### Two models of sparse and clustered dynamic networks

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### Abstract

We present two models of sparse dynamic networks that display transitivity - the tendency for vertices sharing a common neighbour to be neighbours of one another. Our first network is a continuous time Markov chain  $G = \{G_t = (V, E_t), t \ge 0\}$  whose states are graphs with the common vertex set  $V = \{1, \ldots, n\}$ . The transitions are defined as follows. Given t, the vertex pairs  $\{i, j\} \subset V$  are assigned independent exponential waiting times  $A_{ij}$ . At time  $t + \min_{ij} A_{ij}$  the pair  $\{i_0, j_0\}$  with  $A_{i_0j_0} = \min_{ij} A_{ij}$  toggles its adjacency status. To mimic clustering patterns of sparse real networks we set intensities  $a_{ij}$  of exponential times  $A_{ij}$  to be negatively correlated with the degrees of the common neighbours of vertices i and j in  $G_t$ . Another dynamic network is based on a latent Markov chain  $H = \{H_t = (V \cup W, E_t), t \ge 0\}$ whose states are bipartite graphs with the bipartition  $V \cup W$ , where  $W = \{1, \ldots, m\}$  is an auxiliary set of attributes/affiliations. Our second network  $G' = \{G'_t = (E'_t, V), t \ge 0\}$  is the affiliation network defined by H: vertices  $i_1, i_2 \in V$  are adjacent in  $G'_t$  whenever  $i_1$  and  $i_2$ have a common neighbour in  $H_t$ . We analyze geometric properties of both dynamic networks at stationarity and show that networks possess high clustering. They admit tunable degree distribution and clustering coefficients.

### 1 Introduction

Many real networks, especially those depicting human interaction, like social networks of friendships, collaboration networks, citation networks and other show clustering, the propensity of nodes to cluster together by forming relatively small groups with a high density of ties within a group. Clustering is closely related to network transitivity, the tendency for two nodes sharing a common neighbour to be neighbors of one another thus forming a triangle of connections. Locally, in a vicinity of a node, this tendency can be quantified by the probability that two randomly selected neighbours of the node are adjacent. The network average of this probability, called the (average) local clustering coefficient, is used to quantify the network transitivity. Another popular measure of network transitivity, the global clustering coefficient, is the probability that two randomly selected neighbours of a randomly selected node are adjacent. In many social networks both clustering coefficients are on the order of tens of persent while the edge density, the probability that two randomly selected nodes are adjacent, is of much smaller order. Often the edge density scales as  $n^{-1}$ , where n is the number of nodes in the network. We call networks with such edge densities sparse.

Mathematical modelling of sparse networks displaying clustering/transitivity has attracted considerable attention in the literature, see e.g., [14] and references therein. We briefly review several approaches to modeling of clustered networks. In order to enhance the number of triangles in an evolving locally tree-like network Holme and Kim [15] suggested inserting additional edges that close desired fraction of open triangles (paths of lenght two). Newman [20] generalised the configuration random graph model by prescribing network nodes numbers of triangles they participate in. In this way a predefined number of triangles can be introduced into configuration random graph. Bollobás et al. [6] built a clustered network by taking a union of randomly located small dense subgraphs of variable sizes. Guillaume and Latapy [11] noted an underlying bipartite structure present in many social networks, where nodes (actors) sharing a common hobby or affiliation are more likely to become friends, and where each hobby/affiliation defines a tightly connected cluster of actors related to it. They suggested modelling a clustered network by first linking actors to affiliations and then connecting actors that share common affiliations, see also [2], [13]. We call such networks affiliation networks.

The present paper is devoted to the modelling of sparse and clustered dynamic networks using Markov chains. By dynamic network we mean a collection of random graphs  $\{G_t =$  $(V, E_t), t \ge 0$  sharing the same vertex set  $V = \{1, \ldots, n\}$  and having random edge sets  $E_t, t \geq 0$ . We present two stationary random processes  $\{G_t, t \geq 0\}$  with tunable degree distribution and tunable non-vanishing clustering coefficients. Our study is build upon earlier work on dynamic network Markov chains [10], [24], [26]. We mention that network Markov chain of [26] is composed of  $\binom{n}{2}$  independent Markov chains defining the adjacency status of each vertex pair  $\{i, j\} \subset V$  individually (we refer to Section 2 for details). The network admits tunable edge density and degree distribution, but since the edges are inserted/deleted independently of each other it does not show clustering. Grindrod et al [10] and Užupytė and Witt [24] introduced transitivity into the network Markov chain by relating the birth/death rate of an edge to the number of triangles it participates in (cf. [15]). More preciselly, they set the birth (death) rate of an edge  $i \sim i$  to be an affine function of the number of the common neighbours of vertices i and j. Here  $i \sim j$  means that i and j are adjacent. A drawback of the models of [10], [24] is that for large n they have a little control over the edge density and clustering strength.

In the present paper we suggest a remedy to this drawback. Inspired by clustering patterns observed in real networks, where the number of closed triangles incident to a vertex negatively correlates with the degree of the vertex ([7], [21], [22], [25]) we set the birth rate of an edge  $i \sim j$  to be negatively correlated with the degrees of the common neighbours of i and j. We show below that such a modification leads to a stationary dynamic network model admiting tunable edge density and clustering coefficients.

Another dynamic clustered network considered in this paper is a stationary affiliation network built upon an underlying bipartite graph valued Markov chain with independent edges. Now the clustering property is caused by the bipartite structure as noted in [11]. We analyse the degree sequence and global clustering coefficient at stationarity using the tools developed for random intersection graphs [2]. We note that earlier work on dynamic affiliation network models ([3], [4], [12]) addresses the case where the network size n = n(t)increases with time. Clearly, such networks do not admit stationary distributions.

Finally, we mention the recent work by Milewska et al. [19], where a sparse and clustered dynamic network is constructed by taking unions of small dense subgraphs that are inserted/deleted at random times (cf. [6]).

The rest of the paper is organized as follows. In section 2 we formally define the network Markov chain and analyze geometric properties of the network analytically and by numerical simulations. In section 3 we define stationary affiliation network and show the degree distribution and global clustering coefficient. Proofs of the results of section 3 are given in Appendix.

## 2 Network Markov chain

Let  $\mathbf{G} = \{G_t = (V, E_t), t \ge 0\}$  be a continuous time Markov chain, whose states are graphs on the vertex set V and transitions are defined as follows. Given  $G_t$  (the state occupied at time t), the update takes place at time  $t' := t + \min_{ij} A_{ij}$ , where  $A_{ij} = A_{ij}(G_t)$ ,  $\{i, j\} \in V$ are independent exponential waiting times with intensities  $a_{ij} = a_{ij}(G_t)$  defined below. The pair  $\{i_0, j_0\}$  with  $A_{i_0j_0} = \min_{ij} A_{ij}$  changes its adjacency status: the edge  $i_0 \sim j_0$  is inserted if it is not present at time t; the edge  $i_0 \sim j_0$  is removed if it is present at time t. Thus, at time t' the Markov chain jumps to the state  $G_{t'} = (V, E_{t'})$ , where the edge sets  $E_t$  and  $E_{t'}$ differ in the single edge  $i_0 \sim j_0$ .

Let us define the intensities  $a_{ij}$  for  $\{i, j\} \subset V$ . Let  $\alpha, \beta, \lambda, \mu \geq 0$  and let  $\lambda_i, \mu_i, 1 \leq i \leq n$ , be positive numbers. Given graph G = (V, E) we assign clustering weights  $\nu_{ij}(G, \alpha)$  and  $\nu_{ij}(G, \beta)$  to each vertex pair  $\{i, j\} \subset V$ , where

$$\nu_{ij}(G,s) = \sum_{v \in N_{ij}} (d_v(G))^{-s}, \qquad s \ge 0.$$
(1)

For s = 0 we have  $\nu_{ij}(G, 0) = |N_{ij}|$ . Here  $N_{ij} = N_{ij}(G)$  stands for the set of common neighbours of vertices i and j in G;  $d_v(G)$  denotes the degree of vertex v in G. Furthermore, each vertex pair  $\{i, j\}$  is assigned intensity

$$a_{ij}(G) = \begin{cases} \lambda_i \lambda_j + \lambda \nu_{ij}(G, \alpha) & \text{for } \{i, j\} \notin E, \\ (\mu_i \mu_j - \mu \nu_{ij}(G, \beta))_+ & \text{for } \{i, j\} \in E. \end{cases}$$
(2)

Here  $x_+$  stands for max $\{x, 0\}$ . A standard argument shows that the chain **G** has unique stationary distribution. Chain **G** starting with random graph  $G_0$  having such a distribution is called stationary network in what follows.

For  $\lambda = \mu = 0$  transitions of the chain **G** are defined by the transitions of  $\binom{n}{2}$  independent Markov chains describing adjacency dynamic of each vertex pair  $\{i, j\} \subset V$  separately. (The Markov chain of the vertex pair  $\{i, j\}$  has two states  $i \sim j$  and  $i \not\sim j$ , where state  $i \sim j$ (i and j are adjacent) has exponential holding time with the intensity  $\mu_i \mu_j$  and the state  $i \not\sim j$  (i and j aren't adjacent) has exponential holding time with the intensity  $\lambda_i \lambda_j$ .) The stationary network of **G** has independent edges and, hence, it lacks the clustering property. Assuming, in addition, that  $\mu_i$  is the same for each vertex  $i \in V$  ( $\mu_i \equiv const$ ) we obtain a dynamic network considered in [26]. Let us mention that weights  $\lambda_i$  strongly correlate with respective vertex degrees  $d_i(G_t)$ ,  $i \in V$ , and are useful in modeling the degree distribution of  $G_t$  for large t. Furthermore, large values of  $\lambda_i, \mu_i$  enhance the variability (over time) of links incident to vertex  $i \in V$ .

Grindrod et al. [10] introduced the term  $\lambda \nu_{ij}(G,0)$  to enhance the triadic closure effect. We mention that [10] considers the (discrete) jump chain  $\mathbf{G}^* = \{G_k^* = (V, E_k^*), k = 0, 1, 2, ...\}$  related to  $\mathbf{G}$  defined by (2), where  $\lambda_i = const_1$ ,  $\mu_i = const_2$  do not depend on i and where  $\mu = 0$ . More precisely,  $\mathbf{G}^*$  represents the list of distinct states visited by the chain  $\mathbf{G}$  arranged in the chronological order. That is,  $G_0^* = G_0$ ,  $G_1^* = G_{t_1}$ ,  $G_2^* = G_{t_2}$ , ..., where  $t_1 < t_2 < \ldots$  are the subsequent jump times of continuous chain  $\mathbf{G}$ . Užupytė and Wit [24] complemented the model of [10] by adding the "triadic protection" term  $\mu \nu_{ij}(G,0)$  aimed at reducing the deletion rate of the edges belonging to the closed triangles. They consider the continuous chain  $\mathbf{G}$  defined by (2) with  $\lambda_i = const_1$ ,  $\mu_i = const_2$ .

It has already been mentioned that for large n dynamic networks of [10], [24] permit little control over the edge density, which becomes very sensitive to parameters  $\mu$  and  $\lambda$ . To overcome such disadvantage we suggest choosing clustering weights  $\nu_{ij}(G, s)$  that correlate negatively with degrees of the common neigbours of i and j. An intuition behind this choice is based on the plausible assumption that for i, j being friends of an individual with a large number of acquaintances makes less impact on the mutual relations between i, j than beying friends with a person having just a few contacts. Moreover, [21], [22], [25], see also [7], note that in some sparse and clustered real networks the fraction of closed triangles incident to a vertex scales as a negative power of the degree of that vertex. Findings of [7], [21], [22], [25] motivated our choice of the clustering weights (1).

We are most interested in sparse networks, where the number of vertices n is large. In the simplest case, where  $\lambda = \mu = 0$  and where  $\lambda_i = const_1$  and  $\mu_i = const_2$  are the same for each i (we write, for short,  $\lambda_i \lambda_j = \lambda_0$  and  $\mu_i \mu_j = \mu_0$ ) each vertex pair toggles its adjacency status independently and the expected holding time of an edge (respectively, non-edge) is  $\mu_0^{-1}$  (respectively,  $\lambda_0^{-1}$ ). By the law of large numbers the probability that i and j are adjacent in  $G_t$  is asymptotically  $\mu_0^{-1}/(\mu_0^{-1} + \lambda_0^{-1}) = \lambda_0/(\mu_0 + \lambda_0)$  as  $t \to \infty$ . Hence a snapshot  $G_t$ of the stationary network has the distribution of the binomial random graph with the edge density  $\lambda_0/(\mu_0 + \lambda_0)$ . Furthermore, a sparse network is obtained if one chooses  $\mu_0 = n$  and  $\lambda_0 = c$ , where c > 0 denotes a number independent of n (think of a sequence of network Markov chains with vertex number  $n \to \infty$ ). More generally, for  $\lambda = \mu = 0$ ,  $\mu_i \mu_j \equiv n$  and  $\sum_{i=1}^n \lambda_i \leq cn$  uniformly in n one can obtain a sparse stationary network having independent edges and the degree sequence strongly correlated with the sequence of weights { $\lambda_i$ }, [26].

The simulation study of subsection 2.1 below shows that network Markov chain (2) with clustering weights  $\nu_{ij}(G, \alpha)$ ,  $\nu_{ij}(G, \beta)$ , where  $\alpha, \beta > 0$ , can produce highly clustered sparse stationary dynamic networks with tunable edge density and clustering coefficients. These empirical findings are supported by a limited analytical study (given in subsection 2.2 below) showing upper and lower bounds of the order  $n^{-1}$  on the average edge density. In addition, we establish a lower bound of the order n on the average number of triangles and in a special case of  $\alpha = 2$  we relate the average edge density to the average local clustering coefficient.

Before proceeding further, we introduce some notation. We use terms vertex and node interchangeably. Given a graph G = (V, E) we denote by  $\Delta_v(G)$  the number of triangles incident to a vertex  $v \in V$ . The total number of triangles is denoted  $N_{\Delta}(G) = \frac{1}{3} \sum_{v \in V} \Delta_v(G)$ The total number of 2-paths is denoted  $N_{\Lambda}(G) = \sum_{v \in V} {\binom{d_v(G)}{2}}$ . For a vertex  $v \in V$  of degree  $d_v(G) \geq 2$  we denote  $C_v^{\mathrm{L}}(G) = \Delta_v(G) {\binom{d_v(G)}{2}}^{-1}$  the local clustering coefficient of v(= probability that two randomly selected neighbours of v are neighbours to each other). In the case where  $d_v(G) \leq 1$  we put  $C_v^{\mathrm{L}}(G) = 0$ . The average local clustering coefficient and the global clustering coefficient are denoted

$$\bar{C}^{\mathrm{L}}(G) = \frac{1}{n} \sum_{v \in V} C_v^{\mathrm{L}}(G) \quad \text{and} \quad C^{\mathrm{GL}}(G) = \frac{3N_{\Delta}(G)}{N_{\Lambda}(G)}.$$

We put  $C^{\text{GL}}(G) = 0$  when  $N_{\Delta}(G) = 0$ . The average degree and the average edge density are denoted  $\bar{d}(G) = n^{-1} \sum_{v \in V} d_v(G)$  and  $e(G) = {n \choose 2}^{-1} |E|$  respectively. Finally, we denote by  $\mathbb{I}_A$  the indicator function of an event (or set) A.

### 2.1 Numerical Simulations

The aim of the simulation study is twofold: testing the clustering properties of sparse network (2) equipped with clustering weights  $\nu_{ij}(G, \alpha)$ ,  $\nu_{ij}(G, \beta)$ , where  $\alpha, \beta > 0$  and comparison of the clustering properties for  $\alpha, \beta > 0$  and  $\alpha = \beta = 0$  (the case  $\alpha = \beta = 0$  corresponds to the setup of [10], [24]).

To address both questions simultaneously we consider a simplified model (2), where we assume that  $\lambda_i \lambda_j \equiv const_1 := \lambda_0$  and  $\mu_i \mu_j \equiv const_2 := \mu_0$ , see (3) below. Recall that for  $\lambda = \mu = 0$  the edges are inserted/deleted independently of each other and the ratio  $\mu_0/\lambda_0$  defines the network edge density  $1/(1 + \mu_0/\lambda_0)$  at stationarity. Hence, tuning the ratio  $\mu_0/\lambda_0$  one can achieve the desired edge density. Here we assume that the ratio  $\mu_0/\lambda_0$  is fixed and address the question about tuning parameters  $\lambda$  and  $\mu$  for achieving desired values of clustering coefficients.

In the simulations we put the vertex number n = 1000,  $\mu_0 = n$  and  $\lambda_0 = 1$  (for  $\lambda = \mu = 0$  such network is sparse at stationarity). We only consider two instances of values of the pair  $(\alpha, \beta)$ : the choice of parameters  $\alpha = 2.75$  and  $\beta = 2.5$  is referred to as "general triadic model" below; the choice of parameters  $\alpha = \beta = 0$  is referred to as "simple triadic model". Given

 $(\alpha, \beta)$  we generate network Markov chains for different values of  $(\mu, \lambda)$  from the range that features variability of the local clustering coefficient (our target parameter). For each choice of  $(\mu, \lambda)$  we sample network snaphot  $G_t$  out of (approximately) stationary distribution and evaluate the edge density  $e(G_t)$  (Figure 1) and local clustering coefficient  $\bar{C}^{L}(G_t)$  (Figure 2). To generate an approximately stationary network we run the respective Markov chain starting from an empty graph until  $3n^2$  jumps (edge changes) occur. Further simulation steps do not change values of  $e(G_t)$  and  $\bar{C}^{L}(G_t)$  beyond the rounding error.



Figure 1: Edge densities in stationary graphs



Figure 2: Average local clustering coefficients in stationary graphs

In Figures 1 and 2 values of parameters  $\mu$  and  $\lambda$  are depicted on the vertical and horizontal axis respectively. Evenly spaced labels on each axis depict values of geometric sequences

with the common ratio 1.35. The colours are put on logarithmic scale and the same scale is applied across different images.

As we can see from Figure 1, "general triadic model" admits tunable (average) local clustering coefficient while the edge density remains reasonably small (recall that the ratio  $\mu_0/\lambda_0$  remains fixed). On the other hand, "simple triadic model" shows a swift jump from a sparse graph to the complete graph. Hence while trying to achieve the desired values of the clustering coefficient we are losing cotrol over the edge density.

In Figure 3 (a) we examine several clustering characteristics of the stationary network generated by the "general triadic model" with  $\mu = 15000$  and  $\lambda = 20000$ . Given integer  $k \ge 2$ , let g(k) denote the number of vertices v of degree d(v) = k. Let  $f(k) = \frac{1}{g(k)} \sum_{v: d(v) = k} C_v^{L}(G)$  denote the average value of the local clustering coefficient over the set of vertices of degree k. We put f(k) = 0 for g(k) = 0. We call f the "local clustering coefficient curve". The fact that f is decreasing tells us that the local clustering coefficient negatively correlates with vertex degree, a phenomenon observed in many sparse real networks ([7], [21], [22], [25]). The "general triadic model" reproduces this network property. We also mention that the edge density 0.004 is by two orders less than the average local clustering coefficient. Hence the network is sparse and highly clustered.

Lastly, we touch on the question of the component structure. One may wonder whether the high values of the clustering coefficients are caused by a few (perhaps one) relatively small, but dense subgraphs. Figure 3 (b) shows that this is not the case. The stationary network generated by "general triadic model" admits a large connected component collecting a fraction of nodes. For simplicity we put  $\mu \equiv 0$  (no triad protection). Hence the only remaining parameter to vary is  $\lambda$ . On the horizontal axis we depict values of  $\frac{\lambda}{n}$ . We recall that the number of vertices n = 1000 remains fixed.



Figure 3: Clustering versus degree and the largest component size

### 2.2 Rigorous results

Let f be a real valued function defined on the set of graphs with the vertex set V. For example, it can be the number of edges f(G) = |E| of graph G = (V, E), or the number of triangles  $f(G) = N_{\Delta}(G)$ , etc. For a stationary Markov chain **G** the function  $t \to \mathbf{E}f(G_t)$  is a constant. Hence  $\frac{\partial}{\partial t}\mathbf{E}f(G_t) = 0$ . This identity, when applied to properly chosen function f, can give useful information about average characteristics of the network at stationarity. We explore two instances. Choosing f(G) = |E| we show lower and upper bounds for the average edge density  $e_t := \mathbf{E}e(G_t)$ ; choosing  $f(G) = N_{\Delta}(G)$  we infer about the number of triangles.

Since for stationary **G** the average edge density  $e_t$  and average clustering coefficient  $\mathbf{E}\bar{C}^{\mathrm{L}}(G_t)$  do not depend on t, we sometimes drop the subscript t and write  $e = e_t$  and

 $\overline{C}^{\mathrm{L}} = \mathbf{E}\overline{C}^{\mathrm{L}}(G_t)$ . We observe that, by symmetry, the probability distribution of bivariate random variable  $(d_v(G_t), \Delta_v(G_t))$  is the same for all  $v \in V$ . Furthermore, for a stationary network this distribution does not depend on t either. We denote by  $(d, \Delta)$  a bivariate random variable having the same distribution as  $(d_v(G_t), \Delta_v(G_t))$ .

To make calculations feasible we assume for the rest of the section that the products  $\lambda_i \lambda_j$ and  $\mu_i \mu_j$  in (2) do not depend on i, j. In this case (2) reads as follows

$$a_{ij}(G) = \begin{cases} \lambda_0 + \lambda \nu_{ij}(G, \alpha) & \text{for } \{i, j\} \notin E, \\ (\mu_0 - \mu \nu_{ij}(G, \beta))_+ & \text{for } \{i, j\} \in E, \end{cases}$$
(3)

where  $\lambda, \lambda_0 > 0$  and  $\mu, \mu_0 > 0$ . Below **G** denotes a stationary Markov chain defined by (3). Edge density. We have that

$$e \ge \frac{\lambda_0}{\lambda_0 + \mu_0}.\tag{4}$$

For  $\alpha, \beta \geq 2$  we have

$$e \le \frac{\lambda_0 + \frac{1}{n-1} \max\{\lambda, \mu\}}{\lambda_0 + \mu_0}.$$
(5)

For  $\alpha, \beta \geq 1$  and  $\lambda_0 + \mu_0 > \max{\lambda, \mu}$  we have that

$$e \le \frac{\lambda_0}{\lambda_0 + \mu_0 - \max\{\lambda, \mu\}}.$$
(6)

An important conclussion to draw from inequalities (4), (5), (6) is that for  $\frac{\mu_0}{\lambda_0}$  of the order n and max{ $\lambda, \mu$ } of the order  $\mu_0$  the network  $G_t$  is sparse and has average edge density of the order  $n^{-1}$  as  $n \to +\infty$ .

Proof of (4), (5), and (6). Equation  $\frac{\partial}{\partial t}\mathbf{E}|E_t| = 0$  implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t} a_{ij}(G_t) = \mathbf{E}\sum_{\{i,j\}\in E_t} a_{ij}(G_t).$$
(7)

In view of (3) we can write the latter identity in the form

$$\mathbf{E}\left(\lambda_0\left(\binom{n}{2}-|E_t|\right)+\nu_t'+\nu_t''-\mu_0|E_t|\right)=0.$$
(8)

where

$$\nu_t' = \lambda \sum_{\{i,j\} \notin E_t} \sum_{v \in N_{ij}} \frac{1}{d_v^{\alpha}} \quad \text{and} \quad \nu_t'' = \sum_{\{i,j\} \in E_t} \min\left\{\mu_0, \mu \sum_{v \in N_{ij}} \frac{1}{d_v^{\beta}}\right\}$$
(9)

account for the contribution of the clustering weights  $\lambda \nu_{ij}(G_t, \alpha)$  and  $\mu \nu_{ij}(G_t, \beta)$ . Here we write, for short,  $N_{ij} = N_{ij}(G_t)$  and  $d_w = d_w(G_t)$ . By the linearity of expectation, we obtain from (8) that

$$\lambda_0 - (\lambda_0 + \mu_0)e_t + {\binom{n}{2}}^{-1} \mathbf{E}(\nu'_t + \nu''_t) = 0.$$
(10)

The inequalities  $\nu'_t \ge 0$ ,  $\nu''_t \ge 0$  imply  $\lambda_0 - (\lambda_0 + \mu_0)e_t \ge 0$ . We arrived to lower bound (4). Let us show upper bounds (5), (6). We denote  $\tau := \min\{\alpha, \beta\}$  and estimate

$$\nu_t' \leq \lambda \sum_{\{i,j\} \notin E_t} \sum_{v \in N_{ij}} \frac{1}{d_v^{\tau}},$$
$$\nu_t'' \leq \mu \sum_{\{i,j\} \in E_t} \sum_{v \in N_{ij}} \frac{1}{d_v^{\beta}} \leq \mu \sum_{\{i,j\} \in E_t} \sum_{v \in N_{ij}} \frac{1}{d_v^{\tau}}.$$

Combining these inequalities we obtain

$$\begin{split} \nu'_t + \nu''_t &\leq \max\{\lambda, \mu\} \sum_{1 \leq i < j \leq n} \sum_{v \in N_{ij}} \frac{1}{d_v^\tau} = \max\{\lambda, \mu\} \sum_{v \in V: \, d_v \geq 2} \frac{1}{d_v^\tau} \binom{d_v}{2} \\ &\leq \max\{\lambda, \mu\} \frac{1}{2} \sum_{v \in V} d_v^{2-\tau}. \end{split}$$

For  $\tau = 2$  we have  $\nu'_t + \nu''_t \leq \frac{n}{2} \max\{\lambda, \mu\}$ . Invoking this inequality in (10) we obtain (5). For  $\tau = 1$  we have  $\nu'_t + \nu''_t \leq \frac{1}{2} \max\{\lambda, \mu\} |E_t|$ . Now (10) yields (6). Special case of  $\alpha = 2$ . In this special case we consider a slightly modified version of (3)

that includes the "correction term"

$$\varkappa_{ij}(G) = \frac{1}{n-1} \left( \mathbb{I}_{\{d_i(G)=0\}} + \mathbb{I}_{\{d_j(G)=0\}} \right) + \frac{1}{n-2} \left( \mathbb{I}_{\{d_i(G)=1\}} + \mathbb{I}_{\{d_j(G)=1\}} \right)$$

In addition, we replace  $\nu_{ij}(G,2)$  by related quantity  $\nu_{ij}^*(G) = \sum_{w \in N_{ij}(G)} {\binom{d_w(G)}{2}}^{-1}$ . We set

$$a_{ij}(G) = \begin{cases} \lambda_0 + \lambda \nu_{ij}^*(G) + \lambda \varkappa_{ij}(G) & \text{for } \{i, j\} \notin E, \\ \mu_0 & \text{for } \{i, j\} \in E. \end{cases}$$
(11)

The reason for such a modification is that it admits a closed form expression for the average edge density.

For a stationary network Markov chain defined by (11) we have that

$$e = \frac{\lambda_0 + \frac{2}{n-1}\lambda(1 - \bar{C}^{\rm L})}{\lambda_0 + \mu_0}$$
(12)

Noting that  $\bar{C}^{L} \leq 1$  we obtain from (12) the upper and lower bounds for the average edge density

$$\frac{\lambda_0}{\lambda_0 + \mu_0} \le e \le \frac{\lambda_0 + \frac{2}{n-1}\lambda}{\lambda_0 + \mu_0}.$$

Letting  $n \to +\infty$  and choosing  $\frac{\mu_0}{\lambda_0}$ ,  $\frac{\lambda}{\lambda_0}$  and  $\frac{\mu}{\lambda_0}$  of the order n we have that e is of the order  $n^{-1}$ . Hence the model produces a sparse dynamic network.

Proof of (12). Equation (7) implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t} \left(\lambda_0 + \lambda\varkappa_{ij} + \lambda\sum_{w\in N_{ij}}\frac{1}{\binom{d_w}{2}}\right) = \mu_0 \mathbf{E}|E_t|.$$
(13)

Here we write, for short,  $\varkappa_{ij} = \varkappa_{ij}(G_t)$ ,  $N_{ij} = N_{ij}(G_t)$  and  $d_w = d_w(G_t)$ . Invoking the identities

$$\sum_{\{i,j\}\notin E_t} 1 = \binom{n}{2} - |E_t|,$$

$$\sum_{\{i,j\}\notin E_t} \varkappa_{ij} = \sum_{w\in V} \mathbb{I}_{\{d_w=0\}} + \sum_{w\in V} \mathbb{I}_{\{d_w=1\}},$$

$$\sum_{\{i,j\}\notin E_t} \sum_{w\in N_{ij}} \frac{1}{\binom{d_w}{2}} = \sum_{w\in V: d_w \ge 2} \frac{\binom{d_w}{2} - \Delta_w(G_t)}{\binom{d_w}{2}} = \sum_{w\in V: d_w \ge 2} (1 - C_w^{\mathrm{L}}(G_t))$$

and dividing both sides of (13) by  $\binom{n}{2}$  we have

$$\lambda_0(1-e) + \frac{2\lambda}{n-1} \left( \mathbf{P}\{d \le 1\} + \mathbf{P}\{d \ge 2\} - \bar{C}^{\mathrm{L}} \right) = \mu_0 e,$$

where d denotes the degree of a randomly selected vertex. We have arrived to (12).

Number of triangles. For a stationary network Markov chain defined by (3) where  $0 < \alpha \leq 2$  we show that

$$\mathbf{E}\Delta \ge \frac{\lambda}{4(\lambda_0 + \mu_0 + \lambda)} \mathbf{P}\{d \ge 2\}.$$
(14)

An important conclussion to draw from inequality (14) is that choosing  $\frac{\mu_0}{\lambda_0}$ ,  $\frac{\lambda}{\lambda_0}$  and  $\frac{\mu}{\lambda_0}$  of the order *n* one can obtain a sparse stationary dynamic network with the property that the average number of triangles incident to a vertex of degree at least two (formally, the conditional expectation  $\mathbf{E}(\Delta | d \ge 2) = \frac{\mathbf{E}\Delta}{\mathbf{P}\{d\ge 2\}}$ ) is bounded from below by a constant. Note that  $d_v(G_t) \le 1$  implies  $\Delta_v(G_t) = 0$ . Hence  $\Delta_v(G_t) = \Delta_v(G_t)\mathbb{I}_{\{d_v(G_t)\ge 2\}}$  and  $\Delta = \Delta \mathbb{I}_{\{d\ge 2\}}$ . Proof of (14) Equation  $\frac{\partial}{\partial N_t}(C) = 0$  implies

Proof of (14). Equation  $\frac{\partial}{\partial t}N_{\Delta}(G_t) = 0$  implies

$$\mathbf{E}\sum_{\{i,j\}\notin E_t} |N_{ij}(G_t)| a_{ij}(G_t) = \mathbf{E}\sum_{\{i,j\}\in E_t} |N_{ij}(G_t)| a_{ij}(G_t).$$
(15)

Here the left sum evaluates the average birth rate of triangles: connecting a pair of nonadjacent vertices i, j by an edge creates  $|N_{ij}(G_t)|$  new triangles. The right sum evaluates the average death rate of triangles: deletion of an edge  $\{i, j\} \in E_t$  eliminates  $|N_{ij}(G_t)|$  triangles from  $G_t$ . Furthermore, for  $\{i, j\} \in E_t$  we have  $a_{ij}(G_t) \leq \mu_0$ . Hence the sum on the right of (15)

$$\sum_{\{i,j\}\in E_t} |N_{ij}(G_t)| a_{ij}(G_t) \le \sum_{\{i,j\}\in E_t} |N_{ij}(G_t)| \mu_0 = \sum_{v\in V: \, d_v(G_t)\ge 2} \Delta_v(G_t) \mu_0.$$
(16)

In the last identity we use the observation that  $\Delta_v(G_t)$  counts edges whose both endpoints are adjacent to v. Similarly for the sum on the left of (15)

$$\sum_{\{i,j\}\notin E_t} |N_{ij}(G_t)| a_{ij}(G_t) \ge \sum_{v \in V: \, d_v(G_t) \ge 2} \left( \binom{d_v(G_t)}{2} - \Delta_v(G_t) \right) \left( \lambda_0 + \frac{\lambda}{d_v^{\alpha}(G_t)} \right) \tag{17}$$

Here we use the observation that  $\binom{d_v(G_t)}{2} - \Delta_v(G_t)$  counts pairs  $\{i, j\}$  of neighbours of v that are non-adjacent  $(\{i, j\} \notin E_t)$ . Inequality (17) follows from the fact that  $a_{ij}(G_t) \ge \lambda_0 + \frac{\lambda}{d_v^{\alpha}}$  for each  $v \in N_{ij}(G_t)$ .

Invoking (16) and (17) in (15) we obtain

$$\mathbf{E}\sum_{v\in V:\,d_v(G_t)\geq 2} \left( \binom{d_v(G_t)}{2} - \Delta_v(G_t) \right) \left( \lambda_0 + \frac{\lambda}{d_v^{\alpha}(G_t)} \right) \leq \mathbf{E}\sum_{v\in V:\,d_v(G_t)\geq 2} \Delta_v(G_t)\mu_0.$$

Recall that the probability distribution of bivariate random variable  $(d_v(G_t), D_v(G_t))$  is the same for all  $v \in V$ . Collecting the terms  $\Delta_v(G_t)$  on the right and dividing both sides by n we have

$$\frac{\lambda_0}{2} \mathbf{E} \big( d(d-1) \big) + \frac{\lambda}{2} \mathbf{E} (d^{1-\alpha}(d-1) \mathbb{I}_{\{d \ge 2\}} \le (\lambda_0 + \mu_0) \mathbf{E} \Delta + \lambda \mathbf{E} \left( \Delta_v \frac{\mathbb{I}_{\{d_v \ge 2\}}}{d_v^{\alpha}} \right).$$

Next we upper bound  $\lambda \mathbf{E} \left( \Delta_v d_v^{-\alpha} \mathbb{I}_{\{d_v \geq 2\}} \right) \leq \lambda \mathbf{E} \Delta$  and obtain

$$\frac{\lambda}{2} \mathbf{E} \left( d^{1-\alpha} (d-1) \mathbb{I}_{\{d \ge 2\}} \right) \le (\lambda_0 + \mu_0 + \lambda) \mathbf{E} \Delta.$$

Furthermore, using inequality  $\frac{1}{2} \leq \frac{d-1}{d} \leq \frac{d-1}{d^{\alpha-1}}$ , which holds for  $0 < \alpha \leq 2$  and  $d \geq 2$  we lower bound the left side by  $\frac{\lambda}{4}$  and obtain inequality equivalent to (14)

$$\frac{\lambda}{4}\mathbf{P}\{d\geq 2\}\leq (\lambda_0+\mu_0+\lambda)\mathbf{E}\Delta.$$

### 3 Dynamic affiliation network

Let  $\mathbf{H} = \{H_t = (V \cup W, E_t), t \ge 0\}$  be a continuous time Markov chain, whose states are bipartite graphs with the bipartition  $V \cup W$ , where  $V = \{1, \ldots, n\}$  and  $W = \{1, \ldots, m\}$ . Transitions of  $\mathbf{H}$  are defined as follows. Given  $H_t$  (the state occupied at time t), the update takes place at time  $t' := t + \min_{(i,u) \in V \times W} B_{iu}$  when the pair  $(i_0, u_0)$  with  $B_{i_0 u_0} = \min_{(i,u) \in V \times W} B_{iu}$ changes its adjacency status. Here  $B_{iu} = B_{iu}(H_t)$ , are independent exponential waiting times with intensities  $b_{iu} = b_{iu}(H_t)$  defined below. Thus, at time t' chain  $\mathbf{H}$  jumps to the state  $H_{t'} = (V \cup W, E_{t'})$ , where the edge sets  $E_t$  and  $E_{t'}$  differ in the single edge  $i_0 \sim u_0$ . Markov chain  $\mathbf{H}$  defines dynamic affiliation network  $\mathbf{G}' = \{G'_t = (E'_t, V), t \ge 0\}$ : for each t any two vertices  $i, j \in V$  are adjacent in  $G'_t$  whenever i and j have a common neighbour in  $H_t$ .

Now we define intensities  $b_{iu}$ . We fix  $\mu > 0$  and assign positive weights  $y_i$  and  $x_u$  to  $i \in V$ and  $u \in W$  that model activity of actors and attrativeness of attributes. For a bipartite graph  $H = (V \cup W, E)$  we set

$$b_{iu}(H) = \begin{cases} y_i x_u & \text{for } (i, u) \notin E, \\ \mu & \text{for } (i, u) \in E. \end{cases}$$
(18)

Clearly, **H** has a unique stationary distribution defined by the weight sequences  $\{y_i\}_{i=1}^n$ ,  $\{x_u\}_{u=1}^m$  and  $\mu$ . Furthermore, **H** comprises of  $n \times m$  independent continuous Markov chains describing adjacency dynamic of each vertex pair  $(i, u) \subset V \times W$  separately, where the Markov chain of a pair (i, j) has two states  $i \sim u$  and  $i \not\sim u$  whose exponential holding times have intensities  $\mu$  and  $y_i x_u$  respectively. Thus, at stationarity, a snaphot  $H_t$  represents a random bipartite graph, where edges are inserted independently with probabilities

$$\mathbf{P}\{i \sim u\} = \frac{y_i x_u}{y_i x_u + \mu} =: p_{iu},$$
(19)

for  $(i, u) \in V \times W$ . We assume in what follows that dynamic affiliation network  $\mathbf{G}'$  is defined by a stationary Markov chain  $\mathbf{H}$  satisfying (19). In this case probability distributions of random graphs  $G'_t$  and  $H'_t$  do not depend on t and with a little abuse of notation we write, for short,  $G' = G'_t$  and  $H = H_t$ . We show that G' admits tunable degree distribution and nonvanishing global clustering coefficient.

We will use the following notation. By  $P_{y,n} = \sum_{i=1}^{n} \delta_{y_i}$  and  $P_{x,m} = \sum_{u=1}^{m} \delta_{x_u}$  we denote empirical distributions of the sequences  $\{y_1, \ldots, y_n\}$  and  $\{x_1, \ldots, x_m\}$ . Here  $\delta_t$  stands for the degenerate distribution that assigns mass 1 to point t. Furthermore we denote

$$\langle x^s \rangle = \frac{1}{m} \sum_{u \in [m]} x^s_u, \quad \langle y^s \rangle = \frac{1}{n} \sum_{i \in [n]} y^s_i, \quad \gamma^2 = \frac{m}{n}, \quad \varkappa = \frac{nm}{\mu^2}.$$

It is important to mention that the ratio  $\gamma^2 = \frac{m}{n}$  correlates negatively with the clustering strength. More precisely, the global clustering coefficient of G' is asymptotically inversely proportional to  $\gamma$  for large n and m, see (25) below.

Degrees of G'. Here we show that the expected value  $\mathbf{E}d_i$  of the degree  $d_i = d_i(G')$  of vertex *i* is approximately proportional to its weight  $y_i$ . Moreover,  $d_i$  has asymptotic compound Poisson distribution as the network size  $n \to +\infty$ .

**Theorem 1.** For each  $i \in V$  we have

$$0 \le y_i \varkappa \langle x^2 \rangle \langle y \rangle - \mathbf{E} d_i \le \frac{\varkappa}{\mu} y_i \left( \langle x^3 \rangle \langle y^2 \rangle + y_i \langle x^3 \rangle \langle y \rangle \right) + \frac{\varkappa^2}{n} y_i^2 \langle x^2 \rangle^2 \langle y^2 \rangle + \frac{1}{n} y_i \langle x^2 \rangle.$$
(20)

It follows from Theorem 1 that for large n, m and  $\mu = \mu(n, m)$  of the order  $\sqrt{nm}$  the expected degree of a vertex *i* in *G'* is asymptotically proportional its "activity" weight  $y_i$ .

**Corollary 1.** Let  $n, m \to +\infty$ . Put  $\mu = \sqrt{nm}$  and assume that for some c > 0 we have  $\langle x^3 \rangle \leq c$  and  $\langle y^2 \rangle \leq c$  uniformly in n, m. Then for each i

$$\mathbf{E}d_i = y_i \langle x^2 \rangle \langle y \rangle + O\left(\frac{y_i^2}{\sqrt{nm}}\right).$$
(21)

In the sparse regime, when  $\mathbf{E}d_i$  remains bounded as  $n, m \to +\infty$ , the probability distribution of  $d_i$  does not concentrate around the expected value  $\mathbf{E}d_i$ . Theorem 2 below shows that  $d_i$  has a compound Poisson asymptotic distribution. Recall that compound Poisson distribution is the probability distribution of a randomly stopped sum  $\sum_{k=1}^{\Lambda} \xi_k$ , where  $\xi_1, \xi_2, \ldots$  are independent and identically distributed random variables, which are independent of Poisson random variable  $\Lambda$ . We write  $\Lambda \sim \mathcal{P}(\lambda)$ , where  $\lambda := \mathbf{E}\Lambda$  denotes the expected value and denote by  $\mathcal{CP}(\lambda, P_{\xi})$  the (compound Poisson) distribution of  $\sum_{k=1}^{\Lambda} \xi_k$ . Here  $P_{\xi}$  denotes the (common) probability distribution of  $\xi_k$ .

Let  $x_1, x_2, \ldots$  and  $y_1, y_2, \ldots$  be positive infinite sequences of weights. In Theorem 2 we consider random affiliation networks  $G'_{n,m}$ ,  $n, m = 1, 2, \ldots$ , based on respective bipartite random graphs  $H_{n,m}$  whose edges are inserted independently with probabilities (19). Note that each  $H_{n,m}$  is defined by truncated (finite) sequences  $\{x_1, \ldots, x_m\}$  and  $\{y_1, \ldots, y_n\}$ .

To formulate our next result we need the following conditions: for  $n, m \to +\infty$  we have

(i)  $P_{x,m}$  converges weakly to some probability distribution, say  $P_X$ , having a finite first moment  $\int sP_X(ds) < \infty$  and  $\langle x \rangle$  converges to  $\int sP_X(ds)$ ;

(ii) the family of distributions  $\{P_{y,n}, n = 1, 2, ...\}$  is uniformly integrable and  $\langle y \rangle$  converges to some number  $a_y > 0$ .

**Theorem 2.** Let  $\mu = \sqrt{nm}$ . Let  $n \to +\infty$ . Assume that m = m(n) is such that m/n converges to some  $\gamma_o > 0$ . Assume that (i) and (ii) hold. Denote  $a_x = \int sP_X(ds)$  and introduce function  $s \to \lambda_s = sa_y \gamma_o^{-1}$ . For each i = 1, 2, ... the probability distribution of  $d_i$  converges weakly to the compound Poisson distribution  $\mathcal{CP}(y_i a_x \gamma_o, Q)$ , where the discrete probability distribution Q assigns probabilities

$$Q(\lbrace t \rbrace) = \int \frac{s}{a_x} e^{-\lambda_s} \frac{\lambda_s^t}{t!} P_X(ds).$$
(22)

to integers t = 0, 1, 2, ...

We note that Q is a mixture of Poisson distributions. To sample from Q one can use the two step procedure: 1) generate a (size biased) random variable  $\tilde{X}$  according to the distribution  $\mathbf{P}{\{\tilde{X} = s\}} = \frac{s}{a_x} \mathbf{P}{\{X = s\}}, s = 0, 1, ...; 2)$  sample Poisson random variable with rate  $\tilde{X}a_y\gamma_o^{-1}$ .

Clustering in G'. We recall that  $N_{\Delta}(G')$  denotes the number of triangles in G' and  $N_{\Lambda}(G')$  denotes the number of 2- paths in G'.

**Theorem 3.** Let  $\mu = \sqrt{nm}$ . Let  $n \to +\infty$ . Assume that m = m(n) is such that m/n converges to some  $\gamma_o > 0$ . Assume that for some constant c > 0 we have  $\langle x^5 \rangle < c$  and  $\langle y^4 \rangle < c$  for all n. Then

$$N_{\Delta}(G') = \frac{n}{6\gamma} \langle x^3 \rangle \langle y \rangle^3 + o_P(\sqrt{n}), \qquad (23)$$

$$N_{\Lambda}(G') = \frac{n}{2\gamma} \langle x^3 \rangle \langle y \rangle^3 + \frac{n}{2} \langle x^2 \rangle^2 \langle y^2 \rangle \langle y \rangle^2 + O_P(\sqrt{n}).$$
(24)

In particular, the global clustering coefficient

$$C^{GL}(G) = \frac{\langle x^3 \rangle \langle y \rangle^3}{\langle x^3 \rangle \langle y \rangle^3 + \gamma_o \langle x^2 \rangle^2 \langle y^2 \rangle \langle y \rangle^2} + o_P(1).$$
(25)

We remark that conditions  $\langle x^5 \rangle < c$  and  $\langle y^4 \rangle < c$  of Theorem 3 can be relaxed. We expect that the minimal conditions  $\langle x^3 \rangle < c$  and  $\langle y^2 \rangle < c$  plus the uniform integrability of  $t^3 P_{x,m}(dt)$  and  $t^2 P_{y,n}(dt)$  would suffice.

## 4 Concluding remarks

We presented two dynamic network models that generate sparse and clustered stationary networks. Both models seems natural as they mimic dynamics of real network processes. Luckily, for rigorous analysis of dynamic affiliation network we can use techniques developed for random intersection graphs [2], [9], [13], [14], [18], [19]. On the other hand we have only a few rigorous results for stationary Markov chains with clustering like (2), (3). It would be interesting to learn more about network structure and properties of this model via rigorous analysis.

Acknowledgement. The diagrams were generated using Matplotlib [16]. Authors thank Information technology research center of Vilnius University for a high performance computing resources.

# 5 Appendix

Here we prove Theorems 1, 2, 3. Before proofs we introduce some notation.

For  $v \in V \cup W$  we denote by  $N_v$  the set of neighbours of v in H. Note that for  $i \in V$ and  $u \in W$  we have  $N_i \subset W$  and  $N_u \subset V$ . For  $i \in V$ ,  $u \in W$  we denote by  $\mathbb{I}_{iu} = \mathbb{I}_{\{i \sim u\}}$  the indicator function of the event  $i \sim u$  (meaning that i and u are adjacent in H). For  $i \in V$ we denote by  $N_i^G$  the set of neighbours of i in G';  $d_i = |N_i^G|$  denotes the degree of i in G'.

We write, for short,

$$S_a(x) = \sum_{i \in [m]} x_i^a = m \langle x^a \rangle, \qquad S_a(y) = \sum_{j \in [n]} y_j^a = n \langle y^a \rangle, \qquad p_{ik}^\star = \frac{y_i x_k}{\mu}.$$

In the proof (sometimes without mentioning) we apply inequalities

$$p_{iu}^{\star} \ge p_{iu} \ge (1 - p_{iu}^{\star}) p_{iu}^{\star}, \quad \prod_{j} p_{i_{j}u_{j}}^{\star} \ge \prod_{j} p_{i_{j}u_{j}} \ge \left(1 - \sum_{j} p_{i_{j}u_{j}}^{\star}\right) \prod_{j} p_{i_{j}u_{j}}^{\star}.$$
 (26)

The first (and third) inequality is obvious. The second one follows from the inequalities  $\frac{a}{b} \geq \frac{a}{a+b} \geq (1-\frac{a}{b})\frac{a}{b}$ , for a > 0 and b > 0. The fourth one is obtained by iterating the second inequality.

Proof of Theorem 1. We only prove (20). Let

$$\begin{aligned} &d_i^{(1)} = \sum_{u \in N_i} |N_u \setminus \{i\}| = \sum_{u \in W} \mathbb{I}_{iu} \sum_{j \in [n] \setminus \{i\}} \mathbb{I}_{ju}, \\ &R_i = \sum_{\{u,v\} \subset N_i} \left| (N_u \setminus \{i\}) \cap (N_v \setminus \{i\}) \right| = \sum_{\{u,v\} \subset W} \mathbb{I}_{iu} \mathbb{I}_{iv} \sum_{j \in [n] \setminus \{i\}} \mathbb{I}_{ju} \mathbb{I}_{jv}. \end{aligned}$$

Using inclusion-exclusion inequalities we estimate the number of elements of the set  $N_i^G = \bigcup_{u \in N_i} (N_u \setminus \{i\})$ . We have

$$d_i^{(1)} - R_i \le |N_i^G| \le d_i^{(1)}.$$
(27)

To prove the left inequality of (20) we use  $\mathbf{E}d_i \leq \mathbf{E}d_i^{(1)}$  (see (27)) and invoke bound (28) shown below. To show the right inequality of (20) we use  $\mathbf{E}d_i^{(1)} - \mathbf{E}R_i \leq \mathbf{E}d_i$  (see (27)) and

invoke bounds (29), (30) below. Finally, using (26) we estimate  $\mathbf{E}d_i^{(1)}$  and  $\mathbf{E}R_i$ .

$$\mathbf{E}d_{i}^{(1)} = \sum_{u \in [m]} p_{iu} \sum_{j \in [n] \setminus \{i\}} p_{ju} \leq \sum_{u \in [m]} p_{iu} \sum_{j \in [n]} p_{ju} \leq \frac{mn}{\mu^{2}} y_{i} \langle x^{2} \rangle \langle y \rangle,$$
(28)  
$$\mathbf{E}d_{i}^{(1)} = \sum_{u \in [m]} \sum_{j \in [n]} p_{iu} p_{ju} - \sum_{u \in [m]} p_{iu}^{2}$$
$$\geq \sum_{u \in [m]} \sum_{j \in [n]} p_{iu}^{\star} p_{ju}^{\star} \left(1 - p_{iu}^{\star} - p_{ju}^{\star}\right) - \sum_{u \in [m]} (p_{iu}^{\star})^{2}$$
$$= \frac{mn}{\mu^{2}} y_{i} \langle x^{2} \rangle \langle y \rangle - \frac{mn}{\mu^{3}} y_{i}^{2} \langle x^{3} \rangle \langle y \rangle - \frac{mn}{\mu^{3}} y_{i} \langle x^{3} \rangle \langle y^{2} \rangle - \frac{m}{\mu^{2}} y_{i}^{2} \langle x^{2} \rangle$$
(29)

and

$$\mathbf{E}R_{i} \leq \sum_{\{u,v\}\subset[m]} p_{iu}p_{iv}\sum_{j\in[n]} p_{ju}p_{jv} \leq \sum_{\{u,v\}\subset[m]} p_{iu}^{\star}p_{iv}^{\star}\sum_{j\in[n]} p_{ju}^{\star}p_{jv}^{\star}$$
$$\leq \frac{1}{2}\frac{m^{2}n}{\mu^{4}}y_{i}^{2}\langle x^{2}\rangle^{2}\langle y^{2}\rangle.$$
(30)

Proof of Theorem 1 is complete.

Proof of Theorem 2. Before the proof we introduce some notation and collect auxiliary results. Given two random variables  $\xi$  and  $\zeta$  we denote by  $d_{TV}(P_{\xi}, P_{\zeta})$  the total variation distance between the probability distributions  $P_{\xi}$  and  $P_{\zeta}$  of  $\xi$  and  $\zeta$ . With a little abuse of notation we also write  $d_{TV}(\xi, \zeta)$ .

notation we also write  $d_{TV}(\xi, \zeta)$ . We fix vertex  $i \in V$ . For  $u \in W$  we write, for short,  $\zeta_u = \mathbb{I}_{iu}$  and  $\xi_u = \sum_{j \in V \setminus \{i\}} \mathbb{I}_{ju}$ . Let  $\zeta_u^*, \zeta_u^*, \zeta_u^o$  and  $\xi_u^*, \xi_u^o$ ,  $\xi_u^o$  be Poisson random variables with expected values

$$\begin{split} \mathbf{E}\zeta_{u}^{*} &= p_{iu}, \quad \mathbf{E}\zeta_{u}^{\star} = p_{iu}^{\star}, \quad \mathbf{E}\zeta_{u}^{o} = \frac{\gamma_{o}}{\gamma}p_{iu}^{\star}, \\ \mathbf{E}\xi_{u}^{*} &= \sum_{j \in V \setminus \{i\}} p_{ju}, \quad \mathbf{E}\xi_{u}^{\star} = \sum_{j \in V \setminus \{i\}} p_{ju}^{\star}, \quad \mathbf{E}\xi_{u}^{o} = \gamma_{o}^{-1}x_{u}a_{y} \end{split}$$

Note that  $\mathbf{E}\zeta_u^* = \mathbf{E}\zeta_u$  and  $\mathbf{E}\xi_u^* = \mathbf{E}\xi_u$ . Let  $(\xi_u(k), \xi_u^*(k), \xi_u^*(k), \xi_u^o(k)), k \ge 1$ , be iid copies of  $(\xi_u, \xi_u^*, \xi_u^*, \xi_u^o)$ . We assume that each collection

$$\{ \zeta_u, \xi_u(k), u \in W, k \in \mathbb{N} \}, \qquad \{ \zeta_u^*, \xi_u(k), u \in W, k \in \mathbb{N} \}, \qquad \{ \zeta_u^*, \xi_u^*(k), u \in W, k \in \mathbb{N} \}, \\ \{ \zeta_u^*, \xi_u^*(k), u \in W, k \in \mathbb{N} \}, \qquad \{ \zeta_u^o, \xi_u^o(k), u \in W, k \in \mathbb{N} \}, \qquad \{ \zeta_u^o, \xi_u^o(k), u \in W, k \in \mathbb{N} \}.$$

consists of independent random variables. We introduce random variables

$$d_i^{(2)} = \sum_{u \in W} \sum_{k=1}^{\zeta_u^*} \xi_u(k), \qquad d_i^{(3)} = \sum_{u \in W} \sum_{k=1}^{\zeta_u^*} \xi_u^*(k),$$
$$d_i^{(4)} = \sum_{u \in W} \sum_{k=1}^{\zeta_u^*} \xi_u^*(k), \qquad d_i^{(5)} = \sum_{u \in W} \sum_{k=1}^{\zeta_u^o} \xi_u^o(k).$$

When estimating the total variation distance between sums of random variables, say,  $\sum_{k \in [m]} \eta_k =: \eta$  and  $\sum_{k \in [m]} \kappa_k =: \kappa$  we will often apply the following device. We define intermediate sums  $\varphi_r = \sum_{k=1}^r \kappa_k + \sum_{k=r+1}^m \eta_k$  so that  $\eta = \varphi_0$  and  $\kappa = \varphi_m$  and note that  $d_{TV}(\varphi_r, \varphi_{r+1}) \leq d_{TV}(\eta_r, \kappa_r)$ . Combining this inequality with the triangle inequality we have

$$d_{TV}(\eta,\kappa) \le \sum_{r=0}^{m-1} d_{TV}(\varphi_r,\varphi_{r+1}) \le \sum_{r=1}^m d_{TV}(\eta_r,\kappa_r).$$
(31)

Let a, b > 0 and let  $\tau, \nu$  be random variables. We will use the following inequality for the total variation distance between compound Poisson distributions  $C\mathcal{P}(a, P_{\tau})$  and  $C\mathcal{P}(b, P_{\nu})$ , see [5] formula (25),

$$d_{TV}\left(\mathcal{CP}(a, P_{\tau}), \mathcal{CP}(b, P_{\nu})\right) \le 2|a-b| + bd_{TV}(P_{\tau}, P_{\nu}).$$
(32)

Now we are ready to prove the theorem. We first assume that the sequence  $x_1, x_2, \ldots$  is bounded, that is, for some M > 0 we have  $x_j \leq M \forall j$ . The proof of the general case (where this assumption is waived) is given afterwards. The assumption implies that

$$p_{ju} \le \frac{y_j M}{y_j M + \mu} \le \frac{y_j M}{\mu}, \quad \text{for} \quad (j, u) \in V \times W.$$
 (33)

Here the first inequality follows from the fact that the function  $x \to \frac{y_j x}{y_j x + \mu}$  is increasing.

The proof consist of two parts. In the first part (steps 1 - 5 below) we show that  $d_{TV}(d_i, d_i^{(5)}) = o(1)$ . In the second part (step 6 below) we show that the Fourier transform (characteristic function) of the distribution of  $d_i^{(5)}$  converges to that of  $\mathcal{CP}(y_i a_x \gamma_o, Q)$ .

Let us show that  $d_{TV}(d_i, d_i^{(5)}) = o(1)$ . We first apply triangle inequality

$$d_{TV}(d_i, d_i^{(5)}) \le d_{TV}(d_i, d_i^{(1)}) + \sum_{k=1}^4 d_{TV}(d_i^{(k)}, d_i^{(k+1)})$$
(34)

and then show that each term on the right is of the order o(1).

Step 1. Here we show that  $d_{TV}(d_i, d_i^{(1)}) = o(1)$ . We have, see (27), (30),

$$d_{TV}(d_i, d_i^{(1)}) \leq \mathbf{P}\{d_i \neq d_i^{(1)}\} \leq \mathbf{P}\{R_i \geq 1\} \leq \mathbf{E}R_i$$

$$\leq \sum_{\{u,v\} \subset W} p_{iu}p_{iv} \sum_{j \in V} p_{ju}p_{jv}.$$
(35)

Using (33) we estimate  $p_{iu}p_{iv} \leq \frac{y_i^2 M^2}{\mu^2}$  and  $p_{ju}p_{jv} \leq \frac{y_j M}{\mu} \frac{y_j M}{y_j M + \mu}$  and upper bound the right side of (35) by

$$M^{3}\binom{m}{2}\frac{y_{i}^{2}}{\mu^{3}}S, \quad \text{where} \quad S := \sum_{j \in V} y_{j}\frac{y_{j}M}{y_{j}M + \mu}.$$
(36)

It remains to show that S = o(n). To this aim we fix  $\varepsilon \in (0, 1)$  and estimate

$$\frac{y_j M}{y_j M + \mu} \le \begin{cases} 1, & \text{for } y_j > \varepsilon \mu, \\ \frac{y_j M}{\mu} \le \varepsilon M, & \text{for } y_j \le \varepsilon \mu. \end{cases}$$
(37)

We have

$$S \leq \varepsilon M \sum_{j: y_j \leq \varepsilon \mu} y_j + \sum_{j: y_j > \varepsilon \mu} y_j \leq \varepsilon M S_1(y) + \sum_{j: y_j > \varepsilon \mu} y_j$$
(38)  
= $\varepsilon M n \langle y \rangle + n \int_{s > \varepsilon \mu} s P_{y,n}(ds).$ 

Choosing  $\varepsilon = \varepsilon_n \downarrow 0$  such that  $\varepsilon_n \mu \to \infty$  as  $n \to \infty$  we obtain  $\int_{s > \varepsilon_n \mu} s P_{y,n}(ds) = o(1)$  by the uniform integrability condition (ii). Hence S = o(n).

Step 2. Here we show that  $d_{TV}\left(d_i^{(1)}, d_i^{(2)}\right) = o(1)$ . To this aim we write  $d_i^{(1)}$  in the form  $d_i^{(1)} = \sum_{u \in W} \sum_{k=1}^{\zeta_u} \xi_u(k)$  and apply (31). In this case  $\eta_r = \sum_{k=1}^{\zeta_r} \xi_r(k)$  and  $\kappa_r = 0$ 

 $\sum_{k=1}^{\zeta_r^*} \xi_r(k)$ . Invoking inequalities  $d_{TV}(\eta_r, \kappa_r) \leq d_{TV}(\zeta_r^*, \zeta_r) \leq p_{ir}^2$  (the last inequality follows by LeCam's inequality [23]) we obtain

$$d_{TV}\left(d_i^{(1)}, d_i^{(2)}\right) \le \sum_{r \in [m]} d_{TV}(\zeta_r, \zeta_r^*) \le \sum_{r \in [m]} p_{ir}^2.$$

We note that the sum on the right  $\sum_{r \in [m]} p_{ir}^2 \leq M^2 y_i^2 \frac{m}{\mu^2} = O(n^{-1}).$ 

Step 3. Here we show that  $d_{TV}(d_i^{(2)}, d_i^{(3)}) = o(1)$ . To this aim we apply (31) with  $\eta_r = \sum_{k=1}^{\zeta_r^*} \xi_r^*(k)$  and  $\kappa_r = \sum_{k=1}^{\zeta_r^*} \xi_r(k)$ . Invoking inequalities

$$d_{TV}(\eta_r, \kappa_r) \le p_{ir} d_{TV}(\xi_r^*, \xi_r) \le p_{ir} \sum_{j \in V \setminus \{i\}} p_{jr}^2$$

(the first inequality follows by (32), the second one follows by LeCam's inequality) we obtain

$$d_{TV}(d_i^{(2)}, d_i^{(3)}) \le \sum_{r \in [m]} d_{TV} \left( \sum_{k=1}^{\zeta_r^*} \xi_r^*(k), \sum_{k=1}^{\zeta_r^*} \xi_r(k) \right) \le \sum_{r \in [m]} p_{ir} \sum_{j \in V \setminus \{i\}} p_{jr}^2.$$

To show that the quantity on the right is of the order o(1) we proceed as in (37), (38) above. We have

$$\sum_{r\in[m]} p_{ir} \sum_{j\in V\setminus\{i\}} p_{jr}^2 \leq \sum_{r\in[m]} \frac{y_i M}{\mu} \sum_{j\in V\setminus\{i\}} \frac{y_j M}{\mu} \frac{y_j M}{y_j M + \mu}$$
$$= M^2 \frac{y_i}{n} \sum_{j\in V\setminus\{i\}} y_j \frac{y_j M}{y_j M + \mu} \leq M^2 \frac{y_i}{n} S = o(1).$$

Step 4. Here we show that  $d_{TV}(d_i^{(3)}, d_i^{(4)}) = o(1)$ . To this aim we apply (31) with  $\eta_r = \sum_{k=1}^{\zeta_r^*} \xi_r^*(k)$  and  $\kappa_r = \sum_{k=1}^{\zeta_r^*} \xi_r^*(k)$ . Invoking inequalities

$$d_{TV}(\eta_r, \kappa_r) \le 2(p_{ir}^{\star} - p_{ir}) + p_{ir}d_{TV}(\xi_r^{\star}, \xi_r^{\star}) \le 2(p_{ir}^{\star} - p_{ir}) + 2p_{ir}\sum_{j \in V \setminus \{i\}} (p_{jr}^{\star} - p_{jr})$$

(the first inequality follows by (32), the second one follows from (32) applied to Poisson random variables  $\xi_r^*$  and  $\xi_r^*$ ) we obtain

$$d_{TV}(d_i^{(3)}, d_i^{(4)}) \le 2 \sum_{r \in [m]} (p_{ir}^{\star} - p_{ir}) + 2 \sum_{r \in [m]} p_{ir} \sum_{j \in V \setminus \{i\}} (p_{jr}^{\star} - p_{jr}).$$
(39)

Using inequality  $0 \le p_{ir}^{\star} - p_{ir} \le (p_{ir}^{\star})^2$ , which follows from the first inequality of (26), we upper bound the first term on the right of (39)

$$\sum_{r \in [m]} (p_{ir}^{\star} - p_{ir}) \le \sum_{r \in [m]} (p_{ir}^{\star})^2 \le \sum_{r \in [m]} \frac{y_i^2 M^2}{\mu^2} = M^2 \frac{y_i^2}{n} = O(n^{-1}).$$
(40)

To show that the second term on the right of (39) is of the order o(1) we proceed as in (37), (38) above. Denote  $S' = \sum_{r \in [m]} p_{ir} \sum_{j \in V \setminus \{i\}} (p_{jr}^* - p_{jr})$ . Given  $\varepsilon > 0$  we estimate

$$p_{jr}^{\star} - p_{jr} \leq \begin{cases} p_{jr}^{\star} \leq y_j \frac{M}{\mu}, & \text{for } y_j > \varepsilon \mu, \\ (p_{jr}^{\star})^2 \leq \frac{y_j^2 M^2}{\mu^2} \leq \varepsilon y_j \frac{M^2}{\mu}, & \text{for } y_j \leq \varepsilon \mu. \end{cases}$$

Furthermore, we estimate  $p_{ir} \leq p_{ir}^{\star}$ . These inequalities imply

$$S' \leq \sum_{r \in [m]} \frac{y_i M}{\mu} \left( \varepsilon \frac{M^2}{\mu} \sum_{j: y_j \leq \varepsilon \mu} y_j + \frac{M}{\mu} \sum_{j: y_j > \varepsilon \mu} y_j \right)$$
$$\leq y_i M^2 \frac{nm}{\mu^2} \left( \varepsilon M \langle y \rangle + \int_{s > \varepsilon \mu} s P_{y,n}(ds) \right).$$

Choosing  $\varepsilon = \varepsilon_n \downarrow 0$  such that  $\varepsilon_n \mu \to \infty$  as  $n \to \infty$  we obtain  $\int_{s > \varepsilon_n \mu} s P_{y,n}(ds) = o(1)$  by the uniform integrability condition (ii). Hence S' = o(1).

Step 5. Here we show that  $d_{TV}(d_i^{(4)}, d_i^{(5)}) = o(1)$ . To this aim we apply (31) with  $\eta_r = \sum_{k=1}^{\zeta_r^o} \xi_r^o(k)$  and  $\kappa_r = \sum_{k=1}^{\zeta_r^\star} \xi_r^\star(k)$ . Invoking inequalities

$$d_{TV}(\eta_r, \kappa_r) \le 2 \left| \frac{\gamma_o}{\gamma} - 1 \right| p_{ir}^{\star} + p_{ir}^{\star} d_{TV}(\xi_r^{\star}, \xi_r^o) \\ \le 2 \left| \frac{\gamma_o}{\gamma} - 1 \right| p_{ir}^{\star} + p_{ir}^{\star} 2 \left| \mathbf{E} \xi_r^{\star} - \mathbf{E} \xi_r^o \right|$$

(the first inequality follows by (32), the second one follows from (32) applied to Poisson random variables  $\xi_r^*$  and  $\xi_r^o$ ) we obtain

$$d_{TV}(d_i^{(4)}, d_i^{(5)}) \le 2 \left| \frac{\gamma_o}{\gamma} - 1 \right| \sum_{r \in [m]} p_{ir}^{\star} + 2 \sum_{r \in [m]} p_{ir}^{\star} \left| \mathbf{E} \xi_r^{\star} - \mathbf{E} \xi_r^o \right| =: R_1 + 2R_2.$$
(41)

The first term on the right  $R_1 = o(1)$  because  $\sum_{r \in [m]} p_{ir}^* = y_i \gamma \langle x \rangle \leq y_i \gamma M$  is bounded and  $\gamma \to \gamma_o$ . To show that  $R_2 = o(1)$  we write  $\mathbf{E} \xi_r^o - \mathbf{E} \xi_r^*$  in the form

$$\mathbf{E}\xi_r^o - \mathbf{E}\xi_r^\star = x_r\delta + p_{ir}^\star, \qquad \delta := \gamma_o^{-1}a_y - \gamma^{-1}\langle y \rangle$$

and note that  $\delta = o(1)$ . We have

$$R_2 \le |\delta| \sum_{r \in [m]} x_r p_{ir}^{\star} + \sum_{r \in [m]} (p_{ir}^{\star})^2 = o(1).$$

Here we used  $\sum_{r \in [m]} x_r p_{ir}^* \le y_i M^2 \gamma = O(1)$  and  $\sum_{r \in [m]} (p_{ir}^*)^2 = O(n^{-1})$ , see (40).

Step 6. We write the Fourier transform of the probability distribution  $\mathcal{CP}(y_i a_x \gamma_o, Q)$ in the form  $f(t) = e^{y_i a_x \gamma_o(f_Q(t)-1)}, t \in \mathbb{R}$ . Here  $f_Q(t) = \int e^{\mathbf{i}ts} Q(ds)$  denotes the Fourier transform of the probability distribution Q;  $\mathbf{i} = \sqrt{-1}$  denotes the imaginary unit. We write the characteristic function of  $d_i^{(5)}$  in the form (recal that  $S_1(x) = \sum_{r=1}^m x_r$ )

$$\mathbf{E}e^{\mathbf{i}td_i^{(5)}} = \prod_{r=1}^m \exp\left\{ \left( e^{(e^{\mathbf{i}t}-1)\mathbf{E}\xi_r^o} - 1 \right) \mathbf{E}\zeta_r^o \right\}$$
$$= \exp\left\{ \gamma_o y_i \langle x \rangle \sum_{r=1}^m \frac{x_r}{S_1(x)} \left( e^{(e^{\mathbf{i}t}-1)\mathbf{E}\xi_r^o} - 1 \right) \right\}$$

We denote by  $\tilde{P}_X(ds) := \frac{s}{a_x} P_X(ds)$  the size biased distribution  $P_X$ . We denote by  $\tilde{P}_{x,m}(ds) = \sum_{r=1}^m \frac{x_r}{S_1(x)} \delta_{x_r}$  the size biased distribution  $P_{x,m}$ . Condition (i) implies that  $\tilde{P}_{x,m}$  converges weakly to  $\tilde{P}_X$  as  $m \to +\infty$ . Hence

$$\sum_{r=1}^{m} \frac{x_r}{S_1(x)} \left( e^{(e^{\mathbf{i}t} - 1)\mathbf{E}\xi_r^o} - 1 \right) = \int \left( e^{(e^{\mathbf{i}t} - 1)\lambda_s} - 1 \right) \tilde{P}_{x,m}(ds)$$

converges to

$$\int \left(e^{(e^{\mathbf{i}t}-1)\lambda_s}-1\right)P_X(ds) = f_Q(t) - 1.$$

This fact together with the convergence of the first moments  $\langle x \rangle \to a_x$  yield the pointwise convergence of the Fourier transforms  $\mathbf{E}e^{itd_i^{5}} \to f(t)$  as  $m \to \infty$ .

We have proved Theorem 2 in the case, where the sequence  $x_k, k \ge 1$  is bounded.

Now we waive the assumption of boundedness of  $x_k, k \ge 1$ . Let  $x_r, r \ge 1$  be a weight sequence satisfying condition (i). Given M > 0 define the truncated sequence  $x_{r,M} = x_r \mathbb{I}_{\{x_r \le M\}}, r \ge 1$ . Let  $P_{X,M}$  denote the probability distribution of  $X\mathbb{I}_{\{X \le M\}}$ . Condition (i) implies that  $\frac{1}{m} \sum_{r \in [m]} \delta_{x_{r,M}}$  converges weakly to  $P_{X,M}$  and  $\frac{1}{m} \sum_{r \in [m]} x_{r,M}$  converges to  $\mathbf{E}(X\mathbb{I}_{\{X \le M\}}) =: a_{x,M}$  as  $m \to \infty$ . Let  $d_{i,M}$  denote the degree of vertex  $i \in V$  in the affiliation random graph  $G'_M$  defined by the sequences  $y_k, k \ge 1$  and  $x_{k,M}, k \ge 1$ . Let  $D_M$ and D be random variables with the distributions  $\mathcal{CP}(y_i a_{x,M} \gamma_o, Q_M)$  and  $\mathcal{CP}(y_i a_x \gamma_o, Q)$ . We have

$$d_{TV}(d_i, D) \le d_{TV}(d_i, d_{i,M}) + d_{TV}(d_{i,M}, D_M) + d_{TV}(D_M, D)$$

Note that the first term on the right

$$\begin{aligned} d_{TV}(d_i, d_{i,M}) &\leq \mathbf{P}\{d_{i,M} \neq d_i\} \leq \mathbf{P}\{\exists u \in W : i \sim u, x_u > M\} \\ &\leq \mathbf{E}\left(\sum_{u \in W} \mathbb{I}_{\{i \sim u\}} \mathbb{I}_{\{x_u > M\}}\right) = \sum_{u \in W} p_{iu} \mathbb{I}_{\{x_u > M\}} \\ &\leq \sum_{u \in W} p_{iu}^* \mathbb{I}_{\{x_u > M\}} = y_i \gamma \int_{s > M} s P_{x,m}(ds) \end{aligned}$$

converges to 0 uniformly in m as  $M \to +\infty$ , by the uniform integrability condition (i). Furthermore, the second term  $d_{TV}(d_{i,M}, D_M) = o(1)$  as  $n, m \to \infty$  because the convergence in distribution of integer valued random variables implies the convergence in the total variation distance. Finally, for  $M \to +\infty$  we have  $d_{TV}(D_M, D) = o(1)$ , because  $a_{x,M} \to a_x$  and  $Q_M \to Q$ .

Now we show that  $d_{TV}(d_i, D) \to 0$  as  $n, m \to \infty$ . We fix  $\varepsilon > 0$  and choose large M such that  $d_{TV}(d_i, d_{i,M}) < \varepsilon$  and  $d_{TV}(D_M, D) < \varepsilon$ . Then, given M, we let  $n, m \to \infty$ . We obtain  $d_{TV}(d_i, D) \le 2\varepsilon + o(1)$ . Proof of Theorem 2 is complete.

Before the proof of Theorem 3 below we introduce some notation and state an auxiliary result. Let  $V_0^3$  denote the set of ordered triples (i, j, k) of distinct elements  $i, j, k \in V$ ; by  $\binom{V}{3}$  we denote the collection of subsets of V of size 3. For  $w \in W$  we denote

$$U_w = \sum_{i \in V} p_{iw}^{\star}, \qquad T(w) = \sum_{\{i,j,k\} \in \binom{V}{3}} p_{iw}^{\star} p_{jw}^{\star} p_{kw}^{\star}.$$

Furthermore, we denote

$$T'(u,v) = \sum_{(i,j,k) \in V_0^3} p_{ju}^{\star} p_{jv}^{\star} p_{kv}^{\star} \quad \text{and} \quad T' = \sum_{\{u,v\} \subset W} T'(u,v)$$

**Lemma 1.** Let  $\mu^2 = nm$ . Denote  $L_y = \langle y^2 \rangle \langle y \rangle^2 - \frac{2}{n} \langle y^3 \rangle \langle y \rangle - \frac{1}{n} \langle y^2 \rangle^2 + \frac{2}{n^2} \langle y^4 \rangle$ . We have

$$0 \le \frac{1}{6} \frac{n^3}{\mu^3} x_w^3 \langle y \rangle^3 - T(w) \le \frac{1}{2} \frac{n^2}{\mu^3} x_w^3 \langle y^2 \rangle \langle y \rangle, \tag{42}$$

$$T'_{3} = \frac{n}{2} \left( \langle x^{2} \rangle^{2} - \frac{\langle x^{4} \rangle}{m} \right) L_{y}.$$

$$\tag{43}$$

Proof of Lemma 1. Recall that V = [n]. For numbers  $a_i, b_i, c_i, i \in [n]$  we use notation  $S(a) = \sum_{i \in [n]} a_i, S(ab) = \sum_{i \in [n]} a_i b_i, S(abc) = \sum_{i \in [n]} a_i b_i c_i.$ 

Proof of (42). We apply identity

$$S^{3}(a) = \sum_{i \in [n]} a_{i}^{3} + 3 \sum_{i \in [n]} a_{i}^{2} \sum_{j \in [n] \setminus i} a_{j} + 6 \sum_{1 \le i < j < k \le n} a_{i} a_{j} a_{k}$$

to  $a_i = p_{iw}^{\star}$  and obtain inequalities that are equivalent to (42).

$$0 \le U_w^3 - 6T(w) \le 3\sum_{i \in V} (p_{iw}^{\star})^2 \sum_{j \in V} p_{jw}^{\star}$$

Proof of (43). We apply identity

$$\sum_{(i,j,k)\in V_0^3} a_i b_j c_k = S(a)S(b)S(c) - S(a)S(bc) - S(b)S(ac) - S(c)S(ab) + 2S(abc)$$

to  $a_i = p_{iu}^{\star}, b_j = p_{ju}^{\star} p_{jv}^{\star}, c_k = p_{kv}^{\star}$  and obtain  $T'(u, v) = \frac{n^3}{\mu^4} x_u^2 x_v^2 L_y$ .

Proof of Theorem 3. Let  $G_c = (V, E_c)$  be the multigraph with colored edges defined by the bipartite graph  $H = (V \cup W, E)$ : vertices  $i, j \in V$  are connected by an edge of color  $w \in W$  in  $G_c$  (denoted  $i \stackrel{w}{\sim} j$ ) whenever i, j are neighbours of w in H. A subgraph of  $G_c$  is projected to subgraph of G' by removing edge colors and merging obtained parallel edges.

Triangle count. Here we show (23). Note that each triangle of G' is either projection of a monochromatic triangle or projection of a triangle with all edges of different colors (we call such triangle of  $G_c$  polychromatic). Let  $\Delta_w$  be the set of monochromatic triangles of color w. Let  $\Delta_{u,v,z}$  be the set of polychromatic triangles with edge colors u, v, z. Let  $\Delta_w^*$ (respectively  $\Delta_{u,v,z}^*$ ) denote the set of triangles in G' obtained as projections of triangles from  $\Delta_w$  (respectively  $\Delta_{u,v,z}$ ). We denote

$$\Delta' = \cup_{w \in W} \Delta_w^* \qquad \text{and} \qquad \Delta'' = \cup_{\{u,v,z\} \subset W} \Delta_{u,v,z}^*.$$

The union  $\Delta' \cup \Delta''$  contains all triangles of G'. Hence  $N_{\Delta} = |\Delta' \cup \Delta''|$  and (23) follows from the relations shown below

$$|\Delta'| = \frac{n}{6\gamma} \langle x^3 \rangle \langle y \rangle^3 + O_P(\sqrt{n}), \tag{44}$$

$$\mathbf{E}|\Delta''| \le \frac{1}{6} \langle x^2 \rangle^3 \langle y^2 \rangle^3 = O_P(1).$$
(45)

Proof of (44). Denote  $T_1 = \sum_{w \in W} |\Delta_w^*|$  and  $T_2 = \sum_{\{u,v\} \subset W} |\Delta_u^* \cap \Delta_v^*|$ . By inclusion-exclusion inequalities we have

$$T_1 - T_2 \le |\Delta'| \le T_1.$$
 (46)

We show below that  $\mathbf{E}T_2 = o(1)$ . Hence  $|\Delta'| = T_1 + o_P(1)$ . Furthermore, we show that

$$\mathbf{E}T_1 = \frac{n}{6\gamma} \langle x^3 \rangle \langle y \rangle^3 + O(1) \tag{47}$$

and  $\operatorname{Var} T_1 = O(n)$ . These two relations imply  $T_1 = \mathbf{E} T_1 + O_P(\sqrt{n})$ , by Chebyshev's inequality. We have arrived to (44).

Let us show that  $\mathbf{E}T_2 = O(m^{-1})$ . We have  $|\Delta_u^* \cap \Delta_v^*| = \sum_{\{i,j,k\} \subset V} \mathbb{I}_{iu} \mathbb{I}_{ju} \mathbb{I}_{ku} \mathbb{I}_{iv} \mathbb{I}_{jv} \mathbb{I}_{kv}$  and

$$\mathbf{E}T_{2} = \mathbf{E}\left(\sum_{\{u,v\}\subset W}\sum_{\{i,j,k\}\subset V} \mathbb{I}_{iu}\mathbb{I}_{ju}\mathbb{I}_{ku}\mathbb{I}_{iv}\mathbb{I}_{jv}\mathbb{I}_{kv}\right)$$
$$\leq \sum_{\{u,v\}\subset W}\sum_{\{i,j,k\}\subset V} p_{iu}^{\star}p_{ju}^{\star}p_{ku}^{\star}p_{iv}^{\star}p_{jv}^{\star}p_{kv}^{\star}$$
(48)

$$\leq \frac{1}{12m} \langle x^3 \rangle^2 \langle y^2 \rangle^3. \tag{49}$$

Let us prove (47). Note that  $T_1 = \sum_{w \in W} |\Delta_w|$  since  $|\Delta_w^*| = |\Delta_w| \ \forall w \in W$ . We evaluate

$$\mathbf{E}|\Delta_w| = \sum_{\{i,j,k\} \subset V} \mathbb{I}_{iw} \mathbb{I}_{jw} \mathbb{I}_{kw} = \sum_{\{i,j,k\} \subset V} p_{iw} p_{jw} p_{kw}.$$

Using (26) we approximate  $\mathbf{E}|\Delta_w|$  by  $T(w) := \sum_{\{i,j,k\} \subset V} p_{iw}^{\star} p_{jw}^{\star} p_{kw}^{\star}$ ,

$$T(w) - R_1(w) \le \mathbf{E} |\Delta_w| \le T(w),$$

$$R_1(w) := \sum_{\{i,j,k\} \subset V} p_{iw}^* p_{jw}^* p_{kw}^* \left( p_{iw}^* + p_{jw}^* + p_{kw}^* \right).$$
(50)

A straightforward calculation shows

$$R_1(w) = \frac{1}{6} \sum_{(i,j,k) \in V_0^3} p_{iw}^{\star} p_{jw}^{\star} p_{kw}^{\star} \left( p_{iw}^{\star} + p_{jw}^{\star} + p_{kw}^{\star} \right) \le \frac{1}{2} \frac{n^3}{\mu^4} x_w^4 \langle y^2 \rangle \langle y \rangle^2.$$

Combining these inequalities with (42) and (50) we obtain

$$0 \le \frac{1}{6} \frac{n^3}{\mu^3} x_w^3 \langle y \rangle^3 - \mathbf{E} |\Delta_w| \le \frac{1}{2} \frac{n^3}{\mu^4} x_w^4 \langle y^2 \rangle \langle y \rangle^2 + \frac{1}{2} \frac{n^2}{\mu^3} x_w^3 \langle y^2 \rangle \langle y \rangle.$$
(51)

Summing over  $w \in W$  we obtain for  $\mu = \sqrt{nm}$ 

$$0 \le \frac{n}{6\gamma} \langle x^3 \rangle \langle y \rangle^3 - \mathbf{E}T_1 \le \frac{1}{2\gamma^2} \langle x^4 \rangle \langle y^2 \rangle \langle y \rangle^2 + \frac{1}{2\gamma} \langle x^3 \rangle \langle y^2 \rangle \langle y \rangle.$$
(52)

We have arrived to (47).

It remains to show that  $\operatorname{Var} T_1 = O(n)$ . By the independence of  $\Delta_w, w \in W$ , we have  $\operatorname{Var} T_1 = \sum_{w \in W} \operatorname{Var} |\Delta_w|$ . We show below that for each  $w \in W$ 

$$\mathbf{Var}|\Delta_w| \le \sum_{r=3}^5 \frac{x_w^r}{\gamma^r} \langle y \rangle^r.$$
(53)

This bound implies

$$\operatorname{Var} T_1 \leq \sum_{r=3}^5 \frac{m}{\gamma^r} \langle x^r \rangle \langle y \rangle^r = n \sum_{r=3}^5 \gamma^{2-r} \langle x^r \rangle \langle y \rangle^r = O(n).$$

Proof of (53). We fix  $w \in W$ . For  $A \subset V$  we denote  $\mathbb{I}_A = \prod_{i \in A} \mathbb{I}_{iw}$  and  $\overline{\mathbb{I}}_A = \mathbb{I}_A - \mathbb{E}\mathbb{I}_A$ . Note that  $|\Delta_w| = \sum_{A \in \binom{V}{3}} \mathbb{I}_A$ . Since  $\mathbb{E}(\overline{\mathbb{I}}_A \overline{\mathbb{I}}_{A'}) = 0$  for  $A \cap A' = \emptyset$ , we have

$$\mathbf{Var}|\Delta_{w}| = \mathbf{E} \left( \sum_{A \in \binom{V}{3}} \bar{\mathbb{I}}_{A} \right)^{2} = S_{1} + 2S_{2} + 2S_{3},$$

$$S_{1} := \sum_{A \in \binom{V}{3}} \mathbf{E} \bar{\mathbb{I}}_{A}^{2}, \qquad S_{2} := \sum_{\{A,A'\} \subset \binom{V}{3}: |A \cap A'| = 2} \mathbf{E} (\bar{\mathbb{I}}_{A} \bar{\mathbb{I}}_{A'}),$$

$$S_{3} := \sum_{\{A,A'\} \subset \binom{V}{3}: |A \cap A'| = 1} \mathbf{E} (\bar{\mathbb{I}}_{A} \bar{\mathbb{I}}_{A'}).$$
(54)

We upper bound the sums  $S_1, S_1, S_3$  using simple inequality  $\mathbf{E}(\bar{\mathbb{I}}_A \bar{\mathbb{I}}_{A'}) \leq \mathbf{E} \mathbb{I}_{A \cup A'}$ . A straightforward calculation shows that

$$S_{1} \leq \sum_{A \in \binom{V}{3}} \mathbf{E} \mathbb{I}_{A} = \sum_{\{i,j,k\} \subset V} p_{iw} p_{jw} p_{kw} \leq \sum_{\{i,j,k\} \subset V} p_{iw}^{\star} p_{jw}^{\star} p_{kw}^{\star} \leq \frac{U_{w}^{3}}{6} = \frac{x_{w}^{3}}{6\gamma^{3}} \langle y \rangle^{3}$$

$$S_{2} \leq \sum_{\{i,j\} \subset V} \sum_{\{k,l\} \subset V \setminus \{i,j\}} \mathbf{E} \mathbb{I}_{\{i,j,k\} \cup \{i,j,l\}} = \sum_{\{i,j\} \subset V} \sum_{\{k,l\} \subset V \setminus \{i,j\}} p_{iw} p_{jw} p_{kw} p_{lw} \leq \frac{U_{w}^{4}}{4} = \frac{x_{w}^{4}}{4\gamma^{4}} \langle y \rangle^{4},$$

$$2S_{3} \leq \sum_{i \in V} \sum_{\{j,k\} \subset V \setminus \{i\}} \sum_{\{s,t\} \subset V \setminus \{i,j,k\}} \mathbf{E} \mathbb{I}_{\{i,j,k\} \cup \{i,s,t\}}$$

$$= \sum_{i \in V} \sum_{\{j,k\} \subset V \setminus \{i\}} \sum_{\{s,t\} \subset V \setminus \{i,j,k\}} p_{iw} p_{jw} p_{kw} p_{sw} p_{tw}$$

$$\leq \frac{1}{4} \left(\sum_{i \in V} p_{iw}\right)^{5} \leq \frac{U_{w}^{5}}{4} = \frac{x_{w}^{5}}{4\gamma^{5}} \langle y \rangle^{5}.$$

Invoking these bounds in (54) and then summing (54) over  $w \in W$  we obtain (53).

Proof of (45). Given  $\{i, j, k\} \subset V$  and  $\{u, v, w\} \subset W$  the expected number of polychromatic triangles in  $G_c$  on vertices i, j, k and with edge colors u, v, w is the sum

$$\sum_{\pi} p_{i\pi_u} p_{j\pi_u} p_{i\pi_v} p_{k\pi_v} p_{j\pi_w} p_{k\pi_w}$$

that runs over permutations  $\pi = (\pi_u, \pi_v, \pi_w)$  of (u, v, w). Using  $p_{ab} \leq p_{ab}^{\star}$  we upper bound the sum by

$$\sum_{\pi} p_{i\pi_u}^{\star} p_{j\pi_u}^{\star} p_{i\pi_v}^{\star} p_{k\pi_v}^{\star} p_{j\pi_w}^{\star} p_{k\pi_w}^{\star} = \frac{6}{\mu^6} (x_u x_v x_w y_i y_j y_k)^2.$$

Consequently, we have

$$\mathbf{E}|\Delta''| \le \sum_{\{i,j,k\} \subset V} \sum_{\{u,v,w\} \subset W} \frac{6}{\mu^6} (x_u x_v x_w y_i y_j y_k)^2 \le \frac{1}{6} \langle x^2 \rangle^3 \langle y^2 \rangle^3.$$
(55)

Count of 2-paths. Here we show (24). For  $u, v \in W$  we denote by  $\Lambda_u$  (respectively  $\Lambda_{uv}$ ) the set of 2-paths of  $G_c$  with both edges colored u (respectively with edges receiving different colors u and v). Every 2-path of  $G_c$  is projected to a path in G' by removing edge colors. Let  $\Lambda_u^*$  (respectively  $\Lambda_{uv}^*$ ) denote the sets of 2-paths in G' obtained as projections of 2-paths of  $\Lambda_u$  (respectively  $\Lambda_{uv}$ ). Denote  $\Lambda' = \bigcup_{u \in W} \Lambda_u^*$  and  $\Lambda'' = \bigcup_{\{u,v\} \subset W} \Lambda_{uv}^*$ . The union  $\Lambda' \cup \Lambda''$ contains all 2-paths of G'. Hence  $N_{\Lambda} = |\Lambda' \cup \Lambda''|$ . To show (24) we use inclusion-exclusion identity

$$N_{\Lambda} = |\Lambda' \cup \Lambda''| = |\Lambda'| + |\Lambda''| - |\Lambda' \cap \Lambda''|$$

and evaluate each term on the right

$$|\Lambda'| = 3|\Delta'| = \frac{n}{2\gamma} \langle x^3 \rangle \langle y \rangle^3 + O_P(\sqrt{n}), \tag{56}$$

$$|\Lambda''| = \frac{n}{2} \langle x^2 \rangle^2 \langle y \rangle^2 \langle y^2 \rangle + O_P(\sqrt{n}).$$
(57)

$$\mathbf{E}|\Lambda' \cap \Lambda''| \le \frac{2}{\gamma} \langle x^3 \rangle \langle x^2 \rangle \langle y^2 \rangle^2 \langle y \rangle = O(1).$$
(58)

The first identity of (56) is obvious, the second relation is shown in (44). It remains to prove (57), (58).

Proof of (58). Given a path  $i \sim j \sim k \in \Lambda' \cap \Lambda''$  there exists  $\{u, v\} \in V$  such that the monochromatic 2-path  $i \stackrel{u}{\sim} j \stackrel{u}{\sim} k$  of color u is present in  $G_c$  and at least one of the edges  $i \stackrel{v}{\sim} j, j \stackrel{v}{\sim} k$  of color v is present in  $G_c$ . In the inequality below  $\mathbb{I}_{iu}\mathbb{I}_{ju}\mathbb{I}_{ku}$  represents the indicator of the monochromatic triangle and  $\mathbb{I}_{iv}\mathbb{I}_{jv}$  represents the indicator of additional edge of color v connecting i and j in  $G_c$ . We have

$$\begin{aligned} \mathbf{E}|\Lambda' \cap \Lambda''| &\leq \mathbf{E}\left(\sum_{u \in W} \sum_{v \in W \setminus \{u\}} \sum_{\{i,j,k\} \subset V} 2\mathbb{I}_{iu}\mathbb{I}_{ju}\mathbb{I}_{ku} \left(\mathbb{I}_{iv}\mathbb{I}_{jv} + \mathbb{I}_{iv}\mathbb{I}_{kv} + \mathbb{I}_{jv}\mathbb{I}_{kv}\right)\right) \\ &= 2\sum_{u \in W} \sum_{v \in W \setminus \{u\}} \sum_{\{i,j,k\} \subset V} p_{iu}p_{ju}p_{ku} \left(p_{iv}p_{jv} + p_{iv}p_{kv} + p_{jv}p_{kv}\right) \\ &\leq 2\sum_{u \in W} \sum_{v \in W \setminus \{u\}} \sum_{(i,j,k) \in V^3} p_{iu}p_{ju}p_{ku}p_{iv}p_{jv} \\ &\leq 2\sum_{u \in W} \sum_{v \in W \setminus \{u\}} \sum_{(i,j,k) \in V^3} p_{iu}^*p_{ju}^*p_{ku}^*p_{jv}^* \leq \frac{2}{\gamma} \langle x^3 \rangle \langle x^2 \rangle \langle y^2 \rangle^2 \langle y \rangle. \end{aligned}$$

*Proof of (57).* In the proof we use relations

$$|\Lambda_{uv}^*| \le |\Lambda_{uv}| = \sum_{(i,j,k) \in V_0^3} \mathbb{I}_{iu} \mathbb{I}_{jv} \mathbb{I}_{kv},$$
(59)

$$|\Lambda_{uv}| - |\Lambda_{uv}^*| = |\Lambda_u^* \cap \Lambda_v^*| = 3|\Delta_u^* \cap \Delta_v^*|.$$

$$\tag{60}$$

We only comment on (60) since (59) is obvious. The difference  $|\Lambda_{uv}| - |\Lambda_{uv}^*|$  counts 2-paths  $i \sim j \sim k \in \Lambda_{uv}^*$  such that both colored paths  $i \sim j \sim k$  and  $i \sim j \sim k$  are present in  $\Lambda_{uv}$ . This means that  $i, j, k \in N_u \cap N_v$ . In particular, we have  $|\Lambda_{uv}| - |\Lambda_{uv}^*| = |\Lambda_u^* \cap \Lambda_v^*|$ . Hence the first identity of (60). The second identity of (60) follows from the fact that in G' a triangle gives rise to three 2-paths.

Denote  $T_3 = \sum_{\{u,v\} \subset W} |\Lambda_{uv}|$ . We establish (57) in three steps: we show that

$$|\Lambda''| = \sum_{\{u,v\} \subset W} |\Lambda_{uv}^*| + O_P(1), \tag{61}$$

$$\sum_{\{u,v\} \subset W} |\Lambda_{uv}^*| = T_3 + O_P(1), \tag{62}$$

$$T_3 = \frac{n}{2} \langle x^2 \rangle^2 \langle y \rangle^2 \langle y^2 \rangle + O_P(\sqrt{n}).$$
(63)

Step 1. Here we prove (61). To this aim we apply inclusion-exclusion inequality

$$0 \le \sum_{\{u,v\} \subset W} |\Lambda_{uv}^*| - |\Lambda''| \le R_2, \qquad R_2 := \sum_{\substack{\{u,v\}, \{s,t\} \subset W: \\ \{u,v\} \ne \{s,t\}}} |\Lambda_{uv}^* \cap \Lambda_{st}^*|$$
(64)

and show that

$$\mathbf{E}R_2 \leq \frac{1}{n} \langle x^2 \rangle^4 \langle y^4 \rangle \langle y^2 \rangle^2 + \langle x^2 \rangle^3 \langle y \rangle \langle y^2 \rangle \langle y^3 \rangle + \frac{1}{\mu} \langle x^2 \rangle^2 \langle x^3 \rangle \langle y^2 \rangle^2 \langle y^3 \rangle.$$

To estimate  $\mathbf{E}R_2$  we split

$$R_{2} = R_{2.0} + R_{2.1}, \quad \text{where} \quad R_{2.k} = \sum_{\substack{\{u,v\}, \{s,t\} \subset W: \\ |\{u,v\} \cap \{s,t\}| = k}} |\Lambda_{uv}^{*} \cap \Lambda_{st}^{*}|$$

and upper bound  $\mathbf{E}R_{2.0}$ ,  $\mathbf{E}R_{2.1}$ . We start with  $\mathbf{E}R_{2.0}$ . For  $\{u, v\} \cap \{s, t\} = \emptyset$  we have

$$\begin{split} \mathbf{E}|\Lambda_{uv}^* \cap \Lambda_{st}^*| &\leq \mathbf{E} \sum_{(i,j,k) \in V_0^3} \mathbb{I}_{iu} \mathbb{I}_{ju} \mathbb{I}_{jv} \mathbb{I}_{kv} \left( \mathbb{I}_{is} \mathbb{I}_{js} \mathbb{I}_{jt} \mathbb{I}_{kt} + \mathbb{I}_{it} \mathbb{I}_{jt} \mathbb{I}_{js} \mathbb{I}_{ks} \right) \\ &= \mathbf{E} \sum_{(i,j,k) \in V_0^3} p_{iu} p_{ju} p_{jv} p_{kv} \left( p_{is} p_{js} p_{jt} p_{kt} + p_{it} p_{jt} p_{js} p_{ks} \right) \\ &\leq 2 \frac{n^3}{\mu^8} x_u^2 x_v^2 x_s^2 x_t^2 \langle y^4 \rangle \langle y^2 \rangle^2. \end{split}$$

Here the sum  $\sum_{(i,j,k)\in V_0^3} \mathbb{I}_{iu}\mathbb{I}_{jv}\mathbb{I}_{kv}$  runs over colored 2-paths  $i \stackrel{v}{\sim} j \stackrel{v}{\sim} k \in \Lambda_{uv}$  and  $\mathbb{I}_{is}\mathbb{I}_{js}\mathbb{I}_{jt}\mathbb{I}_{kt} + \mathbb{I}_{it}\mathbb{I}_{jt}\mathbb{I}_{js}\mathbb{I}_{ks}$  accounts for the matching 2-paths from  $\Lambda_{st}$ . In the very last inequality we used  $p_{pq} \leq p_{pq}^*$ , see (26). We obtain

$$\mathbf{E}R_{2.0} \leq \sum_{\substack{\{u,v\},\{s,t\}\subset W:\\|\{u,v\}\cap\{s,t\}|=0}} 2\frac{n^3}{\mu^8} x_u^2 x_v^2 x_s^2 x_t^2 \langle y^4 \rangle \langle y^2 \rangle^2 \leq \frac{1}{n} \langle x^2 \rangle^4 \langle y^4 \rangle \langle y^2 \rangle^2.$$

We similarly upper bound  $\mathbf{E}R_{2.1}$ . For u = s we have

$$\begin{split} \mathbf{E}|\Lambda_{uv}^* \cap \Lambda_{ut}^*| &\leq \mathbf{E} \sum_{(i,j,k) \in V_0^3} \mathbb{I}_{iu} \mathbb{I}_{ju} \mathbb{I}_{jv} \mathbb{I}_{kv} \left( \mathbb{I}_{iu} \mathbb{I}_{ju} \mathbb{I}_{jt} \mathbb{I}_{kt} + \mathbb{I}_{it} \mathbb{I}_{jt} \mathbb{I}_{ju} \mathbb{I}_{ku} \right) \\ &= \mathbf{E} \sum_{(i,j,k) \in V_0^3} p_{iu} p_{ju} p_{jv} p_{kv} \left( p_{jt} p_{kt} + p_{it} p_{jt} p_{ku} \right) \\ &\leq \frac{n^3}{\mu^6} x_u^2 x_v^2 x_t^2 \langle y \rangle \langle y^2 \rangle \langle y^3 \rangle + \frac{n^3}{\mu^7} x_u^3 x_v^2 x_t^2 \langle y^2 \rangle^2 \langle y^3 \rangle. \end{split}$$

Hence

$$\begin{split} \mathbf{E}R_{2.1} &\leq \sum_{\{u,v\} \subset W} \sum_{t \in W \setminus \{u,v\}} \mathbf{E} \left( |\Lambda_{uv}^* \cap \Lambda_{ut}^*| + |\Lambda_{uv}^* \cap \Lambda_{vt}^*| \right) \\ &\leq \langle x^2 \rangle^3 \langle y \rangle \langle y^2 \rangle \langle y^3 \rangle + \frac{1}{\mu} \langle x^2 \rangle^2 \langle x^3 \rangle \langle y^2 \rangle^2 \langle y^3 \rangle. \end{split}$$

Step 2. Here we show (63). From (60) and (49) we obtain

$$T_3 - \sum_{\{u,v\} \subset W} |\Lambda_{uv}^*| = 3 \sum_{\{u,v\} \subset W} |\Delta_u^* \cap \Delta_v^*| = 3T_2 = o_P(1).$$
(65)

Step 3. Here we prove (63). To this aim we show that

$$\mathbf{E}T_3 = \frac{n}{2} \langle x^2 \rangle^2 \langle y \rangle^2 \langle y^2 \rangle + O(1) \quad \text{and} \quad \mathbf{Var} \, T_3 = O(n).$$
(66)

Note that (66) combined with Chebyshev's inequality implies  $T_3 = \mathbf{E}T_3 + O_P(\sqrt{\mathbf{Var}T_3})$ .

Let us evaluate  $\mathbf{E}T_3$ . We have  $\mathbf{E}T_3 = \sum_{\{u,v\} \subset W} \sum_{(i,j,k) \in V_0^3} p_{iu} p_{ju} p_{jv} p_{kv}$ , by the second relation of (59). We approximate  $\mathbf{E}T_3$  by  $T' = \sum_{\{u,v\} \subset W} \sum_{(i,j,k) \in V_0^3} p_{iu}^* p_{jv}^* p_{kv}^*$ . Inequalities (26) imply  $0 \leq T' - \mathbf{E}T_3 \leq R'$ , where

$$\begin{aligned} R' &= \sum_{\{u,v\} \subset W} \sum_{(i,j,k) \in V_0^3} p_{iu}^{\star} p_{jv}^{\star} p_{kv}^{\star} (p_{iu}^{\star} + p_{ju}^{\star} + p_{jv}^{\star} + p_{kv}^{\star}) \\ &\leq \frac{1}{\gamma} \langle x^3 \rangle \langle x^2 \rangle \left( \langle y^2 \rangle^2 \langle y \rangle + \langle y^3 \rangle \langle y \rangle^2 \right). \end{aligned}$$

Combining this bound with (43) we obtain the first relation of (66).

It remains to prove the second relation of (66). Note that  $|\Lambda_{uv}|$  and  $|\Lambda_{s,t}|$  are independent for  $\{u, v\} \cap \{s, t\} = \emptyset$ . Hence

$$\operatorname{Var} T_3 = S_1 + S_2, \qquad S_k = \sum_{\substack{\{u,v\}, \{s,t\} \subset W: \\ |\{u,v\} \cap \{s,t\}| = k}} \operatorname{Cov}(|\Lambda_{uv}|, |\Lambda_{st}|), \qquad k = 1, 2.$$

In the rest part of the proof we show that  $S_1 = O(n)$  and  $S_2 = O(n)$ .

Let us evaluate  $S_2 = \sum_{\{u,v\} \subset W} \operatorname{Var} |\Lambda_{uv}|$ . Given  $\{u,v\} \subset W$  and  $(i,j,k) \in V_0^3$  we denote

Here the sum run over permutations  $\pi = (\pi_i, \pi_j, \pi_k)$  of i, j, k. We write  $|\Lambda_{\underline{uv}}| - \mathbf{E}|\Lambda_{\underline{uv}}|$  in the form  $\sum_{A \subset V, |A|=3} \overline{\mathbb{I}}_A$ , where  $\overline{\mathbb{I}}_A = \overline{\mathbb{I}}_{\{i,j,k\}}$  for  $A = \{i, j, k\}$ , and note that  $\overline{\mathbb{I}}_A$  and  $\overline{\mathbb{I}}_{A'}$  are uncorrelated unless  $|A \cap A'| \ge 1$ . Hence

$$\begin{aligned} \mathbf{Var}|\Lambda_{uv}| &= \mathbf{E} \left( \sum_{\substack{A \subset V, |A| = 3}} \bar{\mathbb{I}}_A \right)^2 = S_{2.1}(u, v) + S_{2.2}(u, v) + S_{2.3}(u, v), \\ S_{2.k}(u, v) &= \sum_{\substack{A \subset V: \\ |A| = 3}} \sum_{\substack{A' \subset V: |A'| = 3, \\ |A \cap A'| = k}} \mathbf{E}(\bar{\mathbb{I}}_A \bar{\mathbb{I}}_{A'}), \qquad k = 1, 2, 3. \end{aligned}$$

Now we upper bound the terms  $S_{2,r}(u, v)$ , r = 1, 2, 3. We use the simple inequality for the covariance of non-negative random variables X, Y,

$$\mathbf{E}(XY) - \mathbf{E}X\mathbf{E}Y \le \mathbf{E}(XY). \tag{68}$$

We apply (68) to sums of Bernoulli random variables. We have for distinct  $i, j, k, q, r \in V$ 

$$\mathbf{E}\bar{\mathbb{I}}^{2}_{(i,j,k)} \le \mathbf{E}\mathbb{I}_{(i,j,k)} = p_{iu}p_{ju}p_{jv}p_{kv} \le p_{iu}^{\star}p_{ju}^{\star}p_{jv}^{\star}p_{kv}^{\star} = \frac{1}{\mu^{4}}x_{u}^{2}x_{v}^{2}y_{i}y_{j}^{2}y_{k}, \tag{69}$$

$$\mathbf{E}\left(\bar{\mathbb{I}}_{\{i,j,k\}}\left(\bar{\mathbb{I}}_{\{i,j,q\}} + \bar{\mathbb{I}}_{\{i,q,k\}} + \bar{\mathbb{I}}_{\{q,j,k\}}\right)\right) \leq \mathbf{E}\left(\mathbb{I}_{\{i,j,k\}}\left(\mathbb{I}_{\{i,j,q\}} + \mathbb{I}_{\{i,q,k\}} + \mathbb{I}_{\{q,j,k\}}\right)\right) \\
\leq \mathbf{E}\left(\mathbb{I}_{\{i,j,k\}}9\left(\mathbb{I}_{qu} + \mathbb{I}_{qv}\right)\right) = 9p_{iu}p_{ju}p_{jv}p_{kv}\left(p_{qu} + p_{qv}\right) \tag{70}$$

$$\leq 9p_{iu}^{\star}p_{ju}^{\star}p_{jv}^{\star}p_{kv}^{\star}\left(p_{qu}^{\star}+p_{qv}^{\star}\right) = \frac{9}{\mu^{5}}x_{u}^{2}x_{v}^{2}(x_{u}+x_{v})y_{i}y_{j}^{2}y_{k}y_{q},\tag{71}$$

$$\mathbf{E}\left(\bar{\mathbb{I}}_{(i,j,k)}\left(\bar{\mathbb{I}}_{\{i,r,q\}} + \bar{\mathbb{I}}_{\{r,j,q\}} + \bar{\mathbb{I}}_{\{r,q,k\}}\right)\right) \leq \mathbf{E}\left(\mathbb{I}_{(i,j,k)}\left(\mathbb{I}_{\{i,r,q\}} + \mathbb{I}_{\{r,j,q\}} + \mathbb{I}_{\{r,q,k\}}\right)\right) \\
\leq \mathbf{E}\left(\mathbb{I}_{(i,j,k)}9\left(\mathbb{I}_{qu}\mathbb{I}_{rv} + \mathbb{I}_{qv}\mathbb{I}_{ru}\right)\right) = 9p_{iu}p_{ju}p_{jv}p_{kv}\left(p_{qu}p_{rv} + p_{ru}p_{qv}\right) \tag{72}$$

$$\leq 9p_{iu}^{\star}p_{ju}^{\star}p_{jv}^{\star}p_{kv}^{\star}\left(p_{qu}^{\star}p_{rv}^{\star}+p_{ru}^{\star}p_{qv}^{\star}\right) = \frac{18}{\mu^6}x_u^3x_v^3y_iy_j^2y_ky_qy_r.$$
(73)

In the first inequality of (70) (respectively (72)) we used inequality  $\mathbb{I}_{\{l,r,q\}} \leq 3(\mathbb{I}_{qu} + \mathbb{I}_{qv})$ (respectively  $\mathbb{I}_{\{l,r,q\}} \leq 3(\mathbb{I}_{qu}\mathbb{I}_{rv} + \mathbb{I}_{qv}\mathbb{I}_{ru})$ ) which hold for any distinct  $l, q, r \in V$ . Estimation of  $S_{2,3}(u, v)$ . Combining the inequality  $(\sum_{i=1}^{6} a_i)^2 \leq 6\sum_{i=1}^{6} a_i^2$ , which follows

by Hölder's inequality, with (69) we obtain

$$\mathbf{E}\bar{\mathbb{I}}^{2}_{\{i,j,k\}} \leq 6\sum_{\pi} \mathbf{E}\bar{\mathbb{I}}^{2}_{(\pi_{i},\pi_{j},\pi_{k})} \leq \frac{6}{\mu^{4}} x_{u}^{2} x_{v}^{2} y_{i} y_{j} y_{k} (2y_{i} + 2y_{j} + 2y_{k}).$$

Summing over  $\{i, j, k\} \subset V$  we have

$$S_{2.3}(u,v) = \sum_{\{i,j,k\} \subset V} \mathbf{E}\bar{\mathbb{I}}^2_{\{i,j,k\}} \le 6 \frac{n}{m^2} x_u^2 x_v^2 \langle y^2 \rangle \langle y \rangle^2.$$
(74)

Estimation of  $S_{2,2}(u, v)$ . Here we use inequality (71). We have

$$S_{2.2}(u,v) = \sum_{(i,j,k)\in V_0^3} \sum_{q\in V\setminus\{i,j,k\}} \mathbf{E} \left( \bar{\mathbb{I}}_{\{i,j,k\}} \left( \bar{\mathbb{I}}_{\{i,j,q\}} + \bar{\mathbb{I}}_{\{i,q,k\}} + \bar{\mathbb{I}}_{\{q,j,k\}} \right) \right)$$
(75)  
$$\leq \frac{9}{\mu^5} \sum_{(i,j,k)\in V_0^3} \sum_{q\in V\setminus\{i,j,k\}} x_u^2 x_v^2 (x_u + x_v) y_i y_j^2 y_k y_q$$
  
$$\leq \frac{9}{\gamma} \frac{n}{m^2} x_u^2 x_v^2 (x_u + x_v) \langle y^2 \rangle \langle y \rangle^3.$$

Estimation of  $S_{2,1}(u, v)$ . Here we use inequality (73). We have

$$S_{2.1}(u,v) = \sum_{(i,j,k)\in V_0^3} \sum_{\{q,r\}\subset V\setminus\{i,j,k\}} \mathbf{E}\left(\bar{\mathbb{I}}_{\{i,j,k\}}\left(\bar{\mathbb{I}}_{\{i,r,q\}} + \bar{\mathbb{I}}_{\{r,j,q\}} + \bar{\mathbb{I}}_{\{r,q,k\}}\right)\right)$$
(76)  
$$\leq \frac{18}{\mu^6} \sum_{(i,j,k)\in V_0^3} \sum_{\{q,r\}\subset V\setminus\{i,j,k\}} x_u^3 x_v^3 y_i y_j^2 y_k y_q y_r$$
  
$$\leq \frac{9}{\gamma^2} \frac{n}{m^2} x_u^3 x_v^3 \langle y^2 \rangle \langle y \rangle^4.$$

Combining (74), (75) and (76) we obtain

$$S_{2} = \sum_{\{u,v\} \subset W} \operatorname{Var}|\Lambda_{uv}|$$
  
$$\leq n \left( 3\langle x^{2} \rangle^{2} \langle y^{2} \rangle \langle y \rangle^{2} + \frac{9}{\gamma} \langle x^{3} \rangle \langle x^{2} \rangle \langle y^{2} \rangle \langle y \rangle^{3} + \frac{9}{2\gamma^{2}} \langle x^{3} \rangle^{2} \langle y^{2} \rangle \langle y \rangle^{4} \right) = O(n).$$

Let us evaluate  $S_1$ . To this aim we write  $S_1$  in the form

$$S_1 = \sum_{\{u,v\} \subset W} \sum_{t \in W \setminus \{u,v\}} \left( \mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{ut}|) + \mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{vt}|) \right)$$

and evaluate  $\mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{ut}|)$  and  $\mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{vt}|)$ . We now focus on  $\mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{ut}|)$ . We fix u, v, t and use notation (67). In addition, we denote

$$\mathbf{I}_{(i,j,k)} = \mathbb{I}_{iu} \mathbb{I}_{ju} \mathbb{I}_{jt} \mathbb{I}_{kt}, \qquad \qquad \bar{\mathbf{I}}_{(i,j,k)} = \mathbf{I}_{(i,j,k)} - \mathbf{E} \mathbf{I}_{(i,j,k)}.$$

In view of the second relation of (59) we have

$$|\Lambda_{uv}| - \mathbf{E}|\Lambda_{uv}| = \sum_{(i,j,k) \in V_0^3} \overline{\mathbb{I}}_{(i,j,k)}, \qquad |\Lambda_{ut}| - \mathbf{E}|\Lambda_{ut}| = \sum_{(p,q,r) \in V_0^3} \overline{\mathbf{I}}_{(p,q,r)}.$$

Furthermore, since for  $\{i, j\} \cap \{p, q\} = \emptyset$  random variables  $\overline{\mathbb{I}}_{(i,j,k)}, \overline{\mathbf{I}}_{(p,q,r)}$  are independent we have

$$\mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{ut}|) = \sum_{(i,j,k) \in V_0^3} \sum_{(p,q,r) \in V_0^3} \mathbf{E}\left(\bar{\mathbb{I}}_{(i,j,k)}\bar{\mathbf{I}}_{(p,q,r)}\right)$$
(77)  
$$= \sum_{(i,j,k) \in V_0^3} \sum_{\substack{(p,q,r) \in V_0^3:\\ \{p,q\} \cap \{i,j\} \neq \emptyset}} \mathbf{E}\left(\bar{\mathbb{I}}_{(i,j,k)}\bar{\mathbf{I}}_{(p,q,r)}\right)$$
$$=: S_1'(u,t;v) + S_2'(u,t;v).$$

Here, for h = 1, 2, we denote

$$S'_{h}(u,t;v) = \sum_{\substack{(i,j,k) \in V_{0}^{3} \\ |\{p,q\} \cap \{i,j\}| = h}} \sum_{\substack{(p,q,r) \in V_{0}^{3}: \\ |\{p,q\} \cap \{i,j\}| = h}} \mathbf{E}\left(\bar{\mathbb{I}}_{(i,j,k)}\bar{\mathbf{I}}_{(p,q,r)}\right).$$

Now we estimate  $S'_h(u,t;v)$  using inequalities (68) and  $p_{ab} \leq p^{\star}_{ab} \leq x_b y_a/\mu$ , see (26). Estimation of  $S'_2(u,t;v)$ . We have

$$\begin{split} S_2'(u,t;v) &= \sum_{(i,j,k)\in V_0^3} \sum_{r\in V\setminus\{i,j\}} \mathbf{E} \left( \bar{\mathbb{I}}_{(i,j,k)}(\bar{\mathbf{I}}_{(i,j,r)} + \bar{\mathbf{I}}_{(j,i,r)}) \right) \\ &\leq \sum_{(i,j,k)\in V_0^3} \sum_{r\in V\setminus\{i,j\}} \mathbf{E} \left( \mathbb{I}_{(i,j,k)}(\mathbf{I}_{(i,j,r)} + \mathbf{I}_{(j,i,r)}) \right) \\ &= \sum_{(i,j,k)\in V_0^3} \sum_{r\in V\setminus\{p,q\}} p_{iu} p_{jv} p_{kv}(p_{jt} p_{rt} + p_{it} p_{rt}) \\ &\leq \frac{n^4}{\mu^6} x_u^2 x_v^2 x_t^2 \left( \langle y^3 \rangle \langle y \rangle^3 + \langle y^2 \rangle^2 \langle y \rangle^2 \right). \end{split}$$

Estimation of  $S'_1$ . Denote for short  $a_{(i,j,q,r)} = \mathbf{I}_{(i,q,r)} + \mathbf{I}_{(q,i,r)} + \mathbf{I}_{(j,q,r)} + \mathbf{I}_{(q,j,r)}$  and  $\bar{a}_{(i,j,q,r)} = a_{(i,j,q,r)} - \mathbf{E} a_{(i,j,q,r)}$ . We have

$$S'_{1}(u,t;v) = \sum_{\substack{(i,j,k) \in V_{0}^{3} \\ \{q,r\} \cap \{i,j\} = \emptyset}} \sum_{\substack{(i,j,k) \in V^{2}: q \neq r, \\ \{q,r\} \cap \{i,j\} = \emptyset}} \mathbf{E} \left( \bar{\mathbb{I}}_{(i,j,k)} \bar{a}_{(i,j,q,r)} \right).$$

Invoking inequalities

$$\begin{split} \mathbf{E} \left( \bar{\mathbb{I}}_{(i,j,k)} \bar{a}_{(i,j,q,r)} \right) &\leq \mathbf{E} \left( \mathbb{I}_{(i,j,k)} a_{(i,j,q,r)} \right) \\ &= p_{iu} p_{ju} p_{jv} p_{kv} \left( p_{qu} p_{qt} p_{rt} + p_{qu} p_{it} p_{rt} + p_{qu} p_{jt} p_{rt} \right) \\ &\leq \frac{1}{\mu^6} x_u^3 x_v^2 x_t^2 \left( 2y_i y_j^2 y_k y_q^2 y_t + y_i^2 y_j^2 y_k y_q y_r + y_i y_j^3 y_k y_q y_r \right) \end{split}$$

we obtain

$$S_1'(u,t;v) \le \frac{n^5}{\mu^7} x_u^3 x_v^2 x_t^2 \left( 3 \langle y^2 \rangle^2 \langle y \rangle^3 + \langle y^3 \rangle \langle y \rangle^4 \right).$$

Combining the upper bounds for  $S'_1(u, v; t)$ ,  $S'_2(u, v; t)$  above with (77) we obtain

$$\sum_{\{u,v\}\subset W} \sum_{t\in W\setminus\{u,v\}} \mathbf{Cov}(|\Lambda_{uv}|,|\Lambda_{ut}|) \le n\langle x^2\rangle^3 \left(\langle y^3\rangle\langle y\rangle^3 + \langle y^2\rangle^2\langle y\rangle^2\right) \\ + \frac{n}{\gamma} \langle x^3\rangle\langle x^2\rangle^2 \left(3\langle y^2\rangle^2\langle y\rangle^3 + \langle y^3\rangle\langle y\rangle^4\right)$$

The same bound holds for  $\sum_{\{u,v\} \subset W} \sum_{t \in W \setminus \{u,v\}} \mathbf{Cov}(|\Lambda_{uv}|, |\Lambda_{vt}|)$ . Hence  $S_1 = O(n)$ . Proof of Theorem 3 is complete.

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