Civita connection (also denoted by  $\nabla$  for simplicity), the action of  $\nabla$  may be further extended to tensorial combinations of  $TM$  and  $E$  as well as their duals. Then it holds

(2.9) 
$$
D\omega(e_0, ..., e_p) = \sum_{k=0}^p (-1)^k \nabla_{e_k} \omega(e_0, ..., \hat{e_k}, ..., e_p),
$$

where  $e_1, ..., e_p \in \Gamma(TM)$  and symbols covered by  $\wedge$  are omitted. For a metric H on E, we have the pointwise inner-product

(2.10) 
$$
\langle \cdot, \cdot \rangle: A^p(M, E) \times A^p(M, E) \to C^\infty(M),
$$

(2.11) 
$$
(\omega, \theta) \mapsto \sum_{i_1 < ... < i_p} H(\omega(e_{i_1}, ..., e_{i_p}), \theta(e_{i_1}, ..., e_{i_p})),
$$

where  $\{e_i\}_{i=1}^{\dim M}$  is an orthogonal unit basis of TM. Henceforth, we will omit the subscript H if there is no ambiguity. The generalized inner-product

(2.12) 
$$
\langle \cdot, \cdot \rangle: A^p(M, E) \times A^q(M, E) \to A^{p+q}(M)
$$

is defined by requiring  $\langle \alpha \otimes u, \beta \otimes v \rangle = \langle u, v \rangle \alpha \wedge \beta$  for any  $\alpha \in A^p(M)$ ,  $\beta \in A^q(M)$  and  $u, v \in \Gamma(E)$ . In particular, we have  $\langle \omega, *{\theta} \rangle = \langle \omega, {\theta} \rangle$  dvolg for  $\omega, {\theta} \in A^p(M, E)$ , where ∗ is the Hodge star operator acting on the form component. If the base space is a K¨ahler manifold, it is emphasized that the inner product on the complication tangent bundle we shall use is Hermitian. The connection  $\nabla$  is called unitary if

$$
(2.13) \t d < u, v> = <\nabla u, v> + ,
$$

where  $u, v \in \Gamma(E)$ . The co-differential operator

(2.14) 
$$
D_H^* : A^p(M, E) \to A^{p-1}(M, E)
$$

is determined by  $D_H$  (the exterior differential operator of  $\nabla_H$ ) via

(2.15) 
$$
\int_M (D_H \omega, \theta) \, \text{dvol}_g = \int_M (\omega, D_H^* \theta) \, \text{dvol}_g,
$$

where  $\omega \in A^{p-1}(M, E), \theta \in A^p(M, E)$  and one of them is compactly supported. It holds

(2.16) 
$$
D_H^* \theta(e_1, ..., e_{p-1}) = -\operatorname{tr}_g \nabla_{H, \bullet} \theta(\bullet, e_1, ..., e_{p-1}),
$$

where for notational simplicity,  $\nabla_H$  also denotes the connection on  $\Lambda^pT^*M \otimes E$  which acts on  $\Lambda^p T^*M$  by the Levi-Civita connection  $\nabla$  and acts on E by  $\nabla_H$ .

2.3. Flat vector bundles. If  $\nabla$  is a flat connection on a vector bundle E, decomposing  $F_{\nabla}$ into Hermitian part and anti-Hermitian part, we have

(2.17) 
$$
D_H \psi_H = 0, \ F_{\nabla_H} + \psi_H \wedge \psi_H = 0.
$$

In particular, we have

(2.18) 
$$
D_H^{1,0} \psi_H^{1,0} = 0, \ D_H^{1,0} \psi_H^{0,1} + D_H^{0,1} \psi_H^{1,0} = 0,
$$

(2.19) 
$$
D_H^{1,0} D_H^{1,0} + \psi_H^{1,0} \wedge \psi_H^{1,0} = 0, \ [D_H^{1,0}, D_H^{0,1}] + [\psi_H^{1,0}, \psi_H^{0,1}] = 0.
$$

Note  $\psi_H \in A^1(M, \text{End}(E))$ , hence  $\text{tr } \psi_H \in A^1(M)$  and  $\text{tr } \psi_H \in \Gamma(TM)$ , where  $\hat{\bullet}$  is the dual of • via g. Let p be a point of the base manifold M and  $\pi_1(M)$  be the fundamental group with base point p. Since  $\nabla$  is flat, the parallel transport along a closed curve starting at p depends only on its homotopy class and it gives a representation  $\rho : \pi_1(M) \to GL(r)$  or  $\rho : \pi_1(M) \to U(r)$  if  $\nabla$  is unitary, where  $r = \text{rank}(E)$ . Conversely, given a representation  $\rho: \pi_1(M) \to GL(r)$ , it corresponds to  $E = \tilde{M} \times_{\rho} \mathbb{C}^r$  given by the universal covering space  $\tilde{M}$ 

and the action of  $[c] \in \pi_1(M)$  on  $(q, v) \in \tilde{M} \times \mathbb{C}^r$  via  $[c] \cdot (q, v) = ([c] \cdot q, \rho([c])(v))$ . Then we say the flat vector bundle E arises from the fundamental group representation  $\rho$ .

In the following proof, we let  $\eta : M \to [0,1]$  be a cut-off function such that  $\eta = 1$  on a  $B_p(R)$ ,  $\eta = 0$  outside  $B_p(2R)$  and  $|\nabla \eta| \leq CR^{-1}$  for a constant C.

## 2.4. Proof of Theorem [1.1.](#page-1-0) Firstly, we apply the divergence theorem to get

<span id="page-4-4"></span>
$$
0 = \int_M \text{div}(\eta^2 < df, \text{tr } \psi_H > \text{tr } \hat{\psi}_H) \, \text{dvol}_g
$$
\n
$$
(2.20) \qquad = \int_M < df, \text{tr } \psi_H > \sqrt{\eta^2}, \text{tr } \hat{\psi}_H > \text{dvol}_g + \int_M \eta^2 < \sqrt{\eta^2}, \text{tr } \psi_H > \text{dvol}_g + \int_M \eta^2 < \sqrt{\eta^2}, \text{tr } \psi_H > \text{dvol}_g + \int_M \eta^2 < \sqrt{\eta^2}, \text{tr } \psi_H > \text{dvol}_g + \int_M \eta^2 < df, \text{tr } \psi_H > \text{dvol}_g.
$$

Since  $H$  is a harmonic metric, we have

<span id="page-4-1"></span>(2.21)  
\n
$$
0 = \text{tr}\,\nabla_H^* \psi_H
$$
\n
$$
= -\text{tr}\,\text{tr}_g \nabla_H \psi_H
$$
\n
$$
= -\text{div}\,\text{tr}\,\psi_H
$$
\n
$$
= -\text{div}\,\text{tr}\,\hat{\psi}_H.
$$

Using the steady soliton equation Ric<sub>g</sub> =  $\nabla df$ , we get

<span id="page-4-2"></span>(2.22) 
$$
\langle \nabla df, \operatorname{tr} \psi_H \rangle, \operatorname{tr} \psi_H \rangle = \langle \nabla \operatorname{Ric}_g, \operatorname{tr} \psi_H \rangle, \operatorname{tr} \psi_H \rangle
$$

$$
= \operatorname{Ric}_g(\operatorname{tr} \hat{\psi}_H, \operatorname{tr} \hat{\psi}_H).
$$

We observe that  $D_H \psi_H$  is the anti-symmetrization of the covariant derivative on  $\Lambda^2 T^* M \otimes$ End(E) and thus the flatness of  $\nabla$  yields for any two vector fields  $X, Y$ ,

<span id="page-4-0"></span>(2.23) 
$$
0 = D_H \psi_H(X, Y)
$$

$$
= \nabla_{H,X} \psi_H(Y) - \nabla_{H,Y} \psi_H(X).
$$

On the other hand, we have

(2.24) 
$$
\langle \langle df, \nabla \operatorname{tr} \psi_H \rangle, \operatorname{tr} \psi_H \rangle = \langle \langle df, \operatorname{tr} \nabla_H \psi_H \rangle, \operatorname{tr} \psi_H \rangle,
$$

(2.25) 
$$
\langle \nabla \operatorname{tr} \psi_H, \operatorname{tr} \psi_H \rangle, df \rangle = \langle \nabla \operatorname{tr} \psi_H, \operatorname{tr} \nabla_H \psi_H \rangle, df \rangle.
$$

By the symmetry (2.[23\)](#page-4-0), it follows

<span id="page-4-3"></span>(2.26) 
$$
\langle \langle \, d f, \nabla \, \text{tr} \, \psi_H \rangle, \text{tr} \, \psi_H \rangle = \langle \langle \, \nabla \, \text{tr} \, \psi_H, \text{tr} \, \psi_H \rangle, \text{d} f \rangle.
$$

Using (2.[21\)](#page-4-1), (2.[22\)](#page-4-2), (2.[26\)](#page-4-3) on (2.[20\)](#page-4-4), we arrive at

<span id="page-4-5"></span>(2.27) 
$$
0 = \int_M  <\nabla \eta^2, \text{tr } \hat{\psi}_H > \text{dvol}_g + \int_M \eta^2 \text{Ric}_g(\text{tr } \hat{\psi}_H, \text{tr } \hat{\psi}_H) \text{dvol}_g + \int_M \eta^2 << \nabla \text{ tr } \psi_H, \text{tr } \psi_H >, df > \text{dvol}_g.
$$

Secondly, applying the divergence theorem again implies

<span id="page-4-6"></span>(2.28) 
$$
0 = \int_M \text{div}(\eta^2 |\operatorname{tr} \psi_H|^2 \nabla f) \, \text{dvol}_g
$$

$$
= 2 \int_M \eta^2 \langle \nabla \operatorname{tr} \psi_H, \operatorname{tr} \psi_H \rangle, df \rangle \, \text{dvol}_g + \int_M \eta^2 |\operatorname{tr} \psi_H|^2 \Delta_g f \, \text{dvol}_g
$$

$$
+ \int_M |\operatorname{tr} \psi_H|^2 \langle \nabla \eta^2, \nabla f \rangle \, \text{dvol}_g.
$$

Denote by  $Ric^{\#}_{g}$  the Ricci transformation, now we derive the Bochner formula

<span id="page-5-0"></span>
$$
\Delta_g |\psi_H|^2 = -2 < \nabla_H^* \nabla_H \psi_H, \psi_H > +2|\nabla_H \psi_H|^2
$$
  
\n
$$
= -2g^{ij} < F_{\nabla_H}(\cdot, \frac{\partial}{\partial x^i})(\psi_H(\frac{\partial}{\partial x^j})), \psi_H >
$$
  
\n
$$
+ 2g^{ij} < \psi_H(F_{\nabla}(\cdot, \frac{\partial}{\partial x^i})(\frac{\partial}{\partial x^j})), \psi_H > +2|\nabla_H \psi_H|^2
$$
  
\n(2.29)  
\n
$$
= g^{ij} < [[\psi_H, \psi_H(\frac{\partial}{\partial x^i})], \psi_H(\frac{\partial}{\partial x^j})], \psi_H >
$$
  
\n
$$
+ 2 < \psi_H \circ \text{Ric}_g^{\#}, \psi_H > +2|\nabla_H \psi_H|^2
$$
  
\n
$$
= |[\psi_H, \psi_H]|^2 + 2 < \psi_H \circ \text{Ric}_g^{\#}, \psi_H > +2|\nabla_H \psi_H|^2
$$
  
\n
$$
\geq 2 \text{Ric}_g(\text{tr}\,\hat{\psi}_H, \text{tr}\,\hat{\psi}_H) + 2|\nabla|\psi_H||^2,
$$

where we have used the harmonicity of H, the flatness of  $\nabla$  and rank $(L) = 1$ . Gathering (2.[27\)](#page-4-5), (2.[28\)](#page-4-6), (2.[29\)](#page-5-0), we deduce

$$
2\int_{M} \eta^{2} |\nabla |\psi_{H}||^{2} \text{dvol}_{g}
$$
  
\n
$$
\leq -\int_{M} \langle \nabla \eta^{2}, \nabla |\psi_{H}|^{2} \rangle \text{dvol}_{g} - 2\int_{M} \eta^{2} \text{Ric}_{g}(\text{tr}\,\hat{\psi}_{H}, \text{tr}\,\hat{\psi}_{H}) \text{dvol}_{g}
$$
  
\n
$$
= -\int_{M} \langle \nabla \eta^{2}, \nabla |\psi_{H}|^{2} \rangle \text{dvol}_{g} + 2\int_{M} \langle \text{df}, \text{tr}\,\psi_{H} \rangle \langle \nabla \eta^{2}, \text{tr}\,\hat{\psi}_{H} \rangle \text{dvol}_{g}
$$
  
\n
$$
+ 2\int_{M} \eta^{2} \langle \nabla \nabla \text{tr}\,\psi_{H}, \text{tr}\,\psi_{H} \rangle, df \rangle \text{dvol}_{g}
$$
  
\n
$$
= -\int_{M} \langle \nabla \eta^{2}, \nabla |\psi_{H}|^{2} \rangle \text{dvol}_{g} + 2\int_{M} \langle \text{df}, \text{tr}\,\psi_{H} \rangle \langle \nabla \eta^{2}, \text{tr}\,\hat{\psi}_{H} \rangle \text{dvol}_{g}
$$
  
\n
$$
- \int_{M} \eta^{2} |\text{tr}\,\psi_{H}|^{2} \Delta_{g} f \text{dvol}_{g} - \int_{M} |\text{tr}\,\psi_{H}|^{2} \langle \nabla \eta^{2}, \nabla f \rangle \text{dvol}_{g}
$$
  
\n
$$
\leq 4 \int_{M} \eta |\nabla \eta| |\psi_{H}| |\nabla |\psi_{H}| |\text{dvol}_{g} + C' \int_{M} |\nabla \eta| |\nabla f| |\psi_{H}|^{2} \text{dvol}_{g}
$$
  
\n
$$
- \int_{M} \eta^{2} |\text{tr}\,\psi_{H}|^{2} \text{Sca}_{g} \text{dvol}_{g},
$$

for a constant  $C'$ . The Cauchy-Schwarz inequality gives

$$
(2.31) \qquad 4\int_M \eta |\nabla \eta| |\psi_H| |\nabla |\psi_H| | \, \mathrm{dvol}_g \le \int_M \eta^2 |\nabla |\psi_H| |^2 \, \mathrm{dvol}_g + 4 \int_M |\nabla \eta|^2 |\psi_H|^2 \, \mathrm{dvol}_g.
$$

The above two inequalities imply

$$
\int_M \eta^2 |\nabla |\psi_H||^2 \operatorname{dvol}_g + \int_M \eta^2 |\operatorname{tr} \psi_H|^2 \operatorname{Sca}_g \operatorname{dvol}_g
$$
\n
$$
\leq 4 \int_M |\nabla \eta|^2 |\psi_H|^2 \operatorname{dvol}_g + C' \int_M |\nabla \eta| |\nabla f| |\psi_H|^2 \operatorname{dvol}_g
$$
\n
$$
\leq R^{-1} C (4 + C' \sqrt{\lambda}) \int_{B_p(2R)\backslash B_p(R)} |\psi_H|^2 \operatorname{dvol}_g,
$$

where we have used the uniform bound of  $|\nabla f|$  (see (2.[3\)](#page-2-0)) and assumed R large enough. By letting R goes infinity and noting  $Scag \geq 0$  (see [\(2](#page-2-0).3)), we find  $|\psi_H|$  is constant. Since the volume is infinite (see (2.[4\)](#page-2-1)), we conclude  $\psi_H = 0$  as  $|\psi_H| \in L^2$ .

# <span id="page-6-0"></span>2.5. Proof of Theorem [1.2.](#page-1-1)

**Proposition 2.1.** Let  $(E, \nabla)$  be a flat vector bundle on a noncompact complete Kähler manifold  $(M, g)$  and H be a harmonic metric with  $\int_{B_p(R)} |\psi_H|^2 \, dvol_g = o(R^2)$  as R goes infinity, then H must be pluri-harmonic.

Proof. Firstly, by 
$$
D_H^{0,1} D_H^{0,1} + \psi_H^{0,1} \wedge \psi_H^{0,1} = 0
$$
, we have  
\n
$$
\operatorname{tr} G_H \wedge G_H = 2 \operatorname{tr} (G_H^{2,0} \wedge G_H^{0,2}) + \operatorname{tr} (G_H^{1,1} \wedge G_H^{1,1})
$$
\n
$$
= 2 \operatorname{tr} (\psi_H^{1,0} \wedge \psi_H^{1,0} \wedge D_H^{0,1} D_H^{0,1}) + \operatorname{tr} (D_H^{0,1} \psi_H^{1,0} \wedge D_H^{0,1} \psi_H^{1,0})
$$
\n
$$
= -2 \operatorname{tr} (\psi_H^{1,0} \wedge \psi_H^{1,0} \wedge \psi_H^{0,1} \wedge \psi_H^{0,1}) + \overline{\partial} \operatorname{tr} (\psi_H^{1,0} \wedge D_H^{0,1} \psi_H^{1,0})
$$
\n(2.33)\n
$$
+ \operatorname{tr} (\psi_H^{1,0} \wedge D_H^{0,1} D_H^{0,1} \psi_H^{1,0})
$$
\n
$$
= -2 \operatorname{tr} (\psi_H^{1,0} \wedge \psi_H^{1,0} \wedge \psi_H^{0,1} \wedge \psi_H^{0,1}) + \overline{\partial} \operatorname{tr} (\psi_H^{1,0} \wedge D_H^{0,1} \psi_H^{1,0})
$$
\n
$$
- \operatorname{tr} (\psi_H^{1,0} \wedge [\psi_H^{0,1} \wedge \psi_H^{0,1}, \psi_H^{1,0}])
$$
\n
$$
= \overline{\partial} \operatorname{tr} (\psi_H^{1,0} \wedge D_H^{0,1} \psi_H^{1,0}).
$$

Similar calculation was also used in [\[Co\]](#page-8-2). Set  $\omega_g$  to be the Kähler form, we arrive at

<span id="page-6-1"></span>
$$
\int_{M} \eta^{2} \operatorname{tr}(G_{H} \wedge G_{H}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!} = \int_{M} \eta^{2} \overline{\partial} \operatorname{tr}(\psi_{H}^{1,0} \wedge D_{H}^{0,1} \psi_{H}^{1,0}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!}
$$
\n
$$
= \int_{M} \eta^{2} \overline{\partial} \left( \operatorname{tr}(\psi_{H}^{1,0} \wedge D_{H}^{0,1} \psi_{H}^{1,0}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!} \right)
$$
\n
$$
= - \int_{M} 2\eta \overline{\partial} \eta \wedge \operatorname{tr}(\psi_{H}^{1,0} \wedge D_{H}^{0,1} \psi_{H}^{1,0}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!}
$$
\n
$$
\leq C'' \int_{M} |\eta| |\nabla \eta| |\psi_{H}^{1,0}| |D_{H}^{0,1} \psi_{H}^{1,0}| \operatorname{d} \mathrm{vol}_{g}
$$
\n
$$
\leq C'' \int_{M} (\epsilon \eta^{2} |D_{H}^{0,1} \psi_{H}^{1,0}|^{2} + \frac{1}{4\epsilon} |\nabla \eta|^{2} |\psi_{H}^{1,0}|^{2}) \operatorname{d} \mathrm{vol}_{g}
$$
\n
$$
\leq C'' \int_{M} (\epsilon \eta^{2} |G_{H}|^{2} + \frac{1}{\epsilon} |\nabla \eta|^{2} |\psi_{H}|^{2}) \operatorname{d} \mathrm{vol}_{g},
$$

for a constant  $C''$  and any constant  $\epsilon > 0$ .

On the other hand, we also have

(2.35) 
$$
(G_H^{1,1})^* = D_H^{1,0} \psi_H^{0,1} = -G_H^{1,1}, \ (G_H^{0,2})^* = -D_H^{1,0} D_H^{1,0} = G_H^{2,0}.
$$

Hence it holds

<span id="page-6-2"></span>
$$
\int_{M} \eta^{2} \operatorname{tr}(G_{H} \wedge G_{H}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!}
$$
\n
$$
= -\int_{M} \eta^{2} \operatorname{tr}(G_{H}^{1,1} \wedge (G_{H}^{1,1})^{*}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!} + \int_{M} 2\eta^{2} \operatorname{tr}(G_{H}^{2,0} \wedge (G_{H}^{2,0})^{*}) \wedge \frac{\omega_{g}^{n-2}}{(n-2)!}
$$
\n(2.36)\n
$$
= \int_{M} \eta^{2} (|G_{H}^{1,1}|^{2} - |\Lambda G_{H}^{1,1}|^{2}) \operatorname{dvol}_{g} + \int_{M} 2\eta^{2} \operatorname{tr}(G_{H}^{2,0} \wedge \ast (G_{H}^{2,0})^{*})
$$
\n
$$
= \int_{M} \eta^{2} (|G_{H}^{1,1}|^{2} - |\Lambda G_{H}|^{2}) \operatorname{dvol}_{g} + \int_{M} 2\eta^{2} |G_{H}^{2,0}|^{2} \operatorname{dvol}_{g}
$$
\n
$$
= \int_{M} \eta^{2} |G_{H}|^{2} \operatorname{dvol}_{g}.
$$

Taking  $\epsilon = (2C^{''})^{-1}$ , it follows from (2.[34\)](#page-6-1) and (2.[36\)](#page-6-2) that

(2.37) 
$$
\int_M \eta^2 |G_H|^2 \, \text{dvol}_g \le 4(C'')^2 \int_M |\nabla \eta|^2 |\psi_H|^2 \, \text{dvol}_g
$$

$$
\le 4R^{-2} (CC'')^2 \int_{B_p(2R)\backslash B_p(R)} |\psi_H|^2 \, \text{dvol}_g,
$$

By letting r goes infinity, we conclude  $G_H = 0$ .

Thanks to Proposition [2.1,](#page-6-0) it holds

<span id="page-7-0"></span>(2.38) 
$$
D_H^{1,0} \psi_H^{0,1} = 0, \ D_H^{0,1} \psi_H^{1,0} = 0.
$$

The property of Ricci curvature of a Kähler manifold and the equation  $Ric_g + \nabla df = g$  imply

 $\sim$ 

<span id="page-7-1"></span>(2.39) 
$$
\nabla^{1,0}\partial f = 0, \ \nabla^{0,1}\overline{\partial} f = 0.
$$

We set  $F = g(df, tr \psi_H)$  and it can be written as

(2.40) 
$$
F = g(\partial f, \text{tr}\,\psi_H^{0,1}) + g(\overline{\partial} f, \text{tr}\,\psi_H^{1,0})
$$

$$
\triangleq F_1 + F_2.
$$

For  $Z \in T^{1,0}M$ , using (2.[38\)](#page-7-0) and (2.[39\)](#page-7-1), we deduce

(2.41)  
\n
$$
\partial_Z F_1 = g(\nabla_Z \partial f, \text{tr } \psi_H^{0,1}) + g(\partial f, \nabla_Z \text{ tr } \psi_H^{0,1})
$$
\n
$$
= g(\nabla_Z \partial f, \text{tr } \psi_H^{0,1}) + g(\partial f, \text{tr } \nabla_{H,Z} \psi_H^{0,1})
$$
\n
$$
= g(\nabla_Z \partial f, \text{tr } \psi_H^{0,1}) + g(\partial f, \text{tr } D_{H,Z} \psi_H^{0,1})
$$
\n
$$
= 0,
$$
\n
$$
\overline{\partial_Z} F_2 = g(\nabla_Z \overline{\partial} f, \text{tr } \psi_H^{1,0}) + g(\overline{\partial} f, \nabla_Z \text{tr } \psi_H^{1,0})
$$
\n
$$
= g(\nabla_Z \overline{\partial} f, \text{tr } \psi_H^{1,0}) + g(\overline{\partial} f, \text{tr } \nabla_{H,Z} \psi_H^{1,0})
$$
\n
$$
= g(\nabla_Z \overline{\partial} f, \text{tr } \psi_H^{1,0}) + g(\overline{\partial} f, \text{tr } D_{H,Z} \psi_H^{1,0})
$$

 $= 0.$ 

Therefore  $\partial \overline{\partial} F = -\overline{\partial} \partial F_1 + \partial \overline{\partial} F_2 = 0$  and it yields

$$
\int_M |\nabla F|^2 \eta^2 d\text{vol}_g = -\int_M \Delta_g F F \eta^2 d\text{vol}_g - \int_M F < \nabla F, \nabla \eta^2 > \text{dvol}_g
$$
\n
$$
\leq \int_M 2\eta |F| |\nabla F| |\nabla \eta| d\text{vol}_g
$$
\n
$$
\leq \frac{1}{2} \int_M |\nabla F|^2 \eta^2 d\text{vol}_g + 2 \int_M |F|^2 |\nabla \eta|^2 d\text{vol}_g.
$$

From this and the fact that  $|\nabla f|$  grows linearly (see [\(2](#page-2-2).7)), we arrive at

$$
\int_M |\nabla F|^2 \eta^2 d\mathrm{vol}_g \le C'''\int_M |\nabla \eta|^2 |\psi_H|^2 |\nabla f|^2 d\mathrm{vol}_g
$$
\n
$$
\le \frac{CC'''(\mathrm{dist}(\bullet, p) + c_3)^2}{4R^2} \int_{B_p(2R)\backslash B_p(R)} |\psi_H|^2 d\mathrm{vol}_g,
$$

for a constant  $C'''$ . By letting R goes infinity we conclude  $\nabla F = 0$  and hence F is a constant. In fact, the asymptotic behavior  $(2.6)$  $(2.6)$  of f guarantees that it attains its minimum somewhere on a compact subset of  $M$ , so  $F = 0$  somewhere and hence everywhere.

Furthermore, recall that

(2.45) 
$$
\Delta_{g,f} |\psi_H|^2 = 2 < \psi_H \circ \text{Ric}_g^{\#}, \psi_H > + |[\psi_H, \psi_H]|^2 + 2|\nabla_H \psi_H|^2 - < \nabla f, \nabla |\psi_H|^2 > .
$$
 Note for line bundles, it holds

$$
2 < \psi_H \circ \text{Ric}_{g}^{\#}, \psi_H > - < \nabla f, \nabla |\psi_H|^2 >
$$
  
\n
$$
= 2 < \psi_H \circ \text{Ric}_{g,f}^{\#}, \psi_H > -2 < \psi_H \circ (\nabla df)^{\#}, \psi_H > - < \nabla f, \nabla |\psi_H|^2 >
$$
  
\n
$$
= |\psi_H|^2 - 2 < \psi_H \circ (\nabla df)^{\#}, \psi_H > -2 < df, < \nabla_H \psi_H, \psi_H >>
$$
  
\n
$$
= |\psi_H|^2 - 2 < \text{tr } \psi_H \circ (\nabla df)^{\#}, \text{tr } \psi_H > -2 < df, < \text{tr } \nabla_H \psi_H, \text{tr } \psi_H >>
$$
  
\n
$$
= |\psi_H|^2 - 2 < \text{tr } \psi_H, \nabla df > + < \nabla \text{tr } \psi_H, df >, \text{tr } \psi_H >
$$
  
\n
$$
= |\psi_H|^2 - 2 < df, \text{tr } \psi_H >
$$
  
\n
$$
= |\psi_H|^2,
$$

where we have used  $\text{Ric}_{g,f} = \frac{1}{2}g$  and (2.[26\)](#page-4-3). It follows

(2.47) 
$$
\Delta_{g,f} |\psi_H|^2 = 2|\psi_H|\Delta_{g,f}|\psi_H| + 2|\nabla|\psi_H||^2 = |\psi_H|^2 + 2|\nabla_H \psi_H|^2 \ge |\psi_H|^2 + 2|\nabla|\psi_H||^2,
$$

so we arrive at  $2\Delta_{q,f} |\psi_H| \geq |\psi_H|$ . We find  $\psi_H = 0$  by noting

(2.48) 
$$
\int_M |\psi_H|^2 e^{-f} \operatorname{dvol}_g \le \int_M |\psi_H|^2 \operatorname{dvol}_g < \infty,
$$

and applying Yau's Liouville theorem with respect to drifted Laplacian (see [\[Ya2](#page-9-1), [Na](#page-8-12), [PW\]](#page-9-9)).

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Chenghong He, School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, People's Republic of China

Email address: Hech@njust.edu.cn

Di Wu, School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, People's Republic of China

Email address: wudi@njust.edu.cn

Xi Zhang, School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, People's Republic of China

Email address: mathzx@njust.edu.cn