

# DUALITY FOR $p$ -ADIC GEOMETRIC PRO-ÉTALE COHOMOLOGY I: A FARGUES-FONTAINE AVATAR

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ABSTRACT.  $p$ -adic geometric pro-étale cohomology of smooth partially proper rigid analytic varieties over  $p$ -adic fields can be represented by solid quasi-coherent sheaves on the Fargues-Fontaine curve. We prove that these sheaves satisfy a Poincaré duality. This is done by passing, via comparison theorems, to analogous sheaves representing syntomic cohomology and then reducing to Poincaré duality for  $\mathbf{B}_{\text{st}}^+$ -twisted Hyodo-Kato and filtered  $\mathbf{B}_{\text{dR}}^+$ -cohomologies that, in turn, reduce to Serre duality for smooth Stein varieties – a classical result. A similar computation yields a Künneth formula.

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## 1. INTRODUCTION

Let  $K$  be a finite extension of  $\mathbf{Q}_p$  and  $C = \widehat{\mathbf{Q}}_p$ . In [10, Th. 1.1], we have established a Poincaré duality for the arithmetic  $p$ -adic pro-étale cohomology of dagger curves over  $K$  and stated a conjecture for the existence of such a duality for partially proper rigid analytic varieties of arbitrary dimension. In this paper, we investigate the possibility of existence of geometric (i.e., for pro-étale cohomology over  $C$ ) Poincaré duality for such varieties. The case of the open unit disc  $D$  of dimension 1 shows that such a duality cannot be the naive one (except in the case of

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proper varieties). Indeed, using syntomic methods as in [12, 15], we obtain that the only nontrivial cohomology groups are as follows:

$$(1.1) \quad \begin{aligned} H^0(D_C, \mathbf{Q}_p(1)) &\simeq \mathbf{Q}_p(1), & H^1(D_C, \mathbf{Q}_p(1)) &\simeq \mathcal{O}(D_C)/C, \\ H_c^2(D_C, \mathbf{Q}_p(1)) &\simeq \mathbf{Q}_p \oplus \mathcal{O}(\partial D_C)/\mathcal{O}(D_C), \end{aligned}$$

where  $\partial D_C$  denotes the "boundary of  $D_C$ ". Since we have the isomorphisms

$$\mathcal{O}(D_C)/C \xrightarrow{\sim} \Omega^1(D_C), \quad \mathcal{O}(\partial D_C)/\mathcal{O}(D_C) \xrightarrow{\sim} H_c^1(D_C, \mathcal{O}),$$

we see in (1.1) a Serre duality as well as a simple  $\mathbf{Q}_p$ -duality but they do not fit together into an obvious duality.

We will show in this paper that if we see the pro-étale cohomology as living on the Fargues-Fontaine curve then we do have a Poincaré duality. Recall that the  $p$ -adic geometric pro-étale cohomology of a smooth partially proper rigid analytic variety  $X$  over  $K$  can be represented by a solid quasi-coherent sheaf on the Fargues-Fontaine curve, i.e., the pro-étale cohomology can be computed as

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \simeq \mathrm{R}\Gamma(X_{\mathrm{FF}, C^b}, \mathcal{E}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p)),$$

for a (nuclear) solid quasi-coherent sheaf  $\mathcal{E}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p)$  on the Fargues-Fontaine curve  $X_{\mathrm{FF}, C^b}$  defined using relative period sheaves. Similarly, geometric compactly supported pro-étale cohomology  $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}, c}(X_C, \mathbf{Q}_p)$  can be represented by solid quasi-coherent sheaf  $\mathcal{E}_{\mathrm{pro\acute{e}t}, c}(X_C, \mathbf{Q}_p)$  on  $X_{\mathrm{FF}, C^b}$ . See Section 4.1.2 for the definitions.

Via comparison theorems, we see that, if  $* \in \{, c\}$ ,

$$\mathcal{E}_{\mathrm{pro\acute{e}t}, *}(X_C, \mathbf{Q}_p(r)) \simeq \mathcal{E}_{\mathrm{syn}, *}(X_C, \mathbf{Q}_p(r)), \quad r \geq 2d,$$

where  $d$  is the dimension of  $X$  and  $\mathcal{E}_{\mathrm{syn}, *}(X_C, \mathbf{Q}_p(r))$  is the syntomic cohomology sheaf (a solid quasi-coherent sheaf on the Fargues-Fontaine curve representing syntomic cohomology; see Section 3.2 for a definition). This is equivalent to proving a comparison theorem between corresponding Frobenius equivariant sheaves on the Fargues-Fontaine curve  $Y_{\mathrm{FF}, C^b}$ , which amounts to untwisting Frobenius from classical comparison theorems. Luckily for us, the proofs of comparison theorems in [13] and [1] do actually (implicitly) prove the untwisted versions (see Theorem 4.6 for details).

Recall that classical syntomic cohomology is built from  $(\varphi, N)$ -eigenspaces of  $\mathbf{B}_{\mathrm{st}}^+$ -twisted Hyodo-Kato cohomology and from filtered  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology. Representing it (in a stable range) by the sheaf  $\mathcal{E}_{\mathrm{syn}, *}(X_C, \mathbf{Q}_p(r))$  on the Fargues-Fontaine curve separates these terms: heuristically speaking, the (completed) stalks of  $\mathcal{E}_{\mathrm{syn}, *}(X_C, \mathbf{Q}_p(r))$  at points outside  $\infty$  are  $N$ -eigenspaces of  $\mathbf{B}_{\mathrm{st}}^+$ -twisted Hyodo-Kato cohomology and the (completed) stalk at  $\infty$  is the  $r$ -th filtration level of  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology.

Now, the stalk cohomology sheaves satisfy Poincaré duality: Poincaré duality for Hyodo-Kato cohomology reduces, via the Hyodo-Kato isomorphism, to that for de Rham cohomology and Poincaré duality for filtered de Rham cohomology, in turn, reduces to Serre duality for smooth Stein varieties – a classical result (see [1] and Section 5.1 for details). These dualities are inherited by the sheaves  $\mathcal{E}_{\mathrm{syn}, *}(X_C, \mathbf{Q}_p(r))$ , for  $r \geq 2d$ , and then by the sheaves  $\mathcal{E}_{\mathrm{pro\acute{e}t}, *}(X_C, \mathbf{Q}_p(r))$  yielding the main result of this paper:

**Theorem 1.2.** (Poincaré duality for pro-étale sheaves) *We have a natural, Galois equivariant, quasi-isomorphisms in  $\mathrm{QCoh}(X_{\mathrm{FF}, C^b})$*

$$(1.3) \quad \mathcal{E}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF}, C^b})}(\mathcal{E}_{\mathrm{pro\acute{e}t}, c}(X_C, \mathbf{Q}_p(d))[2d], \mathcal{O}).$$

The proof of the theorem does not proceed as sketched above though, due to the difficulties of passing to stalks in the theory of solid quasi-coherent sheaves. Instead we argue in a similar vein with  $\varphi$ -modules on the  $Y_{\mathrm{FF}, C^b}$ -curve. In the last part of the paper, we sketch an alternative proof of Theorem 1.2 that, instead of passing to the  $Y_{\mathrm{FF}, C^b}$ -curve, uses dual modifications.

Analogous argument, with splitting into Hyodo-Kato and de Rham terms, yields a Künneth formula:

**Theorem 1.4.** (Künneth formula) *Let  $X, Y$  be smooth partially proper varieties over  $K$ . Then the canonical map*

$$\kappa : \mathcal{E}_{\text{proét}}(X_C, \mathbf{Q}_p) \otimes_{\mathcal{O}}^{\mathbb{L}} \mathcal{E}_{\text{proét}}(Y_C, \mathbf{Q}_p) \rightarrow \mathcal{E}_{\text{proét}}((X \times_K Y)_C, \mathbf{Q}_p)$$

*is a quasi-isomorphism in  $\text{QCoh}(X_{\text{FF}, C^b})$ .*

**Remark 1.5.** In Theorem 1.2 and Theorem 1.4, we can replace  $C$ , functorially, with any affinoid perfectoid over  $C$ .

**Remark 1.6.** This paper<sup>1</sup>, is the first one in a series of two papers, the second of which will descend the duality (1.3) to the "real" world, which for us is the world of Topological Vector Spaces (that is, topologically enriched Vector Spaces). It is most likely that for publication the two papers will be combined.

**Remark 1.7.** (*Related work*) There is an ongoing project of Johannes Anschütz, Arthur-César Le Bras, and Lucas Mann on developing 6-functor formalism for solid quasi-coherent sheaves on the Fargues-Fontaine curve. They have announced that it includes Poincaré duality of the type proved in this paper for smooth rigid analytic varieties.

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*Notation and conventions.* Let  $K$  be a finite extension of  $\mathbf{Q}_p$ . Let  $\mathcal{O}_K$  be the ring of integers in  $K$ , and  $k$  be its residue field. Let  $W(k)$  be the ring of Witt vectors of  $k$  and let  $F$  be its fraction field (i.e.,  $W(k) = \mathcal{O}_F$ ).

Let  $\overline{K}$  be an algebraic closure of  $K$  and let  $\mathcal{O}_{\overline{K}}$  denote the integral closure of  $\mathcal{O}_K$  in  $\overline{K}$ . Let  $C = \widehat{\overline{K}}$  be the  $p$ -adic completion of  $\overline{K}$ . Set  $\mathcal{G}_K = \text{Gal}(\overline{K}/K)$  and let  $\varphi$  be the absolute Frobenius on  $W(\overline{k})$ .

We will denote by  $\mathbf{B}_{\text{cr}}, \mathbf{B}_{\text{st}}, \mathbf{B}_{\text{dR}}$  the crystalline, semistable, and de Rham period rings of Fontaine.

All rigid analytic spaces and dagger spaces considered will be over  $K$  or  $C$ ; we assume that they are separated, taut, and countable at infinity. Huber pairs will always be sheafy. The category of affinoid perfectoid spaces over an affinoid perfectoid space  $S$  over  $C$  will be denoted by  $\text{Perf}_S$ .

We will use condensed mathematics as developed in [7], [8]. We fix an implicit cut-off cardinal  $\kappa$  (in the sense of [17, Sec. 4]), and assume all our perfectoid spaces, and condensed sets to be  $\kappa$ -small.

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<sup>1</sup>It was written in connection with W. N.'s lectures at a workshop in Singapore, in November 2024.

We will use the bracket notation for certain limits:  $[C_1 \xrightarrow{f} C_2]$  denotes the mapping fiber of  $f$  and we set

$$\left[ \begin{array}{ccc} C_1 & \xrightarrow{f_1} & K_1 \\ \downarrow & & \downarrow \\ C_2 & \xrightarrow{f_2} & K_2 \end{array} \right] := [[C_1 \xrightarrow{f_1} K_1] \rightarrow [C_2 \xrightarrow{f_2} K_2]].$$

## 2. QUASI-COHERENT SHEAVES ON THE FARGUES-FONTAINE CURVE

Here, we will review briefly basic facts concerning quasi-coherent sheaves on the Fargues-Fontaine curve. This is partly based on [2], [3], and [6, Sec. 6.2].

**2.1. Fargues-Fontaine curve.** Recall the definition of the relative Fargues-Fontaine curve (see [18, Lecture 12]). Let  $S = \mathrm{Spa}(R, R^+)$  be an affinoid perfectoid space over the finite field  $\mathbf{F}_p$ . Let

$$Y_{\mathrm{FF},S} := \mathrm{Spa}(W(R^+), W(R^+)) \setminus V(p[p^b])$$

be the relative mixed characteristic punctured unit disc. It is an analytic adic space over  $\mathbf{Q}_p$ . The Frobenius on  $R^+$  induces the Witt vector Frobenius and hence a Frobenius  $\varphi$  on  $Y_{\mathrm{FF},S}$  with free and totally discontinuous action. The Fargues-Fontaine curve relative to  $S$  (and  $\mathbf{Q}_p$ ) is defined as

$$X_{\mathrm{FF},S} := Y_{\mathrm{FF},S}/\varphi^{\mathbf{Z}}.$$

For an interval  $I = [s, r] \subset (0, \infty)$  with rational endpoints, we have the open subset

$$Y_{\mathrm{FF},S,I} := \{|\cdot| : |p|^r \leq |[p^b]| \leq |p|^s\} \subset Y_{\mathrm{FF},S}.$$

It is a rational open subset of  $\mathrm{Spa}(W(R^+), W(R^+))$  hence an affinoid space,

$$Y_{\mathrm{FF},S,I} := \mathrm{Spa}(\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+).$$

One can form  $X_{\mathrm{FF},S}$  as the quotient of  $Y_{\mathrm{FF},S,[1,p]}$  via the identification  $\varphi : Y_{\mathrm{FF},S,[1,1]} \xrightarrow{\sim} Y_{\mathrm{FF},S,[p,p]}$ . If  $S = \mathrm{Spa}(C^b, \mathcal{O}_{C^b})$ , we will write  $Y_{\mathrm{FF}}, X_{\mathrm{FF}}, Y_{\mathrm{FF},I}, \mathbf{B}_I, \mathbf{B}_I^+$ .

We will denote by  $x_\infty$  the  $(C, \mathcal{O}_C)$ -point of the curve  $X_{\mathrm{FF}}$  corresponding to Fontaine's map  $\theta : W(\mathcal{O}_C) \rightarrow \mathcal{O}_C$ , by  $y_\infty$  the corresponding point on  $Y_{\mathrm{FF}}$ , and by  $\iota_\infty : \mathrm{Spa}(C, \mathcal{O}_C) \rightarrow T_{\mathrm{FF}}$ ,  $T = X, Y$ , the corresponding closed immersions. More generally, if  $S$  is the tilt of a perfectoid space  $S^\sharp$  over  $\mathrm{Spa}(\mathbf{Q}_p)$ , there is an induced closed immersion  $\theta : S^\sharp \hookrightarrow Y_{\mathrm{FF},S}$  which is locally given by Fontaine's map  $\theta : W(R^+) \rightarrow R^{\sharp,+}$ . We will denote by  $\iota_\infty : S^\sharp \xrightarrow{\theta} T_{\mathrm{FF},S}$  the induced closed immersions and by  $y_\infty, x_\infty$ , the corresponding divisors.

We set

$$\mathbf{B}_S := \lim_{I \subset (0, \infty)} \mathbf{B}_{S,I},$$

where  $I$  varies over all the compact intervals of  $(0, \infty)$  with rational endpoints. We will denote by  $\mathbf{B}_{S,\log}$  the log-crystalline period ring (see [14, Sec. 10.3.1]). We have  $\mathbf{B}_S[U] \xrightarrow{\sim} \mathbf{B}_{S,\log}$ ,  $U \mapsto \log([p^b]/p)$ , with  $\varphi(U) = pU, \sigma(U) = U + \log[\sigma(p^b)/p^b]$ , for  $\sigma \in \mathcal{G}_K$ , and  $N = -d/dU$ . We define  $\mathbf{B}_{S,I,\log}$  in a similar manner.

**2.2. Quasi-coherent sheaves on the Fargues-Fontaine curve.** We will present now quasi-coherent sheaves on  $X_{\mathrm{FF}}$  as  $\varphi$ -modules on a convenient chart of  $Y_{\mathrm{FF}}$ .

**2.2.1. Solid quasi-coherent sheaves.** We start with a brief survey of solid quasi-coherent sheaves. Let  $Y$  be an analytic adic space over  $\mathbf{Q}_p$ . We denote by  $\mathrm{QCoh}(Y)$  the  $\infty$ -category of solid quasi-coherent sheaves on  $Y$ , and by  $\mathrm{Nuc}(Y)$  the full  $\infty$ -subcategory of solid nuclear sheaves on  $Y$ . See [2], [3] for the definitions of these categories and their basic properties. We will often drop the word "solid" if this does not cause confusion. If  $Y = \mathrm{Spa}(R, R^+)$ , then we have an equivalence [2, Th. 1.6]

$$(2.1) \quad \mathrm{QCoh}(Y) \simeq \mathcal{D}((R, R^+)_{\square}),$$

where the latter is the derived category of solid  $(R, R^+)$ -modules, i.e., modules over the analytic ring  $(R, R^+)_{\square}$ . In what follows, if this does not confusion, we will write  $R_{\square}^{\text{an}} := (R, R^+)_{\square}$ . For a general  $Y$ , the category  $\text{QCoh}(Y)$  is obtained by gluing the categories  $\mathcal{D}((R, R^+)_{\square})$  in the analytic topology.

By  $\text{Perf}(Y)$ , we denote the full  $\infty$ -subcategory of perfect sheaves on  $Y$ ; that is, complexes which locally for the analytic topology are quasi-isomorphic to a bounded complex of finite, locally free  $\mathcal{O}_Y$ -modules. If  $Y = \text{Spa}(R, R^+)$  is affinoid, then the natural functor

$$\text{Perf}(R) \rightarrow \text{Perf}(Y)$$

is an equivalence, where the left-hand side denotes the  $\infty$ -category of perfect complexes of  $R$ -modules (i.e., bounded complexes of finite projective  $R$ -modules).

The categories  $\text{QCoh}(Y)$ ,  $\text{Nuc}(Y)$ , and  $\text{Perf}(Y)$  are (compatibly) symmetric monoidal. In the definition of the  $\infty$ -category  $\text{QCoh}(Y)$  we will bound everything by a fixed uncountable cardinal so that the category is presentable; it is then also closed symmetric monoidal. The  $\infty$ -category  $\text{Nuc}(Y)$  is as well presentable and closed symmetric monoidal. Similarly for the  $\infty$ -category  $\text{Perf}(Y)$ .

**Remark 2.2.** The categories  $\text{QCoh}(Y)$ ,  $\text{Nuc}(Y)$ , and  $\text{Perf}(Y)$  can be defined in a more general setting, where  $Y = (R, R^+)$  is a pair such that  $R$  is a complete Huber ring and  $R^+ \subset R^0$  is an arbitrary subring (see [2, Sec. 3.3] for details). We will most often use the case when  $R^+ = \mathbf{Z}$ .

2.2.2. *Quasi-coherent  $\varphi$ -sheaves on  $Y_{\text{FF}}$ .* The  $\infty$ -category of quasi-coherent  $\varphi$ -equivariant sheaves over  $Y_{\text{FF}, S}$  (in short:  $\varphi$ -sheaves over  $Y_{\text{FF}, S}$ ) is the equalizer

$$\text{QCoh}(Y_{\text{FF}, S})^{\varphi} := \text{eq} \left( \text{QCoh}(Y_{\text{FF}, S}) \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{\text{Id}} \end{array} \text{QCoh}(Y_{\text{FF}, S}) \right).$$

It is the  $\infty$ -category of pairs  $(\mathcal{E}, \varphi_{\mathcal{E}})$ , where  $\mathcal{E}$  is a quasi-coherent sheaf on  $Y_{\text{FF}, S}$  and  $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$  is a quasi-isomorphism<sup>2</sup>. The category  $\text{Nuc}(Y_{\text{FF}, S})^{\varphi}$  (resp.  $\text{Perf}(Y_{\text{FF}, S})^{\varphi}$ ) is the full  $\infty$ -subcategory of  $\text{QCoh}(Y_{\text{FF}, S})^{\varphi}$  spanned by the pairs  $(\mathcal{E}, \varphi_{\mathcal{E}})$ , where  $\mathcal{E}$  is a nuclear (resp. perfect) sheaf on  $Y_{\text{FF}, S}$ .

In what follows we will set  $u = (p-1)/p, v = p-1$  if  $p \neq 2$ ; for  $p = 2$  we take  $u = 3/4, v = 3/2$ . If  $S$  is the tilt of a perfectoid space  $S^{\natural}$  over  $\text{Spa}(\mathbf{Q}_p)$ , then the divisor on  $Y_{S, [u, v]}$  associated to  $t$  is  $y_{\infty}$  and  $t$  is a unit in  $\mathbf{B}_{S, [u, v/p]}$ . Via analytic descent, we like to describe the above categories of  $\varphi$ -equivariant sheaves using the chart  $Y_{\text{FF}, S, [u, v]}$  (via Frobenius we have  $\varphi : Y_{\text{FF}, S, [u/p, v/p]} \xrightarrow{\sim} Y_{\text{FF}, S, [u, v]}$ ):

$$\text{QCoh}(Y_{\text{FF}, S})^{\varphi} \simeq \text{eq} \left( \text{QCoh}(Y_{\text{FF}, S, [u, v]}) \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{j^*} \end{array} \text{QCoh}(Y_{\text{FF}, S, [u, v/p]}) \right).$$

We wrote here  $\varphi, j$  for the Frobenius and the open embedding maps from  $Y_{\text{FF}, S, [u, v/p]}$  to  $Y_{\text{FF}, S, [u, v]}$ , respectively. That is,  $\text{QCoh}(Y_{\text{FF}, S})^{\varphi}$  is the  $\infty$ -category of pairs  $(\mathcal{E}, \varphi_{\mathcal{E}})$ , where  $\mathcal{E}$  is a quasi-coherent sheaf on  $Y_{\text{FF}, S, [u, v]}$  and  $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E} \xrightarrow{\sim} j^* \mathcal{E}$  is a quasi-isomorphism. The categories  $\text{Nuc}(Y_{\text{FF}, S})^{\varphi}$ ,  $\text{Perf}(Y_{\text{FF}, S})^{\varphi}$  can be described in an analogous way.

We note that, since we have the equivalence (2.1), we can also write<sup>3</sup>

$$\text{QCoh}(Y_{\text{FF}, S})^{\varphi} \simeq \mathcal{D}(\mathbf{B}_{S, \square}^{\text{FF}})^{\varphi} := \text{eq} \left( \mathcal{D}(\mathbf{B}_{S, [u, v], \square}^{\text{an}}) \begin{array}{c} \xrightarrow{\varphi^*} \\ \xrightarrow{j^*} \end{array} \mathcal{D}(\mathbf{B}_{S, [u, v/p], \square}^{\text{an}}) \right).$$

(Frobenius  $\varphi$  maps  $\mathbf{B}_{S, [u, v], \square}^{\text{an}}$  to  $\mathbf{B}_{S, [u, v/p], \square}^{\text{an}}$ .) It is the  $\infty$ -category of pairs  $M_S = (M_{S, [u, v]}, \varphi_M)$ , where  $M_{S, [u, v]}$  is a complex of  $\mathbf{B}_{S, [u, v], \square}^{\text{an}}$ -modules and the Frobenius  $\varphi_M$  is a quasi-isomorphism of complexes of  $\mathbf{B}_{S, [u, v/p], \square}^{\text{an}}$ -modules

$$\varphi_M : \varphi^* M_{S, [u, v]} \xrightarrow{\sim} M_{S, [u, v/p]} := M_{S, [u, v]} \otimes_{\mathbf{B}_{S, [u, v], \square}^{\text{an}}}^{\mathbf{L}} \mathbf{B}_{S, [u, v/p], \square}^{\text{an}}.$$

<sup>2</sup>We will call isomorphisms in the  $\infty$ -categories  $\text{QCoh}(-)$  quasi-isomorphisms to be compatible with more classical set-ups.

<sup>3</sup>We stress here that  $\mathcal{D}(\mathbf{B}_{S, \square}^{\text{FF}})$  and  $\mathbf{B}_{S, \square}^{\text{FF}}$  is just a notation; the ring  $\mathbf{B}_{S, \square}^{\text{FF}}$  does not exist.

**Remark 2.3.** In what follows it will be convenient to consider the following variant of the  $\infty$ -category  $\mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$  (where we drop the superscript  $(-)^{\text{an}}$  from the rings):

$$\mathcal{D}(\mathbf{B}_S^{\text{FF}})^\varphi := \text{eq}\left( \mathcal{D}(\mathbf{B}_{S,[u,v],\square}) \xrightarrow[\underline{f}^*]{\varphi^*} \mathcal{D}(\mathbf{B}_{S,[u,v/p],\square}) \right).$$

It is the  $\infty$ -category of pairs  $M_S = (M_{S,[u,v]}, \varphi_M)$ , where  $M_{S,[u,v]}$  is a complex of solid  $\mathbf{B}_{S,[u,v]}$ -modules and the Frobenius  $\varphi_M$  is a quasi-isomorphism of complexes of solid  $\mathbf{B}_{S,[u,v/p]}$ -modules

$$\varphi_M : \varphi^* M_{S,[u,v]} \xrightarrow{\sim} M_{S,[u,v/p]} := M_{S,[u,v]} \otimes_{\mathbf{B}_{S,[u,v],\square}}^{\mathbf{L}} \mathbf{B}_{S,[u,v/p]}.$$

We call  $\mathcal{D}(\mathbf{B}_S^{\text{FF}})^\varphi$  *the category of  $\varphi$ -complexes of  $\mathbf{B}_S^{\text{FF}}$ -modules*. Since we have the equivalences of symmetric monoidal categories  $\mathcal{D}((\mathbf{B}_{S,I}, \mathbf{Z})_\square) = \mathcal{D}(\mathbf{B}_{S,I,\square})$  (see [5, Lemma A.16]), this corresponds to using the analytic structure with respect to  $\mathbf{Z}$  in place of  $\mathbf{B}_{S,I}^+$ . In particular, we have a canonical monoidal functor  $\mathcal{D}(\mathbf{B}_S^{\text{FF}})^\varphi \rightarrow \mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$ .

**2.2.3. Monoidal structure on quasi-coherent sheaves on  $Y_{\text{FF}}$ .** The category  $\text{QCoh}(Y_{\text{FF},S})^\varphi$  is closed symmetric monoidal. We will now present how the closed symmetric monoidal structure can be seen on the level of the category  $\mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$ . In what follows we have set  $\mathbf{B}_1 := \mathbf{B}_{S,[u,v],\square}^{\text{an}}$ ,  $\mathbf{B}_2 := \mathbf{B}_{S,[u,v/p],\square}^{\text{an}}$ .

The (derived) tensor product in  $\mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$ , denoted by  $(-) \otimes_{\mathbf{B}_{S,\square}^{\text{FF}}}^{\mathbf{L}} (-)$ , is inherited from the one of the category  $\mathcal{D}(\mathbf{B}_1)$ . More precisely, for  $(M, \varphi_M), (N, \varphi_N) \in \mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$ , their tensor product is defined by:

$$M \otimes_{\mathbf{B}_{S,\square}^{\text{FF}}}^{\mathbf{L}} N := (M_{S,[u,v]} \otimes_{\mathbf{B}_1}^{\mathbf{L}} N_{S,[u,v]}, \varphi_{M \otimes N}),$$

$$\varphi_{M \otimes N} = \varphi_M \otimes \varphi_N : (M_{S,[u,v]} \otimes_{\mathbf{B}_1}^{\mathbf{L}} N_{S,[u,v]}) \otimes_{\mathbf{B}_{1,\varphi}}^{\mathbf{L}} \mathbf{B}_2 \rightarrow (M_{S,[u,v]} \otimes_{\mathbf{B}_1}^{\mathbf{L}} N_{S,[u,v]}) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 = (M_{S,[u,v/p]} \otimes_{\mathbf{B}_2}^{\mathbf{L}} N_{S,[u,v/p]}).$$

Frobenius  $\varphi_{M \otimes N}$  is a quasi-isomorphism because so are Frobeniuses  $\varphi_M$  and  $\varphi_N$ .

The internal RHom, denoted by  $\text{RHom}_{\mathbf{B}_{S,\square}^{\text{FF}}}(-, -)$ , in the category  $\mathcal{D}(\mathbf{B}_{S,\square}^{\text{FF}})^\varphi$  is defined by:

$$\text{RHom}_{\mathbf{B}_{S,\square}^{\text{FF}}}(M, N) := (\text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}), \varphi_{M,N}),$$

$$\varphi_{M,N} := (\varphi_M^{-1}, \varphi_N) : \text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}) \otimes_{\mathbf{B}_{1,\varphi}}^{\mathbf{L}} \mathbf{B}_2 \rightarrow \text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2.$$

In the definition of Frobenius  $\varphi_{M,N}$  we have used the following (non-obvious) fact:

**Lemma 2.4.** *The canonical maps*

$$\text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}) \otimes_{\mathbf{B}_{1,\varphi}}^{\mathbf{L}} \mathbf{B}_2 \rightarrow \text{RHom}_{\mathbf{B}_2}(M_{S,[u,v]} \otimes_{\mathbf{B}_{1,\varphi}}^{\mathbf{L}} \mathbf{B}_2, N_{S,[u,v]} \otimes_{\mathbf{B}_{1,\varphi}}^{\mathbf{L}} \mathbf{B}_2),$$

$$\text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 \rightarrow \text{RHom}_{\mathbf{B}_2}(M_{S,[u,v/p]}, N_{S,[u,v/p]})$$

*are quasi-isomorphisms.*

*Proof.* To start, note that, since the first map is induced by the composition of the maps

$$\varphi : \mathbf{B}_{S,[u,v],\square}^{\text{an}} \rightarrow \mathbf{B}_{S,[u/p,v/p],\square}^{\text{an}}, \quad \text{can} : \mathbf{B}_{S,[u/p,v/p],\square}^{\text{an}} \rightarrow \mathbf{B}_{S,[u,v/p],\square}^{\text{an}}$$

where the first map is an isomorphism, it suffices to argue for the second quasi-isomorphism in the lemma.

Write  $M_{S,[u,v]} = \text{colim}_{i \in I} M_{S,[u,v]}^i$  as a filtered colimit of compact objects  $\{M_{S,[u,v]}^i\}$ ,  $i \in I$ . Then

$$\begin{aligned} \text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}, N_{S,[u,v]}) &= \text{RHom}_{\mathbf{B}_1}(\text{colim}_{i \in I} M_{S,[u,v]}^i, N_{S,[u,v]}) \\ &\simeq \text{Rlim}_I \text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}^i, N_{S,[u,v]}) \end{aligned}$$

and similarly for  $[u, v/p]$ . It follows that it suffices to show that

$$(\text{Rlim}_I \text{RHom}_{\mathbf{B}_1}(M_{S,[u,v]}^i, N_{S,[u,v]})) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 \xrightarrow{\sim} \text{Rlim}_I \text{RHom}_{\mathbf{B}_2}(M_{S,[u,v/p]}^i, N_{S,[u,v/p]}).$$

But, by [2, Prop. 5.38], we have

$$\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{\mathbf{B}_1}(M_{S,[u,v]}^i, N_{S,[u,v]}) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 \xrightarrow{\sim} \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{\mathbf{B}_2}(M_{S,[u,v/p]}^i, N_{S,[u,v/p]}).$$

Hence it suffices to show that

$$(\mathbf{R}\lim_I \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{\mathbf{B}_1}(M_{S,[u,v]}^i, N_{S,[u,v]})) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 \xrightarrow{\sim} \mathbf{R}\lim_I (\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{\mathbf{B}_1}(M_{S,[u,v]}^i, N_{S,[u,v]}) \otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2).$$

That is, that the functor  $(-)\otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2$  commutes with derived limits.

To show this write  $\mathbf{B}_{S,[u,v/p]} = \mathbf{B}_{S,[u,v]} \langle f \rangle$ , where  $f = (p/[p^b]^{p/v}) \in \mathbf{B}_{S,[u,v]}$ . By [2, Prop. 4.11], we have

$$(2.5) \quad (-)\otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2 \simeq (-)\otimes_{(\mathbf{Z}[T], \mathbf{Z})_{\square}}^{\mathbf{L}} (\mathbf{Z}[T], \mathbf{Z}[T])_{\square},$$

where the map  $(\mathbf{Z}[T], \mathbf{Z})_{\square} \rightarrow (\mathbf{B}_{S,[u,v]}, \mathbf{B}_{S,[u,v]}^+)_{\square}$  is induced by  $T \mapsto f$ . But, by [2, Prop. 3.12], for  $M \in \mathcal{D}((\mathbf{Z}[T], \mathbf{Z})_{\square})$ , we have

$$M \otimes_{(\mathbf{Z}[T], \mathbf{Z})_{\square}}^{\mathbf{L}} (\mathbf{Z}[T], \mathbf{Z}[T])_{\square} \simeq \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_R(R_{\infty}/R, M)[1],$$

where  $R = \mathbf{Z}[T]$ ,  $R_{\infty} = \mathbf{Z}((T^{-1}))$ . It follows that the functor  $(-)\otimes_{\mathbf{B}_1}^{\mathbf{L}} \mathbf{B}_2$  commutes with derived limits, as wanted.  $\square$

Finally, we note that Frobenius  $\varphi_{M,N}$  is a quasi-isomorphism because so are Frobeniuses  $\varphi_M$  and  $\varphi_N$ .

**Remark 2.6.** (1) Everything above is valid for the category  $\mathcal{D}(\mathbf{B}_S^{\text{FF}})^{\varphi}$  with the same proofs.

(2) Let  $M, N \in \mathcal{D}((R, \mathbf{Z})_{\square})$ . We note that the natural map

$$\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{(R, \mathbf{Z})_{\square}}(M, N) \otimes_{(R, \mathbf{Z})_{\square}}^{\mathbf{L}} (R, R^+)_{\square} \rightarrow \mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{(R, R^+)_{\square}}(M \otimes_{(R, \mathbf{Z})_{\square}}^{\mathbf{L}} (R, R^+)_{\square}, N \otimes_{(R, \mathbf{Z})_{\square}}^{\mathbf{L}} (R, R^+)_{\square})$$

is a quasi-isomorphism in the case  $N$  is  $(R, R^+)_{\square}$ -complete. It follows that  $\mathbf{R}\underline{\mathbf{H}}\mathbf{om}_{(R, \mathbf{Z})_{\square}}(M, N)$  is then also  $(R, R^+)_{\square}$ -complete. For example, this is the case when  $N$  is nuclear.

**2.2.4. Quasi-coherent  $\varphi$ -sheaves on  $Y_{\text{FF}}$  and  $\varphi$ -modules.** We will now describe the categories  $\text{Nuc}(Y_{\text{FF}, S})^{\varphi}$  and  $\text{Perf}(Y_{\text{FF}, S})^{\varphi}$  using complexes of (usual) solid modules.

Recall that the natural maps of analytic rings  $(\mathbf{B}_{S,I}, \mathbf{Z})_{\square} \rightarrow (\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+)_{\square}$  induce base change functors

$$(2.7) \quad (-)\otimes_{(\mathbf{B}_{S,I}, \mathbf{Z})_{\square}}^{\mathbf{L}} (\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+)_{\square} : \mathcal{D}((\mathbf{B}_{S,I}, \mathbf{Z})_{\square}) \rightarrow \mathcal{D}((\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+)_{\square}).$$

By [6, (6.13)], the functors (2.7) induce equivalences on the full subcategories of nuclear and perfect complexes:

$$(2.8) \quad \begin{aligned} \text{Nuc}(\mathbf{B}_{S,I}) &:= \text{Nuc}((\mathbf{B}_{S,I}, \mathbf{Z})_{\square}) \xrightarrow{\sim} \text{Nuc}((\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+)_{\square}), \\ \text{Perf}(\mathbf{B}_{S,I}) &\simeq \text{Perf}((\mathbf{B}_{S,I}, \mathbf{Z})_{\square}) \xrightarrow{\sim} \text{Perf}((\mathbf{B}_{S,I}, \mathbf{B}_{S,I}^+)_{\square}). \end{aligned}$$

We define the category  $\text{Nuc}(\mathbf{B}_S^{\text{FF}})^{\varphi}$  (resp.  $\text{Perf}(\mathbf{B}_S^{\text{FF}})^{\varphi}$ ) as the full  $\infty$ -subcategory of  $\mathcal{D}(\mathbf{B}_S^{\text{FF}})^{\varphi}$  spanned by the pairs  $(M_{S,[u,v]}, \varphi_M)$ , where  $M_{S,[u,v]}$  is a nuclear (resp. perfect) complex over  $\mathbf{B}_{S,[u,v]}$ . That is, the  $\infty$ -category  $\text{Nuc}(\mathbf{B}_S^{\text{FF}})^{\varphi}$  of nuclear  $\varphi$ -complexes of  $\mathbf{B}_S^{\text{FF}}$ -modules, is defined as the equalizer:

$$\text{Nuc}(\mathbf{B}_S^{\text{FF}})^{\varphi} := \text{eq} \left( \text{Nuc}(\mathbf{B}_{S,[u,v]}) \xrightarrow[\text{can}]{\varphi^*} \text{Nuc}(\mathbf{B}_{S,[u,v/p]}) \right).$$

Similarly, for the category  $\text{Perf}(\mathbf{B}_S^{\text{FF}})^{\varphi}$  of  $\varphi$ -complexes of perfect  $\mathbf{B}_S^{\text{FF}}$ -modules.

We have the following simple fact:

**Lemma 2.9.** *The canonical functor*

$$\mathcal{D}(\mathbf{B}_S^{\text{FF}})^{\varphi} \rightarrow \text{QCoh}(Y_{\text{FF}, S})$$

*induces equivalences of  $\infty$ -categories:*

$$(2.10) \quad \text{Nuc}(\mathbf{B}_S^{\text{FF}})^{\varphi} \xrightarrow{\sim} \text{Nuc}(Y_{\text{FF}, S})^{\varphi}, \quad \text{Perf}(\mathbf{B}_S^{\text{FF}})^{\varphi} \xrightarrow{\sim} \text{Perf}(Y_{\text{FF}, S})^{\varphi}.$$

*Proof.* Our claim follows from equivalences (2.8).  $\square$

The categories  $\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^\varphi$ , and  $\mathrm{Perf}(\mathbf{B}_S^{\mathrm{FF}})^\varphi$  are symmetric monoidal: the (derived) tensor products (denoted by  $(-) \otimes_{\mathbf{B}_S^{\mathrm{FF}}}^{\mathrm{L}} (-)$ ) are inherited from the ones of the categories  $\mathrm{Nuc}(\mathbf{B}_{S,[u,v]})$ , and  $\mathrm{Perf}(\mathbf{B}_{S,[u,v]})$ , respectively. The canonical functor to the category  $\mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^\varphi$  is symmetric monoidal. The functors in Lemma 2.9 are compatible with these structures.

2.2.5. *Quasi-coherent sheaves on  $X_{\mathrm{FF}}$ .* The action of  $\varphi$  on  $Y_{\mathrm{FF},S}$  being free and totally discontinuous, by the analytic descent for solid quasi-coherent sheaves, we obtain an equivalence of  $\infty$ -categories

$$\mathcal{E}_{\mathrm{FF},S} : \mathrm{QCoh}(Y_{\mathrm{FF},S})^\varphi \xrightarrow{\sim} \mathrm{QCoh}(Y_{\mathrm{FF},S}/\varphi^{\mathbf{Z}}) = \mathrm{QCoh}(X_{\mathrm{FF},S}).$$

Similarly, we get equivalences of closed symmetric monoidal  $\infty$ -categories

$$(2.11) \quad \mathrm{Nuc}(Y_{\mathrm{FF},S})^\varphi \xrightarrow{\sim} \mathrm{Nuc}(X_{\mathrm{FF},S}), \quad \mathrm{Perf}(Y_{\mathrm{FF},S})^\varphi \xrightarrow{\sim} \mathrm{Perf}(X_{\mathrm{FF},S}).$$

By Lemma 2.9, this yields a functor

$$(2.12) \quad \mathcal{E}_{\mathrm{FF},S} : \mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^\varphi \rightarrow \mathrm{QCoh}(X_{\mathrm{FF},S}).$$

We will often skip the subscript  $S$  from  $\mathcal{E}_{\mathrm{FF},S}$  if this does not cause confusion. Restricting to nuclear or perfect complexes we get the following result (see [6, Th. 6.8] for a similar statement):

**Lemma 2.13.** (1) *The functor  $\mathcal{E}_{\mathrm{FF},S}$ , from (2.12), induces equivalences of  $\infty$ -categories*

$$(2.14) \quad \mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^\varphi \xrightarrow{\sim} \mathrm{Nuc}(X_{\mathrm{FF},S}), \quad \mathrm{Perf}(\mathbf{B}_S)^\varphi \xrightarrow{\sim} \mathrm{Perf}(X_{\mathrm{FF},S}).$$

(2) *Let  $\mathcal{E} \in \mathrm{Nuc}(X_{\mathrm{FF},S})$ . Let  $(M(\mathcal{E})_{[u,v]}, \varphi_M)$  be the nuclear  $\varphi$ -complex of  $\mathbf{B}_S^{\mathrm{FF}}$ -modules corresponding to  $\mathcal{E}$  via (2.14). Then, there is a natural quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S}, \mathcal{E}) \simeq [M(\mathcal{E})_{[u,v]} \xrightarrow{\varphi^{-1}} M(\mathcal{E})_{[u,v/p]}].$$

*Proof.* The first claim is a combination of (2.10) and (2.11). For the second claim, we compute

$$\begin{aligned} \mathrm{R}\Gamma(X_{\mathrm{FF},S}, \mathcal{E}) &\simeq \mathrm{R}\Gamma(\varphi^{\mathbf{Z}}, \mathrm{R}\Gamma(Y_{\mathrm{FF}}, \mathcal{E}_{|Y_{\mathrm{FF}}})) \simeq [\Gamma(Y_{\mathrm{FF},S,[u,v]}, \mathcal{E}_{|Y_{\mathrm{FF}}}) \xrightarrow{\varphi^{-1}} \Gamma(Y_{\mathrm{FF},S,[u,v/p]}, \mathcal{E}_{|Y_{\mathrm{FF}}})] \\ &\simeq [M(\mathcal{E})_{[u,v]} \xrightarrow{\varphi^{-1}} M(\mathcal{E})_{[u,v/p]}]. \end{aligned}$$

$\square$

### 3. SYNTOMIC COMPLEXES ON THE FARGUES-FONTAINE CURVE

In this section we define quasi-coherent sheaves on the Fargues-Fontaine curve representing syntomic cohomology of smooth partially proper rigid analytic varieties.

3.1. **Hyodo-Kato and de Rham cohomologies.** We start with the cohomologies of de Rham type. We use [1, Sec. 4, Sec. 5] as the basic reference.

3.1.1. *De Rham cohomology.* Let  $X$  be a partially proper rigid analytic variety over  $K$ . We have the (filtered) de Rham complexes in  $\mathcal{D}(K_\square)$  and (filtered)  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology complexes in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$ , respectively:

$$F^r \mathrm{R}\Gamma_{\mathrm{dR},*}(X), \quad F^r \mathrm{R}\Gamma_{\mathrm{dR},*}(X_C/\mathbf{B}_{\mathrm{dR}}^+), \quad r \in \mathbf{N},$$

as well as the quotients

$$\mathrm{R}\Gamma_{\mathrm{dR},*}(X_C, r) := \mathrm{R}\Gamma_{\mathrm{dR},*}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r.$$

The latter complexes can be represented by quasi-coherent sheaves on  $X_{\mathrm{FF}}$ . For  $r \in \mathbf{N}$ , we define the *de Rham modules*

$$\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_C, r) := \mathrm{R}\Gamma_{\mathrm{dR},*}(X_C, r).$$



Since  $\mathbf{B}_{[u,v]}/t^i = \mathbf{B}_{\mathrm{dR}}^+/t^i$ , these are  $\mathbf{B}_{[u,v]}$ -modules. Since  $\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_C, r) \otimes_{\mathbf{B}_{[u,v]}}^{\mathrm{L}\square} \mathbf{B}_{[u,v/p]} = 0$  (recall that  $t$  is invertible in  $\mathbf{B}_{[u,v/p]}$ ), these complexes taken as pairs  $\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_C, r) = (\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_C, r), 0)$  define nuclear  $\varphi$ -complexes over  $\mathbf{B}^{\mathrm{FF}}$ .

We denote by

$$\mathcal{E}_{\mathrm{dR},*}(X_C, r) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}}(X_C, r))$$

the corresponding nuclear quasi-coherent sheaves on  $X_{\mathrm{FF}}$ . We will call them *de Rham sheaves*. We record the following simple fact:

**Lemma 3.1.** *Let  $r \in \mathbf{N}$ . We have a natural quasi-isomorphism in  $\mathrm{QCoh}(X_{\mathrm{FF}})$*

$$\mathcal{E}_{\mathrm{dR},*}(X_C, r) \simeq i_{\infty,*}\mathrm{R}\Gamma_{\mathrm{dR},*}(X_C, r).$$

For  $S \in \mathrm{Perf}_C$ , by replacing  $\mathbf{B}, \mathbf{B}_{\mathrm{dR}}^+, X_{\mathrm{FF}}$  with  $\mathbf{B}_{S^b}, \mathbf{B}_{\mathrm{dR}}^+(S), X_{\mathrm{FF},S^b}$  in the above, we obtain de Rham modules and sheaves on  $X_{\mathrm{FF},S^b}$ :  $\mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}}(X_S, r), \mathcal{E}_{\mathrm{dR},*}(X_S, r)$ . These are functors on  $\mathrm{Perf}_C$ .

**3.1.2. De Rham cohomology – Stein case.** Let  $X$  be a smooth Stein rigid analytic variety over  $K$ . In this case the above cohomology complexes can be made more explicit.

(•) *De Rham cohomology.* Let  $r \in \mathbf{N}$ . Since coherent cohomology of  $X$  is trivial in nonzero degrees and we have Serre duality, the (filtered) de Rham cohomology of  $X$  can be computed by the following complexes in  $\mathcal{D}(K_{\square})$ :

$$\begin{aligned} F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X) &\simeq (\Omega^r(X) \rightarrow \cdots \rightarrow \Omega^d(X))[-r], \\ F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X) &\simeq (H_c^d(X, \Omega^r) \rightarrow H_c^d(X, \Omega^{r+1}) \rightarrow \cdots \rightarrow H_c^d(X, \Omega^d))[-d-r]. \end{aligned}$$

The second quasi-isomorphism follows from the fact that  $H_c^i(X, \Omega^j) = 0$ , for  $i \neq d$ . The terms of the first complex are nuclear Fréchet over  $K$  and those of the second complex are of compact type over  $K$  (in classical terminology).

(•)  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology. Let  $r \in \mathbf{N}$ . The (filtered)  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology of  $X$  can be computed by the following complexes in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$ :

$$\begin{aligned} (3.2) \quad F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+) &\simeq \mathcal{O}(X) \otimes_K^{\square} t^r \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \Omega^1(X) \otimes_K^{\square} t^{r-1} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \cdots \rightarrow \Omega^d(X) \otimes_K^{\square} t^{r-d} \mathbf{B}_{\mathrm{dR}}^+, \\ F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+) &\simeq (H_c^d(X, \mathcal{O}) \otimes_K^{\square} t^r \mathbf{B}_{\mathrm{dR}}^+ \rightarrow H_c^d(X, \Omega^1) \otimes_K^{\square} t^{r-1} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \cdots \rightarrow H_c^d(X, \Omega^d) \otimes_K^{\square} t^{r-d} \mathbf{B}_{\mathrm{dR}}^+)[-d]. \end{aligned}$$

The tensor products are actually derived because  $\mathbf{B}_{\mathrm{dR}}^+$  is Fréchet hence flat.

This yields the quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$ :

$$\begin{aligned} (3.3) \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X_C, r) &\simeq \mathcal{O}(X) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^r) \rightarrow \Omega^1(X) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^{r-1}) \rightarrow \cdots \rightarrow \Omega^d(X) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^{r-d}), \\ \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C, r) &\simeq (H_c^d(X, \mathcal{O}) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^r) \rightarrow H_c^d(X, \Omega^1) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^{r-1}) \rightarrow \cdots \rightarrow H_c^d(X, \Omega^d) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/t^{r-d}))[-d]. \end{aligned}$$

We will denote the respective cohomology groups by  $H_{\mathrm{dR}}^i(X, r)$  and  $H_{\mathrm{dR},c}^i(X, r)$ .

For  $i \geq 0$ , we have short exact sequences in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$  (see [9, Example 3.30], [1, Lemma 4.15])

$$\begin{aligned} (3.4) \quad 0 &\rightarrow \Omega^i(X_C)/\mathrm{Im} d \rightarrow H_{\mathrm{dR}}^i(X_C, r) \rightarrow H_{\mathrm{dR}}^i(X) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/F^{r-i-1}) \rightarrow 0, \\ 0 &\rightarrow (H_c^d(X, \Omega^{i-d})/\mathrm{Im} d) \otimes_K^{\square} \mathrm{gr}_F^{r-i+d-1} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow H_{\mathrm{dR},c}^i(X_C, r) \rightarrow H_{\mathrm{dR},c}^i(X) \otimes_K^{\square} (\mathbf{B}_{\mathrm{dR}}^+/F^{r-i+d-1}) \rightarrow 0. \end{aligned}$$

**3.1.3. Hyodo-Kato cohomology.** Let  $X$  be a smooth rigid analytic variety over  $C$ . Let  $\mathrm{R}\Gamma_{\mathrm{HK}}(X) \in \mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$  be the Hyodo-Kato cohomology defined in [13, Sec. 4] (see also [6, Sec. 3]). Here  $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$  is the derived  $\infty$ -category of solid  $(\varphi, N, \mathcal{G}_K)$ -modules over  $\check{C}$ .

Let  $r \in \mathbf{Z}$ . Consider the twisted Hyodo-Kato cohomology in  $\mathcal{D}_{\varphi, \mathcal{G}_K}(\check{C}_{\square})$

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}_I}(X_C, r) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{I, \log}]^{N=0},$$

where the twist  $\{r\}$  means Frobenius divided by  $p^r$  and  $I \subset (0, \infty)$  is a compact interval with rational endpoints. We define  $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_C, r)$  in a similar way. We claim that, for compact intervals  $I \subset J \subset (0, \infty)$  with rational endpoints, we have the canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}_J}(X_C, r) \otimes_{\mathbf{B}_J}^{\mathrm{L}\square} \mathbf{B}_I \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}_I}(X_C, r).$$

Indeed, for that, it suffices to show that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{J, \log} \otimes_{\mathbf{B}_J}^{\mathrm{L}\square} \mathbf{B}_I \rightarrow \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{I, \log}$$

is a quasi-isomorphism. But this is clear since the solid tensor product commutes with direct sums.

We define the pair<sup>4</sup>

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_C, r) := (\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_C, r), \varphi), \quad \varphi : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_C, r) \rightarrow \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v/p]}}(X_C, r),$$

where the Frobenius  $\varphi$  is induced from the Hyodo-Kato Frobenius and the Frobenius  $\varphi : \mathbf{B}^{[u, v]} \rightarrow \mathbf{B}^{[u, v/p]}$ . It yields a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}^{[u, v/p], \square})$

$$\varphi : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_C, r) \otimes_{\mathbf{B}^{[u, v], \varphi}}^{\mathrm{L}\square} \mathbf{B}^{[u, v/p]} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v/p]}}(X_C, r)$$

The pair  $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_C, r)$  defines a nuclear  $\varphi$ -complex (actually  $(\varphi, \mathcal{G}_K)$ -complex) over  $\mathbf{B}^{\mathrm{FF}}$ , which we will call *Hyodo-Kato module*.

We define *Hyodo-Kato sheaves* on  $X_{\mathrm{FF}}$  as

$$\mathcal{E}_{\mathrm{HK}}(X_C, r) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_C, r)).$$

By Lemma 2.13, these are nuclear quasi-coherent sheaves on  $X_{\mathrm{FF}}$ . If the cohomology groups of  $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$  are of finite rank over  $\mathcal{C}$  then the sheaf  $\mathcal{E}_{\mathrm{HK}}(X_C, r)$  is perfect. By Lemma 2.13 and [6, Th. 6.3], we have natural quasi-isomorphisms in  $\mathcal{D}(\mathbf{Q}_p, \square)$

$$(3.5) \quad \begin{aligned} \mathrm{R}\Gamma(X_{\mathrm{FF}}, \mathcal{E}_{\mathrm{HK}}(X_C, r)) &\simeq [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v], \log}]^{N=0, \varphi=1} \\ &\xleftarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{\log}]^{N=0, \varphi=1}, \end{aligned}$$

where we set

$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v], \log}]^{N=0, \varphi=1} := \left[ \begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v], \log} & \xrightarrow{\varphi-1} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v/p], \log} \\ \downarrow N & & \downarrow N \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v], \log} & \xrightarrow{p\varphi-1} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}^{[u, v/p], \log} \end{array} \right].$$

And similarly for  $[\mathrm{R}\Gamma_{\mathrm{HK}}(X)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{\log}]^{N=0, \varphi=1}$ .

For  $S \in \mathrm{Perf}_C$ , by changing  $\mathbf{B}, \mathbf{B}_I, \mathbf{B}_{I, \log}$  to  $\mathbf{B}_{S^\flat}, \mathbf{B}_{S^\flat, I}, \mathbf{B}_{S^\flat, I, \log}$ , we obtain Hyodo-Kato modules and sheaves:

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_S, r), \quad \mathcal{E}_{\mathrm{HK}}(X_S, r).$$

These are functors on  $\mathrm{Perf}_C$ . In the case  $X$  is partially proper, we have analogs  $\mathrm{R}\Gamma_{\mathrm{HK}, c}^{\mathbf{B}}(X_S, r), \quad \mathcal{E}_{\mathrm{HK}, c}(X_S, r)$  for Hyodo-Kato cohomology with compact support<sup>5</sup> and the following analog of quasi-isomorphism (3.5):

**Lemma 3.6.** *Let  $r \in \mathbf{Z}$ . We have a natural quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$*

$$\mathrm{R}\Gamma(X_{\mathrm{FF}, S^\flat}, \mathcal{E}_{\mathrm{HK}, *}(X_S, r)) \simeq [\mathrm{R}\Gamma_{\mathrm{HK}, *}^{\mathbf{B}}(X_S, r)]^{\varphi=1}.$$

<sup>4</sup>There is a certain doubling of notation with the previous paragraph but we hope that this will not cause confusion in what follows.

<sup>5</sup>See [1, Sec. 4, Sec. 5] for the definition and basic properties of compactly supported Hyodo-Kato cohomology.

3.1.4. *Hyodo-Kato map.* Let  $X$  be a smooth partially proper rigid analytic variety over  $K$ . Recall that we have the natural Hyodo-Kato maps (see [13, Sec. 4]) in  $\mathcal{D}(\check{C}_\square)$  and  $\mathcal{D}(\mathbf{B}_{\text{dR},\square}^+)$ , respectively:

$$\iota_{\text{HK}} : \text{R}\Gamma_{\text{HK}}(X_C) \rightarrow \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{dR}}^+), \quad \iota_{\text{HK}} : \text{R}\Gamma_{\text{HK}}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{dR}}^+ \xrightarrow{\sim} \text{R}\Gamma_{\text{dR}}(X_C/\mathbf{B}_{\text{dR}}^+).$$

Combined with the canonical map  $\iota : \mathbf{B}_{[u,v],\log} \rightarrow \mathbf{B}_{[u,v]}/t^i$ , it defines a map between complexes of solid  $\mathbf{B}_{[u,v]}$ -modules:

$$(3.7) \quad \iota_{\text{HK}} : \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u,v]}}(X_C, r) = [\text{R}\Gamma_{\text{HK}}(X_C)\{r\} \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{[u,v],\log}]^{N=0} \rightarrow \text{R}\Gamma_{\text{dR}}^{\mathbf{B}^{[u,v]}}(X_C, r).$$

Since we have a commutative diagram

$$(3.8) \quad \begin{array}{ccc} \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u,v]}}(X_C, r) & \xrightarrow{\varphi \otimes \varphi} & \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u,v/p]}}(X_C, r) \\ \downarrow \iota_{\text{HK}} & & \downarrow \\ \text{R}\Gamma_{\text{dR}}^{\mathbf{B}^{[u,v]}}(X_C, r) & \longrightarrow & 0, \end{array}$$

the map (3.7) clearly lifts to a map of  $\varphi$ -modules over  $\mathbf{B}^{\text{FF}}$ :

$$\iota_{\text{HK}} : \text{R}\Gamma_{\text{HK}}^{\mathbf{B}}(X_C, r) \rightarrow \text{R}\Gamma_{\text{dR}}^{\mathbf{B}}(X_C, r).$$

This Hyodo-Kato map descends to the level of nuclear quasi-coherent sheaves on  $X_{\text{FF}}$ :

$$\iota_{\text{HK}} : \mathcal{E}_{\text{HK}}(X_C, r) \rightarrow \mathcal{E}_{\text{dR}}(X_C, r).$$

Everything above has a version for compactly supported cohomologies (see [1, Sec. 4.2.2] for Hyodo-Kato morphisms), as well as for  $S$ -cohomologies, for  $S \in \text{Perf}_C$  (varying functorially in  $S$ ).

3.2. **Syntomic cohomology.** We pass now to syntomic cohomology.

3.2.1. *Classical syntomic cohomology.* Let  $X$  be a smooth partially proper rigid analytic variety over  $K$ . Let  $r \in \mathbf{N}$ . Consider the classical syntomic cohomology (ala Bloch-Kato) (see [13, Sec. 5.4])

$$\text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_C, \mathbf{Q}_p(r)) := [[\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{st}}^+]^{N=0, \varphi=p^r} \xrightarrow{\iota_{\text{HK}} \otimes \iota} \text{R}\Gamma_{\text{dR},*}(X_C/\mathbf{B}_{\text{dR}}^+)/F^r].$$

It satisfies the following comparison theorem:

**Theorem 3.9.** (Period isomorphism, [13, Th. 6.9]) *Let  $r \in \mathbf{N}$ . There is a natural quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p, \square)$*

$$(3.10) \quad \alpha_r : \tau_{\leq r} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_C, \mathbf{Q}_p(r)) \simeq \tau_{\leq r} \text{R}\Gamma_{\text{proét},*}(X_C, \mathbf{Q}_p(r)).$$

Moreover, it yields a natural quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p, \square)$

$$\alpha_r : \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_C, \mathbf{Q}_p(r)) \simeq \text{R}\Gamma_{\text{proét},*}(X_C, \mathbf{Q}_p(r)), \quad r \geq 2d.$$

*Proof.* Only the second claim requires justification. For the usual cohomology, this follows from quasi-isomorphism (3.10) and the fact that the complexes  $\text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_C, \mathbf{Q}_p(r))$ ,  $\text{R}\Gamma_{\text{proét}}(X_C, \mathbf{Q}_p(r))$  live in the  $[0, 2d]$ -range. To see the latter fact in the case  $X$  is Stein, note that using (3.3) we get  $H_{\text{syn}}^{\mathbf{B}_{\text{cr}}^+, i}(X_C, \mathbf{Q}_p(r)) = 0$ , for  $i \geq d+1$ . From this and (3.10) we get that  $H_{\text{proét}}^i(X_C, \mathbf{Q}_p(d+j)) = 0$ , for  $d+j \geq i \geq d+1, j \geq 1$ , and then, by twisting, that  $H_{\text{proét}}^i(X_C, \mathbf{Q}_p(r)) = 0$ , for  $i \geq d+1$ , as wanted. Now, for a general partially proper  $X$ , we need to add  $d$  for the analytic dimension of cohomology yielding the range  $[0, 2d]$ , as wanted.

For the cohomology with compact support, we argue similarly but using (3.4) instead of (3.3) in the case  $X$  is Stein. The case of partially proper  $X$  follows from that by a (co)-Čech argument.  $\square$

The above has a version in families. Let  $S \in \text{Perf}_C$  and let  $r \in \mathbf{N}$ . We have the classical (crystalline) syntomic cohomology in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :

$$(3.11) \quad \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)) := [[\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{st}}^+(S)]^{N=0, \varphi=p^r} \xrightarrow{\iota_{\text{HK}} \otimes \iota} \text{R}\Gamma_{\text{dR},*}(X_C/\mathbf{B}_{\text{dR}}^+(S))/F^r].$$

It satisfies the following comparison theorem:

**Theorem 3.12.** (Period isomorphism in families, [13, Cor. 7.37], [1, Th. 7.11]) *Let  $r \in \mathbf{N}$ . There is a natural, functorial in  $S$ , quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$*

$$(3.13) \quad \alpha_r : \tau_{\leq r} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)) \simeq \tau_{\leq r} \text{R}\Gamma_{\text{proét},*}(X_S, \mathbf{Q}_p(r)).$$

Moreover, it yields a natural, functorial in  $S$ , quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$

$$\alpha_r : \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)) \simeq \text{R}\Gamma_{\text{proét},*}(X_S, \mathbf{Q}_p(r)), \quad r \geq 2d.$$

*Proof.* The argument is analogous to the one used in the proof of Theorem 3.9.  $\square$

3.2.2. *Variants of syntomic cohomology.* We will need the following variant of syntomic cohomology in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :

$$(3.14) \quad \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u,v]}}(X_S, \mathbf{Q}_p(r)) := [[\text{R}\Gamma_{\text{HK},*}^{\mathbf{B}^{[u,v]}}(X_S, r)]^{\varphi=1} \xrightarrow{\iota_{\text{HK}}} \text{R}\Gamma_{\text{dR},*}^{\mathbf{B}^{[u,v]}}(X_S, r)], \quad r \in \mathbf{N}.$$

**Lemma 3.15.** *Let  $r \in \mathbf{N}$ . There is a natural, functorial in  $S$ , quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :*

$$(3.16) \quad \tau_{\leq r} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u,v]}}(X_S, \mathbf{Q}_p(r)) \simeq \tau_{\leq r} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)).$$

Moreover, it yields a quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :

$$\text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u,v]}}(X_S, \mathbf{Q}_p(r)) \simeq \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)), \quad r \geq d.$$

*Proof.* Let  $\mathbf{B}_{S^b, [u, \infty]} := W(R^{b,+}) \langle [p^b]/p^u \rangle [1/p]$ . Define yet another variant of syntomic cohomology in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :

$$\text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u, \infty]}}(X_S, \mathbf{Q}_p(r)) := [[\text{R}\Gamma_{\text{HK},*}^{\mathbf{B}^{[u, \infty]}}(X_S, r)]^{\varphi=1} \xrightarrow{\iota_{\text{HK}}} \text{R}\Gamma_{\text{dR},*}(X_S, r)].$$

The three different variants of syntomic cohomology introduced above are linked via maps

$$\text{R}\Gamma_{\text{syn},*}^{\mathbf{B}_{\text{cr}}^+}(X_S, \mathbf{Q}_p(r)) \xrightarrow{f_1} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u, \infty]}}(X_S, \mathbf{Q}_p(r)) \xrightarrow{f_2} \text{R}\Gamma_{\text{syn},*}^{\mathbf{B}^{[u,v]}}(X_S, \mathbf{Q}_p(r))$$

induced by canonical maps  $\mathbf{B}_{\text{cr}}^+(S) \rightarrow \mathbf{B}_{S^b, [u, \infty]}$ ,  $\mathbf{B}_{S^b, [u, \infty]} \rightarrow \mathbf{B}_{S^b, [u, v]}$ , and  $\mathbf{B}_{S^b, [u, \infty]} \rightarrow \mathbf{B}_{S^b, [u, v/p]}$  (see [11, Sec. 2.4.2]). We claim that the map  $f_1$  is a quasi-isomorphism and the map  $f_2$  is a quasi-isomorphism after truncation  $\tau_{\leq r}$ . To show that, it suffices to prove that the related maps

$$\begin{aligned} f'_1 : [\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{st}}^+(S)]^{N=0, \varphi=p^r} &\rightarrow [\text{R}\Gamma_{\text{HK},*}^{\mathbf{B}^{[u, \infty]}}(X_S, r)]^{\varphi=1}, \\ f'_2 : [\text{R}\Gamma_{\text{HK},*}^{\mathbf{B}^{[u, \infty]}}(X_S, r)]^{\varphi=1} &\rightarrow [\text{R}\Gamma_{\text{HK},*}^{\mathbf{B}^{[u,v]}}(X_S, r)]^{\varphi=1} \end{aligned}$$

are quasi-isomorphisms in the wanted ranges. Or, first dropping (naively)  $N=0$  and then log on both sides, that so are the maps

$$\begin{aligned} f'_1 : [\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{cr}}^+(S)]^{\varphi=p^j} &\rightarrow [\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{S^b, [u, \infty]}]^{\varphi=p^j}, \quad j \in \mathbf{Z}; \\ f'_2 : \tau_{\leq r} [\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{S^b, [u, \infty]}]^{\varphi=p^s} &\rightarrow \tau_{\leq r} [\text{R}\Gamma_{\text{HK},*}(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{S^b, [u, v]}]^{\varphi=p^s}, \quad s = r-1, r. \end{aligned}$$

Let us first look at the map  $f'_1$ . Taking cohomologies in degree  $i \geq 0$ , we get maps

$$f'_1 : (H_{\text{HK},*}^i(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{\text{cr}}^+(S))^{\varphi=p^j} \rightarrow (H_{\text{HK},*}^i(X_C) \otimes_{\check{C}}^{\text{L}\square} \mathbf{B}_{S^b, [u, \infty]}))^{\varphi=p^j}.$$

We used here [13, Prop. 5.8]. Since  $H_{\text{HK}}^i(X_C)$  and  $H_{\text{HK},c}^i(X_C)$  are a countable limit, resp. colimit, of finite rank  $\varphi$ -isocrystals over  $\check{C}$ , we may assume that the Hyodo-Kato cohomology groups are finite rank. But then, since  $\varphi(\mathbf{B}_{S^b, [u, \infty]}) \subset \mathbf{B}_{\text{cr}}^+(S) \subset \mathbf{B}_{S^b, [u, \infty]}$ , it is clear that  $f'_1$  is an isomorphism, as wanted.

Concerning the map  $f'_2$ , we first pass to cohomology in degree  $i$  and then assume that the Hyodo-Kato cohomology has finite rank as above. Let  $j \in \mathbf{N}$ . We then claim that the map

$$(3.17) \quad H_{\mathrm{HK}}^i(X_C)\{j\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v]} \xrightarrow{1-\varphi} H_{\mathrm{HK}}^i(X_C)\{j\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v/p]}$$

is surjective for  $i \leq j$ . Indeed, by Lemma 2.13, the complex (3.17) computes the cohomology of the vector bundle  $\mathcal{E}_i$  on  $X_{\mathrm{FF}, S^b}$  associated to  $H_{\mathrm{HK}}^i(X_C)\{j\}$ . Our claim now follows from the fact that the slopes of Frobenius on  $H_{\mathrm{HK}}^i(X_C)$  are  $\leq i$  (see [13, proof of Prop. 5.20]) hence the slopes of  $\mathcal{E}_i$  are  $\geq 0$  and  $H^1(X_{\mathrm{FF}, S^b}, \mathcal{E}_i) = 0$ , as wanted.

Similarly, we see that the map

$$(3.18) \quad H_{\mathrm{HK}, c}^i(X_C)\{j\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v]} \xrightarrow{1-\varphi} H_{\mathrm{HK}, c}^i(X_C)\{j\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v/p]}$$

is surjective for  $j \geq d$  using the fact that the slopes of Frobenius on  $H_{\mathrm{HK}, c}^i(X_C)$  are in the  $[i - d, d]$  range (use Poincaré duality for Hyodo-Kato cohomology to flip to the usual cohomology).

Now, it suffices to show that, for  $i \in \mathbf{N}, j \geq -1$ , the map

$$f'_2 : (H_{\mathrm{HK}, * }^i(X_C) \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, \infty]})^{\varphi=p^j} \rightarrow (H_{\mathrm{HK}, * }^i(X_C) \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v]})^{\varphi=p^j}$$

is an isomorphism. But in the case  $S = C$  this follows from [4, Prop. 3.2] and the general case reduces to that one using the fact that all our algebras are spectral.

The above arguments prove the quasi-isomorphism in (3.16) for the usual cohomology and we get the statement for the compactly supported cohomology from the case of usual cohomology by a colim argument. Concerning the last sentence of our lemma, the above argument shows the case of compactly supported cohomology. For the usual cohomology, since the complex  $\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}_{\mathrm{cr}}^+}(X_S, \mathbf{Q}_p(r))$  lives in the  $[0, 2d]$  range (see the proof of Theorem 3.9) it suffices to show that so does the complex  $\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}_{[u, v]}}(X_S, \mathbf{Q}_p(r))$ . But here we can use the same argument as in the proof of Theorem 3.9.  $\square$

**Remark 3.19.** Bosco in [6, Th. 6.3] considered the following variant of syntomic cohomology in  $\mathcal{D}(\mathbf{Q}_p(S)_{\square})$ :

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{FF}}(X_S, \mathbf{Q}_p(r)) := [[\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_S, r)]^{\varphi=1} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_S, r)], \quad r \in \mathbf{N}.$$

**Lemma 3.20.** *The canonical map  $\mathbf{B}_{S^b} \rightarrow \mathbf{B}_{S^b, [u, v]}$  induces a morphism in  $\mathcal{D}(\mathbf{Q}_p(S)_{\square})$*

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathrm{FF}}(X_S, \mathbf{Q}_p(r)) \rightarrow \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}_{[u, v]}}(X_S, \mathbf{Q}_p(r)).$$

*This is a quasi-isomorphism.*

*Proof.* Arguing as in the proof of Lemma 3.15, it suffices to show that the induced morphism

$$[H_{\mathrm{HK}}^i(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b} \xrightarrow{1-\varphi} H_{\mathrm{HK}}^i(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b}] \rightarrow [H_{\mathrm{HK}}^i(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v]} \xrightarrow{1-\varphi} H_{\mathrm{HK}}^i(X_C)\{r\} \otimes_{\mathcal{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u, v/p]}]$$

is a quasi-isomorphism in the case  $H_{\mathrm{HK}}^i(X_C)$  is of finite rank. But this follows from Lemma 3.6.  $\square$

**3.2.3. Syntomic  $\varphi$ -modules over  $\mathbf{B}^{\mathrm{FF}}$ .** Let  $X$  be a smooth partially proper rigid analytic variety over  $K$ .

**Definition 3.21.** Let  $r \in \mathbf{N}$ . Let  $S \in \mathrm{Perf}_C$ .

(1) Set

$$\mathrm{R}\Gamma_{\mathrm{syn}, * }^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma_{\mathrm{HK}, * }^{\mathbf{B}}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}, * }^{\mathbf{B}}(X_S, r)].$$

This is a nuclear  $\varphi$ -module over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$ . We call it a *syntomic module*.

(2) The (nuclear) *syntomic sheaves* on  $X_{\mathrm{FF}, S}$  are defined by

$$\mathcal{E}_{\mathrm{syn}, * }(X_S, \mathbf{Q}_p(r)) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma_{\mathrm{syn}, * }^{\mathbf{B}}(X_S, \mathbf{Q}_p(r))).$$

We have a distinguished triangle in  $\mathrm{QCoh}(X_{\mathrm{FF}, S^b})$

$$(3.22) \quad \mathcal{E}_{\mathrm{syn}, * }(X_S, \mathbf{Q}_p(r)) \rightarrow \mathcal{E}_{\mathrm{HK}, * }(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathcal{E}_{\mathrm{dR}, * }(X_S, r).$$

**Lemma 3.23.** *Let  $r \geq 2d$ . We have natural, functorial in  $S$ , quasi-isomorphisms in  $\mathcal{D}(\mathbf{Q}_p(S)_\square)$ :*

$$\begin{aligned} \mathrm{R}\Gamma(X_{\mathrm{FF}, S^b}, \mathcal{E}_{\mathrm{syn}, *}(X_S, \mathbf{Q}_p(r))) &\simeq \mathrm{R}\Gamma_{\mathrm{syn}, *}^{\mathbf{B}^{[u, v]}}(X_S, \mathbf{Q}_p(r)), \\ \mathrm{R}\Gamma(X_{\mathrm{FF}, S^b}, \mathcal{E}_{\mathrm{syn}, *}(X_S, \mathbf{Q}_p(r))) &\simeq \mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbf{Q}_p(r)). \end{aligned}$$

*Proof.* The first quasi-isomorphism follows from Lemma 2.13. The second quasi-isomorphism follows from the first one, the quasi-isomorphism (3.16), and Theorem 3.12.  $\square$

#### 4. PRO-ÉTALE COMPLEXES ON THE FARGUES-FONTAINE CURVE

In this section we define quasi-coherent sheaves on the Fargues-Fontaine curve representing  $p$ -adic (geometric) pro-étale cohomology of smooth partially proper rigid analytic varieties and prove a comparison theorem with the quasi-coherent sheaves representing syntomic cohomology.

**4.1. Definitions.** We start with definitions.

**4.1.1. Twisted coefficients.** Let  $S \in \mathrm{Perf}_C$ . Let  $n, k \geq 0$ . Define the line bundle  $\mathcal{O}(n, k)$  on  $X_{\mathrm{FF}, S^b}$  by the exact sequence of  $\mathcal{O}_{\mathrm{FF}, S^b}$ -modules

$$0 \rightarrow \mathcal{O}(n, k) \rightarrow \mathcal{O}(n) \rightarrow i_{\infty, *}(\mathcal{O}/t^k) \rightarrow 0,$$

where the first map is an inclusion. The sheaf  $\mathcal{O}(n, n)$  will be the target of our trace maps. Note that  $\mathcal{O}(n, k)$  is just  $\mathcal{O}(n - k)$  with (Galois-)Tate twist  $k$ ; in particular, we have  $H^0(X_{\mathrm{FF}, S^b}, \mathcal{O}(n, n)) = \underline{\mathbf{Q}}_p(S)(n)$ .

On the level of  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$ , the sheaf  $\mathcal{O}(n, k)$  is the module  $\mathbf{B}_{S^b}\{n, k\}$  represented by the module  $\mathbf{B}_{S^b, [u, v]}\{n, k\}$  defined by the exact sequence

$$(4.1) \quad 0 \rightarrow \mathbf{B}_{S^b, [u, v]}\{n, k\} \rightarrow \mathbf{B}_{S^b, [u, v]}\{n\} \rightarrow \mathbf{B}_{S^b, [u, v]}\{n\}/t^k \rightarrow 0,$$

where the first map is an inclusion. We have  $\mathbf{B}_{S^b, [u, v]}\{n, k\} \simeq \mathbf{B}_{S^b, [u, v]}\{n - k\}(k)$  as a Frobenius, Galois module. Note that the Frobenius map:

$$\varphi : \mathbf{B}_{S^b, [u, v]}\{n, k\} \otimes_{\mathbf{B}_{S^b, [u, v], \varphi}^{\mathrm{L}\square}} \mathbf{B}_{S^b, [u, v/p]} \rightarrow \mathbf{B}_{S^b, [u, v]}\{n, k\} \otimes_{\mathbf{B}_{S^b, [u, v]}^{\mathrm{L}\square}} \mathbf{B}_{S^b, [u, v/p]}$$

is an isomorphism because it is isomorphic to the Frobenius on  $\mathbf{B}_{S^b, [u, v/p]}\{n - k\}$ .

**4.1.2. Pro-étale modules and sheaves.** Let  $X$  be a smooth partially proper dagger variety over  $K$ . For  $r \in \mathbf{N}$ ,  $v' = v, v/p$ , and  $S \in \mathrm{Perf}_C$ , we set

$$\mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v']}}(X_S, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbb{B}_{[u, v']})(r),$$

where  $\mathbb{B}_{[u, v']}$  denotes the relative period sheaf corresponding to  $\mathbf{B}_{[u, v']}$  (see [6, Sec. 2.3.1] for a description of condensed structure on these modules). We note that, by [6, Lemma 4.8], we have a canonical quasi-isomorphism  $\mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbb{B}_{[u, v]}) \otimes_{\mathbf{B}_{S^b, [u, v/p]}^{\mathrm{L}\square}} \mathbf{B}_{S^b, [u, v/p]} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbb{B}_{[u, v/p]})$ .

We define the *pro-étale modules* as the pairs

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) &:= (\mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v]}}(X_S, \mathbf{Q}_p(r)), \varphi), \\ \varphi : \mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v]}}(X_S, \mathbf{Q}_p(r)) &\rightarrow \mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v/p]}}(X_S, \mathbf{Q}_p(r)), \end{aligned}$$

where the Frobenius  $\varphi$  is induced by the Frobenius  $\varphi : \mathbb{B}_{[u, v]} \rightarrow \mathbb{B}_{[u, v/p]}$ . It yields a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}_{S^b, [u, v/p], \square})$

$$\varphi : \mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v]}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^b, [u, v], \varphi}^{\mathrm{L}\square}} \mathbf{B}_{S^b, [u, v/p]} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{proét}, *}^{\mathbf{B}^{[u, v/p]}}(X_S, \mathbf{Q}_p(r)).$$

Indeed, it suffices to show that the Frobenius map

$$\varphi : \mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbb{B}_{[u, v]}) \otimes_{\mathbf{B}_{S^b, [u, v], \varphi}^{\mathrm{L}\square}} \mathbf{B}_{S^b, [u, v/p]} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{proét}, *}(X_S, \mathbb{B}_{[u, v']})$$

is a quasi-isomorphism. But this follows directly from [6, Lemma 4.8].

The pairs  $\mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r))$  defines nuclear  $\varphi$ -complexes (actually  $(\varphi, \mathcal{L}_K)$ -complexes) over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$ , which we will call *pro-étale modules*. For the nuclear property see [6, Lemma 6.15]. We will denote by

$$\mathcal{E}_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r)) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)))$$

the corresponding nuclear quasi-coherent sheaves on  $X_{\mathrm{FF},S^b}$ . We will call them *pro-étale sheaves*. Pro-étale modules and sheaves are functors on  $\mathrm{Perf}_C$ .

**Lemma 4.2.** *We have a natural, functorial in  $S$ , quasi-isomorphism in  $\mathcal{D}(\mathbf{Q}_p(S)_{\square})$ :*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S^b}, \mathcal{E}_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r))) \simeq \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r)).$$

*Proof.* By Lemma 2.13 we have natural, functorial in  $S$ , quasi-isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma(X_{\mathrm{FF},S^b}, \mathcal{E}_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r))) &\simeq [\mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}_{[u,v]}}(X_S, \mathbf{Q}_p(r)) \xrightarrow{\varphi^{-1}} \mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}_{[u,v/p]}}(X_S, \mathbf{Q}_p(r))] \\ &\simeq [\mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbb{B}_{[u,v]})(r) \xrightarrow{\varphi^{-1}} \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbb{B}_{[u,v/p]})(r)] \\ &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r)). \end{aligned}$$

Here, in the last quasi-isomorphism, we have used the exact sequence (see [11, Lemma 2.23])

$$0 \rightarrow \mathbf{Q}_p \rightarrow \mathbb{B}_{[u,v]} \xrightarrow{\varphi^{-1}} \mathbb{B}_{[u,v/p]} \rightarrow 0$$

□

**4.2. Comparison theorem on the Fargues-Fontaine curve.** We move now to the comparison theorem. Let  $X$  be a smooth partially proper variety over  $K$ , of dimension  $d$ .

**Proposition 4.3.** *Let  $r \geq 2d$ . There is a natural, functorial in  $S$ , quasi-isomorphism in  $\mathrm{QCoh}(X_{\mathrm{FF},S^b})$ :*

$$(4.4) \quad \alpha_r : \mathcal{E}_{\mathrm{syn},*}(X_S, \mathbf{Q}_p(r)) \simeq \mathcal{E}_{\mathrm{proét},*}(X_S, \mathbf{Q}_p(r)).$$

*Proof.* It suffices to construct a natural quasi-isomorphism of  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$

$$\mathrm{R}\Gamma_{\mathrm{syn},*}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)).$$

That is, a natural quasi-isomorphism of pairs

$$(\mathrm{R}\Gamma_{\mathrm{syn},*}^{\mathbf{B}_{[u,v]}}(X_S, \mathbf{Q}_p(r)), \varphi) \simeq (\mathrm{R}\Gamma_{\mathrm{proét},*}^{\mathbf{B}_{[u,v]}}(X_S, \mathbf{Q}_p(r)), \varphi).$$

But this follows from a "Frobenius untwisted" version of Theorem 3.12 presented in Theorem 4.6 below. We just have to argue that we can drop truncations in 4.7: but this follows from the fact that both sides live in degrees  $[0, 2d]$ , which can be seen as in the proof of Theorem (3.9)). □

**Remark 4.5.** We did not list the truncated version of Theorem 3.12 in Proposition 4.3 because the issue of truncation vis a vis localization is a subtle one.

**Theorem 4.6.** (Comparison theorem on the  $Y_{\mathrm{FF}}$ -curve) *Let  $X$  be a smooth partially proper variety over  $K$ . We have natural, functorial in  $S$ , and compatible with Frobenius quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}_{S^b, [u,v], \square})$  and  $\mathcal{D}(\mathbf{B}_{S^b, [u,v/p], \square})$  respectively:*

$$(4.7) \quad \begin{aligned} \tau_{\leq r} \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbb{B}_{[u,v]})(r) &\simeq \tau_{\leq r} [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}_{[u,v]}}(X_S, r) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_S, r)], \\ \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, \mathbb{B}_{[u,v/p]})(r) &\simeq \mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}_{[u,v/p]}}(X_S, r). \end{aligned}$$

*Proof.* For  $v' = v, v/p$ , we define  $F^r \mathbb{B}_{[u,v']} := t^r \mathbb{B}_{[u,v']}$ . We note that  $F^r \mathbb{B}_{[u,v/p]} \simeq \mathbb{B}_{[u,v/p]}$ . We clearly have the isomorphism  $t^r : \mathbb{B}_{[u,v']}(r) \xrightarrow{\sim} F^r \mathbb{B}_{[u,v']}\{r\}$ . We want to construct natural, functorial in  $S$  and compatible with Frobenius, quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}_{S^b, [u,v], \square})$  and  $\mathcal{D}(\mathbf{B}_{S^b, [u,v/p], \square})$ , respectively:

$$\begin{aligned} \tau_{\leq r} \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, F^r \mathbb{B}_{[u,v]}\{r\}) &\simeq \tau_{\leq r} [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}_{[u,v]}}(X_S, r) \rightarrow \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}_{[u,v]}}(X_S, r)], \\ \mathrm{R}\Gamma_{\mathrm{proét},*}(X_S, F^r \mathbb{B}_{[u,v/p]}\{r\}) &\simeq \mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}_{[u,v/p]}}(X_S, r). \end{aligned}$$

For the usual cohomology, these quasi-isomorphisms were constructed in [13, Sec. 7]. They are not explicitly stated there because we almost always carry through the constructions the eigenspaces of Frobenius but, in fact, the latter can be dropped as they are only used to pass between various period rings and here we work with one fixed period ring. For the gist of the construction the interested reader should consult the diagram (7.16) (with the top row moved a step lower and with added  $[u, v]$ -decoration), its refinement (7.31), Section 7.4 in general, and diagram (7.36) (with decoration changed again to  $[u, v]$ ) in particular.

The case of compactly supported cohomology follows now easily from the case of usual cohomology by taking colimits and finite limits.  $\square$

## 5. POINCARÉ DUALITIES ON THE FARGUES-FONTAINE CURVE

We are now ready to state and prove pro-étale duality on the Fargues-Fontaine curve. The same techniques allow us to prove also pro-étale Künneth formula.

**5.1. Hyodo-Kato and de Rham dualities.** Let  $X$  be a smooth partially proper rigid analytic variety over  $K$ , of dimension  $d$ .

5.1.1. *De Rham dualities.* Recall the following dualities (see [1, Cor. 6.20, Th. 6.24, Cor. 6.26]).

**Proposition 5.1.** *Let  $L = K, C$ .*

(1) (Serre duality) *There is a trace map of solid  $L$ -modules*

$$\mathrm{tr}_{\mathrm{coh}} : \mathrm{R}\Gamma_c(X_L, \Omega^d)[d] \rightarrow L.$$

*The pairing*

$$\mathrm{R}\Gamma(X_L, \Omega^j) \otimes_L^{\mathrm{L}\square} \mathrm{R}\Gamma_c(X_L, \Omega^{d-j})[d] \rightarrow \mathrm{R}\Gamma_c(X_L, \Omega^d)[d] \xrightarrow{\mathrm{tr}_{\mathrm{coh}}} L$$

*is perfect, i.e., it yields the quasi-isomorphism in  $\mathcal{D}(L_\square)$ :*

$$\mathrm{R}\Gamma(X_L, \Omega^j) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_c(X_L, \Omega^{d-j})[d], L).$$

(2) (Filtered de Rham duality) *There are natural trace maps in  $\mathcal{D}(L_\square)$  and  $L_\square$ , respectively:*

$$\mathrm{tr}_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \rightarrow L, \quad \mathrm{tr}_{\mathrm{dR}} : H_{\mathrm{dR},c}^{2d}(X_L) \rightarrow L.$$

(a) *The pairing in  $\mathcal{D}(L_\square)$*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L) \otimes_L^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \rightarrow \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} L$$

*is a perfect duality, i.e., we have induced quasi-isomorphism in  $\mathcal{D}(L_\square)$*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L) \xrightarrow{\simeq} \mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d], L).$$

(b) *More generally, let  $r, r' \in \mathbf{N}, r + r' = d$ . The pairing in  $\mathcal{D}(L_\square)$*

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)/F^{r'+1}) \otimes_L^{\mathrm{L}\square} F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \rightarrow \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \xrightarrow{\mathrm{tr}_{\mathrm{dR}}} L$$

*is a perfect duality, i.e., we have induced quasi-isomorphisms in  $\mathcal{D}(L_\square)$*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)/F^{r'+1} \xrightarrow{\simeq} \mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X)[2d], L),$$

$$F^{r'+1} \mathrm{R}\Gamma_{\mathrm{dR}}(X_L) \xrightarrow{\simeq} \mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_{\mathrm{dR},c}(X)/F^r[2d], L).$$



5.1.2.  $\mathbf{B}_{\mathrm{dR}}^+$ -dualities. The duality for  $\mathbf{B}_{\mathrm{dR}}^+$ -cohomology has a slightly different form. For  $r \geq d$ , a natural trace map in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$  can be defined by the composition

$$\mathrm{tr}_{\mathbf{B}_{\mathrm{dR}}^+} : F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d] \rightarrow \mathrm{R}\Gamma_c(X, \Omega^d) \otimes_K^{\mathrm{L}\square} F^{r-d} \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{\mathrm{tr}_{\mathrm{coh}} \otimes \mathrm{Id}} F^{r-d} \mathbf{B}_{\mathrm{dR}}^+.$$

**Corollary 5.2.** (Filtered  $\mathbf{B}_{\mathrm{dR}}^+$ -duality [1, Cor. 6.28]) *Let  $r, r' \geq d, s = r + r' - d$ . The pairing in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$*

$$F^{r'} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+) \otimes_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{L}\square} F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d] \rightarrow F^{r'+r} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d] \xrightarrow{\mathrm{tr}_{\mathbf{B}_{\mathrm{dR}}^+}} F^s \mathbf{B}_{\mathrm{dR}}^+$$

is a perfect duality, i.e., we have an induced quasi-isomorphism in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$

$$F^{r'} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+) \xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\mathrm{dR},\square}^+}(F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d], F^s \mathbf{B}_{\mathrm{dR}}^+).$$

We will need a version of the above result. To state it, take  $r, r' \geq d, s = r + r' - d$  and consider the pairing in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$

$$(5.3) \quad (\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/F^{r'}) \otimes_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{L}\square} (F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/t^s)[2d-1] \rightarrow F^s \mathbf{B}_{\mathrm{dR}}^+$$

defined as the composition

$$\begin{aligned} (\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/F^{r'}) \otimes_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{L}\square} (F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/t^s)[2d-1] &\xrightarrow{\cup} F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/t^s[2d-1] \\ &\rightarrow \mathrm{R}\Gamma_c(X_C, \Omega^d) \otimes_K^{\mathrm{L}\square} (F^{r-d} \mathbf{B}_{\mathrm{dR}}^+/t^s)[-1] \xrightarrow{\partial} \mathrm{R}\Gamma_c(X_C, \Omega^d) \otimes_K^{\mathrm{L}\square} F^s \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{\mathrm{tr}_{\mathrm{coh}} \otimes \mathrm{Id}} F^s \mathbf{B}_{\mathrm{dR}}^+ \end{aligned}$$

Here the third morphism is the boundary map induced by the exact sequence

$$0 \rightarrow F^s \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{\mathrm{can}} F^{r-d} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow F^{r-d} \mathbf{B}_{\mathrm{dR}}^+/t^s \rightarrow 0$$

**Corollary 5.4.** *The pairing (5.3) is a perfect duality, i.e., we have an induced quasi-isomorphism in  $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$*

$$(5.5) \quad \gamma : \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/F^{r'} \xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\mathrm{dR},\square}^+}(F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/t^s[2d-1], F^s \mathbf{B}_{\mathrm{dR}}^+).$$

*Proof.* Consider the following map of distinguished triangles

$$\begin{array}{ccc} F^{r'} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+) & \xrightarrow{\sim} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\mathrm{dR},\square}^+}(F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d], F^s \mathbf{B}_{\mathrm{dR}}^+) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+) & \xrightarrow{\sim} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\mathrm{dR},\square}^+}(t^s \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)[2d], F^s \mathbf{B}_{\mathrm{dR}}^+) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/F^{r'} & \xrightarrow{\gamma} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\mathrm{dR},\square}^+}(F^r \mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}_{\mathrm{dR}}^+)/t^s[2d-1], F^s \mathbf{B}_{\mathrm{dR}}^+), \end{array}$$

where the middle arrow is the de Rham duality map ( $\mathbf{B}_{\mathrm{dR}}^+$ -linearized) and the top arrow is the  $\mathbf{B}_{\mathrm{dR}}^+$ -duality map from Corollary 5.2. Both are quasi-isomorphisms (see Proposition 5.1). Hence so is the bottom duality map, as wanted.  $\square$

The duality map (5.5) can be lifted to the Fargues-Fontaine curve: the pairing (5.3) induces a pairing of  $\mathbf{B}_{S^b, [u, v]}$ -modules

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathbf{B}_{[u, v]}}(X_S, r) \otimes_{\mathbf{B}_{S^b, [u, v]}}^{\mathrm{L}\square} (F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}_{[u, v]}}(X_S)/t^s) \rightarrow \mathbf{B}_{S^b, [u, v]} \{s, s\}[-2d+1],$$

which, in turn, induces a pairing of nuclear  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{\mathbf{B}}(X_S, r) \otimes_{\mathbf{B}_{S^b}^{\mathrm{FF}}}^{\mathrm{L}} (F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}}(X_S)/t^s) \rightarrow \mathbf{B}_{S^b} \{s, s\}[-2d+1],$$

where we set  $F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}}(X_S)/t^s := (F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}_{[u, v]}}(X_S)/t^s, 0)$ . This descends to a pairing on  $X_{\mathrm{FF}, S^b}$ :

$$(5.6) \quad \mathcal{E}_{\mathrm{dR}}(X_S, r) \otimes_{\mathcal{O}}^{\mathrm{L}} i_{\infty, *} (F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^s) \rightarrow \mathcal{O}(s, s)[-2d+1],$$

where we set  $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}_{\mathrm{dR}}^+) := \mathrm{R}\Gamma_{\mathrm{dR},c}(X_K) \otimes_K^{\mathrm{L}\square} \mathbf{B}_{\mathrm{dR}}^+(S)$ . The pairing (5.6) induces a duality map in  $\mathrm{QCoh}(X_{\mathrm{FF},S^b})$ :

$$(5.7) \quad \gamma_{X_S} : \mathcal{E}_{\mathrm{dR}}(X_S, r) \rightarrow \mathbb{D}(i_{\infty,*}(F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^s)[2d-1], \mathcal{O}(s, s)),$$

where we set  $\mathbb{D}(-, -) := \mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^b})}(-, -)$ .

**Lemma 5.8.** *The duality map (5.7) is a quasi-isomorphism.*

*Proof.* We need to show that the duality map

$$\gamma_{X_S}^{\mathrm{FF}} : \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathbf{B}}(X_S, r) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{S^b}^{\mathrm{FF}}}(F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}}(X_S)/t^s[2d-1], \mathbf{B}_{S^b}\{s, s\}).$$

is a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}_{S^b}^{\mathrm{FF}})$ . Or, passing to solid  $\mathbf{B}' := \mathbf{B}_{S^b, [u,v]}$ -modules, that the duality map

$$(5.9) \quad \gamma_{X_S} : \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathbf{B}^{[u,v]}}(X_S, r) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}^{\mathbf{B}^{[u,v]}}(X_S)/t^s[2d-1], \mathbf{B}')$$

is a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}'_{\square})$ . But this is Corollary 5.4 (strictly speaking, its  $S$ -version but it holds by the same arguments).  $\square$

5.1.3. *Hyodo-Kato duality.* This is based on [1, Sec. 6.4.2]. There exists a natural trace map in  $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$ :

$$\mathrm{Tr}_X : \mathrm{R}\Gamma_{\mathrm{HK},c}(X_C) \rightarrow \check{C}\{-d\}[-2d].$$

The pairing in  $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$  ( $s = r + r' - d$ )

$$(5.10) \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r'\} \rightarrow \mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r + r'\} \xrightarrow{\mathrm{Tr}_X} \check{C}\{s\}[-2d]$$

is perfect, i.e., it induces a quasi-isomorphism in  $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$

$$(5.11) \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})}(\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r'\}, \check{C}\{s\}[-2d]),$$

where the internal Hom is just  $\mathrm{R}\mathcal{H}om_{\check{C}_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C), \check{C}[-2d])$  – the internal Hom in  $\mathcal{D}(\check{C}_{\square})$  – equipped with  $(\varphi, N, \mathcal{G}_K)$ -actions via  $\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r' - s\}$ .

The above duality can be lifted to the Fargues-Fontaine curve: the pairing (5.10) induces a pairing of  $\mathbf{B}_{S^b, [u,v]}$ -modules

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u,v]}}(X_S, r) \otimes_{\mathbf{B}_{S^b, [u,v]}^{\mathrm{L}\square}} \mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}^{[u,v]}}(X_S, r') \rightarrow \check{C}\{s\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}_{S^b, [u,v]}[-2d],$$

which, in turn, induces a pairing of nuclear  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_S, r) \otimes_{\mathbf{B}_{S^b}^{\mathrm{L}\square}} \mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}}(X_S, r') \rightarrow \mathbf{B}_{S^b}\{s\}[-2d].$$

This descends to a pairing on  $X_{\mathrm{FF},S^b}$ :

$$\mathcal{E}_{\mathrm{HK}}(X_S, r) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathcal{E}_{\mathrm{HK}}(X_S, r') \rightarrow \mathcal{O}(s)[-2d],$$

which induces a duality map in  $\mathrm{QCoh}(X_{\mathrm{FF},S^b})$ :

$$(5.12) \quad \gamma_{X_S} : \mathcal{E}_{\mathrm{HK}}(X_S, r) \rightarrow \mathbb{D}(\mathcal{E}_{\mathrm{HK},c}(X_S, r')[2d], \mathcal{O}(s)).$$

**Lemma 5.13.** *The map  $\gamma_{X_S}$  above is a quasi-isomorphism in  $\mathrm{QCoh}(X_{\mathrm{FF},S^b})$ .*

*Proof.* Since  $\mathbf{B}_{S^b, [u,v]}$  is  $\mathbf{B}_{S^b, [u,v], \square}^{\mathrm{an}}$ -complete (see [2, Lemma 3.24]), by Remark 2.6, we may pass from  $\mathbf{B}_{S^b, [u,v]}^+$  to  $\mathbf{Z}$ , i.e., to  $\mathbf{B}_{S^b}^{\mathrm{FF}}$ -modules. Hence we need to show that the duality map

$$\gamma_{X_S}^{\mathrm{FF}} : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}}(X_S, r) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{S^b}^{\mathrm{FF}}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}}(X_S, r')[2d], \mathbf{B}_{S^b}\{s\}).$$

is a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}_{S^b}^{\mathrm{FF}})$ . Or, passing to solid  $\mathbf{B}' := \mathbf{B}_{S^b, [u,v]}$ -modules, that the duality map

$$(5.14) \quad \gamma_{X_S} : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u,v]}}(X_S, r) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}^{[u,v]}}(X_S, r')[2d], \mathbf{B}')$$

is a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}'_{\square})$ . We claim that, for that, it suffices to check that, for  $j \in \mathbf{N}$ , the duality map on cohomology groups level

$$(5.15) \quad \gamma_{X_S}^j : H_{\mathrm{HK}}^{\mathbf{B}^{[u,v],j}}(X_S, r) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(H_{\mathrm{HK},c}^{\mathbf{B}^{[u,v],2d-j}}(X_S, r'), \mathbf{B}')$$

is an isomorphism in  $\mathbf{B}'_{\square}$ . Indeed, passing to cohomology in (5.14), we need to check that the duality map

$$\gamma_{X_S}^j : H_{\mathrm{HK}}^{\mathbf{B}^{[u,v],j}}(X_S, r) \rightarrow H^j(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}^{[u,v]}}(X_S, r')[2d], \mathbf{B}'))$$

is an isomorphism in  $\mathbf{B}'_{\square}$ . But  $H_{\mathrm{HK},c}^i(X_C)$  is a direct sum of copies of  $\check{C}$  hence we have

$$H^j(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}^{[u,v]}}(X_S, r')[2d], \mathbf{B}')) \simeq \underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(H_{\mathrm{HK},c}^{\mathbf{B}^{[u,v],2d-j}}(X_S, r'), \mathbf{B}'),$$

as wanted.

To prove (5.15), we observe that, for  $i \in \mathbf{Z}$ , we have the natural isomorphisms<sup>6</sup>

$$(5.16) \quad H_{\mathrm{HK},*}^{\mathbf{B}^{[u,v],j}}(X_S, i) \simeq (H_{\mathrm{HK},*}^j(X_C)\{i\} \otimes_{\check{C}}^{\square} \mathbf{B}'_{\log})^{N=0} \xleftarrow{\sim} H_{\mathrm{HK},*}^j(X_C)\{i\} \otimes_{\check{C}}^{\square} \mathbf{B}'.$$

Here the second quasi-isomorphism is defined by the map  $\exp(NU)$  (this makes sense because the monodromy operator on the Hyodo-Kato cohomology  $H_{\mathrm{HK},*}^j(X_C)$  is nilpotent). For the first quasi-isomorphism

$$H_{\mathrm{HK},*}^{\mathbf{B}^{[u,v],j}}(X_S, i) = H^j([\mathrm{R}\Gamma_{\mathrm{HK},*}(X_C)\{i\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log}]^{N=0}) \simeq (H_{\mathrm{HK},*}^j(X_C)\{i\} \otimes_{\check{C}}^{\square} \mathbf{B}'_{\log})^{N=0}$$

we used the fact that

$$H^j(\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}^{[u,v]}}(X_C)\{i\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log}) \simeq H_{\mathrm{HK},*}^{\mathbf{B}^{[u,v],j}}(X_C)\{i\} \otimes_{\check{C}}^{\square} \mathbf{B}'_{\log},$$

that  $N$  is nilpotent on  $H_{\mathrm{HK},*}^{\mathbf{B}^{[u,v],j}}(X_C)$  (so we can do devissage by the kernels of the action of  $N$ ), and that  $\mathbf{B}' \xrightarrow{\sim} [\mathbf{B}'_{\log}]^{N=0}$ .

It is easy to check that the maps in (5.16) are compatible with products. Hence we can write the duality map (5.15) as the Hyodo-Kato pairing

$$\gamma_{X_S}^j : H_{\mathrm{HK}}^j(X_C) \otimes_{\check{C}}^{\square} \mathbf{B}' \rightarrow \underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(H_{\mathrm{HK},c}^{2d-j}(X_C) \otimes_{\check{C}}^{\square} \mathbf{B}', \mathbf{B}').$$

To show that it is an isomorphism in  $\mathbf{B}'_{\square}$  it suffices thus to evoke the Hyodo-Kato duality (5.11) and to show that the natural map

$$\underline{\mathrm{Hom}}_{\check{C}_{\square}}(H_{\mathrm{HK},c}^{2d-j}(X_C), \check{C}) \otimes_{\check{C}}^{\square} \mathbf{B}' \rightarrow \underline{\mathrm{Hom}}_{\check{C}_{\square}}(H_{\mathrm{HK},c}^{2d-j}(X_C), \mathbf{B}')$$

is an isomorphism in  $\mathbf{B}'_{\square}$ . But this is an isomorphism by [16, Th. 3.40] since  $\mathbf{B}'$  is a Banach space over  $\check{C}$ .  $\square$

**5.2. Syntomic duality.** Let  $X$  be a smooth partially proper rigid analytic variety over  $K$  of dimension  $d$ . Let  $S \in \mathrm{Perf}_C$ . Recall that syntomic  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\mathrm{FF}}$  are defined as (see Sec. 3.2.3)

$$\mathrm{R}\Gamma_{\mathrm{syn},*}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}}(X_S, r)],$$

where the Hyodo-Kato map is described by diagram (3.8). The Hyodo-Kato and de Rham cup products are compatible with this diagram hence yield a cup product on the syntomic  $\varphi$ -modules:

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^b}^{\mathrm{FF}}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r')) \rightarrow \mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r+r')).$$

This product can be described by an analogous product on the  $\mathbf{B}' := \mathbf{B}_{S^b, [u,v]}$ -chart:

$$(5.17) \quad \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r')) \rightarrow \mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r+r')),$$

where we set

$$(5.18) \quad \mathrm{R}\Gamma_{\mathrm{syn},*}^{[u,v]}(X_S, \mathbf{Q}_p(i)) := [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}^{[u,v]}}(X_S, i) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}^{[u,v]}}(X_S, i)].$$

<sup>6</sup>We can ignore the Galois action here.

It is compatible with the products on  $\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}^{[u,v]}}(X_S, i)$  and  $F^i \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}}(X_S/\mathbf{B}')$ . Here we defined  $F^i \mathrm{R}\Gamma_{\mathrm{dR},*}(X_S/\mathbf{B}')$  as  $F^i \mathrm{R}\Gamma_{\mathrm{dR},*}(X_C/\mathbf{B}_{\mathrm{dR}}^+)$  with  $\mathbf{B}_{\mathrm{dR}}^+$  replaced by  $\mathbf{B}'$ .

Let  $s \geq d$ . There is a trace map

$$\mathrm{Tr}_X : \mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(s)) \rightarrow \mathbf{B}_{S^\flat}^{\mathrm{FF}}\{s-d, s-d\}[-2d]$$

defined on the  $\mathbf{B}'$ -chart via the trace map

$$(5.19) \quad \mathrm{Tr}_X^{[u,v]} : \mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(s)) \rightarrow \mathbf{B}'\{s-d, s-d\}[-2d],$$

which is compatible with the Hyodo-Kato and de Rham trace maps. The map  $\mathrm{Tr}_X^{[u,v]}$  is defined using the exact sequence

$$H_{\mathrm{syn},c}^{[u,v],2d}(X_S, \mathbf{Q}_p(s)) \rightarrow H_{\mathrm{HK},c}^{2d}(X_S, s) \xrightarrow{\iota_{\mathrm{HK}}} H_{\mathrm{dR},c}^{2d}(X_S, s),$$

which can be written more explicitly as the exact sequence

$$(5.20) \quad H_{\mathrm{syn},c}^{[u,v],2d}(X_S, \mathbf{Q}_p(s)) \rightarrow (H_{\mathrm{HK},c}^{2d}(X_C)\{s\} \otimes_{\check{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log})^{N=0} \xrightarrow{\iota_{\mathrm{HK}}} H_{\mathrm{dR},c}^{2d}(X) \otimes_K^{\mathrm{L}\square} (\mathbf{B}'\{s-d\}/F^{s-d}).$$

Using the (compatible) Hyodo-Kato and de Rham trace maps

$$\mathrm{Tr}_X : H_{\mathrm{HK},c}^{2d}(X_C)\{s\} \xrightarrow{\sim} \check{C}\{s-d\}, \quad \mathrm{Tr}_X : H_{\mathrm{dR},c}^{2d}(X) \xrightarrow{\sim} K,$$

(5.20) yields a map

$$H_{\mathrm{syn},c}^{[u,v],2d}(X_S, \mathbf{Q}_p(s-d)) \rightarrow \mathrm{Ker}(\mathbf{B}'\{s-d\} \rightarrow \mathbf{B}'\{s-d\}/F^{s-d}) = \mathbf{B}'\{s-d, s-d\},$$

hence the trace (5.19), as wanted.

For  $s := r + r' - d$ , the above can be lifted to the Fargues-Fontaine curve: the cup product (5.17) and trace map (5.19) induce a pairing of  $\mathbf{B}'$ -modules

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X^{[u,v]}} \mathbf{B}'\{s, s\}[-2d],$$

which, in turn, induces a pairing of nuclear  $\varphi$ -modules over  $\mathbf{B}_{S^\flat}^{\mathrm{FF}}$

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^\flat}^{\mathrm{FF}}}^{\mathrm{L}} \mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathbf{B}_{S^\flat}\{s, s\}[-2d].$$

This descends to a pairing in  $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ :

$$\mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathcal{O}(s, s)[-2d],$$

which induces a natural map  $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$

$$(5.21) \quad \gamma_{X_S} : \mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \rightarrow \mathbb{D}(\mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s, s)).$$

**Theorem 5.22.** (Syntomic Poincaré duality on the Fargues-Fontaine curve)

Let  $r, r' \geq 2d, s := r + r' - d$ . The map  $\gamma_{X_S}$  is a quasi-isomorphism in  $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ .

*Proof.* It is enough to show this in  $\varphi$ -modules over  $\mathbf{B}_{S^\flat}^{\mathrm{FF}}$  for the corresponding map

$$(5.23) \quad \gamma_{X_S} : \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{S^\flat}^{\mathrm{FF}}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r'))[2d], \mathbf{B}_{S^\flat}\{s, s\}).$$

Or in  $\mathcal{D}(\mathbf{B}_{S^\flat}^{[u,v],\square})$  for the induced map

$$\gamma_{X_S}^{[u,v]} : \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_{\square}^{[u,v]}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r'))[2d], \mathbf{B}'(s)).$$

But for that, it is enough to check that base changes of  $\gamma_{X_S}^{[u,v]}$  to both  $\mathbf{B}'[1/t]$  and  $\mathbf{B}'/t$  are quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$ . This last claim requires a bit of justification. We have the exact sequence of solid  $\mathbf{B}'$ -modules

$$0 \rightarrow \mathbf{B}' \rightarrow \mathbf{B}'[1/t] \rightarrow \mathbf{B}'[1/t]/\mathbf{B}' \rightarrow 0.$$

Hence it suffices to check that base changes of  $\gamma_{X_S}^{[u,v]}$  to both  $\mathbf{B}'[1/t]$  and  $\mathbf{B}'[1/t]/\mathbf{B}'$  are quasi-isomorphisms. Writing  $\mathbf{B}'[1/t]/\mathbf{B}' = \mathrm{colim}_n(\mathbf{B}'/t^n)$  and using the fact that the tensor products commute with filtered colimits, we see that it suffices to check that base changes of  $\gamma_{X_S}^{[u,v]}$  to both  $\mathbf{B}'[1/t]$  and  $\mathbf{B}'[1/t]/t^i$  are quasi-isomorphisms. Finally, by devissage, we can drop  $i$  to 1, as wanted.

For the first base change, we have quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r))[1/t] &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}[u,v]}(X_S, r)[1/t], \\ \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r'))[2d], \mathbf{B}') [1/t] &\xleftarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}[u,v]}(X_S, r')[2d], \mathbf{B}'(s))[1/t]. \end{aligned}$$

And  $\gamma_{X_S}^{[u,v]}$  is just the canonical map

$$\gamma_{X_S} : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}[u,v]}(X_S, r)[1/t] \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}[u,v]}(X_S, r'), \mathbf{B}')[1/t]$$

induced by the Hyodo-Kato pairing (5.10). Since it is compatible with  $t$ -action, it suffices to show that the canonical map

$$\gamma_{X_S} : \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}[u,v]}(X_S, r) \rightarrow \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{HK},c}^{\mathbf{B}[u,v]}(X_S, r'), \mathbf{B}')$$

is a quasi-isomorphism in  $\mathcal{D}(\mathbf{B}'_{\square})$ . But this was shown in (5.14), in the proof of Lemma 5.13.

For the base change to  $\mathbf{B}'/t$ , write  $S = \mathrm{Spa}(R, R^+)$ ; then  $\mathbf{B}'/t = R$ . We claim that we have a compatible with product quasi-isomorphism in  $\mathcal{D}(\mathbf{B}'_{\square})$

$$(5.24) \quad \mathrm{R}\Gamma_{\mathrm{syn},*}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \simeq F^r \mathrm{R}\Gamma_{\mathrm{dR},*}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R.$$

To show (5.24) we compute:

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn},*}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &= [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}[u,v]}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}[u,v]}(X_S, r)] \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \\ &\xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \xrightarrow{\iota_{\mathrm{HK}} \otimes \mathrm{Id}} \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R] \end{aligned}$$

Then we use the following commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \\ \downarrow \iota_{\mathrm{HK}} & & \parallel \\ \mathrm{R}\Gamma_{\mathrm{dR},*}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{dR},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \\ \uparrow & & \uparrow \\ F^r \mathrm{R}\Gamma_{\mathrm{dR},*}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R & \longrightarrow & 0 \end{array}$$

It defines quasi-isomorphisms between the mapping fibers of the rows yielding (5.24). The quasi-isomorphism in the above diagram needs a justification: take the composition

$$(\mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}[u,v]}(X_S) \{s\} \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK},*}^{\mathbf{B}[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR},*}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R$$

It is equal to  $\iota_{\mathrm{HK}}$  hence a quasi-isomorphism, as wanted.

From (5.24), we get the quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r'))[2d], \mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r'))[2d], R) \\ &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}') [2d], R). \end{aligned}$$

We have, compatible with products, quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$  (see [1, Prop. 4.7, Prop. 4.11])

$$(5.25) \quad \begin{aligned} F^r \mathrm{R}\Gamma_{\mathrm{dR}}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\xrightarrow{\sim} \bigoplus_{i=0}^d \mathrm{R}\Gamma(X, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r-i)[-i], \\ F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\xrightarrow{\sim} \bigoplus_{i=0}^d \mathrm{R}\Gamma_c(X, \Omega^i) \otimes_K^{\square} R(r'-i)[-i]. \end{aligned}$$

Putting (5.24) and (5.25) together, we get, compatible with products, quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\simeq \bigoplus_{i=0}^d \mathrm{R}\Gamma(X, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r-i)[-i], \\ \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_{\square}}(\mathrm{R}\Gamma_{\mathrm{syn},c}^{[u,v]}(X_S, \mathbf{Q}_p(r'))[2d], \mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{R}\square}(\bigoplus_{i=0}^d \mathrm{R}\Gamma_c(X, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r'-i)[2d-i], R). \end{aligned}$$

And our result follows from Serre duality<sup>7</sup> (see Proposition 5.1) which yields the quasi-isomorphisms in  $\mathcal{D}(R_{\square})$ .

$$\begin{aligned} \mathrm{R}\Gamma(X, \Omega^i) \otimes_K^{\mathrm{L}\square} R &\xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{K_{\square}}(\mathrm{R}\Gamma_c(X, \Omega^{d-i})[d], K) \otimes_K^{\mathrm{L}\square} R \\ &\xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{R_{\square}}(\mathrm{R}\Gamma_c(X, \Omega^{d-i}) \otimes_K^{\mathrm{L}\square} R[d], R). \end{aligned}$$

The second quasi-isomorphism holds by the same argument as the one used at the end of the proof of Lemma 5.13.  $\square$

**5.3. Syntomic duality: an alternative argument.** We present here an alternative proof of Theorem 5.22 (conditional on the compatibilities in Lemma 5.29 below). It uses dual modifications to inverse the arrows in the defining syntomic distinguished triangles (3.22).

More precisely, let  $j \geq i \geq 0$ . We will construct a distinguished triangle in  $\mathrm{QCoh}(X_{\mathrm{FF}, S^{\flat}})$

$$(5.26) \quad \mathcal{E}_{\mathrm{HK}, c}(X_S, i) \otimes_{\mathcal{O}}^{\mathrm{L}\square} \mathcal{O}(0, j) \rightarrow \mathcal{E}_{\mathrm{syn}, c}(X_S, \mathbf{Q}_p(i)) \rightarrow i_{\infty, *} F^i \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^j,$$

which is a twisted version of (3.22). To do that, consider the following map of distinguished triangles

$$(5.27) \quad \begin{array}{ccccc} \mathcal{E}_{\mathrm{HK}, c}(X_S, i) \otimes_{\mathcal{O}}^{\mathrm{L}\square} \mathcal{O}(0, j) & \longrightarrow & \mathcal{E}_{\mathrm{HK}, c}(X_S, i) & \xrightarrow{\iota_{\mathrm{HK}}} & i_{\infty, *} \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^j \\ \downarrow & & \parallel & & \downarrow \mathrm{can} \\ \mathcal{E}_{\mathrm{syn}, c}(X_S, \mathbf{Q}_p(i)) & \longrightarrow & \mathcal{E}_{\mathrm{HK}, c}(X_S, i) & \xrightarrow{\iota_{\mathrm{HK}}} & \mathcal{E}_{\mathrm{dR}, c}(X_S, i) \end{array}$$

Here, the bottom distinguished triangle is (3.22); the top one is induced from the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_S, i) \otimes_{\mathbf{B}_{S^{\flat}, [u, v]}^{\mathrm{L}\square}} \mathbf{B}_{S^{\flat}, [u, v]}^{\{0, j\}} \rightarrow \mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_S, i) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}^{\mathbf{B}^{[u, v]}}(X_S, i)/t^j$$

obtained by tensoring the exact sequence (4.1) for  $0, j$  with  $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_S, i)$ . (Recall that  $\mathrm{R}\Gamma_{\mathrm{HK}}^{\mathbf{B}^{[u, v]}}(X_C, r) = [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes_C^{\mathrm{L}\square} \mathbf{B}_{S^{\flat}, [u, v], \log}^{\mathrm{L}\square}]^{N=0}$ ). The dashed arrow in diagram (5.27) is defined to make the diagram a map of distinguished triangles. The diagram yields quasi-isomorphisms

$$\begin{aligned} [\mathcal{E}_{\mathrm{HK}, c}(X_S, i) \otimes_{\mathcal{O}}^{\mathrm{L}\square} \mathcal{O}(0, j) \rightarrow \mathcal{E}_{\mathrm{syn}, c}(X_S, \mathbf{Q}_p(i))][1] &\xrightarrow{\sim} [i_{\infty, *} \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^j \rightarrow \mathcal{E}_{\mathrm{dR}, c}(X_S, i)] \\ &\xrightarrow{\sim} [i_{\infty, *} F^i \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^j]. \end{aligned}$$

That is, we get a distinguished triangle (5.26), as wanted.

Now, let  $r, r' \geq 2d, s = r + r' - d$ . Consider the following diagram in  $\mathrm{QCoh}(X_{\mathrm{FF}, S^{\flat}})$  (note that  $s \geq r'$ )

$$(5.28) \quad \begin{array}{ccc} \mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) & \xrightarrow{\gamma_{X_S}^{\mathrm{syn}}} & \mathbb{D}(\mathcal{E}_{\mathrm{syn}, c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s, s)) \\ \downarrow & & \downarrow \\ \mathcal{E}_{\mathrm{HK}}(X_S, r) & \xrightarrow[\sim]{\gamma_{X_S}^{\mathrm{HK}}} & \mathbb{D}(\mathcal{E}_{\mathrm{HK}, c}(X_S, r') \otimes_{\mathcal{O}}^{\mathrm{L}\square} \mathcal{O}(0, s)[2d], \mathcal{O}(s, s)) \\ \downarrow & & \downarrow \\ \mathcal{E}_{\mathrm{dR}}(X_S, r) & \xrightarrow[\sim]{\gamma_{X_S}^{\mathrm{dR}}} & \mathbb{D}(i_{\infty, *} F^{r'} \mathrm{R}\Gamma_{\mathrm{dR}, c}(X_S/\mathbf{B}_{\mathrm{dR}}^+)/t^s[2d-1], \mathcal{O}(s, s)), \end{array}$$

where the horizontal maps are defined by the syntomic, Hyodo-Kato, and  $\mathbf{B}_{\mathrm{dR}}^+$ -pairings, respectively (see (5.21), (5.12), (5.7)).

Let us assume Lemma 5.29 below. To prove that the top horizontal arrow in diagram 5.28 is a quasi-isomorphism it suffices to show that so are the two lower arrows. But this follows from Lemma 5.13 (we used the isomorphism  $\mathcal{O}(0, s) \otimes_{\mathcal{O}}^{\mathrm{L}\square} \mathcal{O}(s) \simeq \mathcal{O}(s, s)$ ) and Lemma 5.8. It remains to prove:

**Lemma 5.29.** *Diagram (5.28) above is a map of distinguished triangles.*

<sup>7</sup>Apply it in degree  $i$ .

**5.4. Pro-étale duality.** Let  $X$  be a smooth partially proper rigid analytic variety over  $K$  of dimension  $d$ . Let  $S \in \text{Perf}_C$ . We define a cup product on the pro-étale  $\varphi$ -modules:

$$(5.30) \quad \text{R}\Gamma_{\text{proét}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{\text{FF}}^{\square}} \text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r')) \rightarrow \text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r+r'))$$

via the cup product on the  $\mathbf{B}' := \mathbf{B}_{S^{\flat}, [u,v]}$ -charts:

$$\text{R}\Gamma_{\text{proét}}(X_S, \mathbb{B}_{[u,v]}(r)) \otimes_{\mathbf{B}'}^{\text{L}\square} \text{R}\Gamma_{\text{proét},c}(X_S, \mathbb{B}_{[u,v]}(r')) \rightarrow \text{R}\Gamma_{\text{proét},c}(X_S, \mathbb{B}_{[u,v]}(r+r'))$$

induced by the cup product on pro-étale cohomology. This product is compatible with the syntomic product (via the comparison quasi-isomorphism from Theorem 4.6): to see this it suffices to argue for the usual cohomology and locally, where the comparison map is known to be compatible with products.

Let  $s \geq 2d$ . We define a trace map

$$(5.31) \quad \text{Tr}_X : \text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(s)) \rightarrow \mathbf{B}_{S^{\flat}}^{\text{FF}}\{s-d, s-d\}[-2d]$$

as the composition

$$\text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(s)) \simeq \text{R}\Gamma_{\text{syn},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(s)) \xrightarrow{\text{Tr}_X} \mathbf{B}_{S^{\flat}}^{\text{FF}}\{s-d, s-d\}[-2d].$$

By [1, Sec. 8.3.3], for  $S = \text{Spa}(C, \mathcal{O}_C)$ , this map is compatible with Huber's trace map.

For  $r, r' \geq d, s := r + r' - d$ , the above can be lifted to the Fargues-Fontaine curve: the cup product (5.30) and trace map (5.31) induce a pairing of nuclear  $\varphi$ -modules over  $\mathbf{B}_{S^{\flat}}^{\text{FF}}$

$$\text{R}\Gamma_{\text{proét}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^{\flat}}^{\text{FF}}}^{\text{L}} \text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \text{R}\Gamma_{\text{proét},c}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\text{Tr}_X} \mathbf{B}_{S^{\flat}}\{s, s\}[-2d].$$

This descends to a pairing in  $\text{QCoh}(X_{\text{FF}, S^{\flat}})$ :

$$\mathcal{E}_{\text{proét}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathcal{O}}^{\text{L}} \mathcal{E}_{\text{proét},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathcal{E}_{\text{proét},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\text{Tr}_X} \mathcal{O}(s, s)[-2d],$$

which induces a natural map  $\text{QCoh}(X_{\text{FF}, S^{\flat}})$

$$(5.32) \quad \gamma_{X_S} : \mathcal{E}_{\text{proét}}(X_S, \mathbf{Q}_p(r)) \rightarrow \mathbb{D}(\mathcal{E}_{\text{proét},c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s, s)).$$

By an abuse of notation, we will write

$$(5.33) \quad \gamma_{X_S} : \mathcal{E}_{\text{proét}}(X_S, \mathbf{Q}_p) \rightarrow \mathbb{D}(\mathcal{E}_{\text{proét},c}(X_S, \mathbf{Q}_p(d))[2d], \mathcal{O}).$$

for the Tate-untwisted version of the map (5.32).

**Corollary 5.34.** (Pro-étale Poincaré duality on the Fargues-Fontaine curve) *The map  $\gamma_{X_S}$  from (5.33) is a quasi-isomorphism in  $\text{QCoh}(X_{\text{FF}, S^{\flat}})$ .*

*Proof.* Choose  $r, r' \geq 2d$  and set  $s := r + r' - d$ . It suffices to prove that the Tate twisted map (5.32) is a quasi-isomorphism. This follows immediately from the syntomic duality from Theorem 5.22 and the comparison result from Proposition 4.3.  $\square$

**5.5. Künneth formula.** Let  $X, Y$  be smooth Stein rigid analytic varieties over  $K$ . The simple observation that we have a quasi-isomorphism in  $\mathcal{D}(K_{\square})$

$$(5.35) \quad \Omega(X) \otimes_K^{\text{L}\square} \mathcal{O}(Y) \oplus \mathcal{O}(X) \otimes_K^{\text{L}\square} \Omega(Y) \xrightarrow{\sim} \Omega(X \times_K Y),$$

which implies the Künneth formula for de Rham cohomology

$$\text{R}\Gamma_{\text{dR}}(X) \otimes_K^{\text{L}\square} \text{R}\Gamma_{\text{dR}}(Y) \xrightarrow{\sim} \text{R}\Gamma_{\text{dR}}(X \times_K Y)$$

leads to the syntomic Künneth formula in  $\text{QCoh}(X_{\text{FF}})$  and hence the pro-étale as well:

**Theorem 5.36.** (Künneth formula) *Let  $X, Y$  be smooth partially proper rigid analytic varieties over  $K$ . Let  $d$  be larger than the dimension of  $X \times_K Y$  and let  $r, r' \geq 2d$ . Let  $S \in \text{Perf}_C$ . The natural maps*

$$\begin{aligned} \kappa : \mathcal{E}_{\text{syn}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathcal{O}}^L \mathcal{E}_{\text{syn}}(Y_S, \mathbf{Q}_p(r')) &\rightarrow \mathcal{E}_{\text{syn}}((X \times_K Y)_S, \mathbf{Q}_p(r+r')), \\ \kappa : \mathcal{E}_{\text{proét}}(X_S, \mathbf{Q}_p) \otimes_{\mathcal{O}}^L \mathcal{E}_{\text{proét}}(Y_S, \mathbf{Q}_p) &\rightarrow \mathcal{E}_{\text{proét}}((X \times_K Y)_S, \mathbf{Q}_p) \end{aligned}$$

are quasi-isomorphisms in  $\text{QCoh}(X_{\text{FF}, S^b})$ .

*Proof.* The pro-étale case follows from the syntomic one via the comparison quasi-isomorphism from Proposition 4.3.

For the syntomic case, it is enough to show that on the level of  $\varphi$ -modules over  $\mathbf{B}_{S^b}^{\text{FF}}$  the corresponding map

$$\kappa : \text{R}\Gamma_{\text{syn}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^b}^{\text{FF}}}^L \text{R}\Gamma_{\text{syn}}^{\mathbf{B}}(Y_S, \mathbf{Q}_p(r')) \rightarrow \text{R}\Gamma_{\text{syn}}^{\mathbf{B}}((X \times_K Y)_S, \mathbf{Q}_p(r+r'))$$

is a quasi-isomorphism. Or that in  $\mathcal{D}(\mathbf{B}'_{\square})$ , for  $\mathbf{B}' := \mathbf{B}_{S^b, [u, v]}$ , the induced map

$$\kappa^{[u, v]} : \text{R}\Gamma_{\text{syn}}^{[u, v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'_{\square}}^L \text{R}\Gamma_{\text{syn}}^{[u, v]}(Y_S, \mathbf{Q}_p(r')) \rightarrow \text{R}\Gamma_{\text{syn}}^{[u, v]}((X \times_K Y)_S, \mathbf{Q}_p(r+r'))$$

is a quasi-isomorphism. See (5.18) for the definition of the individual terms. But for that, as in the proof of Theorem 5.22, it is enough to check that the base changes of  $\kappa^{[u, v]}$  to  $\mathbf{B}'[1/t]$  and to  $\mathbf{B}'/t$  are quasi-isomorphisms.

For the first base change, we use the quasi-isomorphism in  $\mathcal{D}(\mathbf{B}'_{\square})$

$$\text{R}\Gamma_{\text{syn}}^{[u, v]}(X_S, \mathbf{Q}_p(r))[1/t] \xrightarrow{\sim} \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}(X_S, r)[1/t]$$

to write

$$\kappa^{[u, v]}[1/t] : (\text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}(X_S, r) \otimes_{\mathbf{B}'_{\square}}^L \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}(Y_S, r'))[1/t] \rightarrow \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}((X \times_K Y)_S, r+r')[1/t].$$

This map is induced by the Hyodo-Kato pairing

$$\kappa_{\text{HK}}^{[u, v]} : \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}(X_S, r) \otimes_{\mathbf{B}'_{\square}}^L \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}(Y_S, r') \rightarrow \text{R}\Gamma_{\text{HK}}^{\mathbf{B}^{[u, v]}}((X \times_K Y)_S, r+r').$$

To check that this is a quasi-isomorphism we may pass to cohomology. Since  $H_{\text{HK}}^j(X_C)$  is Fréchet (hence flat for the solid tensor product over  $\check{C}$ ), this reduces to checking that the pairing

$$(5.37) \quad \bigoplus_{a=0}^b (H_{\text{HK}}^a(X_C) \otimes_{\check{C}}^L \mathbf{B}'_{\log})^{N=0} \otimes_{\mathbf{B}'_{\square}}^L (H_{\text{HK}}^{j-a}(Y_C) \otimes_{\check{C}}^L \mathbf{B}'_{\log})^{N=0} \rightarrow (H_{\text{HK}}^j((X \times_K Y)_C) \otimes_{\check{C}}^L \mathbf{B}'_{\log})^{N=0}$$

is an isomorphism in  $\mathbf{B}'_{\square}$ .

Now, using the exponential map as in the proof of Lemma 5.13, we can reduce to proving that the pairing

$$\bigoplus_{a=0}^b H_{\text{HK}}^a(X_C) \otimes_{\check{C}}^L \mathbf{B}' \otimes_{\mathbf{B}'_{\square}}^L H_{\text{HK}}^{j-a}(Y_C) \otimes_{\check{C}}^L \mathbf{B}' \rightarrow H_{\text{HK}}^j(X_C \times_C Y_C) \otimes_{\check{C}}^L \mathbf{B}'$$

is an isomorphism in  $\mathbf{B}'_{\square}$ . Or, that so is the pairing in  $\check{C}_{\square}$

$$\bigoplus_{a=0}^b H_{\text{HK}}^a(X_C) \otimes_{\check{C}}^L H_{\text{HK}}^{j-a}(Y_C) \rightarrow H_{\text{HK}}^j(X_C \times_C Y_C).$$

But this follows from the following:

**Lemma 5.38.** (Hyodo-Kato Künneth formula) *Let  $X, Y$  be smooth partially proper rigid analytic varieties over  $C$ . Then the canonical pairing*

$$\kappa_{\text{HK}} : \text{R}\Gamma_{\text{HK}}(X) \otimes_{\check{C}}^L \text{R}\Gamma_{\text{HK}}(Y) \rightarrow \text{R}\Gamma_{\text{HK}}(X \times_C Y).$$

is a quasi-isomorphism in  $\mathcal{D}_{\varphi, N, \mathcal{G}_K}(\check{C}_{\square})$ .



*Proof.* This follows from the comparison (via the Hyodo-Kato morphism) with the Künneth formula for de Rham cohomology

$$\kappa_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_{\mathbf{C}}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR}}(Y) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_{\mathbf{C}} Y).$$

The latter clearly holds if both  $X$  and  $Y$  are Stein. For a general partially proper  $X$  and  $Y$ , we use coverings by a countable number (!) of Stein varieties, the fact that all the complexes in sight are bounded complexes of Fréchet spaces, [5, Prop. 8.33], and the Stein case.  $\square$

For the base change to  $R = \mathbf{B}'/t$ , we get from the proof of Theorem 5.22, compatible with products, quasi-isomorphisms in  $\mathcal{D}(\mathbf{B}'_{\square})$  ( $T = X, Y, s \geq 0$ )

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(T_S, \mathbf{Q}_p(s)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R &\simeq F^s \mathrm{R}\Gamma_{\mathrm{dR}}(T_S/\mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} R \\ &\simeq \bigoplus_{i=0}^{d_T} \mathrm{R}\Gamma(T, \Omega^i) \otimes_K^{\mathrm{L}\square} R(s-i)[-i]. \end{aligned}$$

And the map  $\kappa^{[u,v]}$  can be identified with the map

$$\begin{aligned} (\bigoplus_{i=0}^{d_X} \mathrm{R}\Gamma(X, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r-i)[-i]) \otimes_R^{\mathrm{L}\square} (\bigoplus_{i=0}^{d_Y} \mathrm{R}\Gamma(Y, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r'-i)[-i]) \\ \rightarrow \bigoplus_{i=0}^{d_X+d_Y} \mathrm{R}\Gamma(X \times_K Y, \Omega^i) \otimes_K^{\mathrm{L}\square} R(r+r'-i)[-i]. \end{aligned}$$

If  $X, Y$  are Stein, this map in degree  $i$  is represented by the map

$$\bigoplus_{a=0}^{d_X+d_Y} \Omega^a(X) \otimes_K^{\mathrm{L}\square} R(r-a) \otimes_R^{\mathrm{L}\square} \Omega^{i-a}(Y) \otimes_K^{\mathrm{L}\square} R(r'-i+a) \rightarrow \Omega^i(X \times_K Y) \otimes_K^{\mathrm{L}\square} R(r+r'-i).$$

And the latter map is a quasi-isomorphism in  $R_{\square}$  by (5.35). If  $X, Y$  are general smooth partially proper rigid analytic varieties, we can reduce to the Stein case as in the proof of Lemma 5.38.  $\square$

## REFERENCES

- [1] P. Achinger, S. Gilles, W. Nizioł, *Compactly supported  $p$ -adic pro-étale cohomology of analytic varieties*. Preprint, 2024.
- [2] G. Andreychev, *Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces*. arXiv:2105.12591[math.AG], preprint 2021.
- [3] G. Andreychev,  *$K$ -Theorie adischer Räume*. arXiv:2311.04394[math.KT], preprint 2023.
- [4] L. Berger, *Représentations  $p$ -adiques et équations différentielles*. Invent. Math. 148 (2002), 219–284.
- [5] G. Bosco, *On the  $p$ -adic pro-étale cohomology of Drinfeld symmetric spaces*. arXiv:2110.10683v2[math.NT], preprint 2021.
- [6] G. Bosco, *Rational  $p$ -adic Hodge theory for rigid-analytic varieties*. arXiv:2306.06100[math.AG], preprint 2023.
- [7] D. Clausen, P. Scholze, *Lectures on condensed mathematics*. <https://www.math.uni-bonn.de/people/scholze/Condensed.pdf>, 2019.
- [8] D. Clausen, P. Scholze, *Lectures on analytic geometry*. <https://www.math.uni-bonn.de/people/scholze/Analytic.pdf>, 2020.
- [9] P. Colmez, G. Dospinescu, W. Nizioł, *Cohomology of  $p$ -adic Stein spaces*. Invent. Math. 219 (2020), 873–985.
- [10] P. Colmez, S. Gilles, W. Nizioł, *Arithmetic duality for the pro-étale cohomology of  $p$ -adic analytic curves*. arXiv:2308.07712[math.NT], preprint 2023.
- [11] P. Colmez, W. Nizioł, *Syntomic complexes and  $p$ -adic nearby cycles*. Invent. Math. 208 (2017), 1–108.
- [12] P. Colmez, W. Nizioł, *On the cohomology of the affine space*, in  $p$ -adic Hodge Theory, 71–80, Simons symposia, Springer-Verlag 2020.
- [13] P. Colmez, W. Nizioł, *On the cohomology of  $p$ -adic analytic spaces, I: The basic comparison theorem*. Journal of Algebraic Geometry 34 (2025), 1–108.
- [14] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge  $p$ -adique*. Astérisque 406 (2018), 51–382.
- [15] S. Gilles, *Morphismes de périodes et cohomologie syntomique*. Algebra and Number Theory 17 (2023), 603–666.
- [16] J. Rodrigues Jacinto, J. E. Rodriguez Camargo, *Solid locally analytic representations of  $p$ -adic Lie groups*. Representation Theory 26 (2022), 962–1024.
- [17] P. Scholze, *Étale cohomology of diamonds*. arXiv:1709.07343v3[math.AG], preprint 2017.
- [18] P. Scholze, J. Weinstein, *Berkeley lectures on  $p$ -adic geometry*. Ann. of Math. Stud., 207, 2020.

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