# CLASSIFICATION OF THREEFOLD CANONICAL THRESHOLDS

JHENG-JIE CHEN, JIUN-CHENG CHEN, AND HUNG-YI WU

ABSTRACT. We show that the set  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  of smooth threefold canonical thresholds coincides with  $\mathcal{T}_{2,\mathrm{sm}}^{\mathrm{lc}} = \mathcal{HT}_2$ , where  $\mathcal{HT}_2$  is the 2-dimensional hypersurface log canonical thresholds characterized by Kuwata [Kuw99a, Kuw99b]. We classify the set  $\mathcal{T}_3^{\mathrm{can}}$  of threefold canonical thresholds. More precisely, we prove  $\mathcal{T}_3^{\mathrm{can}} = \{0\} \cup \{\frac{4}{5}\} \cup \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$ .

#### 1. INTRODUCTION

We work over the complex number field  $\mathbb{C}$ . For a log canonical pair (X, S), the log canonical threshold

 $lct(X, S) := \sup\{t \in \mathbb{R} \mid (X, tS) \text{ is log canonical}\}\$ 

is a natural and fundamental invariant which measures the complexity of singularities in algebraic geometry (see [Kol97, Kol08]). The set of n-dimensional log canonical thresholds

$$\mathcal{T}_n^{\mathrm{lc}} := \{ \mathrm{lct}(X, S) | \dim X = n \}$$

has very interesting properties related to the minimal model program (cf. [Sho93, MP04, Bir07, HMX14, LMX24]). Note that it is an interesting and very hard question to describe the set  $\mathcal{T}_n^{\text{lc}}$  explicitly. Certain classifications in  $n \leq 3$  were investigated (cf. [Ale93, Sho93, Kuw99a, Kuw99b, Kol08]).

We consider an analogous notion. Let  $X \ni P$  be a germ of complex algebraic variety with at worst canonical singularities and  $S \ni P$  be a prime  $\mathbb{Q}$ -Cartier divisor. The canonical threshold of the pair (X, S) is defined as

 $ct(X, S) := \sup\{t \in \mathbb{R} \mid \text{the pair } (X, tS) \text{ is canonical}\}.$ 

For every natural number n, the set of canonical thresholds is defined by

$$\mathcal{T}_n^{\operatorname{can}} := \{ \operatorname{ct}(X, S) | \dim X = n \}.$$

It is known that  $\mathcal{T}_2^{\operatorname{can}} = \{0\} \cup \{\frac{1}{k}\}_{k \in \mathbb{N}}$  where  $\mathbb{N}$  denotes the set of positive integers. In this paper, we focus on the classifications in the case n = 3.

Canonical thresholds appear naturally and play crucial roles in the Sarkisov program. Recall that every canonical threshold  $\operatorname{ct}(X,S) \in \mathcal{T}_3^{\operatorname{can}}$  is computed by some divisorial contraction  $\sigma \colon Y \to X$  (cf. [Cor95] or [Mat02]). As the remarkable works of classifications of threefold divisorial contractions to points are completely investigated by Hayakawa, Kawakita, Kawamata, Mori, Yamamoto, and many others (cf. [Mor82, Kaw96, Hay99, Hay00, Kwk01, Kwk02, Kwk05, Yam18]), it is then a natural question to describe the set  $\mathcal{T}_3^{\operatorname{can}}$  explicitly. We collect some known results toward the description of the set  $\mathcal{T}_3^{\text{can}}$  as follows. Note that certain numbers are in this set, e.g.  $\min\{\frac{1}{\alpha} + \frac{1}{\beta}, 1\}$  (by a result of Stepanov [Ste11]) for all positive integers  $\alpha$  and  $\beta$ . It is also known that  $\frac{4}{5} \in \mathcal{T}_3^{\text{can}}$  [Pro08]. Consider the subset

$$\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} := \{ \mathrm{ct}(X, S) \mid \dim X = 3, X \text{ is smooth} \} \subseteq \mathcal{T}_{3}^{\mathrm{can}}.$$

Prokhorov proves that  $\mathcal{T}_3^{\operatorname{can}} \cap (\frac{5}{6}, 1) = \emptyset$  as well as  $\operatorname{ct}(X, S) \leq \frac{4}{5}$  when X is singular [Pro08]. Stepanov proves that  $\mathcal{T}_3^{\operatorname{can}} \cap (\frac{4}{5}, \frac{5}{6}) = \emptyset$  [Ste11]. Another interesting question concerning the set  $\mathcal{T}_3^{\operatorname{can}}$  is to determine if it satisfies the ACC. Stepanov obtains that  $\mathcal{T}_{3,\mathrm{sm}}^{\operatorname{can}}$  satisfies the ACC and establishes the explicit formula for  $\operatorname{ct}_P(X, S)$  when  $P \in S$  is a Brieskorn singularity in [Ste11]. Applying Stepanov's argument, the first named author proves the ACC for  $\mathcal{T}_3^{\operatorname{can}}$  [Che22]. He also shows that  $\mathcal{T}_3^{\operatorname{can}} \cap (\frac{1}{2}, 1)$  coincides with  $\{\frac{1}{2} + \frac{1}{p}\}_{p \in \mathbb{Z}_{\geq 3}} \cup \{\frac{4}{5}\}$ . Moreover, Han, Liu and Luo and the first named author independently prove that the accumulation points of  $\mathcal{T}_3^{\operatorname{can}}$  coincides with  $\mathcal{T}_2^{\operatorname{can}} \setminus \{1\}$  and generalized the ACC to pairs in [HLL22, Che22].

We now state the main results of this paper. The first result is an explicit description of the set  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  of threefold canonical thresholds. We also discover  $\mathcal{T}_{2,\mathrm{sm}}^{\mathrm{lc}} = \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$ . It is interesting to compare this with the well-known theorem that the set of limit points of  $\mathcal{T}_{n+1,\mathrm{sm}}^{\mathrm{lc}}$  is  $\mathcal{T}_{n,\mathrm{sm}}^{\mathrm{lc}}$  [dFM09] [Kol08].

**Theorem 1.1.** (= Theorem 2.2) The set  $\mathcal{T}_{3,sm}^{can}$  consists of  $C \cap [0,1]$  where C is the following set

$$\left\{ \frac{\alpha+\beta}{p_1\alpha+p_2\beta} \middle| \begin{array}{l} \alpha,\beta,p_2\in\mathbb{N} \text{ and } p_1\in\mathbb{Z}_{\geq 0} \text{ such that } \alpha\leq\beta,\\ \gcd(\alpha,\beta)=1 \text{ and either } p_2\geq\max\{\alpha,p_1\} \text{ or } p_2=p_1 \end{array} \right\}.$$

In particular, we have

$$\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} = \mathcal{T}_{2,\mathrm{sm}}^{\mathrm{lc}} (= \mathcal{HT}_2).$$

The next result characterizes the set of 3-dimensional canonical thresholds  $\mathcal{T}_3^{\text{can}}$ . It is plausible that most of the canonical thresholds should come form the smooth case already. We show that  $\frac{4}{5}$  is the only non-trivial exception.

**Theorem 1.2.** (=Theorem 3.15) We have  $\mathcal{T}_{3}^{can} = \{0\} \cup \{\frac{4}{5}\} \cup \mathcal{T}_{3,sm}^{can}$ 

In what follows, we explain the proof of Theorems 1.1. Suppose that  $\operatorname{ct}(X,S) \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  is a canonical threshold. It is known that  $\operatorname{ct}(X,S) \leq 1$  and  $\operatorname{ct}(X,S)$  can be computed as a weighted blow up  $Y \to X = \hat{\mathbb{C}}^3$  at the origin of  $\hat{\mathbb{C}}^3$  with weights  $w := (1, \alpha, \beta)$  for some positive relative prime integers  $\alpha$  and  $\beta$  by Kawakita [Kwk01] such that  $\operatorname{ct}(X,S) = \frac{\alpha+\beta}{m}$  where m = w(f) is the weighted multiplicity of the defining convergent power series f of the prime divisor S. For simplicity, we assume that  $1 < \alpha < \beta$ . Y is then a union of three affine open subsets  $U_1, U_2$  and  $U_3$  such that  $U_1$  is smooth,

$$U_2 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\alpha} (-s, s, 1) \text{ and } U_3 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\beta} (t, 1, -t),$$

where s and t are the positive integers with  $\alpha t = \beta s + 1$ . Denote by  $\bar{s} := \alpha - s$ and  $\bar{t} := \beta - t$  so that  $\alpha \bar{t} = \beta \bar{s} - 1$ . Note that there are two weights  $w_2 = (1, s, t)$  and  $w_3 = (1, \overline{s}, \overline{t})$  over X corresponding to Kawamata blow ups at the origins of  $U_2$  and  $U_3$  respectively. Comparing w with two auxiliary weights  $w_2$  and  $w_3$ , [Che22, Lemma 2.1] yields that

(1) 
$$\left\lfloor \frac{s+t}{\alpha+\beta}m \right\rfloor \ge \left\lceil \frac{s}{\alpha}m \right\rceil \text{ and } \left\lfloor \frac{\bar{s}+\bar{t}}{\alpha+\beta}m \right\rfloor \ge \left\lceil \frac{\bar{t}}{\beta}m \right\rceil$$

By [Che22, Proposition 3.3], we have  $m \ge \alpha\beta$ . In particular, there exist non-negative integers  $p_1$  and  $p_2$  with  $m = p_1\alpha + p_2\beta$  and  $p_1 < \beta$ . From the inequalities in (1) and the above identities  $\alpha t = \beta s + 1$  and  $\alpha \bar{t} = \beta \bar{s} - 1$ , we obtain  $p_2 \ge p_1$  (resp.  $p_2 \ge \alpha$  if  $p_1 \ne p_2$ ). Thus,  $\operatorname{ct}(X, S) \in C \cap [0, 1]$ . Conversely, given a number  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \in C \cap [0,1]$  where  $\alpha, \beta, p_2$  are positive integers and  $p_1$  is a non-negative integer with  $\alpha \le \beta$  and  $\operatorname{gcd}(\alpha, \beta) = 1$  such that either  $p_2 \ge \max\{\alpha, p_1\}$  or  $p_1 = p_2$ . We assume  $\alpha > 1$  for simplicity. Denote by  $m = p_1\alpha + p_2\beta$  and weights  $w = (1, \alpha, \beta), w_2 = (1, s, t)$  and  $w_3 = (1, \bar{s}, \bar{t})$ . We are able to construct a prime divisor  $S := \{f = 0\}$  (near the origin of  $\hat{\mathbb{C}}^3$ ) satisfying the following two conditions:

- the proper transform  $S_Y$  of S in Y is smooth except probably the origins of  $U_2$  and  $U_3$  (near the exceptional divisor of weighted blow up  $\sigma: Y \to X = \hat{\mathbb{C}}^3$  with weights w);
- we have the inequalities

$$\frac{s+t}{\alpha+\beta}m \ge w_2(f)$$
 and  $\frac{\bar{s}+\bar{t}}{\alpha+\beta}m \ge w_3(f)$ 

where  $m = w(f), w_2(f)$  and  $w_3(f)$  are the weighted multiplicities.

Thanks to computations of canonical thresholds for terminal cyclic quotient singularities studied by Kawamata [Kaw96, Lemma 7], it follows that  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \in \mathcal{T}_{3,\text{sm}}^{\text{can}}$  (see Lemma 2.1). Therefore, we have established  $\mathcal{T}_{3,\text{sm}}^{\text{can}} = C \cap [0, 1]$ . It is then straightforward to check that  $C \cap [0, 1]$  coincides with  $\mathcal{HT}_2$  where  $\mathcal{HT}_2$  is the set of 2-dimensional hypersurface log canonical thresholds explicitly classified by Kuwata [Kuw99a, Kuw99b]. Thus, Theorem 1.1 follows.

Theorem 1.2 is basically derived from the argument of Theorem 1.1 and the classifications of divisorial contractions by Kawakita [Kwk05] which we explain as follows. Recall that for every projective threefold X with at worst  $\mathbb{Q}$ -factorial terminal singularities,  $\operatorname{ct}(X, S) \in \mathcal{T}_3^{\operatorname{can}}$  is obtained as  $\operatorname{ct}(X, S) = \frac{a}{m}$  for some divisorial extraction  $\sigma: Y \to X$  extracting only one irreducible divisor E and

$$K_Y = \sigma^* K_X + \frac{a}{n} E$$
 and  $S_Y = \sigma^* S - \frac{m}{n} E$ 

where  $\frac{a}{n}$  (resp.  $\frac{m}{n}$ ) is the discrepancy (resp. multiplicity) and n is the index of the center  $\sigma(E)$  (cf [Cor95] or [Mat02]). We say that ct(X, S) is computed by  $\sigma$  and denote the weighted discrepancy and weighted discrepancy by aand m, respectively. As in [Che22], consider

$$\mathcal{T}_{3,*,\geq 5}^{\operatorname{can}} := \left\{ \operatorname{ct}(X,S) \mid \begin{array}{c} \operatorname{ct}(X,S) \text{ is computed by } \sigma: Y \to X \text{ with weighted} \\ \operatorname{discrepancy} a \geq 5 \text{ contracting a divisor } E \\ \operatorname{to a closed point } \sigma(E) = P \in X \text{ of type } * \end{array} \right\}$$

where the type \* can be cA (resp. cA/m, cD or cD/2) if  $P \in X$  is a singular point of type cA (resp. cA/m, cD or cD/2). According to classifications of threefold divisorial contractions by Kawakita in [Kwk05], we have the following decomposition :

$$\mathcal{T}_{3}^{\operatorname{can}} = \{0\} \cup \aleph_{4} \cup \mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}} \cup \mathcal{T}_{3,cA,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cA/m,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cD,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cD/2,\geq 5}^{\operatorname{can}} \quad (2)$$

where  $\aleph_4 := \mathcal{T}_3^{\operatorname{can}} \cap \{\frac{a}{m}\}_{a,m \in \mathbb{N}, a \leq \max\{4,m\}}$ . From Theorem 1.1 and a computation of Prokhorov [Pro08], one has  $\aleph_4 \cup \mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}} = \{\frac{4}{5}\} \cup \mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}}$  (see Remark 2.5). Theorem 1.2 then follows from the inclusions  $\mathcal{T}_{3,\ast,\geq 5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}]$  for  $\ast = cA, cA/n, cD$  and cD/2 in Proposition 3.1, 3.3, 3.8 and 3.11. Those propositions utilize the argument of the inclusion  $\mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}} \subseteq C \cap [0, 1]$  in Theorem 1.1 with more careful considerations.

1.1. Acknowledgement. The first named author was partially supported by the National Science and Technology Council of Taiwan (Grant Numbers: 112-2115-M-008 -006 -MY2 and 113-2123-M-002-019-). We would like to thank Professors Jungkai Chen, Hsueh-Yung Lin, Jihao Liu, and Dr. Iacopo Brivio for some helpful conversations and encouragement.

## 2. Classification of $\mathcal{T}_{3,\text{sm}}^{\text{can}}$

The aim of this section is to classify the set  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  of smooth threefold canonical thresholds. We begin with the following lemma using the computation of canonical threshold for terminal cyclic quotient singularity by Kawamata in [Kaw96].

**Lemma 2.1.** Let  $\mu_1: Y \to X = \hat{\mathbb{C}}^3 \ni P$  be a weighted blow up with weights  $w = (1, \alpha, \beta)$  and exceptional divisor  $E_1$  where  $\beta > 1$ . Let S be a prime divisor of X defined by a power series f. Let  $\bar{s}$  and  $\bar{t}$  be positive integers with  $\alpha \bar{t} = \beta \bar{s} - 1$  (resp. s and t positive integers with  $\alpha t = \beta s + 1$  if  $\alpha > 1$ ). Define m = w(f) and  $m_3 := w_3(f)$  where  $w_3 = (1, \bar{s}, \bar{t})$  (resp.  $m_2 := w_2(f)$  where  $w_2 = (1, s, t)$ ). Suppose that  $\frac{\bar{s} + \bar{t}}{\alpha + \beta} m \ge m_3$  (resp.  $\frac{s + t}{\alpha + \beta} m \ge m_2$  if  $\alpha > 1$ ) and the proper transform  $S_Y$  is non-singular on Y near  $E_1$  except probably the origin of  $U_3$  (resp.  $U_2$  if  $\alpha > 1$ ). Then  $\operatorname{ct}_P(X, S) = \min\{\frac{\alpha + \beta}{m}, 1\}$ .

*Proof.* The proof actually follows from [Kaw96, Lemma 6 and argument of Lemma 7].

Since  $\operatorname{ct}(X,S) \in [0,1]$ , we may assume  $\frac{\alpha+\beta}{m} < 1$ . Suppose that  $\alpha > 1$ . Recall that Y is a union of three open charts  $U_1 \simeq \hat{\mathbb{C}}^3$ ,  $U_2 \simeq \mathbb{C}^3/\frac{1}{\alpha}(-1,1,-\beta)$ and  $U_3 \simeq \mathbb{C}^3/\frac{1}{\beta}(-1,-\alpha,1)$ . Let  $\mu_2 : Y_2 \to Y$  be the Kawamata blow up (with weights  $v_2 = \frac{1}{\alpha}(\bar{s},s,1)$ ) at the origin of  $U_2$  with exceptional divisor  $E_2$ . Let  $\mu_3 : Y_3 \to Y_2$  be the Kawamata blow up at the origin of  $U_3$  with weights  $v_3 = \frac{1}{\beta}(t,1,\bar{t})$  and exceptional divisor  $E_3$ . Let  $n \ge n_0 \ge 1$  be integers such that  $\mu_i : Y_i \to Y_{i-1}$  are Kawamata blow ups for  $i = 1, 2, ..., n_0$  (resp. usual blow up at smooth point or along smooth curve for  $i = n_0 + 1, ..., n$ ),  $Y_{n_0}$  is non-singular and the composition  $\sigma := \mu_n \circ \cdots \circ \mu_2 \circ \mu_1 \colon Y_n \to X$ is a log resolution of (X, S) near  $P \in X$ . Note that we set  $Y_1 := Y$  and  $Y_0 := X$ . Write  $K_{Y_n} = \sigma^* K_X + \sum_{j=1}^n a_j E_j$  and  $\sigma^* S = S_{Y_n} + \sum_{j=1}^n m_j E_j$ 

with exceptional divisor  $E_j$  of  $\mu_j : Y_j \to Y_{j-1}$  where  $m_1 := m = w(f)$ . For every j = 1, ..., n, denote by  $\sigma_j := \mu_n \circ \cdots \circ \mu_{j+1} : Y_n \to Y_j$  the induced morphism and write

$$K_{Y_j} = \mu_j^* K_{Y_{j-1}} + \bar{a}_j E_j, \ \ \mu_j^* S_{Y_{j-1}} = S_{Y_j} + \bar{m}_j E_j \text{ and } \sigma_j^* E_j = E_j + \sum_{j < k} \alpha_{jk} E_k$$

for non-negative rational numbers  $\alpha_{jk}$ . Then we have  $a_j = \bar{a}_j + \sum_{i < j} a_i \alpha_{ij}$ and  $m_j = \bar{m}_j + \sum_{i < j} m_i \alpha_{ij}$ .

As  $\mu_3 : Y_3 \to Y_2$  is a weighted blow up with weights  $v_3 = \frac{1}{\beta}(t, 1, \bar{t})$  and  $w_3 = \frac{\bar{t}}{\beta}w + \frac{1}{\beta}(t, 1, 0)$ , we see  $\alpha_{13} = \frac{\bar{t}}{\beta}$  and  $\bar{a}_3 = \frac{1}{\beta}$ . From

$$\frac{\bar{s} + \bar{t}}{m_3} = \frac{a_3}{m_3} = \frac{\bar{a}_3 + \alpha_{13}a_1}{\bar{m}_3 + \alpha_{13}m} = \frac{\bar{a}_3 + \alpha_{13}(\alpha + \beta)}{\bar{m}_3 + \alpha_{13}m}$$

and  $\alpha_{13} \ge 0$ , it is easy to see that the assumption  $\frac{\bar{s}+\bar{t}}{\alpha+\beta}m \ge m_3$  is equivalent to  $\frac{\bar{a}_3}{\bar{m}_3} \ge \frac{\alpha+\beta}{m} = \frac{a_1}{m_1}$ . Similarly, the assumption  $\frac{s+t}{\alpha+\beta}m \ge m_2$  is equivalent to  $\frac{\bar{a}_2}{\bar{m}_2} \ge \frac{\alpha+\beta}{m} = \frac{a_1}{m_1}$ .

 $\begin{array}{l} \frac{\bar{a}_2}{\bar{m}_2} \geq \frac{\alpha+\beta}{m} = \frac{a_1}{m_1}.\\ \text{It remains to show } \frac{a_j}{m_j} \geq \frac{a_1}{m_1} \text{ for every } j = 1, ..., n. \text{ It is obvious when } \\ j = 1. \text{ From the assumptions } \frac{\bar{s}+\bar{t}}{\alpha+\beta}m \geq m_3 \text{ and } \frac{s+t}{\alpha+\beta}m \geq m_2, \text{ we may } \\ \text{assume that } j \geq 4. \text{ By induction hypothesis, we assume that } \frac{a_k}{m_k} \geq \frac{a_1}{m_1} \text{ for } \\ k = 1, ..., j - 1. \text{ Denote by } \sigma_3(E_j) \text{ the center of } E_j \text{ on } Y_3. \end{array}$ 

**Case 1:**  $\sigma_3(E_j)$  is contained in  $E_3$ . By [Kaw96, Lemma 6], one has  $\frac{\bar{a}_j}{\bar{m}_j} \geq \frac{\bar{a}_3}{\bar{m}_3}$ . Thus, we see  $\frac{\bar{a}_j}{\bar{m}_j} \geq \frac{\bar{a}_3}{\bar{m}_3} \geq \frac{a_1}{m_1}$ . In particular, we observe

$$\frac{a_j}{m_j} = \frac{\bar{a}_j + \sum_{i < j} a_i \alpha_{ij}}{\bar{m}_j + \sum_{i < j} m_i \alpha_{ij}} \ge \frac{a_1 + \sum_{i < j} a_i \alpha_{ij}}{m_1 + \sum_{i < j} m_i \alpha_{ij}} \ge \frac{a_1 + \sum_{i < j} a_1 \alpha_{ij}}{m_1 + \sum_{i < j} m_1 \alpha_{ij}} = \frac{a_1}{m_1},$$

where the last inequality follows from induction hypothesis. **Case 2:**  $\sigma_3(E_j)$  is contained in  $E_2$ . We may apply the same argument in

Case 1 to obtain  $\frac{a_j}{m_j} \ge \frac{a_1}{m_1}$ .

**Case 3:**  $\sigma_3(E_j)$  is contained in  $E_1$  and not contained in  $E_2 \cup E_3$ . From the assumption that  $S_Y$  is non-singular on Y (near  $E_1$ ) except the origins of  $U_2$  and  $U_3$ , we see that  $\bar{m}_j \in \{1,0\}$  and  $\bar{a}_j = 1$  or 2. Thus  $\frac{\bar{a}_j}{\bar{m}_j} = \bar{a}_j \ge 1 > \frac{\alpha+\beta}{m} = \frac{a_1}{m_1}$  or  $\bar{m}_j = 0$ . In particular,  $\frac{a_j}{m_j} \ge \frac{a_1}{m_1}$  by induction hypothesis.

Suppose that  $\alpha = 1$ . Then  $U_2 \simeq \mathbb{C}^3$  is non-singular. Replacing j by j-1 for  $j \geq 3$  above. We divide it into two cases that  $\sigma_2(E_j)$  is contained in  $E_2$  or  $E_1 \setminus E_2$ . The above arguments yield the same inequality  $\frac{a_j}{m_j} \geq \frac{a_1}{m_1}$ . We finish the proof of Lemma 2.1.

Let f be a non-zero holomorphic function near  $0 \in \mathbb{C}^2$ . Recall that  $c_0(\mathbb{C}^2, f) := \sup\{c \mid |f|^{-c} \text{ is locally } L^2 \text{ near } 0\}$ . Following Kuwata [Kuw99a] and [Kuw99b], we define

$$\mathcal{HT}_2 := \{ c_0(\mathbb{C}^2, f) : f \neq 0 \text{ is holomorphic near } 0 \in \mathbb{C}^2 \}.$$

**Theorem 2.2.** The set  $\mathcal{T}_{3,sm}^{can}$  consists of  $C \cap [0,1]$  where C is the following set

$$\left\{ \frac{\alpha+\beta}{p_1\alpha+p_2\beta} \middle| \begin{array}{l} \alpha,\beta,p_2\in\mathbb{N} \text{ and } p_1\in\mathbb{Z}_{\geq 0} \text{ such that } \alpha\leq\beta, \\ \gcd(\alpha,\beta)=1 \text{ and either } p_2\geq\max\{\alpha,p_1\} \text{ or } p_2=p_1 \end{array} \right\}.$$

In particular,

$$\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} = \mathcal{T}_{2,\mathrm{sm}}^{\mathrm{lc}} (= \mathcal{HT}_2).$$

Proof. We first show  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} \subseteq C \cap [0,1]$ . Suppose that  $\mathrm{ct}(X,S) \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$ . By [Cor95, (2.10) Proposition-definition] and [Kwk01], the canonical threshold  $\mathrm{ct}(X,S) = \frac{\alpha+\beta}{m}$  is realized by a weighted blow up  $\sigma : Y \to X = \hat{\mathbb{C}}^3$  with weights  $w = (1, \alpha, \beta)$  where  $\alpha$  and  $\beta$  are relative prime integers with  $1 \leq \alpha \leq \beta$  and m is the weighted multiplicity of S with respect to  $\sigma$ .

In the case  $\alpha = \beta = 1$ , every positive integer is of the form  $p_1\alpha + p_2\beta$  for some non-negative integers  $p_1$  and  $p_2$ . Thus, we may assume that  $\alpha < \beta$ . As  $gcd(\alpha, \beta) = 1$ , there exists positive integers  $\bar{s}$  and  $\bar{t}$  with  $\alpha \bar{t} = \beta \bar{s} - 1$  (resp, positive integers s and t with  $\alpha t = \beta s + 1$  if  $\alpha > 1$ ). Define  $w_3 = (1, \bar{s}, \bar{t})$ and  $\sigma_3$  to be the weighted blow up at the origin of  $X = \hat{\mathbb{C}}^3$  with the weights  $w_3$ . Similarly, define  $w_2 = (1, s, t)$  and  $\sigma_2$  to be the weighted blow up at the origin of  $X = \hat{\mathbb{C}}^3$  with the weights  $w_2$  if  $\alpha > 1$ . Let  $m_3$  (resp.  $m_2$  if  $\alpha > 1$ ) denote the weighted multiplicity of S with respect to  $\sigma_3$  (resp.  $\sigma_2$  if  $\alpha > 1$ ). From [Che22, Lemma 2.1], one has

$$\lfloor \frac{\bar{s} + \bar{t}}{\alpha + \beta} m \rfloor \ge m_3 \ge \lceil \frac{\bar{t}}{\beta} m \rceil.$$

Similarly, in the case  $\alpha > 1$ , we have

$$\lfloor \frac{s+t}{\alpha+\beta}m \rfloor \ge m_2 \ge \lceil \frac{s}{\alpha}m \rceil.$$

By [Che22, Proposition 3.3] and  $ct(X, S) \leq 1$ , we see  $m \geq \alpha\beta$ . As m is an integer greater than  $\alpha\beta - \alpha - \beta$ , one may write  $m = p_1\alpha + p_2\beta$  for some non-negative integers  $p_1$  and  $p_2$ . Rewriting

$$m = p_1 \alpha + p_2 \beta = (p_1 - \beta \lfloor \frac{p_1}{\beta} \rfloor) \alpha + (p_2 + \lfloor \frac{p_1}{\beta} \rfloor \alpha) \beta$$

one may assume that  $p_1 < \beta$ . It remains to show the following two claims.

Claim 2.3. We have  $p_2 \ge p_1$ .

Proof of the Claim. Since  $\alpha \bar{t} = \beta \bar{s} - 1$  and we assume that  $p_1 < \beta$ , one has

$$\left\lceil \frac{t}{\beta}m\right\rceil = \left\lceil \frac{t}{\beta}(p_1\alpha + p_2\beta)\right\rceil = \left\lceil p_1\bar{s} + p_2\bar{t} - \frac{p_1}{\beta}\right\rceil = p_1\bar{s} + p_2\bar{t}.$$

As  $\lfloor \frac{\bar{s} + \bar{t}}{\alpha + \beta} m \rfloor \ge m_3 \ge \lceil \frac{\bar{t}}{\beta} m \rceil$  where

$$\lfloor \frac{\bar{s} + \bar{t}}{\alpha + \beta} m \rfloor = \lfloor \frac{\bar{s} + \bar{t}}{\alpha + \beta} (p_1 \alpha + p_2 \beta) \rfloor = p_1 \bar{s} + p_2 \bar{t} + \lfloor \frac{p_2 - p_1}{\alpha + \beta} \rfloor,$$

we conclude  $p_2 \ge p_1$ .

Claim 2.4. Either  $p_2 \ge \alpha$  or  $p_1 = p_2$ .

*Proof of the Claim.* The argument is similar to the claim above. We may assume that  $\alpha > 1$ . Since  $\alpha t = \beta s + 1$ , we have

$$\lfloor \frac{s+t}{\alpha+\beta}m \rfloor = \lfloor \frac{s+t}{\alpha+\beta}(p_1\alpha+p_2\beta) \rfloor = p_1s+p_2t-\lceil \frac{p_2-p_1}{\alpha+\beta} \rceil$$
  
and  $\lceil \frac{s}{\alpha}m \rceil = \lceil \frac{s}{\alpha}(p_1\alpha+p_2\beta) \rceil = p_1s+p_2t-\lfloor \frac{p_2}{\alpha} \rfloor.$ 

As we have  $\lfloor \frac{s+t}{\alpha+\beta}m \rfloor \ge m_2 \ge \lceil \frac{s}{\alpha}m \rceil$ , one sees

$$\frac{p_2}{\alpha} \ge \lfloor \frac{p_2}{\alpha} \rfloor \ge \lceil \frac{p_2 - p_1}{\alpha + \beta} \rceil \ge 0,$$

where last inequality follows from Claim 2.3 above. Thus, we conclude  $p_2 \ge \alpha$  or  $p_1 = p_2$ .

Conversely, suppose that we are given  $\alpha, \beta, p_2 \in \mathbb{N}$  and  $p_1 \in \mathbb{Z}_{\geq 0}$  with  $\alpha \leq \beta$ ,  $\gcd(\alpha, \beta) = 1$  satisfying either  $p_2 \geq \max\{\alpha, p_1\}$  or  $p_1 = p_2$ . We shall show that  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  if  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \leq 1$ .

If  $\alpha = \beta = 1$ , we take  $S = (x^m + y^m + z^m = 0)$  and  $\sigma : Y \to X = \hat{\mathbb{C}}^3$  to be the smooth blow up at the origin where m is a positive integer. As  $(Y, S_Y)$  is log smooth for  $m \ge 2$  (resp.  $(\hat{\mathbb{C}}^3, S)$  is log smooth when m = 1), we see  $\operatorname{ct}(X, S) = \min\{\frac{2}{m}, 1\}$  by [Ste11, Theorem 3.6].

From now on, we may assume  $\beta > 1$ . Let  $\sigma: Y \to X = \hat{\mathbb{C}}^3$  be the weighted blow up with weights  $w = (1, \alpha, \beta)$ . Define  $\bar{s}$  and  $\bar{t}$  to be positive integers with  $\alpha \bar{t} = \beta \bar{s} - 1$ . Similarly, in the case  $\alpha > 1$ , we define positive integers  $s = \alpha - \bar{s}$  and  $t = \beta - \bar{t}$  so we have  $\alpha t = \beta s + 1$ . Note that Y is covered by three affine open subsets  $U_1, U_2, U_3$  where  $U_1 \simeq \hat{\mathbb{C}}^3$  and

$$U_3 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\beta} (-1, -\alpha, 1) \simeq \hat{\mathbb{C}}^3 / \frac{1}{\beta} (t, 1, -t) \text{ and}$$
$$U_2 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\alpha} (-1, 1, -\beta) \simeq \hat{\mathbb{C}}^3 / \frac{1}{\alpha} (-s, s, 1) \text{ if } \alpha > 1.$$

Define

$$v_2 = \frac{1}{\alpha}(\bar{s}, s, 1), \ v_3 = \frac{1}{\beta}(t, 1, \bar{t}), \ w_2 = \frac{s}{\alpha}w + \frac{1}{\alpha}(\bar{s}, 0, 1), \\ w_3 = \frac{t}{\beta}w + \frac{1}{\beta}(t, 1, 0).$$

Then  $w_2 = (1, s, t)$ ,  $w_3 = (1, \bar{s}, \bar{t})$  and the weighted blow up  $\sigma'_3 : Z_3 \to U_3$ (resp.  $\sigma'_2 : Z_2 \to U_2$ ) at the origin of  $U_3$  (resp.  $U_2$ ) with weights  $v_3$  (resp.  $v_2$  if  $\alpha > 1$ ) is Kawamata blow up.

Define  $m := p_1 \alpha + p_2 \beta$  and express

$$p_2 = \alpha \lfloor \frac{p_2}{\alpha} \rfloor + q$$

where  $q \in [0, \alpha - 1]$  is an integer.

**Case 1:**  $\alpha | p_2$ . Define S to be the Cartier divisor f = 0 in  $\hat{\mathbb{C}}^3$  where

$$f = x^m + y^{p_1} z^{p_2} + y^{\frac{p_2}{\alpha}\beta + p_1} + z^m$$

Note that m = w(f), and

$$w_3(f) = \min\{w_3(x^m), w_3(y^{p_1}z^{p_2}), w_3(y^{\frac{p_2}{\alpha}\beta + p_1}), w_3(z^m)\} = w_3(y^{p_1}z^{p_2}) = p_1\bar{s} + p_2\bar{t}$$

and S is only singular at the origin if  $p_1 + p_2 \ge 2$ . In particular, the divisor S is prime. In the open chart  $U_1 \simeq \hat{\mathbb{C}}^3$  of Y, the proper transform  $S_Y$  of S in Y is defined by

$$1 + y^{p_1} z^{p_2} + y^{\frac{p_2}{\alpha}\beta + p_1} + x^{(\beta - 1)m} z^m = 0,$$

which is non-singular. Similarly, in the open chart  $U_2 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\alpha} (-1, 1, -\beta)$  if  $\alpha > 1$  (resp.  $U_2 \simeq \hat{\mathbb{C}}^3$  if  $\alpha = 1$ ),  $S_Y$  is defined by

$$x^m + z^{p_2} + 1 + y^{(\beta - 1)m} z^m = 0,$$

which is non-singular. In the open chart  $U_3 \simeq \hat{\mathbb{C}}^3/\frac{1}{\beta}(-1,-\alpha,1), S_Y$  is defined by

$$x^{m} + y^{p_{1}} + y^{\frac{p_{2}}{\alpha}\beta + p_{1}} + z^{(\beta-1)m} = 0,$$

which is only singular at the origin if  $p_1 > 0$  (resp. is nonsingular if  $p_1 = 0$ ). If  $p_1 > 0$ , then the weighted multiplicity of  $S_Y$  with respect to Kawamata blow up  $\sigma'_3: Z_3 \to U_3$  is

$$\beta \cdot v_3(x^m + y^{p_1} + y^{\frac{p_2}{\alpha}\beta + p_1} + z^{(\beta - 1)m}) = p_1$$

By assumption that  $p_2 \ge p_1$ , we have

$$\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \le \frac{1}{p_1},$$

where  $\frac{1}{p_1}$  is the canonical threshold of  $(U_3, S_Y|_{U_3})$  near the origin of  $U_3$  if  $p_1 > 0$  by [Kaw96]. Note that

$$\bar{s} + \bar{t} = \frac{1}{\beta} + \frac{\bar{t}}{\beta}(\alpha + \beta)$$
 and  $w_3(f) = p_1\bar{s} + p_2\bar{t} = \frac{p_1}{\beta} + \frac{\bar{t}}{\beta}(p_1\alpha + p_2\beta).$ 

The above inequality  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \leq \frac{1}{p_1}$  gives  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \leq \frac{\bar{s}+\bar{t}}{w_3(f)}$ . It is then easy to see  $\operatorname{ct}(\hat{\mathbb{C}}^3, S) = \min\{\frac{\alpha+\beta}{p_1\alpha+p_2\beta}, 1\}$  by Lemma 2.1. **Case 2:**  $\alpha \nmid p_2$  and  $p_1 \neq p_2$ . Define S to be the Cartier divisor f = 0 in

 $\hat{\mathbb{C}}^3$  where

$$f = x^m + y^{p_1} z^{p_2} + y^{\lfloor \frac{p_2}{\alpha} \rfloor \beta + p_1} z^q + y^m + z^m.$$

It's not hard to see that S has only isolated singularities, say  $Q_1, ..., Q_n$ . In particular, the divisor S is prime. Without loss of generality, we assume that  $Q_1$  is the origin of  $X = \hat{\mathbb{C}}^3$ . Note that m = w(f) and

$$w_2(f) = w_2(y^{\lfloor \frac{p_2}{\alpha} \rfloor \beta + p_1} z^q) = s(\lfloor \frac{p_2}{\alpha} \rfloor \beta + p_1) + tq \text{ and } w_3(f) = w_3(y^{p_1} z^{p_2}) = p_1 \bar{s} + p_2 \bar{t}$$

As the above computation, outside the singular set  $\{Q_2, ..., Q_n\}$ , the proper transform  $S_Y$  of S in Y is non-singular in  $U_1$  and is only singular at the origin  $U_3$  (resp.  $U_2$ ), where we have

$$\frac{\alpha+\beta}{p_1\alpha+p_2\beta} \le \frac{\bar{s}+\bar{t}}{w_3(f)}$$

In the open chart  $U_2 \simeq \hat{\mathbb{C}}^3 / \frac{1}{\alpha} (-1, 1, -\beta)$ ,  $S_Y$  is defined by

$$x^{m} + z^{p_{2}} + z^{q} + y^{(\alpha-1)m} + y^{(\beta-1)m} z^{m} = 0,$$

which is only singular at the origin of  $U_2$  and  $Q_2, ..., Q_n$ . Then the weighted multiplicity of  $S_Y$  with respect to Kawamata blow up  $\sigma'_2 \colon Z_2 \to U_2$  is

$$\alpha \cdot v_2(x^m + z^{p_2} + z^q + y^{(\alpha - 1)m} + y^{(\beta - 1)m} z^m) = \alpha \cdot v_2(z^q) = q,$$

where  $v_2 = \frac{1}{\alpha}(\alpha - s, s, 1)$  and  $q \in (0, \alpha - 1]$  is an integer. Recall that  $p_2 - q = \alpha \lfloor \frac{p_2}{\alpha} \rfloor$  is a multiple of  $\alpha$ . As  $p_1 \neq p_2$ , we have  $p_2 \geq \max\{\alpha, p_1\}$  by assumption. In particular,  $p_2 - q = \alpha \lfloor \frac{p_2}{\alpha} \rfloor \geq \alpha$  and

$$(p_2 - q)\beta + p_1\alpha \ge \alpha\beta + p_1\alpha > q\alpha$$
 and thus  $\frac{\alpha + \beta}{p_1\alpha + p_2\beta} < \frac{1}{q}$ ,

where  $\frac{1}{q}$  is the canonical threshold of  $(U_2, S_Y|_{U_2})$  near the origin of  $U_2$  by [Kaw96]. Note that

$$s+t = \frac{1}{\alpha} + \frac{s}{\alpha}(\alpha+\beta)$$
 and  $w_2(f) = s(\lfloor \frac{p_2}{\alpha} \rfloor \beta + p_1) + tq = \frac{q}{\alpha} + \frac{s}{\alpha}(p_1\alpha+p_2\beta),$ 

and the inequality  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} < \frac{1}{q}$  implies  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} < \frac{s+t}{w_2(f)}$ . It is then easy to see  $\operatorname{ct}(\hat{\mathbb{C}}^3, S) = \frac{\alpha + \beta}{p_1 \alpha + p_2 \beta}$  by Lemma 2.1.

**Case 3:**  $\alpha \nmid p_2$  and  $p_1 = p_2$ . Define S to be the Cartier divisor f = 0 in  $\hat{\mathbb{C}}^3$  where

$$f = x^m + y^{p_2} z^{p_2} + y^m + z^m$$

Similarly, S is only singular at the origin and thus is prime. Note that

$$m = w(f), w_2(f) = w_2(y^{p_2}z^{p_2}) = (s+t)p_2 \text{ and } w_3(f) = w_3(y^{p_2}z^{p_2}) = (\bar{s}+\bar{t})p_2.$$

Then the proper transform  $S_Y$  is non-singular in the open chart  $U_1$  and is only singular at the origin of  $U_2$  (resp.  $U_3$ ). As  $p_1 = p_2$ , we have  $\frac{\alpha + \beta}{p_1 \alpha + p_2 \beta} =$  $\frac{1}{p_2}$  where  $\frac{1}{p_2}$  is the canonical threshold of  $(U_3, S')$  (resp.  $(U_2, S')$ ) near the origin by [Kaw96]. It is then easy to see  $\operatorname{ct}(\hat{\mathbb{C}}^3, S) = \frac{\alpha + \beta}{p_2 \alpha + p_2 \beta} = \frac{1}{p_2}$  by Lemma 2.1. Thus, we have proved  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} = C \cap [0, 1]$ .

It remains to show  $C \cap [0,1] = \mathcal{HT}_2$ . Recall that  $\mathcal{HT}_2$  consists of 1 and

$$\frac{c_1 + c_2}{c_1 c_2 + a_1 c_2 + a_2 c_1}$$

for some non-negative integers  $a_1, a_2, c_1, c_2$  with  $a_1 + c_1 \ge \max\{2, a_2\}$  and

 $a_2 + c_2 \ge \max\{2, a_1\}$  by [Kol08, (3.2)] (cf. [Kuw99a, Theorem 7.1]). Suppose first that  $1 > \frac{\alpha+\beta}{p_1\alpha+p_2\beta} \in \mathcal{T}_{3,\text{sm}}^{\text{can}}$ . If  $\alpha + \beta$  divides  $p_1\alpha + p_2\beta$ , we put  $c_1 = c_2 = 1, a_1 = \frac{p_1\alpha+p_2\beta}{\alpha+\beta}, a_2 = a_1 - 1 \ge 1$ . One sees  $\frac{\alpha+\beta}{p_1\alpha+p_2\beta} = \frac{1+1}{2}$  $\frac{1+1}{1+a_1+(a_1-1)} \in \mathcal{HT}_2$ . If  $\alpha + \beta$  does not divide  $p_1\alpha + p_2\beta$ , one has  $p_2 > p_1$  and  $p_2 \geq \alpha$ . Let l be the non-negative integer with

$$(\alpha + \beta)l < p_2 - p_1 < (\alpha + \beta)(l+1).$$

We put  $c_1 = \alpha, c_2 = \beta, a_1 = p_2 - \alpha l - \alpha$  and  $a_2 = p_1 + \beta l$ . It is easy to see  $a_1 + c_1 \ge \max\{2, a_2\}$  and  $a_2 + c_2 \ge \max\{2, a_1\}$  and thus  $\frac{\alpha + \beta}{p_1 \alpha + p_2 \beta} \in \mathcal{HT}_2$ . Conversely, suppose that  $0 \ne \frac{c_1 + c_2}{c_1 c_2 + a_1 c_2 + a_2 c_1} \in \mathcal{HT}_2$ . If  $c_i = 0$  for some i = 1, 2, we put  $p_1 = p_2 = a_i$ . Then  $\frac{c_1 + c_2}{c_1 c_2 + a_1 c_2 + a_2 c_1} = \frac{1}{a_i} \in \mathcal{T}_{3,\text{sm}}^{\text{can}}$ . Thus we may assume that both  $c_1$  and  $c_2$  are positive. Let  $d = \gcd(c_1, c_2)$  and  $\alpha = \frac{c_1}{d}$ 

and  $\beta = \frac{c_2}{d}$ . We put  $\alpha = c_1, \beta = c_2, p_1 = a_2$  and  $p_2 = c_1 + a_1$ . Then we see  $\frac{c_1+c_2}{c_1c_2+a_1c_2+a_2c_1} = \frac{\alpha+\beta}{p_1\alpha+p_2\beta} \in \mathcal{T}_{3,\text{sm}}^{\text{can}}$ . This completes the proof of Theorem 2.2.

**Remark 2.5.** Suppose that  $\alpha$  and  $\beta$  are two relative prime positive integers with  $\alpha < \beta$ . If *m* is an integer greater than  $(\beta - 1)\alpha + (\beta - 2)\beta$ , then  $\frac{\alpha+\beta}{m} \in \mathcal{T}_{3,\text{sm}}^{\text{can}}$ . Indeed, as  $(\beta - 1)\alpha + (\beta - 2)\beta \ge \alpha\beta - \alpha - \beta$ , there exist non-negative integers  $p_1$  and  $p_2$  satisfying  $m = p_1\alpha + p_2\beta$  and  $p_1 < \beta$ . We have

$$p_2\beta = m - p_1\alpha > (\beta - 1)\alpha + (\beta - 2)\beta - p_1\alpha \ge \beta(\alpha + \beta - 2) - \beta\alpha = (\beta - 2)\beta,$$
  
whence  $p_2 \ge \beta - 1 \ge \max\{p_1, \alpha\}$ . Therefore,  $\frac{\alpha + \beta}{m} \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  by Theorem 2.2.  
In particular,  $\frac{3}{m} \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  (resp.  $\frac{4}{m} \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$ ) when  $m \ge 3$  (resp.  $m \ge 6$ ). It is  
also easy to see  $\frac{2}{m} = \operatorname{ct}(\hat{\mathbb{C}}^3, S) \in \mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}$  by considering  $S := \{x^m + y^m + z^m = 0\}$  in  $\mathbb{C}^3$  (see [Ste11, Theorem 3.6]) when  $m \ge 2$ . Therefore, every number  
in  $\aleph_4$  with the exception of  $\frac{4}{5}$  is contained in  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}}(=C \cap [0,1])$ . Recall that  $\frac{4}{5}$   
is a canonical threshold indicated in [Pro08, Example 3.11]. We can rewrite  
decomposition in (2) as

$$\mathcal{T}_{3}^{\operatorname{can}} = \{0\} \cup \{\frac{4}{5}\} \cup \mathcal{T}_{3,sm}^{\operatorname{can}} \cup \mathcal{T}_{3,cA,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cA/m,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cD,\geq 5}^{\operatorname{can}} \cup \mathcal{T}_{3,cD/2,\geq 5}^{\operatorname{can}} (3).$$

## 3. Classification of $\mathcal{T}_3^{can}$

In this section, we shall establish the inclusion  $\mathcal{T}_{3,*,\geq 5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}]$  for each \* = cA, cA/n, cD and cD/2 where

$$C = \left\{ \frac{\alpha + \beta}{p_1 \alpha + p_2 \beta} \middle| \begin{array}{l} \alpha, \beta, p_2 \in \mathbb{N} \text{ and } p_1 \in \mathbb{Z}_{\geq 0} \text{ such that } \alpha \leq \beta, \\ \gcd(\alpha, \beta) = 1 \text{ and either } p_2 \geq \max\{\alpha, p_1\} \text{ or } p_2 = p_1 \end{array} \right\}$$

in Propositions 3.1, 3.3, 3.8 and 3.11. The arguments are based on the idea of the inclusion  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} \subseteq C \cap [0,1]$  in the proof of Theorem 2.2 and reducing to coprime weights. Moreover, we will need to consider semi-invariant conditions in the non-Gorenstein cases in Propositions 3.3 and 3.11.

**Proposition 3.1.** We have  $\mathcal{T}_{3,cA,\geq 5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}]$ .

Proof. Let  $\operatorname{ct}(X,S) \in \mathcal{T}_{3,cA}^{\operatorname{can}}$  be a canonical threshold. We have  $\operatorname{ct}(X,S) \leq \frac{4}{5}$  by [Pro08]. By [Cor95, (2.10) Proposition-definition] and Theorem 1.2(i) in [Kwk05], there exists an analytical identification  $P \in X \simeq o \in (\varphi = xy + g(z, u) = 0)$  in  $\hat{\mathbb{C}}^4$  where o denotes the origin of  $\hat{\mathbb{C}}^4$  such that the canonical threshold  $\operatorname{ct}(X, S)$  is computed by a weighted blow up  $\sigma : Y \to X$  of weights  $w = wt(x, y, z, u) = (r_1, r_2, a, 1)$  satisfying the following:

- $w(g(z, u)) = r_1 + r_2 = ad$  where  $r_1, r_2, a, d$  are positive integers with  $a \ge 5$ ;
- $z^{d} \in g(z, u)$  and hence  $w(g(z, u)) = w(z^{d});$
- $gcd(r_1, a) = gcd(r_2, a) = 1;$

•  $ct(X,S) = \frac{a}{m}$  where m = nw(f) and S is defined by the formal power series f = 0 analytically and locally.

We may assume  $d \geq 2$  since otherwise  $P \in X$  is nonsingular and is treated in the previous section. Without loss of generality, we may assume  $r_1 \leq r_2$ . As  $gcd(a, r_1) = gcd(a, r_2) = 1$ , there exist non-negative integers  $s_i^* < r_i$  such that  $1 + a_i r_i = a s_i^*$  for i = 1, 2. Note that  $s_i^* = 0$  if and only if  $r_i = 1$ . From assumption that  $a \ge 5$ , we see  $r_2 \ge (r_1 + r_2)/2 = ad/2 > 1$ . Define

$$w_2 = (r_1 - a_2 d + s_2^*, r_2 - s_2^*, a_1, 1)$$
 (resp.  $w_1 = (r_1 - s_1^*, r_2 - a_1 d + s_1^*, a_2, 1)$  if  $r_1 > 1$ )

Since  $w_1 \succeq \frac{r_1 - s_1^*}{r_1} w$  and  $w_2 \succeq \frac{r_2 - s_2^*}{r_2} w$ , by [Che22, Lemmas 2.1 and 4.1], we see that

$$\lfloor \frac{a_1}{a}m \rfloor \ge \lceil \frac{r_2 - s_2^*}{r_2}m \rceil \text{ (resp. } \lfloor \frac{a_2}{a}m \rfloor \ge \lceil \frac{r_1 - s_1^*}{r_1}m \rceil \text{ if } r_1 > 1).$$

Denote by  $h = \gcd(r_1, r_2)$ . As the positive integer  $r'_i = r_i/h$  has no common divisor with a, there exist non-negative integers  $s_i^{*'} < r'_i$  and  $a'_i < a$  such that  $1 + a'_i r'_i = a s^{*'}_i$  for i = 1, 2. Again, it follows from the assumption  $a \ge 5$ that  $r'_2 > 1$ . Note that  $a_1 + a_2 = a$ .

#### Claim 3.2.

$$\lfloor \frac{s_2^{*'}}{r_2'}m \rfloor \ge \lceil \frac{a_2'}{a}m \rceil \text{ (resp. } \lfloor \frac{s_1^{*'}}{r_1'}m \rfloor \ge \lceil \frac{a_1'}{a}m \rceil \text{ if } r_1 > 1 \text{)}.$$

Proof of Claim 3.2. It follows from

- $r_2 = hr'_2$ , and  $1 + a_2r_2 = as_2^*$  and  $0 < a_2 < a$  and  $0 < s_2^* < r_2$  and  $1 + a'_2r'_2 = as_2^{*'}$  and  $0 < a'_2 < a$  and  $0 < s_2^{*'} < r'_2$

that there exists a non-negative integer  $b_2$  with  $s_2^* = b_2 r_2' + s_2^{*'}$  and thus

$$1 + a_2 r_2 = a s_2^* = a b_2 r_2' + a s_2^{*'} = a b_2 r_2' + 1 + a_2' r_2'.$$

This yields  $a_2hr'_2 = a_2r_2 = r'_2(ab_2 + a'_2)$ . In particular,  $a'_2 = a_2h - ab_2$ . From  $\lfloor \frac{a_1}{a}m \rfloor \geq \lceil \frac{r_2 - s_2^*}{r_2}m \rceil$  and  $a = a_1 + a_2$ , one has  $\lfloor \frac{s_2^*}{r_2}m \rfloor \geq \lceil \frac{a_2}{a}m \rceil$ . Let  $\eta_2$  be an integer with  $\lfloor \frac{s_2^*}{r_2}m \rfloor \ge \eta_2 \ge \lceil \frac{a_2}{a}m \rceil$ . Then we see

$$\frac{s_2^{*'}}{r_2'}m = \frac{s_2^* - b_2r_2'}{r_2'}m = \frac{s_2^*}{r_2'}m - b_2m = \frac{s_2^*}{r_2}mh - b_2m$$
$$\ge \eta_2h - b_2m \ge \frac{a_2}{a}mh - b_2m = \frac{a_2' + ab_2}{a}m - b_2m = \frac{a_2'}{a}m.$$

where  $\eta_2 h - b_2 m$  is an integer. This gives the inequality  $\lfloor \frac{s_2^{*'}}{r_2'}m \rfloor \geq \lceil \frac{a_2'}{a}m \rceil$ .

Similarly, the above argument shows that the inequality  $\lfloor \frac{a_2}{a}m \rfloor \geq \lceil \frac{r_1 - s_1^*}{r_1}m \rceil$ implies  $\lfloor \frac{s_1^{*'}}{r_1'}m \rfloor \geq \lceil \frac{a_1'}{a}m \rceil$  if  $r_1 > 1$ . If  $r_1' = 1$ , then  $a_1' = -1, s_1^{*'} = 0$  and thus  $\lfloor \frac{s_1^{*'}}{r'_1} m \rfloor = 0 \ge \lceil \frac{a'_1}{a} m \rceil$ . We complete the proof of Claim 3.2. 

From [Che22, Proposition 4.2], we have  $dm \ge r_1r_2$  if  $r_1 > 1$ . Note that  $1 \ge \operatorname{ct} = \frac{a}{m} = \frac{r_1+r_2}{dm}$ , we have  $dm \ge r_2 = r_1r_2$  if  $r_1 = 1$ . Now dm is an integer greater than  $r_1r_2 - r_1 - r_2$ , so there exist non-negative integers  $p_1$  and  $p_2$  with  $dm = p_1r_1 + p_2r_2$ . Dividing  $h = \operatorname{gcd}(r_1, r_2)$ , we have

$$d'm = p_1r'_1 + p_2r'_2$$
 where  $d' = \frac{d}{h}$ .

Replacing  $d'm = p_1r'_1 + p_2r'_2$  by  $d'm = (p_1 - r'_2\lfloor\frac{p_1}{r'_2}\rfloor)r'_1 + (p_2 + \lfloor\frac{p_1}{r'_2}\rfloorr'_1)r'_2$ , one may assume that  $0 \le p_1 < r'_2$ . From  $r'_1 + r'_2 = ad'$ , we rewrite  $d'm = p_1ad' + k_1r'_2$  where  $k_1 = p_2 - p_1$  is an integer. We have  $d'|k_1r'_2$ . Since  $gcd(r'_1, r'_2) = 1$  and  $r'_1 + r'_2 = ad'$ , we observe  $gcd(d', r'_2) = 1$  and hence  $d'|k_1$ . Now

$$\frac{a_2'}{a}m = \frac{a_2'}{ad'}(p_1ad' + k_1r_2') = a_2'p_1 + \frac{as_2^{*'} - 1}{ad'}k_1 = a_2'p_1 + \frac{s_2^{*'}k_1}{d'} - \frac{k_1}{ad'}k_1 = a_2'p_1 + \frac{s_2^{*'}k_1}{d'} - \frac{k_1}{ad'}k_1 = \frac{s_2''}{r_2'}m = \frac{as_2^{*'}}{ar_2'}p_1d' + \frac{s_2^{*'}k_1}{d'} = \frac{1 + a_2'r_2'}{r_2'}p_1 + \frac{s_2^{*'}k_1}{d'} = a_2'p_1 + \frac{p_1}{r_2'} + \frac{s_2^{*'}k_1}{d'}$$

and  $d'|k_1$ , one has

$$\lceil \frac{a_2'}{a}m \rceil = a_2'p_1 + \frac{s_2^{*'}k_1}{d'} + \lceil -\frac{k_1}{ad'} \rceil \text{ and } \lfloor \frac{s_2^{*'}}{r_2'}m \rfloor = a_2'p_1 + \frac{s_2^{*'}k_1}{d'} + \lfloor \frac{p_1}{r_2'} \rfloor$$

From Claim 3.2, we have  $\lfloor \frac{s_2^{*'}}{r_2'}m \rfloor \geq \lceil \frac{a'_2}{a}m \rceil$ . It is then the same to  $\lfloor \frac{p_1}{r_2'} \rfloor \geq \lceil -\frac{k_1}{ad'} \rceil$ . Since  $p_1 < r'_2$ , we see  $k_1 \geq 0$  and hence  $p_2 \geq p_1$ .

Suppose that  $r_1 > 1$ . We rewrite  $d'm = k_2r'_1 + p_2ad'$  where  $k_2 = p_1 - p_2 = -k_1$  is a non-positive integer divisible by d'. Note that

$$\frac{a_1'}{a}m = \frac{a_1'}{ad'}(k_2r_1' + p_2ad') = \frac{as_1^{*'} - 1}{ad'}k_2 + a_1'p_2 = \frac{s_1^{*'}k_2}{d'} - \frac{k_2}{ad'} + a_1'p_2,$$
  
$$\frac{s_1^{*'}}{r_1'}m = \frac{s_1^{*'}k_2}{d'} + \frac{s_1^{*'}a}{d'r_1'}p_2d' = \frac{s_1^{*'}k_2}{d'} + \frac{1 + a_1'r_1'}{d'r_1'}p_2d' = \frac{s_1^{*'}k_2}{d'} + a_1'p_2 + \frac{p_2}{r_1'}$$

and  $d'|k_2$ , so

$$\lceil \frac{a_1'}{a}m \rceil = \frac{s_1^{*'}k_2}{d} + a_1'p_2 + \lceil -\frac{k_2}{ad'} \rceil \text{ and } \lfloor \frac{s_1^{*'}}{r_1'}m \rfloor = \frac{s_1^{*'}k_2}{d'} + a_1'p_2 + \lceil \frac{p_2}{r_1'} \rceil.$$

From Claim 3.2, we have  $\lfloor \frac{s_1^{*'}}{r_1'}m \rfloor \geq \lceil \frac{a_1'}{a}m \rceil$ . This is the same with

$$\lfloor \frac{p_2}{r_1'} \rfloor \ge \lceil -\frac{k_2}{ad'} \rceil = \lceil \frac{p_2 - p_1}{ad'} \rceil.$$

From the discussion above that  $p_2 \ge p_1$ , we conclude that either  $p_1 = p_2$  or  $p_2 \ge r'_1$  holds.

Suppose that  $r_1 = 1$ . As we have shown  $p_2 \ge p_1$  is an integer where  $d'm = p_1r'_1 + p_2r'_2$  is positive, we have  $p_2 \ge 1 = r'_1$ . The proof is completed.  $\Box$ 

**Proposition 3.3.** We have  $\mathcal{T}_{3,cA/n,\geq 5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}].$ 

*Proof.* The argument is similar but more subtle than that of Proposition 3.1.

Let  $\operatorname{ct}(X,S) \in \mathcal{T}_{3,cA/n}^{can}$  be a canonical threshold. We have  $\operatorname{ct}(X,S) \leq \frac{4}{5}$ by [Pro08]. By [Cor95, (2.10) Proposition-definition] and Theorem 1.2(i) in [Kwk05], there exist an analytical identification

$$P \in X \simeq o \in (\varphi \colon xy + g(z^n, u) = 0) \subset \mathbb{C}^4 / \frac{1}{n} (1, -1, b, 0)$$

where o denotes the origin of  $\mathbb{C}^4/\frac{1}{n}(1,-1,b,0)$  and a weighted blow up  $\sigma$ :  $Y \to X$  of weights  $w = wt(x, y, z, u) = \frac{1}{n}(r_1, r_2, a, n)$  at the origin satisfying the following:

- $nw(\varphi) = r_1 + r_2 = adn$  where  $r_1, r_2, a, d, n$  are positive integers with  $\begin{array}{l} a \geq 5 \text{ and } n \geq 2; \\ \bullet \ z^{dn} \in g(z^n, u); \end{array}$
- $a \equiv br_1 \pmod{n}$  and 0 < b < n;
- $\operatorname{gcd}(b,n) = \operatorname{gcd}(\frac{a-br_1}{n},r_1) = \operatorname{gcd}(\frac{a+br_2}{n},r_2) = 1.$   $\operatorname{ct}(X,S) = \frac{a}{m}$  where m = nw(f) and S is defined by the formal power series f = 0 analytically and locally.

By interchanging  $r_1$  and  $r_2$ , one may assume  $r_1 \leq r_2$ . As  $a \geq 5$  and  $r_1 + r_2 = adn, r_2 > 1$ . On the open subset  $Y \cap \{\bar{y} \neq 0\}, Y$  is defined by

$$\bar{x} + g(\bar{z}^n \bar{y}^a, \bar{u}\bar{y})/\bar{y}^{ad} = 0 \subset \hat{\mathbb{C}}^4 / \frac{1}{r_2}(-r_1, n, -a, -n),$$

which is isomorphic to  $\hat{\mathbb{C}}^3/\frac{1}{r_2}(n,-a,-n)$ . By terminal lemma, one has

•  $gcd(r_2, an) = 1$  and hence  $gcd(r_1, an) = 1$ .

The conditions  $gcd(b, n) = gcd(r_1, an) = 1$  and  $a \equiv br_1 \pmod{n}$  imply • gcd(a, n) = 1.

In what follows, we construct two auxiliary weights  $w_1$  and  $w_2$  (cf. [Che14, 3.5]). Put  $s_1 := \frac{a-br_1}{n}$  and  $s_2 := \frac{a+br_2}{n}$ . As  $gcd(r_i, s_i) = 1$  for i = 1, 2, we have:

$$\begin{cases} a = br_1 + ns_1; \\ 1 = q_1r_1 + s_1^*s_1; \\ a = -br_2 + ns_2; \\ 1 = q_2r_2 + s_2^*s_2 \end{cases}$$

for some integer  $0 \le s_i^* < r_i$  and some integer  $q_i$ . Denote by

$$\delta_1 := -nq_1 + bs_1^*, \ \delta_2 := -nq_2 - bs_2^*.$$

Then we obtain the following useful identities

•  $\delta_i r_i + n = a s_i^*$ , for i = 1, 2, with each  $\delta_i \neq 0$  by [Che14, Claims 1,2] in 3.5].

Since  $5dn \leq adn = r_1 + r_2$  and we have assumed  $r_1 \leq r_2$ , one sees  $\delta_2 > 0$ . Define

$$w_2 = \frac{1}{n}(r_1 - \delta_2 dn + s_2^*, r_2 - s_2^*, a - \delta_2, n)$$
  
(resp.  $w_1 = \frac{1}{n}(r_1 - s_1^*, r_2 - \delta_1 dn + s_1^*, a - \delta_1, n)$  if  $\delta_1 > 0$ ).

Note that  $w_1 \succeq \frac{r_1 - s_1^*}{r_1} w$  and  $w_2 \succeq \frac{r_2 - s_2^*}{r_2} w$ . By [Che22, Lemmas 2.1 and 5.1], we see that

$$\lfloor \frac{a-\delta_2}{a}m \rfloor \ge m_2 \ge \lceil \frac{r_2-s_2^*}{r_2}m \rceil \text{ (resp.} \lfloor \frac{a-\delta_1}{a}m \rfloor \ge m_1 \ge \lceil \frac{r_1-s_1^*}{r_1}m \rceil \text{ if } \delta_1 > 0).$$

where  $m_2 := nw_2(\mathfrak{m}_2)$  (resp.  $m_1 := nw_1(\mathfrak{m}_1)$ ) is a weighted multiplicity for some monomial  $\mathfrak{m}_2 \in f$  (resp.  $\mathfrak{m}_1 \in f$  if  $\delta_1 > 0$ ). Note that

$$m_2 = nw_2(\mathfrak{m}_2) \equiv (r_2 - s_2^*)r_2^{-1}nw(\mathfrak{m}_2) \equiv (r_2 - s_2^*)r_2^{-1}nw(f)$$
  
$$\equiv (r_2 - s_2^*)r_2^{-1}m \equiv (a - \delta_2)a^{-1}m \pmod{n},$$

where  $a^{-1}$  (resp.  $r_2^{-1}$ ) denotes the inverse of a (resp.  $r_2$ ) modulo n. Similarly,  $m_1 \equiv (r_1 - s_1^*)r_1^{-1}m \pmod{n}$  if  $\delta_1 > 0$ .

Claim 3.4. Suppose that  $a \nmid m$ . Then  $m \geq r_2$ .

Proof of Claim 3.4. Let  $\xi := m - m_2$  where  $m_2$  is an integer above with the properties that

$$\lfloor \frac{a-\delta_2}{a}m \rfloor \ge m_2 \ge \lceil \frac{r_2-s_2^*}{r_2}m \rceil \quad \text{and} \ m_2 \equiv (a-\delta_2)a^{-1}m \pmod{n}.$$

We have

$$\left\lfloor \frac{s_2^*}{r_2}m \right\rfloor \ge \xi \ge \left\lceil \frac{\delta_2}{a}m \right\rceil$$
 and  $\xi \equiv \delta_2 a^{-1}m \pmod{n}$ .

Suppose on the contrary that  $a \nmid m$  and  $m < r_2$ . Denote by  $t_1$  and  $t_2$  the positive integers with  $m = r_2t_1 - at_2$  and  $a - 1 \ge t_1 \ge 1$ . Note that  $t_1$  is the smallest positive integer with  $t_1 \equiv mr_2^{-1} \pmod{a}$  where  $r_2^{-1}$  denotes the inverse of  $r_2$  modulo a. From the equation  $r_2\delta_2 + n = as_2^*$ , one sees

$$\frac{\delta_2}{a}m = \frac{\delta_2}{a}(r_2t_1 - at_2) = \frac{(as_2^* - n)t_1}{a} - \delta_2t_2 = s_2^*t_1 - \delta_2t_2 - \frac{nt_1}{a},$$

which implies

$$\xi \equiv \delta_2 a^{-1} m \equiv a^{-1} (a s_2^* t_1 - a \delta_2 t_2 - n t_1) \equiv s_2^* t_1 - \delta_2 t_2 \pmod{n}.$$

As  $n > \frac{nt_1}{a} > 0$  and  $\xi \ge \lfloor \frac{\delta_2}{a}m \rfloor$  where  $\xi \equiv \delta_2 a^{-1}m \pmod{n}$ , this gives  $\xi \ge s_2^* t_1 - \delta_2 t_2$ . On the other hand, we have

$$\frac{s_2^*}{r_2}m = \frac{s_2^*}{r_2}(r_2t_1 - at_2) = s_2^*t_1 - \frac{(r_2\delta_2 + n)t_2}{r_2} = s_2^*t_1 - \delta_2t_2 - \frac{nt_2}{r_2}$$
$$\leq \xi - \frac{nt_2}{r_2} < \xi,$$

where the last inequality holds as  $r_2t_1 - at_2 = m < r_2$ . This leads to a contradiction that  $\lfloor \frac{s_2^*}{r_2}m \rfloor \ge \xi > \frac{s_2^*}{r_2}m \ge \lfloor \frac{s_2^*}{r_2}m \rfloor$ . The proof of Claim 3.4 is finished.

Claim 3.5. Suppose that  $a \nmid m$ . Then  $m \geq \frac{r_1 r_2}{dn}$ .

Proof of Claim 3.5. Suppose that  $\delta_1 < 0$ . Then  $r_1 \leq (-\delta_1)r_1 + as_1^* = n \leq dn$ and hence  $dnm \geq r_1m \geq r_1r_2$  by Claim 3.4.

We may thus assume that  $\delta_1 > 0$ . It follows from [Che14, Remark 3.3] that  $\delta_1 + \delta_2 = a$ . Since gcd(b, n) = 1 and  $a \nmid m$  where m is an integral combination of  $r_1, r_2, a, n$ , one has that  $a \nmid \delta_1 m$ . One sees that

$$\lfloor \frac{a-\delta_1}{a}m \rfloor + \lfloor \frac{a-\delta_2}{a}m \rfloor = \lfloor \frac{a-\delta_1}{a}m \rfloor + \lfloor \frac{\delta_1}{a}m \rfloor = m-1.$$

Recall that  $m_i \equiv (r_i - s_i^*)r_i^{-1}m \equiv (a - \delta_i)a^{-1}m \pmod{n}$  for i = 1, 2. Therefore

$$m_1 + m_2 \equiv (2a - (\delta_1 + \delta_2))a^{-1}m \equiv aa^{-1}m \equiv m \pmod{n}.$$

Together with  $m-1 = \lfloor \frac{a-\delta_1}{a}m \rfloor + \lfloor \frac{a-\delta_2}{a}m \rfloor \ge m_1 + m_2$ , one observes

$$m-n \ge m_1 + m_2 \ge \lceil \frac{r_1 - s_1^*}{r_1} m \rceil + \lceil \frac{r_2 - s_2^*}{r_2} m \rceil.$$

Note that

$$r_1s_2^* + r_2s_1^* = \frac{a(r_1s_2^* + r_2s_1^*)}{a} = \frac{r_1(r_2\delta_2 + n) + r_2(r_1\delta_1 + n)}{a} = r_1r_2 + dn^2.$$

Suppose on the contrary that  $m < \frac{r_1 r_2}{dn}$ . Then we obtain

$$\lceil \frac{r_1 - s_1^*}{r_1} m \rceil + \lceil \frac{r_2 - s_2^*}{r_2} m \rceil \ge \lceil \frac{r_1 - s_1^*}{r_1} m + \frac{r_2 - s_2^*}{r_2} m \rceil$$
  
=  $\lceil 2m - \frac{r_1 s_2^* + r_2 s_1^*}{r_1 r_2} m \rceil = \lceil m - \frac{dn^2}{r_1 r_2} m \rceil = m - \lfloor \frac{dnm}{r_1 r_2} n \rfloor \ge m - n + 1$ 

which contradicts to  $m-n \ge \lceil \frac{r_1-s_1^*}{r_1}m \rceil + \lceil \frac{r_2-s_2^*}{r_2}m \rceil$ . The proof of Claim 3.5 is finished. 

Denote by  $h := \gcd(r_1, r_2)$  and b' the smallest positive integer with  $b' \equiv bh$ (mod n). Let d' and  $r'_i$  be positive integers with d = d'h and  $r_i = r'_i h$  for i = 1, 2. Put  $s'_1 = \frac{a - b' r'_1}{n}$  and  $s'_2 = \frac{a + b' r'_2}{n}$ . Since the integer *a* is relatively prime to  $r_i = hr'_i$ , we have the following:

$$\begin{cases} a = b'r'_1 + ns'_1; \\ 1 = q'_1r'_1 + s_1^{*'}s'_1; \\ a = -b'r'_2 + ns'_2; \\ 1 = q'_2r'_2 + s_2^{*'}s_2 \end{cases}$$

for some integer  $0 \le s_i^{*'} < r_i'$  and some integer  $q_i'$ . Let

$$\delta'_i := -nq'_i + b's^{*'}_i$$
, for  $i = 1, 2$ .

As above, we observe the following

- $\delta'_i r'_i + n = a s_i^{*'}$  and  $0 \neq \delta'_i < a$  for i = 1, 2.  $\delta'_2 > 0$  and  $a | \delta'_1 + \delta'_2$ . if  $\delta'_1 > 0$ , then  $a = \delta'_1 + \delta'_2$ .

**Claim 3.6.** There exists an integer  $\xi_2$  with

$$\lfloor \frac{s_2^{*'}}{r_2'}m \rfloor \ge \xi_2 \ge \lceil \frac{\delta_2'}{a}m \rceil \text{ and } \xi_2 \equiv s_2^{*'}(r_2'^{-1})m \pmod{n}.$$

Similarly, in the case  $\delta_1 > 0$ , there exists an integer  $\xi_1$  with

$$\lfloor \frac{s_1^{*'}}{r_1'}m \rfloor \ge \xi_1 \ge \lceil \frac{\delta_1'}{a}m \rceil \text{ and } \xi_1 \equiv s_1^{*'}(r_1'^{-1})m \pmod{n}.$$

Proof of Claim 3.6. It follows from

- $r_2 = hr'_2$  and  $gcd(a, r_2) = 1$  and
- $\delta_2 r_2 + \tilde{n} = a s_2^*$  and  $0 < s_2^* < r_2$  and  $\delta'_2 r'_2 + n = a s_2^{*'}$  and  $0 < s_2^{*'} < r'_2$

that there exists an integer  $b_2$  with  $s_2^* = b_2 r_2' + s_2^{*'}$  and thus

$$\delta_2 r_2 + n = as_2^* = ab_2 r_2' + as_2^{*'} = ab_2 r_2' + \delta_2' r_2' + n$$

This yields  $\delta_2 h r'_2 = \delta_2 r_2 = r'_2 (ab_2 + \delta'_2)$ . In particular,  $\delta_2 h = ab_2 + \delta'_2$ . Recall that  $\lfloor \frac{a-\delta_2}{a}m \rfloor \ge m_2 \ge \lceil \frac{r_2-s_2^*}{r_2}m \rceil$  where  $m_2 \equiv (r_2 - s_2^*)r_2^{-1}m \pmod{\frac{a+\delta_2}{a}}$ 

n). Then we see

$$\frac{s_2^{*'}}{r_2'}m = \frac{s_2^* - b_2r_2'}{r_2'}m = \frac{s_2^*}{r_2'}m - b_2m = \frac{s_2^*}{r_2}mh - b_2m$$
$$\ge (m - m_2)h - b_2m \ge \frac{\delta_2}{a}mh - b_2m = \frac{\delta_2' + ab_2}{a}m - b_2m = \frac{\delta_2'}{a}m.$$

Denote by  $\xi_2 = (m - m_2)h - b_2m$ . Then

$$\xi_2 = (m - m_2)h - b_2m \equiv s_2^* r_2^{-1}mh - b_2m \equiv s_2^* r_2'^{-1}m - b_2m \equiv s_2^* r_2'^{-1}m \pmod{n}.$$

Suppose that  $\delta_1 > 0$ . The above argument works by interchanging indices 1 and 2. We complete the proof of Claim 3.6. 

From Claim 3.5, we have  $dmn \ge r_1r_2$ . So there exist non-negative integers  $p_1$  and  $p_2$  with  $dmn = p_1r_1 + p_2r_2$ . Dividing  $h = \gcd(r_1, r_2)$ , we have

$$d'mn = p_1r_1' + p_2r_2'.$$

Replacing by  $d'mn = (p_1 - r'_2 \lfloor \frac{p_1}{r'_2} \rfloor)r'_1 + (p_2 + \lfloor \frac{p_1}{r'_2} \rfloor r'_1)r'_2$ , one may assume that  $0 \leq p_1 < r'_2$ . From  $r'_1 + r'_2 = ad'n$ , we rewrite  $d'mn = p_1ad'n + k_1r'_2$  where  $k_1 = p_2 - p_1$  is an integer. Note that

$$\frac{\delta_2'}{a}m = \frac{\delta_2'}{ad'n}(p_1ad'n + k_1r_2') = \delta_2'p_1 + \frac{r_2'\delta_2'k_1}{ad'n} = \delta_2'p_1 + \frac{k_1(as_2^{*'} - n)}{ad'n}$$
$$= \delta_2'p_1 + \frac{s_2^{*'}k_1}{d'n} - \frac{k_1}{ad'}$$
$$s_2^{*'} = s_2^{*'} + \frac{s_2^{*'}k_1}{ad'n} - \frac{k_1}{ad'}$$

and 
$$\frac{s_2^{*'}}{r_2'}m = \frac{s_2^{*'}}{d'nr_2'}(p_1ad'n + k_1r_2') = \frac{as_2^{*'}p_1}{r_2'} + \frac{s_2^{*'}k_1}{d'n} = \frac{(\delta_2'r_2' + n)p_1}{r_2'} + \frac{s_2^{*'}k_1}{d'n}$$
  
=  $\delta_2'p_1 + \frac{s_2^{*'}k_1}{d'n} + \frac{np_1}{r_2'}.$ 

Claim 3.7.  $d'n|k_1$ .

Proof of Claim 3.7. As  $d'mn = p_1ad'n + k_1r'_2$ , we have  $d'n|k_1r'_2$ . Since  $\gcd(r_1',r_2')\,=\,1$  and  $r_1'\,+\,r_2'\,=\,ad'n,$  it follows that  $\gcd(d'n,r_2')\,=\,1$  and hence  $d'\tilde{n}|k_1$  and the claim is verified. 

From Claims 3.6 and 3.7, there exists an integer  $\xi_2$  with

$$\xi_2 \equiv s_2^{*'}(r_2'^{-1})m \equiv r_2'^{-1}(r_2'(\delta_2'p_1 + \frac{s_2^{*'}k_1}{d'n} + \frac{np_1}{r_2'})) \equiv \delta_2'p_1 + \frac{s_2^{*'}k_1}{d'n} \pmod{n}.$$

Let  $l_2$  be the integer with  $\xi_2 = \delta'_2 p_1 + \frac{s_2^{*'} k_1}{d'n} + nl_2$ . Then the inequalities  $\lfloor \frac{s_2^{*'}}{r'_2}m \rfloor \ge \xi_2 \ge \lceil \frac{\delta'_2}{a}m \rceil$  in Claim 3.6 is then the same to  $\lfloor \frac{np_1}{r'_2} \rfloor \ge nl_2 \ge \lceil -\frac{k_1}{ad'} \rceil$ . As  $p_1 < r'_2$ , we see  $k_1 \ge 0$  and hence  $p_2 \ge p_1$ .

Suppose that  $\delta_1 < 0$ . Then  $r_1 \leq -\delta_1 r_1 = n - as_1^* \leq n$ . If  $p_2 \geq n$ , then we get the desired inequalities  $p_2 \geq n \geq r_1 \geq r'_1$ . We shall rule out the case  $p_1 < p_2 \leq n - 1$ . By Claim 3.4, we have

$$dnr_2 \le dmn = p_1r_1 + p_2r_2 \le (n-1)(r_1 + r_2).$$

So  $((d-1)n+1)r_2 \leq (n-1)r_1$ . As we have assumed  $r_1 \leq r_2$ , one has d = 1. Thus, d' = 1 and  $n|k_1 = p_2 - p_1$  by Claim 3.7. However, this leads to a contradiction  $n \leq k_1 = p_2 - p_1 \leq n - 1 - p_1 \leq n - 1$  provided that  $p_1 < p_2 \leq n - 1$ .

Suppose that  $\delta_1 > 0$ . Write  $d'nm = k_2r'_1 + p_2ad'n$  where  $k_2 = -k_1 = p_1 - p_2$ . Note that

$$\begin{aligned} \frac{\delta_1'}{a}m &= \frac{\delta_1'}{ad'n}(p_2ad'n + k_2r_1') = \delta_1'p_2 + \frac{r_1'\delta_1'k_2}{ad'n} = \delta_1'p_2 + \frac{(as_1^{*'} - n)k_2}{ad'n} \\ &= \delta_1'p_2 + \frac{s_1^{*'}k_2}{d'n} - \frac{k_2}{ad'} \\ \text{and } \frac{s_1^{*'}}{r_1'}m &= \frac{s_1^{*'}}{d'nr_1'}(p_2ad'n + k_2r_1') = \frac{as_1^{*'}p_2}{r_1'} + \frac{s_1^{*'}k_2}{d'n} = \frac{(\delta_1'r_1' + n)p_2}{r_1'} + \frac{s_1^{*'}k_2}{d'n} \\ &= \delta_1'p_2 + \frac{s_1^{*'}k_2}{d'n} + \frac{np_2}{r_1'}. \end{aligned}$$

From Claims 3.6 and 3.7, there exists an integer  $\xi_1$  with

$$\xi_1 \equiv s_1^{*'}(r_1'^{-1})m \equiv r_1'^{-1}(r_1'(\delta_1'p_2 + \frac{s_1^{*'}k_2}{d'n} + \frac{np_2}{r_1'})) \equiv \delta_1'p_2 + \frac{s_1^{*'}k_2}{d'n} \pmod{n}.$$

Let  $l_1$  be the integer with  $\xi_1 = \delta'_1 p_2 + \frac{s_1^{*'} k_2}{d'n} + n l_1$ . Then the inequalities  $\lfloor \frac{s_1^{*'}}{r'_1}m \rfloor \geq \xi_1 \geq \lceil \frac{\delta'_1}{a}m \rceil$  in Claim 3.6 are then the same to  $\lfloor \frac{np_2}{r'_1} \rfloor \geq n l_1 \geq \lceil -\frac{k_2}{ad'} \rceil$ . As  $-k_2 = k_1 \geq 0$ , we see  $np_2/r'_1 \geq n$  and hence  $p_2 \geq r'_1$  provided that  $p_2 > p_1$ . We complete the argument of Proposition 3.3.

**Proposition 3.8.** we have  $\mathcal{T}_{3,cD,>5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}].$ 

*Proof.* Given a canonical threshold  $\operatorname{ct}(X, S) \in \mathcal{T}_{3,cD,\geq 5}^{\operatorname{can}}$ . By [Pro08], we see  $\operatorname{ct}(X, S) \leq \frac{4}{5}$ . By [Cor95, (2.10) Proposition-definition] and the classification of Kawakita [Kwk05, Theorem 1.2], one sees that  $\operatorname{ct}(X, S)$  is realized by a divisorial contraction  $\sigma: Y \to X$  classified in Case 1 and Case 2.

**Case 1.** There exists an analytical identification:

$$(P \in X) \simeq o \in (\varphi : x^2 + xq(z, u) + y^2u + \lambda yz^2 + \mu z^3 + p(y, z, u) = 0) \subset \widehat{\mathbb{C}}^4,$$

where o denotes the origin of  $\hat{\mathbb{C}}^4$  such that  $\sigma: Y \to X$  is a weighted blow up of weights w = wt(x, y, z, u) = (r + 1, r, a, 1) with center  $P \in X$  and

- 2r + 1 = ad where  $d \ge 3$  and the integer  $a \ge 5$  is odd,
- $ct(X, S) = \frac{a}{m}$  where m = w(f) and S is defined by the formal power series f = 0 analytically and locally.

Let  $\sigma_1: Y_1 \to X$  (resp.  $\sigma_2: Y_2 \to X$ ) be the weighted blow up with weights  $w_1 = (r+1-d, r-d, a-2, 1)$  (resp.  $w_2 = (d, d, 2, 1)$ ) at the origin  $P \in X$ . By [Che22, Lemma 6.3] (see also [Che15, Case Ic]), the exceptional set of  $\sigma_1$  is a prime divisor. From the computation in [Che22, Claim 6.6], the defining equation of the exceptional set of  $\sigma_2$  is  $x^2 + \eta z^d$  with odd integer  $d \geq 3$  for some nonzero constant  $\eta$  and hence the exceptional set of  $\sigma_2$  is a prime divisor. As 2r + 1 = ad, it yields

$$w_1 \succeq \frac{r-d}{r}w$$
 and  $w_2 \succeq \frac{d}{r+1}w$ .

It follows from [Che22, Lemma 2.1] that

$$\lfloor \frac{a-2}{a}m \rfloor \ge m_1 \ge \lceil \frac{r-d}{r}m \rceil \quad \text{and} \quad \lfloor \frac{2}{a}m \rfloor \ge m_2 \ge \lceil \frac{d}{r+1}m \rceil \qquad \dagger_1$$

where  $m_1 := w_1(f)$  and  $m_2 := w_2(f)$  denotes the corresponding weighted multiplicities.

**Claim 3.9.** <sup>1</sup> If  $a \nmid m$ , then  $dm \ge r(r+1)$ .

Proof of Claim 3.9. Suppose on the contrary that dm < (r+1)r. We have

$$\lceil \frac{r-d}{r}m\rceil + \lceil \frac{d}{r+1}m\rceil \ge \lceil \frac{r-d}{r}m + \frac{d}{r+1}m\rceil = \lceil m - \frac{dm}{r(r+1)}\rceil = m.$$

However, a is odd and  $a \nmid m$ , hence  $\frac{2m}{a}$  is not an integer. This implies

$$\lfloor \frac{a-2}{a}m \rfloor + \lfloor \frac{2}{a}m \rfloor = m - 1,$$

which contradicts to  $\dagger_1$ . This verifies Claim 3.9.

From Claim 3.9, express  $dm = p_1r + p_2(r+1)$  for some non-negative integers  $p_1$  and  $p_2$  with  $p_1 < r+1$ . Note that

$$\frac{2}{a}m = \frac{2dm}{ad} = \frac{2p_1r + 2p_2(r+1)}{2r+1} = p_1 + p_2 + \frac{p_2 - p_1}{2r+1}$$
$$\frac{d}{r+1}m = \frac{p_1r + p_2(r+1)}{r+1} = p_1 + p_2 - \frac{p_1}{r+1},$$
$$\frac{d}{r}m = \frac{p_1r + p_2(r+1)}{r} = p_1 + p_2 + \frac{p_2}{r}.$$

As  $p_1$  and  $p_2$  are integers, the inequality  $\lfloor \frac{2}{a}m \rfloor \geq \lceil \frac{d}{r+1}m \rceil$  then implies  $\lfloor \frac{p_2-p_1}{2r+1} \rfloor \geq \lceil -\frac{p_1}{r+1} \rceil = -\lfloor \frac{p_1}{r+1} \rfloor = 0$ . In particular,  $p_2 \geq p_1$ . Similarly, the inequality  $\lfloor \frac{a-2}{a}m \rfloor \geq \lceil \frac{r-d}{r}m \rceil$  gives

$$p_1 + p_2 + \lfloor \frac{p_2}{r} \rfloor = \lfloor \frac{d}{r}m \rfloor \ge \lceil \frac{2}{a}m \rceil = p_1 + p_2 + \lceil \frac{p_2 - p_1}{2r + 1} \rceil.$$

<sup>&</sup>lt;sup>1</sup>Claims 3.9 and 3.10 were first obtained in [HLL22].

In particular,  $p_2 \ge r$  provided that  $p_2 - p_1 > 0$ . Case 2. There exists an analytical identification:

$$P \in X \simeq o \in \left(\begin{array}{c} \varphi_1 \colon x^2 + yt + p(y, z, u) = 0;\\ \varphi_2 \colon yu + z^d + q(z, u)u + t = 0 \end{array}\right) \subset \hat{\mathbb{C}}^5$$

where o denotes the origin of  $\hat{\mathbb{C}}^5$  such that  $\sigma: Y \to X$  is a weighted blow up of weights w = (r+1, r, a, 1, r+2) with center  $P \in X$  and

- r+1 = ad where  $d \ge 2$  and  $a \ge 5$ ,
- $ct(X, S) = \frac{a}{m}$  where m = w(f) and S is defined by the formal power series f = 0 analytically and locally.

Compare the weights w with the weights  $w_1 = (r - d + 1, r - d, a - 1, 1, r - d + 2)$  and  $w_2 = (d, d, 1, 1, d)$ . By [Che22, Lemma 2.1, Lemma 6.7 and Lemma 6.8], we have

$$\lfloor \frac{a-1}{a}m \rfloor \ge \lceil \frac{r-d}{r}m \rceil \text{ and } \lfloor \frac{1}{a}m \rfloor \ge \lceil \frac{d}{r+2}m \rceil. \qquad \dagger_2$$

Claim 3.10. If  $a \nmid m$ , then  $2dm \ge (r+2)r$ .

*Proof of Claim 3.10.* Suppose on the contrary that 2dm < (r+2)r. We have

$$\lceil \frac{r-d}{r}m\rceil + \lceil \frac{d}{r+2}m\rceil \ge \lceil \frac{r-d}{r}m + \frac{d}{r+2}m\rceil = \lceil m - \frac{2dm}{r(r+2)}\rceil = m.$$

However,  $a \nmid m$ , hence

$$\lfloor \frac{a-1}{a}m \rfloor + \lfloor \frac{1}{a}m \rfloor = m-1,$$

which contradicts to  $\dagger_2$ . The proof of Claim 3.10 is complete.

From Claim 3.10, express  $2dm = p_1r + p_2(r+2)$  for some non-negative integers  $p_1$  and  $p_2$ . Denote by  $h := \gcd(r, r+2)$ . We have

$$\frac{2dm}{h} = p_1 \frac{r}{h} + p_2 \frac{r+2}{h}.$$

By replacing  $p_2$  by  $p_2 + \lfloor \frac{p_1 h}{r+2} \rfloor \frac{r}{h}$  (resp. replacing  $p_1$  by  $p_1 - \lfloor \frac{p_1 h}{r+2} \rfloor \frac{r+2}{h}$ ), we may assume that  $0 \le p_1 < \frac{r+2}{h}$ . Note that

$$\frac{1}{a}m = \frac{2dm}{2ad} = \frac{p_1r + p_2(r+2)}{2(r+1)} = \frac{p_1 + p_2}{2} + \frac{p_2 - p_1}{2(r+1)},$$
$$\frac{d}{r+2}m = \frac{2dm}{2(r+2)} = \frac{p_1r + p_2(r+2)}{2(r+2)} = \frac{p_1 + p_2}{2} - \frac{p_1}{r+2},$$
$$\frac{d}{r}m = \frac{2dm}{2r} = \frac{p_1r + p_2(r+2)}{2r} = \frac{p_1 + p_2}{2} + \frac{p_2}{r}.$$

Suppose that the integer  $p_1 + p_2$  is even. The inequality  $\lfloor \frac{1}{a}m \rfloor \geq \lceil \frac{d}{r+2}m \rceil$  then implies  $\lfloor \frac{p_2-p_1}{2(r+1)} \rfloor \geq \lceil -\frac{p_1}{r+2} \rceil = 0$ . In particular,  $p_2 \geq p_1$ . Similarly, the inequality  $\lfloor \frac{a-1}{a}m \rfloor \geq \lceil \frac{r-d}{r}m \rceil$  gives  $\lfloor \frac{p_2}{r} \rfloor \geq \lceil \frac{p_2-p_1}{2(r+1)} \rceil$ . In particular,  $p_2 \geq r$  provided that  $p_2 - p_1 > 0$ .

Suppose next that the integer  $p_1 + p_2$  is odd. Since  $(p_1 + p_2)r = 2(dm - p_2)$ , r is even and h = 2. The inequality  $\lfloor \frac{1}{a}m \rfloor \geq \lceil \frac{d}{r+2}m \rceil$  implies  $\lfloor \frac{1}{2} + \frac{p_2 - p_1}{2(r+1)} \rfloor \geq$ 

19

 $\lceil \frac{1}{2} - \frac{p_1}{r+2} \rceil = 1 \text{ as } p_1 < \frac{r+2}{h} = \frac{r+2}{2}. \text{ In particular, } p_2 \ge p_1 + r + 1. \text{ Similarly, } the inequality } \lfloor \frac{a-1}{a}m \rfloor \ge \lceil \frac{r-d}{r}m \rceil \text{ yields } \lfloor \frac{1}{2} + \frac{p_2}{r} \rfloor \ge \lceil \frac{1}{2} + \frac{p_2-p_1}{2(r+1)} \rceil \ge 1. \text{ In particular, } p_2 \ge \frac{r}{2}. \text{ The proof of Proposition 3.8 is complete.}$ 

**Proposition 3.11.** We have  $\mathcal{T}_{3,cD/2,\geq 5}^{\operatorname{can}} \subseteq C \cap [0, \frac{4}{5}]$ .

*Proof.* Given a canonical threshold  $\operatorname{ct}(X, S) \in \mathcal{T}_{3,cD/2}^{\operatorname{can}}$ . We have  $\operatorname{ct}(X, S) \leq \frac{4}{5}$  by [Pro08]. By [Cor95, (2.10) Proposition-definition] and the classification of Kawakita [Kwk05, Theorem 1.2], one sees that  $\operatorname{ct}(X, S)$  is realized by a divisorial contraction  $\sigma: Y \to X$  given in Case 1 and Case 2. **Case 1.** There exists an analytical identification:

$$P \in X \simeq o \in (\varphi : x^2 + xzq(z^2, u) + y^2u + \lambda yz^{2\alpha - 1} + p(z^2, u) = 0) \subset \hat{\mathbb{C}}^4 / \frac{1}{2}(1, 1, 1, 0)$$

where o denotes the origin of  $\hat{\mathbb{C}}^4/\frac{1}{2}(1,1,1,0)$  such that  $\sigma: Y \to X$  is a weighted blow up of weights  $w = \frac{1}{2}(r+2,r,a,2)$  with center  $P \in X$  and

- r + 1 = ad where both  $a \ge 5$  and r are odd;
- $ct(X, S) = \frac{a}{m}$  where m = w(f) and S is defined by the formal power series f = 0 analytically and locally.

Denote by  $\sigma_1: Y_1 \to X$  and  $\sigma_2: Y_2 \to X$  the weighted blow ups with weights  $w_1 = \frac{1}{2}(r+2-2d, r-2d, a-2, 2)$  and  $w_2 = \frac{1}{2}(2d, 2d, 2, 2)$  at the origin  $P \in X$  and  $m_1 := 2w_1(f)$  and  $m_2 := 2w_2(f)$  the weighted multiplicities respectively. By [Che22, Lemma 7.3], the exceptional set of  $\sigma_1$  is a prime divisor. Note that the exceptional set of  $\sigma_2$  is a  $\mathbb{Z}_2$  quotient of

$$\{x^2 + z^{2d} = 0\} \subset \mathbb{P}(2d, 2d, 2, 2).$$

It is a prime divisor. By [Che22, Lemma 2.1] and r + 1 = ad, we have

$$\lfloor \frac{a-2}{a}m \rfloor \ge m_1 \ge \lceil \frac{r-2d}{r}m \rceil \text{ and } \lfloor \frac{2}{a}m \rfloor \ge m_2 \ge \lceil \frac{2d}{r+2}m \rceil$$

with  $m_1 \equiv (a-2)a^{-1}m \equiv m$  and  $m_2 \equiv 2a^{-1}m \equiv 0 \pmod{2}$ . See Remark 3.14 for an alternative explanation of the inequality  $\lfloor \frac{2}{a}m \rfloor \geq m_2$ .

**Claim 3.12.** If  $a \nmid m$ , then  $2dm \ge r(r+2)$ .

Proof of Claim 3.12. Since  $a \nmid m$  and a is odd, we have  $m - 1 = \lfloor \frac{a-2}{a}m \rfloor + \lfloor \frac{2m}{a} \rfloor \geq m_1 + m_2$ . As  $m_1 + m_2 \equiv m \pmod{2}$ , one has  $m - 2 \geq m_1 + m_2$ . Suppose on the contrary that 2dm < r(r+2). We see

$$m_1 + m_2 \ge \lceil \frac{r-2d}{r}m \rceil + \lceil \frac{2d}{r+2}m \rceil \ge \lceil \frac{r-2d}{r}m + \frac{2d}{r+2}m \rceil$$
$$= m - \lfloor \frac{4d}{(r+2)r}m \rfloor \ge m - 1,$$

which leads to a contradiction that  $m-2 \ge m_1 + m_2 \ge m-1$ . This verifies Claim 3.12.

From Claim 3.12, express  $2dm = p_1r + p_2(r+2)$  for some non-negative integers  $p_1$  and  $p_2$  with  $p_1 < r+2$ . Note that

$$\frac{2}{a}m = \frac{2dm}{ad} = \frac{p_1r + p_2(r+2)}{r+1} = p_1 + p_2 + \frac{p_2 - p_1}{r+1},$$
$$\frac{2d}{r+2}m = \frac{p_1r + p_2(r+2)}{r+2} = p_1 + p_2 - \frac{2p_1}{r+2},$$
$$\frac{2d}{r}m = \frac{p_1r + p_2(r+2)}{r} = p_1 + p_2 + \frac{2p_2}{r}.$$

Now the integer  $(p_1 + p_2)r = 2dm - 2p_2$  is even where r is odd. This gives that  $p_1 + p_2$  is even. The inequalities  $\lfloor \frac{2}{a}m \rfloor \ge m_2 \ge \lceil \frac{2d}{r+2}m \rceil$  then imply  $\lfloor \frac{p_2-p_1}{r+1} \rfloor \ge m_2 - (p_1 + p_2) \ge \lceil -\frac{2p_1}{r+2} \rceil$ . From the assumption  $p_1 < r+2$  and that  $m_2 - (p_1 + p_2)$  is even, one sees  $\lfloor \frac{p_2-p_1}{r+1} \rfloor \ge 0$ . In particular,  $p_2 \ge p_1$ .

Similarly, the inequalities  $\lfloor \frac{a-2}{a}m \rfloor \ge m_1 \ge \lceil \frac{r-2d}{r}m \rceil$  are equivalent to  $\lfloor \frac{2d}{r}m \rfloor \ge m-m_1 \ge \lceil \frac{2}{a}m \rceil$ . This implies  $\lfloor \frac{2p_2}{r} \rfloor \ge m-m_1-(p_1+p_2) \ge \lceil \frac{p_2-p_1}{r+1} \rceil$  where the integer  $m-m_1-(p_1+p_2)$  is even. In particular,  $p_2 \ge r$  provided that  $p_2 - p_1 > 0$ .

Case 2. There exists an analytical identification:

$$P \in X \simeq o \in \left(\begin{array}{c} \varphi_1 := x^2 + yt + p(z^2, u) = 0\\ \varphi_2 := yu + z^{2d+1} + q(z^2, u)zu + t = 0 \end{array}\right)$$

in  $\hat{\mathbb{C}}_{x,y,z,u,t}^5/\frac{1}{2}(1,1,1,0,1)$  where o denotes the origin of  $\hat{\mathbb{C}}_{x,y,z,u,t}^5/\frac{1}{2}(1,1,1,0,1)$  such that  $\sigma: Y \to X$  is a weighted blow up of weights  $w = \frac{1}{2}(r+2,r,a,2,r+4)$  with center  $P \in X$  and

- r + 2 = a(2d + 1) where d, r and  $a \ge 5$  are positive integers;
- $ct(X, S) = \frac{a}{m}$  where m = w(f) and S is defined by the formal power series f = 0 analytically and locally.

On the open subset  $U_2 = \{\overline{y} \neq 0\}$ , Y is isomorphic to  $\hat{\mathbb{C}}_{\overline{x},\overline{y},\overline{z}}^2/\frac{1}{r}(-(r+2),2,-a)$ . It follows from terminal lemma that both integers a and r are odd. Denote by  $\sigma_1: Y_1 \to X$  and  $\sigma_2: Y_2 \to X$  the weighted blow up with weights  $w_1 = \frac{1}{2}(r-2d+1, r-2d-1, a-1, 2, r-2d+3)$  and  $w_2 = \frac{1}{2}(2d+1, 2d+1, 1, 2, 2d+1)$  at the origin  $P \in X$  respectively. By [Che22, Lemma 2.1, Lemma 7.6 and Lemma 7.7], there exist two integers  $m_1$  and  $m_2$  satisfying

$$\lfloor \frac{a-1}{a}m \rfloor \ge m_1 \ge \lceil \frac{r-2d-1}{r}m \rceil \text{ and } \lfloor \frac{1}{a}m \rfloor \ge m_2 \ge \lceil \frac{2d+1}{r+4}m \rceil$$

and  $m_1 \equiv (a-1)a^{-1}m$  and  $m_2 \equiv a^{-1}m \pmod{2}$ .

**Claim 3.13.** If  $a \nmid m$ , then  $(4d+2)m \ge r(r+4)$ .

Proof of Claim 3.13. Since  $a \nmid m$ , we have

$$m-1 = \lfloor \frac{1}{a}m \rfloor + \lfloor \frac{a-1}{a}m \rfloor \ge m_1 + m_2.$$

As  $m_1 + m_2 \equiv m \pmod{2}$ , one has  $m - 2 \ge m_1 + m_2$ .

Suppose on the contrary that (4d+2)m < r(r+4). We see

$$m_1 + m_2 \ge \lceil \frac{2d+1}{r+4}m \rceil + \lceil \frac{r-2d-1}{r}m \rceil \ge \lceil \frac{2d+1}{r+4}m + \frac{r-2d-1}{r}m \rceil$$
$$= m - \lfloor \frac{8d+4}{(r+4)r}m \rfloor \ge m - 1.$$

which leads to a contradiction that  $m-2 \ge m_1 + m_2 \ge m-1$ . This verifies the claim.

From Claim 3.13, express  $(4d+2)m = p_1r+p_2(r+4)$  for some non-negative integers  $p_1$  and  $p_2$  with  $p_1 < r + 4$ . Note that

$$\frac{1}{a}m = \frac{(4d+2)m}{(4d+2)a} = \frac{p_1r + p_2(r+4)}{2(r+2)} = \frac{p_1 + p_2}{2} + \frac{p_2 - p_1}{r+2}$$
$$\frac{2d+1}{r+4}m = \frac{(4d+2)m}{2(r+4)} = \frac{p_1r + p_2(r+4)}{2(r+4)} = \frac{p_1 + p_2}{2} - \frac{2p_1}{r+4},$$
$$\frac{2d+1}{r}m = \frac{(4d+2)m}{2r} = \frac{p_1r + p_2(r+4)}{2r} = \frac{p_1 + p_2}{2} + \frac{2p_2}{r}.$$

Now the integer  $(p_1 + p_2)r = (4d + 2)m - 4p_2$  is even where r is odd. In particular, two integers  $p_1 + p_2$  and  $p_2 - p_1$  are even. The inequalities  $\lfloor \frac{1}{a}m \rfloor \ge m_2 \ge \lceil \frac{2d+1}{r+4}m \rceil$  then imply  $\lfloor \frac{p_2 - p_1}{r+2} \rfloor \ge m_2 - \frac{p_1 + p_2}{2} \ge \lceil -\frac{2p_1}{r+4} \rceil$ . From the assumption  $p_1 < r + 4$  and that

$$m_2 - \frac{p_1 + p_2}{2} \equiv a^{-1}m - \frac{p_1 + p_2}{2} \equiv \frac{p_2 - p_1}{r + 2} \equiv p_2 - p_1 \equiv 0 \pmod{2},$$

one sees  $\lfloor \frac{p_2 - p_1}{r+2} \rfloor \ge 0$ . In particular,  $p_2 \ge p_1$ .

Similarly, the inequalities  $\lfloor \frac{a-1}{a}m \rfloor \ge m_1 \ge \lceil \frac{r-2d-1}{r}m \rceil$  are equivalent to  $\lfloor \frac{2d+1}{r}m \rfloor \ge m-m_1 \ge \lceil \frac{1}{a}m \rceil$ . This implies  $\lfloor \frac{2p_2}{r} \rfloor \ge m-m_1 - \frac{p_1+p_2}{2} \ge \lceil \frac{p_2-p_1}{r+2} \rceil$  where

$$m - m_1 - \frac{p_1 + p_2}{2} \equiv m - (a - 1)a^{-1}m - \frac{p_1 + p_2}{2} \equiv \frac{p_2 - p_1}{r + 2} \equiv 0 \pmod{2}.$$

In particular,  $p_2 \ge r$  provided that  $p_2 - p_1 > 0$ . This completes the proof of Proposition 3.11.

**Remark 3.14.** We keep notions in Case 1 of the proof of Proposition 3.11. As  $w_2 \geq \frac{2d}{r+2}w$ , one sees  $m_2 \geq \lceil \frac{2d}{r+2}m \rceil$ . In what follows, we shall provide an alternative argument to establish the inequality  $\lfloor \frac{2}{a}m \rfloor \geq m_2$ . Let  $\mathcal{X} = \mathbb{C}^4/\frac{1}{2}(1,1,1,0)$  and  $\sigma_w : \mathcal{Y} \to \mathcal{X}$  be the weighted blow up with weights  $w = \frac{1}{2}(r+2,r,a,2)$ . Put  $\overline{\varphi}_1 = \varphi(\overline{x}^{\frac{r+2}{2}}, \overline{x}^{\frac{r}{2}}\overline{y}, \overline{x}^{\frac{a}{2}}\overline{z}, \overline{xu})/\overline{x}^{r+1}$ . As  $w(x^2) = r+2$ and  $w(\varphi) = r+1$ ,  $\overline{x} \in \overline{\varphi}_1$  and hence

$$Y \cap U_1 \simeq (\overline{\varphi}_1 = 0) / \frac{1}{r+2} (2, -r, -a, -2) \simeq \mathbb{C}^3_{\overline{y}, \overline{z}, \overline{u}} / \frac{1}{r+2} (-r, -a, -2),$$

where  $U_1 := \{\overline{x} \neq 0\} \simeq \mathbb{C}^4 / \frac{1}{r+2}(2, -r, -a, -2)$  is an open subset of  $\mathcal{Y}$ . Denote by  $\sigma: Y \to X$  be the induced morphism with exceptional divisor E. Let  $\mu: \mathcal{Z}_1 \to \mathcal{Y}$  be the weighted blow up (at the origin of  $U_1$ ) with weights  $w_2 = \frac{1}{r+2}(2d, 2d, 1, r+2-2d)$ . Denote by  $Z_1 = Y_{\mathcal{Z}_1}$  the proper transform. Then the induced map  $\mu_1 = \mu|_{Z_1}: Z_1 \to Y$  is Kawamata blow up. In

particular, the exceptional set of  $\mu_1$ , denoted by  $E_1$ , is a prime divisor of  $Z_1$ . Then we see

$$K_Y = \sigma^* K_X + \frac{a}{2}E, \quad K_{Z_1} = \mu_1^* K_Y + \frac{1}{r+2}E_1,$$
  
$$S_Y = \sigma^* S - \frac{m}{2}E, \quad S_{Z_1} = \mu_1^* S_Y - \frac{m'}{r+2}E_1,$$

for some non-negative integer m'. Note that it follows from  $w_2 = \frac{2d}{r+2}w + \frac{1}{r+2}(0, 2d, 1, r+2-2d)$  that  $E_{Z_1} = \mu_1^* E - \frac{2d}{r+2}E_1$ . By direct toric computations, one has

$$K_{Z_1} = \mu_1^* \sigma^* K_X + \frac{a}{2} E_{Z_1} + \left(\frac{2d}{r+2} \cdot \frac{a}{2} + \frac{1}{r+2}\right) E_1 = \mu_1^* \sigma^* K_X + \frac{a}{2} E_{Z_1} + \frac{2}{2} E_1$$
$$S_{Z_1} = \mu_1^* \sigma^* S - \frac{m}{2} E_{Z_1} - \left(\frac{2d}{r+2} \cdot \frac{m}{2} + \frac{m'}{r+2}\right) E_1 = \mu_1^* \sigma^* S - \frac{m}{2} E_{Z_1} - \frac{m_2}{2} E_1,$$

where  $m_2 = 2w_2(f)$  and the divisor S is defined by the formal power series f = 0 analytically and locally. As  $\operatorname{ct}(X, S)$  is the canonical threshold and  $E_1$  is a prime divisor over  $X, \frac{2}{m_2} \ge \operatorname{ct}(X, S) = \frac{a}{m}$ . In particular,  $\lfloor \frac{2}{a}m \rfloor \ge m_2$ .

**Theorem 3.15.** We have  $\mathcal{T}_3^{\operatorname{can}} = \{0\} \cup \{\frac{4}{5}\} \cup \mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}}$ .

*Proof.* We show non-trivial inclusion  $\mathcal{T}_3^{\operatorname{can}} \subseteq \{0\} \cup \{\frac{4}{5}\} \cup \mathcal{T}_{3,\operatorname{sm}}^{\operatorname{can}}$ . Suppose that  $\operatorname{ct}(X,S) \in \mathcal{T}_3^{\operatorname{can}}$  is non-zero. By taking Q-factorization, X can be assumed to have at worst Q-factorial terminal singularities. From the decomposition in (3) in Remark 2.5, we may assume  $\operatorname{ct}(X,S) \in \mathcal{T}_{3,*,\geq 5}^{\operatorname{can}}$  for some singular type \* = cA, cA/n, cD or cD/2. The rest follows from Theorem 2.2, Propositions 3.1, 3.3, 3.8 and 3.11.

**Remark 3.16.** It is known that the accumulation points of  $\mathcal{T}_3^{can}$  is the set  $\{0\} \cup \{\frac{1}{k}\}_{k \in \mathbb{Z}_{\geq 2}}$  by [Che22a] and [HLL22] independently. This result can be recovered using Theorem 3.15. It is also interesting to study  $(\frac{1}{k-1}, \frac{1}{k}) \cap C$  for any  $k \in \mathbb{Z}_{\geq 2}$ . The k = 2 case was explicitly characterized in [Che22]. The set  $\mathcal{T}_{3,\text{sm}}^{can} \cap (\frac{1}{3}, \frac{1}{2})$  was studied by the third named author in his master thesis [Wu23] and listed in Table 1. These two cases can also be recovered by Theorems 2.2 and 3.15.

$\alpha$	β	$p_1$	$p_2$	Remark
α	β	$p_1$	3	$ct = \frac{1}{3} + \frac{(3-p_1)\alpha}{3(p_1\alpha+3\beta)}$ with $0 \le p_1 \le 2$ , $1 \le \alpha \le 3$ , $(2-p_1)\alpha < \beta$ , $gcd(\alpha, \beta) = 1$
1	1	1	4	ct = 2/5
1	1	0	5	ct = 2/5
1	2	0	4	ct = 3/8
2	3	0	4	ct = 5/12
2	3	1	4	ct = 5/14
2	5	0	4	ct = 7/20
3	4	0	4	ct = 7/16
3	4	1	4	ct = 7/19
3	4	0	5	ct = 7/20
3	5	0	4	ct = 2/5
3	5	1	4	ct = 8/23
3	7	0	4	ct = 5/14
3	8	0	4	ct = 11/32
4	5	0	4	ct = 9/20
4	5	1	4	ct = 3/8
4	5	0	5	ct = 9/25
4	7	0	4	ct = 11/28
4	7	1	4	ct = 11/32
4	9	0	4	ct = 13/36
4	11	0	4	ct = 15/44
5	6	0	5	ct = 11/30
5	7	0	5	ct = 12/35

TABLE 1. canonical thresholds in  $\mathcal{T}_{3,\mathrm{sm}}^{\mathrm{can}} \cap (\frac{1}{3}, \frac{1}{2})$ 

#### References

- [Ale93] V. Alexeev, Two two-dimensional terminations, Duke Math. J.69(1993), no.3, 527-545.
- [Bir07] C. Birkar, Ascending chain condition for log canonical thresholds and termination of log flips, Duke Math. J. 136 (2007), 173–180.
- [Che14] J. A. Chen, Factoring threefold divisorial contractions to points, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V 13 (2014), no. 2, 435-463.
- [Che15] J. A. Chen, Birational maps of 3-folds, Taiwanese J. Math. 19 (2015), no. 6, 1619-1642.
- [Che22] J.-J. Chen, On threefold canonical thresholds, Adv. Math. 404 (2022), Paper No. 108447, 36 pp.
- [Che22a] J.-J. Chen, Accumulation points of 3-fold canonical thresholds, to appear in J. Math. Soc. Japan. arXiv: 2202.06230.
- [Cor95] A. Corti, Factoring birational maps of 3-folds after Sarkisov, J. algebraic Geom. 4 (1995), 223-254.

- [Cor00] A. Corti, Singularities of linear systems and 3-fold birational geometry, Explicit birational geometry of 3-folds, 259-312, London Math. Soc. Lecture Note Ser., 281, Cambridge Univ. Press, Cambridge, 2000.
- [dFM09] T. deFernex and M. Mustaţă, *Limits of log canonical thresholds*, Ann. Sci. Ec. Norm. Supér.(4) 42 (3), 491-515 (2009).
- [HMX14] C. D. Hacon, J. Mckernan and C. Y. Xu, ACC for log canonical thresholds, Annals of Math. (2014).
- [HLL22] J. Han, J. Liu and Y. Luo, ACC for minimal log discrepancies of terminal threefolds, arXiv: 2202.05287v2.
- [Hay99] T. Hayakawa, Blowing ups of 3-dimensional terminal singularities, Publ. Res. Inst. Math. Sci. 35 (1999), no. 3, 515-570.
- [Hay00] T. Hayakawa, Blowing ups of 3-dimensional terminal singularities II, Publ. Res. Inst. Math. Sci. 36 (2000), no. 3, 423-456.
- [Kwk01] M. Kawakita, Divisorial contractions in dimension three which contract divisors to smooth points, Invent. Math. 145 (2001), no. 1, 105-119.
- [Kwk02] M. Kawakita, Divisorial contractions in dimension three which contract divisors to compound A1 points, Compos. Math. 133 (2002), no. 1, 95-116.
- [Kwk05] M. Kawakita, Threefold divisorial contractions to singularities of higher indices, Duke Math. J. 130 (2005), no. 1, 57-126.
- [Kaw96] Y. Kawamata, Divisorial contractions to 3-dimensional terminal quotient singularities, Higher-dimensional complex varieties (Trento, 1994), 241-246, de Gruyter, Berlin 1996.
- [Kol97] J. Kollár, Singularities of pairs, Algebraic geometry-Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997. pp. 221-287.
- [Kol08] J. Kollár, Which powers of holomorphic functions are integrable? arXiv 0805. 0756.
- [Kuw99a] T. Kuwata, On log canonical thresholds of reducible plane curves, Amer. J. Math. 121 (1999), no. 4, 701-721.
- [Kuw99b] T. Kuwata, On Log Canonical Thresholds of Surfaces in  $\mathbb{C}^3$ , Tokyo J. Math. **22(1)**: 245-251 (June 1999).
- [LMX24] J. Liu, , F. Meng; L. Xie, Infinitesimal structure of log canonical thresholds, Doc. Math.29(2024), no.3, 703-732.
- [Mat02] K. Matsuki, Introduction to the Mori program, Universitext. Springer-Verlag, New York, 2002. xxiv+478 pp.
- [MP04] J. M<sup>c</sup>Kernan and Y. Prokhorov, Threefold thresholds, Manuscripta Math., 114 (2004), no. 3, 281-304.
- [Mor82] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. (2)116(1982), no.1, 133-176.
- [Pro08] Y. Prokhorov, Gap conjecture for 3-dimensional canonical thresholds, J. Math. Sci. Univ. Tokyo 15 (2008), no. 4, 449-459.
- [Ste11] D. A. Stepanov, Smooth three-dimensional canonical thresholds, (Russian) Mat. Zametki 90 (2011), no. 2, 285-299; translation in Math. Notes 90 (2011), no. 1-2, 265-278.
- [Sho93] V. V. Shokurov, Three-dimensional log perestroikas. With an appendix in English by YujiroKawamata. Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), 105–203; translation in Russian Acad. Sci. Izv. Math. 40 (1993), 95–202.
- [Wu23] H.-Y. Wu, A survey on the conjecturally minimal volume of klt ample (resp. Fano) varieties and specific classification of 3-folds canonical thresholds in the open interval (1/3, 1/2), NCU master thesis in July 2023.
- [Yam18] Y. Yamamoto, Divisorial contractions to cDV points with discrepancy > 1, Kyoto J. Math. 58 (2018), 529-567.

Department of Mathematics, National Central University, Taoyuan City, 320, Taiwan *Email address*: jhengjie@math.ncu.edu.tw

Department of Mathematics, Third General Building, National Tsing Hua University, No. 101 Sec 2 Kuang Fu Road, Hsinchu, 30043, Taiwan

Email address: jcchen@math.nthu.edu.tw

Department of Mathematics, National Central University, Taoyuan City, 320, Taiwan *Email address:* 112281002@cc.ncu.edu.tw