# HYPERGRAPH *p*-LAPLACIAN EQUATIONS FOR DATA INTERPOLATION AND SEMI-SUPERVISED LEARNING

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ABSTRACT. Hypergraph learning with p-Laplacian regularization has attracted a lot of attention due to its flexibility in modeling higher-order relationships in data. This paper focuses on its fast numerical implementation, which is challenging due to the non-differentiability of the objective function and the non-uniqueness of the minimizer. We derive a hypergraph p-Laplacian equation from the subdifferential of the p-Laplacian regularization. A simplified equation that is mathematically well-posed and computationally efficient is proposed as an alternative. Numerical experiments verify that the simplified p-Laplacian equation suppresses spiky solutions in data interpolation and improves classification accuracy in semi-supervised learning. The remarkably low computational cost enables further applications.

#### 1. INTRODUCTION

Over the past two decades, hypergraphs have become a valuable tool for data processing. It is defined as the generalization of a graph in which a hyperedge can connect more than two vertices. This allows hypergraphs to model higher-order relations involving multiple vertices in data, with applications in areas including image processing [1, 2], bioinformatics [3, 4], social networks [5, 6], etc.

In this paper, we focus on semi-supervised learning on undirected hypergraphs. Let H = (V, E, W) be a hypergraph, where  $V = \{x_i\}_{i=1}^n$  denotes the vertex set,  $E = \{e_k\}_{k=1}^m$  is the hyperedge set, and  $W = \{w_k\}_{k=1}^m$  assigns positive weights for hyperedges. We are given a subset of labeled vertices  $\{x_i\} =: L \subset V$  and the associated labels  $\{y_i\} \subset \mathbb{R}$ . The goal is to assign labels for the remaining vertices  $V \setminus L$  based on the training data  $\{(x_i, y_i), x_i \in L, y_i \in \mathbb{R}\}$ .

We consider the standard approach that minimizes the constraint functional

$$F^{con}(u) = \begin{cases} F(u), & \text{if } u(x_i) = y_i, \ x_i \in L, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here F(u) is the regularization of  $u: V \to \mathbb{R}$  that enforces the smoothness of u. It implicitly assumes that vertices within the same hyperedge tend to have the same label. The constraint in  $F^{con}$  ensures the minimizer of it to satisfy the given

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<sup>2020</sup> Mathematics Subject Classification. 35R02, 65D05.

Key words and phrases. Hypergraph, p-Laplacian, data interpolation, semi-supervised learning.

training data. One of the fundamental algorithms for hypergraph learning is the p-Laplacian regularization [7]

$$F_{CE}(u) = \sum_{k=1}^{m} w_k \sum_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^p,$$
(1)

where p > 1. Notice that the hypergraph H can be approximated by a weighted graph  $G = (V, E_G, W_G)$  with clique expansion [8]. More precisely, for any  $x_i, x_j \in e_k \subset E$ , there exists an edge  $e_{i,j} \in E_G$ . The associated weight  $w_{i,j} = \frac{w_k}{C_{|e_k|}^2}$ , where  $|e_k|$  denotes the cardinality of hyperedge  $e_k$  and  $C_{|e_k|}^2 = |e_k|(|e_k| - 1)/2$ . Then functional  $F_{CE}$  is equivalent to applying the graph *p*-Laplacian [9]

$$F_G(u) = \sum_{i,j=1}^n w_{i,j} |u(x_i) - u(x_j)|^p,$$
(2)

to the weighted graph G. It was shown that the approximation approach can not fully utilize the hypergraph structure [8]. Later in [10], the authors proposed to overcome this limitation with a new hypergraph p-Laplacian regularization

$$F_H(u) = \sum_{k=1}^m w_k \max_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^p,$$
(3)

which is deduced from the Lovász extension of the hypergraph cut.

 $F_H$  is more mathematically appealing than  $F_{CE}$  due to its convexity but nondifferentiability. In [11], the authors defined the hypergraph *p*-Laplacian operator, which is multivalued, as the subdifferential  $\partial F_H$ . Properties of solutions to nonlinear evolution equations governed by  $\partial F_H$  (i.e.,  $\frac{du(t)}{dt} + \partial F_H u(t) \ge 0$  and its variants) were studied [11, 12]. The variational consistency between  $F_H^{con}$  and the continuum *p*-Laplacian

$$\mathcal{F}^{con}(u) = \begin{cases} \int_{\Omega} |\nabla u|^p dx, & \text{if } u \in W^{1,p}(\Omega) \text{ and } u(x_i) = y_i, \ x_i \in L, \\ +\infty, & \text{otherwise,} \end{cases}$$

was established in our previous paper [13] in the setting when the number of vertices n goes to infinity while the number of labeled vertices remains fixed. To avoid the complicated structure of the hypergraph, we considered a class of hypergraphs constructed from point cloud data by the distance-based method. It was shown both theoretically and numerically that  $F_H^{con}$  suppresses spiky solutions in data interpolation better than the graph p-Laplacian  $F_G^{con}$ .

On the other hand, the non-differentiability of  $F_H^{con}$  and the non-uniqueness of its minimizers cause some challenges in the numerical aspect. Unlike the graph functional  $F_G^{con}$ , there exists no straightforward and efficient algorithm for minimizing  $F_H^{con}$ , even in the case p = 2. The primal-dual hybrid gradient (PDHG) algorithm [14] was first considered in [10] for p = 1, 2. A new algorithm [13] that works any  $p \ge 1$  was proposed based on the stochastic PDHG algorithm [15]. To avoid the non-uniqueness of minimizers for  $F_H^{con}$ , the  $\ell^2$ -norm constraint was used in [10]. While in [16], the authors proposed to minimize  $F_H^{con}$  by the subgradient descent method [17] and utilized the confidence interval to ensure the uniqueness. Nevertheless, their high computational cost cannot be neglected for large-scale datasets and hinders further applications of the hypergraph p-Laplacian. The purpose of this paper is to provide an alternative to  $F_H^{con}$  that can be uniquely and efficiently solved. We begin by addressing the non-uniqueness issue of the minimizer for  $F_H^{con}$  and obtain a single-valued *p*-Laplacian operator from the subdifferential of  $F_H^{con}$ . The operator involves unknown parameters dependent on the structure of the hypergraph, preventing us from solving the associated *p*-Laplacian equations for semi-supervised learning. A simplified equation that disregards these parameters is then proposed as an approximation. It is mathematically well-posed: It admits a unique solution and satisfies the comparison principle. There exist hypergraphs on which the solution of the simplified equation coincides with a minimizer of  $F_H^{con}$ . Despite this simplification, we still refer to it as the hypergraph *p*-Laplacian equation.

Through numerical experiments on one-dimensional data interpolation, we observe that the simplified hypergraph p-Laplacian equation substantially inherits the characteristic of  $F_H^{con}$  that suppresses spiky solutions. Experimental results on real-world datasets indicate that it even improves the classification accuracy for semi-supervised learning. The most notable feature of the new equation is its low computational cost. When compared to the aforementioned algorithms, it dramatically reduces the computation time for semi-supervised learning on the selected UCI datasets from dozens of seconds to less than 0.5 seconds. Further applications of the hypergraph p-Laplacian for large-scale datasets become possible.

This paper is organized as follows. In section 2, we establish a hypergraph p-Laplacian equation from the subdifferential of  $F_H^{con}$  and propose a simplified version that is computationally feasible and efficient. The properties of solutions for the equation are also discussed. Numerical experiments are presented in section 3 to demonstrate the performance of the simplified equation for data interpolation and semi-supervised learning. We conclude this paper in section 4.

### 2. Hypergraph *p*-Laplacian equations

Let p > 1. Throughout this paper, we always assume that the hypergraph H is connected. Namely, for any  $x_i, x_j \in V$ , there exist hyperedges  $e_{k_1}, e_{k_1}, \dots, e_{k_l} \in E$ , such that  $x_i \in e_{k_1}, x_j \in e_{k_l}$ , and  $e_{k_s} \cap e_{k_{s+1}} \neq \emptyset$  for any  $s = 1, \dots, l-1$ .

2.1. The property for the minimizer of  $F_H^{con}$ . Notice that  $F_H^{con}$  is coercive and lower semi-continuous [13], it admits at least one minimizer. The non-uniqueness of minimizers can be seen from the fact that the functional depends only on the maximum and minimum values on each hyperedge. We are more concerned with vertices whose values are uniquely determined when minimizing the functional  $F_H^{con}$ .

**Definition 2.1.** Let u be a minimizer of  $F_H^{con}$ . We define  $D(u) \subset V$  to be a subset of vertices such that for any  $x_i \in D(u)$  and any perturbation of u at  $x_i$ , i.e.,

$$v(x_j) := \begin{cases} u(x_i) + \varepsilon, & \text{if } j = i, \\ u(x_j), & \text{otherwise} \end{cases}$$

where  $\varepsilon$  is a constant with small absolute value,  $F_H^{con}(v) > F_H^{con}(u)$  holds.

Clearly,  $L \subset D(u)$ . The following lemma that characterizes the vertex in  $D(u) \setminus L$  follows from the definition directly.

**Lemma 2.2.** Let u be a minimizer of  $F_H^{con}$  and  $x_i \in D(u) \setminus L$ . There exist two hyperedges  $e_k, e_l \in E$  such that

$$u(x_i) = \max_{x_j \in e_k} u(x_j) \quad and \quad u(x_i) = \min_{x_j \in e_l} u(x_j).$$
 (4)

The maximum and minimum values of a minimizer on each hyperedge are uniquely determined by  $F_H^{con}$ .

**Lemma 2.3.** Let  $u_1$  and  $u_2$  be two minimizers of  $F_H^{con}$ . For any  $e_k \in E$ , we have  $\max_{x_j \in e_k} u_1(x_j) = \max_{x_j \in e_k} u_2(x_j) \quad and \quad \min_{x_j \in e_k} u_1(x_j) = \min_{x_j \in e_k} u_2(x_j).$ (5)

*Proof.* Define  $u = \frac{u_1 + u_2}{2}$ . For any  $e_k \in E$ , there exist  $x_{k,i}, x_{k,j} \in V$ , such that

$$\begin{split} \max_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^p &= |u(x_{k,i}) - u(x_{k,j})|^p \\ &\leq \frac{1}{2} |u_1(x_{k,i}) - u_1(x_{k,j})|^p + \frac{1}{2} |u_2(x_{k,i}) - u_2(x_{k,j})|^p \\ &\leq \frac{1}{2} \max_{x_i, x_j \in e_k} |u_1(x_i) - u_1(x_j)|^p + \frac{1}{2} \max_{x_i, x_j \in e_k} |u_2(x_i) - u_2(x_j)|^p. \end{split}$$

If any one of the above inequalities is strict, we have

$$F_H(u) < \frac{1}{2} (F_H(u_1) + F_H(u_2)) = F_H(u_1),$$

which contradicts the assumption that  $u_1$  is a minimizer of  $F_H^{con}$ . Consequently, by the strict convexity of  $|\cdot|^p$ ,

$$\max_{x_i, x_j \in e_k} |u_1(x_i) - u_1(x_j)|^p = \max_{x_i, x_j \in e_k} |u_2(x_i) - u_2(x_j)|^p,$$

for any  $e_k \in E$ . This yields

x

$$\max_{i,x_j \in e_k} \left( u_1(x_i) - u_1(x_j) \right) = \max_{x_i, x_j \in e_k} \left( u_2(x_i) - u_2(x_j) \right)$$

and there exists a constant  $C_k$ , such that

$$\max_{x_j \in e_k} u_1(x_j) = \max_{x_j \in e_k} u_2(x_j) + C_k \text{ and } \min_{x_j \in e_k} u_1(x_j) = \min_{x_j \in e_k} u_2(x_j) + C_k.$$

It is not difficult to see from the fact  $u_1(x_i) = u_2(x_i) = y_i$  for  $x_i \in L$  that  $C_k = 0$ . This proves (5).

**Proposition 2.4.** If  $u_1$  and  $u_2$  are two minimizers of  $F_H^{con}$ , then  $D(u_1) = D(u_2)$ and

$$u_1(x_i) = u_2(x_i),$$
 (6)

for any  $x_i \in D := D(u_1) = D(u_2)$ .

The proposition implies that D is uniquely determined by  $F_H^{con}$ . The nonuniqueness of minimizers for  $F_H^{con}$  comes from  $D^c := V \setminus D$ .

*Proof.* Let 
$$x_i \in D(u_1) \setminus L$$
. By (4), we assume w.l.o.g. that  $x_i \in e_k \cap e_l$  and

$$u_1(x_i) = \max_{x_j \in e_k} u_1(x_j) = \min_{x_j \in e_l} u_1(x_j),$$
(7)

for two hyperedges  $e_k, e_l \in E$ . By (5), to prove (6), we only need to show that

$$u_2(x_i) = \max_{x_j \in e_k} u_2(x_j),$$

which also implies that  $x_i \in D(u_2)$  and proves that  $D(u_1) = D(u_2)$ .

If this is not true, i.e.,

$$u_2(x_i) < \max_{x_j \in e_k} u_2(x_j),$$

it follows from (5) and (7) that

$$u_2(x_i) < \max_{x_j \in e_k} u_2(x_j) = \max_{x_j \in e_k} u_1(x_j) = u_1(x_i) = \min_{x_j \in e_l} u_1(x_j) = \min_{x_j \in e_l} u_2(x_j).$$

This contradicts the assumption that  $x_i \in e_l$  and finishes the proof.

2.2. The subdifferential of  $F_H^{con}$  and the hypergraph *p*-Laplacian equation. Functional  $F_H$  and  $F_H^{con}$  are non-differentiable. We consider the subdifferential for them.

Let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ . For a proper, convex, and lower semi-continuous functional  $J : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$  with effective domain

$$\operatorname{dom}(J) = \{ u \in \mathcal{H} : J(u) < \infty \},\$$

the subdifferential of J at  $u \in \text{dom}(J)$  is defined as

$$\partial J(u) = \{ q \in \mathcal{H} : \langle q, v - u \rangle \le J(v) - J(u), \forall v \in \mathcal{H} \}$$

An element of  $\partial J(u)$  is called a subgradient of J at u. The subgradient coincides with the usual gradient if J is differentiable. We shall use the following proposition of the subdifferential

$$u$$
 is a minimizer of  $J \iff 0 \in \partial J(u)$ , (8)

whose proof is trivial.

The subdifferential of  $F_H$  has been obtained in [11]. More precisely,

$$\partial F_H(u) = p \left\{ \sum_{k=1}^m w_k \max_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^{p-1} b_k, \ b_k \in \arg\max_{b \in B_k} \langle b, u \rangle \right\}, \tag{9}$$

where

$$B_k = \operatorname{conv}\{\mathbb{I}_{x_i} - \mathbb{I}_{x_j} : x_i, x_j \in e_k\},\$$

 $\mathbb{I}_{x_i} \in \mathbb{R}^n$  is an indicator function

$$\mathbb{I}_{x_i}(x_j) = \begin{cases} 1, & \text{if } x_j = x_i, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\operatorname{conv}{S}$  denotes the convex hull of S in  $\mathbb{R}^n$ .

The subdifferential for the constraint functional  $\partial F_H^{con}$  is a corollary of the definition and (9). For any

$$u \in \operatorname{dom}(F_H^{con}) = \{ v \in \mathbb{R}^n : v(x_i) = y_i, x_i \in L \},\$$

we have

$$\partial F_H^{con}(u) = \{ q \in \mathbb{R}^n : q(x_i) = h(x_i), x_i \in V \setminus L \text{ for any } h \in \partial F_H(u) \}.$$
(10)

Namely, a subgradient of  $F_H^{con}(u)$  comes from a subgradient of  $F_H(u)$  by taking arbitrary values at labeled vertices.

By combining the above results (8)–(10), we deduce an equivalent form for the minimizer of  $F_H^{con}$ . More precisely, if u is a minimizer of  $F_H^{con}$ , there exist vectors  $\beta_k \in \arg \max_{b \in B_k} \langle b, u \rangle$  such that

$$\begin{cases} \left(\sum_{k=1}^{m} w_k \max_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^{p-1} \beta_k\right) (x_i) = 0, \quad x_i \in V \setminus L, \\ u(x_i) = y_i, \quad x_i \in L. \end{cases}$$
(11)

Conversely, if a function u satisfies equation (11), where  $\beta_k \in \arg \max_{b \in B_k} \langle b, u \rangle$ , then it is a minimizer of  $F_H^{con}$ .

In the rest of this subsection, we propose a new hypergraph equation based on equation (11). The basic idea is to consider  $|\beta_k(x_i)|$  as a diffusion coefficient that represents the contribution of hyperedge  $e_k$  to vertex  $x_i$ . The proposed equation reads

$$\begin{cases} \sum_{k=1}^{m} w_k \alpha_k(x_i) \left| \max_{x_j \in e_k \cap D} u(x_j) + \min_{x_j \in e_k \cap D} u(x_j) - 2u(x_i) \right|^{p-2} \\ \left( \max_{x_j \in e_k \cap D} u(x_j) + \min_{x_j \in e_k \cap D} u(x_j) - 2u(x_i) \right) = 0, \quad x_i \in D \setminus L, \\ u(x_i) = y_i, \quad x_i \in L, \end{cases}$$
(12)

where

$$\alpha_k(x_i) = \begin{cases} |\beta_k(x_i)|, & \text{if } x_i \in e_k, \\ 0, & \text{if } x_i \notin e_k, \end{cases}$$

,

for  $e_k \in E$  and  $x_i \in D$ . Here we restrict the equation on the subhypergraph  $\tilde{H} = (D, \tilde{E}, W)$ , where  $\tilde{E} = \{e_k \cap D\}_{k=1}^m$ , to avoid the non-uniqueness. The notation  $|0|^{p-2}0 = 0$  is used for the case 1 .

Owing to the following theorem, we call equation (12) the hypergraph *p*-Laplacian equation.

**Theorem 2.5.** Let u be a minimizer of  $F_H^{con}$ . Then  $u|_D$  is a solution of equation (12).

*Proof.* We redefine u in  $D^c$  such that for any  $e_k \in E$ ,

$$\min_{x_i \in e_k} u(x_j) < u(x_i) < \max_{x_i \in e_k} u(x_j), \quad x_i \in e_k \cap D^c.$$

By the assumption, u and  $\beta_k$  satisfy equation (11). Notice that for any  $e_k \in E$  and  $x_i \in D^c$ ,

$$\beta_k(x_i) = 0.$$

Namely, equation (11) is trivial for  $x_i \in D^c$ . Let  $x_i \in D \setminus L$  and  $e_k \in E$ . If  $\beta_k(x_i) > 0$ , we have

$$x_i \in \arg\max_{x_j \in e_k} u(x_j),$$

and

$$-\max_{x_i, x_j \in e_k} |u(x_i) - u(x_j)|^{p-1} \beta_k(x_i)$$
  
=  $\alpha_k(x_i) \left| \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2u(x_i) \right|^{p-2} \left( \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2u(x_i) \right)$ 

The same conclusion can be obtained for the cases  $\beta_k(x_i) < 0$  and  $\beta_k(x_i) = 0$ . Notice that

$$\max_{x_j \in e_k} u(x_j) = \max_{x_j \in e_k \cap D} u(x_j), \quad \min_{x_j \in e_k} u(x_j) = \min_{x_j \in e_k \cap D} u(x_j).$$

This means that  $u(x_i)$  satisfies equation (12) and finishes the proof.

Conversely, if v is a solution of (12), it is not difficult to verify by reversing the proof of Theorem 2.5 that v is also a minimizer of  $F_{\tilde{H}}^{con}$ . Then a unique minimizer of  $F_{H}^{con}$  can be determined, e.g.,

$$u(x_i) = \begin{cases} v(x_i), & \text{if } x_i \in D, \\ \frac{1}{2} \left( \max_{1 \le j \le l} \min_{x_i \in e_{k_j} \cap D} v(x_i) + \min_{1 \le j \le l} \max_{x_i \in e_{k_j} \cap D} v(x_i) \right), & \text{if } x_i \in D^c. \end{cases}$$

Here we use the notation for  $x_i \in D^c$  that  $|x_i| = l$  (i.e., the degree of  $x_i$  is l) and  $x_i \in e_{k_1}, \dots, e_{k_l}$ .

Although a minimizer of  $F_H^{con}$  can be uniquely determined through equation (12). The equation itself is not solvable numerically. Indeed, both the domain D and the diffusion coefficient  $\alpha_k(x_i)$  depend on the structure of the hypergraph and the training set and thus have no general expression.

2.3. A simplified hypergraph *p*-Laplacian equation. The purpose of this subsection is to present a simplified version of equation (12) that does not involve *D* and  $\alpha_k(x_i)$ . To this end, we consider the homogeneous coefficient  $\alpha_k(x_i) \equiv 1$  and the whole domain *V*. The new equation is as follows

$$\begin{cases} L_{H}^{p}u := \sum_{k=1}^{m} w_{k}\chi_{k}(x_{i}) \left| \max_{x_{j} \in e_{k}} u(x_{j}) + \min_{x_{j} \in e_{k}} u(x_{j}) - 2u(x_{i}) \right|^{p-2} \\ \left( \max_{x_{j} \in e_{k}} u(x_{j}) + \min_{x_{j} \in e_{k}} u(x_{j}) - 2u(x_{i}) \right) = 0, \quad x_{i} \in V \setminus L, \\ u(x_{i}) = y_{i}, \quad x_{i} \in L, \end{cases}$$
(13)

where

$$\chi_k(x_i) = \begin{cases} 1, & \text{if } x_i \in e_k, \\ 0, & \text{if } x_i \notin e_k, \end{cases}$$

for  $e_k \in E$  and  $x_i \in V$ .

In general, a solution of equation (13) is no longer a solution of equation (12) (when restricting to D) and is not a minimizer of  $F_H^{con}$ . Figure 1 shows an example of a hypergraph and a function u on it that minimizes  $F_H^{con}$ . Clearly, u is not a solution of equation (13) since in this case  $\alpha_k(x_i) \neq \chi_k(x_i)$ . Nevertheless, there exist specific instances where a solution of equation (13) and a minimizer of  $F_{H}^{con}$  coincide, as demonstrated in Figure 2. For this reason, we still refer to equation (13) as the hypergraph p-Laplacian equation. The theoretical study of the connection between the discrete equation (13) and the classical p-Laplacian equation will be part of our future work.

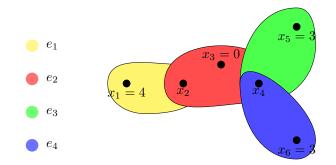


FIGURE 1. A hypergraph with 6 vertices and 4 hyperedges. Let  $p = 2, w_k = 1, k = 1, \dots, 4$ , and  $x_1, x_3, x_5, x_6 \in L$  be labeled vertices. Then  $u = (4, \frac{5}{2}, 0, \frac{5}{2}, 3, 3)^T$  is a minimizer of  $F_H^{con}$  and  $\beta_2(x_2) = \frac{3}{5}, \beta_2(x_4) = \frac{2}{5}$ .

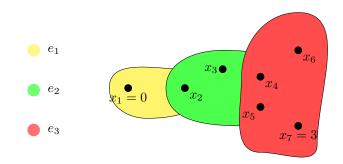


FIGURE 2. A hypergraph with 7 vertices and 3 hyperedges. Let  $p = 2, w_k = 1, k = 1, 2, 3, \text{ and } x_1, x_7 \in L$  be labeled vertices. Then  $u = (0, 1, \frac{3}{2}, 2, 2, \frac{5}{2}, 3)^T$  is both a solution of equation (13) and a minimizer of  $F_H^{con}$ .

The comparison principle and the unique solvability of equation (13) are stated as follows.

**Theorem 2.6** (Comparison principle). If  $u_1, u_2 : V \to \mathbb{R}$  are two functions that satisfy

$$L_{H}^{p}u(x_{j}) = 0, \quad for \; \forall x_{j} \in V \setminus L,$$

and

 $u_1(x_j) \le u_2(x_j), \quad for \ \forall x_j \in L,$ 

then

$$u_1(x_j) \le u_2(x_j), \quad for \ \forall x_j \in V.$$

*Proof.* Assume to the contrary. We claim that there exist a hyperedge  $e_{k_1}$  and vertices  $x_i, x_j \in e_{k_1}$  such that

$$u_1(x_i) - u_2(x_i) = \max_{x \in V} (u_1(x) - u_2(x)) =: c > 0,$$

and

$$u_1(x_j) - u_2(x_j) < c. (14)$$

Otherwise, by the connectivity assumption of the hypergraph H,  $u_1 - u_2 = c$  on every hyperedge, which is a contradiction.

Assume w.l.o.g. that  $|x_i| = l$  and  $x_i \in e_{k_1}, \cdots, e_{k_l}$ . Then we have

 $u_1(x_i) - u_2(x_i) \ge u_1(x) - u_2(x), \quad \forall x \in e_{k_1}, \cdots, e_{k_l}.$ 

Equivalently,

$$u_2(x) - u_2(x_i) \ge u_1(x) - u_1(x_i), \quad \forall x \in e_{k_1}, \cdots, e_{k_l}.$$

Taking the maximum and the minimum for the above inequality respectively and combining the results lead to

$$\max_{x \in e} u_2(x) + \min_{x \in e} u_2(x) - 2u_2(x_i) \ge \max_{x \in e} u_1(x) + \min_{x \in e} u_1(x) - 2u_1(x_i),$$
(15)

for any  $e = e_{k_1}, \dots, e_{k_l}$ . It follows from (14) that the above inequality is strict for hyperedge  $e = e_{k_1}$ . In fact, by

$$\min_{x \in e_{k_1}} u_1(x) - \min_{x \in e_{k_1}} u_2(x) \le \min_{x \in e_{k_1}} (u_1(x) - u_2(x)) \le u_1(x_j) - u_2(x_j) < c$$
$$= u_1(x_i) - u_2(x_i),$$

we have

$$\min_{x \in e_{k_1}} u_2(x) - u_2(x_i) > \min_{x \in e_{k_1}} u_1(x) - u_1(x_i),$$

and consequently,

$$\max_{x \in e_{k_1}} u_2(x) + \min_{x \in e_{k_1}} u_2(x) - 2u_2(x_i) > \max_{x \in e_{k_1}} u_1(x) + \min_{x \in e_{k_1}} u_1(x) - 2u_1(x_i).$$

This together with (15) and the monotonicity of  $|s|^{p-2}s$  imply that

$$L^p_H u_2(x_i) > L^p_H u_1(x_i),$$

which is a contradiction to the assumption.

**Theorem 2.7.** Equation (13) admits a unique solution u that satisfies the estimate

$$\min y_i \le u(x_i) \le \max y_i, \quad x_i \in V.$$

*Proof.* The uniqueness of solutions is a corollary of the comparison principle. We prove the existence of a solution in the following by the Brouwer fixed-point theorem. It can also be proven by Perron's method.

Let

$$X = \{ u \in \mathbb{R}^n : u(x_i) = y_i, x_i \in L \text{ and } \min y_i \le u(x_i) \le \max y_i, x_i \in V \}$$

be a closed and convex subset of  $\mathbb{R}^n$ . For a  $u \in X$ , we consider the auxiliary equation

$$\begin{cases} \sum_{k=1}^{m} w_k \chi_k(x_i) \left| \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2v(x_i) \right|^{p-2} \\ \left( \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2v(x_i) \right) = 0, \quad x_i \in V \setminus L, \\ v(x_i) = y_i, \quad x_i \in L. \end{cases}$$
(16)

Recall that under the notation  $|0|^{p-2}0 = 0$  for  $1 , <math>|s|^{p-2}s$  is continuous and monotone on  $\mathbb{R}$  for p > 1. Consequently, for any  $x_i \in V \setminus L$ , the left-hand side of equation (16) is continuous and monotone with respective to  $v(x_i)$ . Then a zero point exists and equation (16) admits a unique solution  $v : V \to \mathbb{R}$ .

 $\Box$ 

By rewriting equation (16) as

$$v(x_{i}) = \begin{cases} \frac{1}{\sum\limits_{k=1}^{m} w_{k}\chi_{k}(x_{i})g_{k}(u,v(x_{i}))} \sum\limits_{k=1}^{m} w_{k}\chi_{k}(x_{i})g_{k}(u,v(x_{i})) \left(\frac{\max_{x_{j}\in e_{k}} u(x_{j}) + \min_{x_{j}\in e_{k}} u(x_{j})}{2}\right), \\ & \text{if } x_{i} \in V \backslash L, \ \sum\limits_{k=1}^{m} g_{k}(u,v(x_{i})) \neq 0, \\ \sum\limits_{k=1}^{m} \chi_{k}(x_{i})\frac{\max_{x_{j}\in e_{k}} u(x_{j}) + \min_{x_{j}\in e_{k}} u(x_{j})}{2}, & \text{if } x_{i} \in V \backslash L, \ \sum\limits_{k=1}^{m} g_{k}(u,v(x_{i})) = 0, \\ y_{i}, \quad x_{i} \in L, \end{cases}$$

where

$$g_k(u, v(x_i)) = \begin{cases} 0, & \text{if } \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2v(x_i) = 0, \\ \left| \max_{x_j \in e_k} u(x_j) + \min_{x_j \in e_k} u(x_j) - 2v(x_i) \right|^{p-2}, & \text{otherwise,} \end{cases}$$

we further have  $v \in X$ .

Now we define a mapping  $T: X \to X$  by T(u) = v. It is continuous and admits a fixed point u, which is also a solution of equation (13).

The superiority of equation (13) over equation (12) and the functional  $F_H^{con}$  lies in the computational efficiency. It can be solved with fixed-point iteration

$$u^{k+1}(x_i) = u^k(x_i) + \tau \left( \sum_{k=1}^m w_k \chi_k(x_i) \left| \max_{x_j \in e_k} u^k(x_j) + \min_{x_j \in e_k} u^k(x_j) - 2u^k(x_i) \right|^{p-2} \right) \\ \left( \max_{x_j \in e_k} u^k(x_j) + \min_{x_j \in e_k} u^k(x_j) - 2u^k(x_i) \right) \right),$$

for  $x_i \in V \setminus L$ , where  $u^0$  is any initial guess of the solution and  $\tau$  is the step size. The Dirichlet boundary condition  $u(x_i) = y_i, x_i \in L$  is posed at each step. In the case p = 2, we can further drop the step size by iterating

$$u^{k+1}(x_{i}) = \begin{cases} \frac{1}{2\sum_{k=1}^{m} w_{k}\chi_{k}(x_{i})} \sum_{k=1}^{m} w_{k}\chi_{k}(x_{i}) \left(\max_{x_{j} \in e_{k}} u^{k}(x_{j}) + \min_{x_{j} \in e_{k}} u^{k}(x_{j})\right), & \text{if } x_{i} \in V \setminus L, \\ y_{i}, & \text{if } x_{i} \in L. \end{cases}$$
(17)

If

$$u_0 = \begin{cases} \min y_i, & \text{if } x_i \in V \setminus L, \\ y_i, & \text{if } x_i \in L, \end{cases} \quad \left( \text{or } u_0 = \begin{cases} \max y_i, & \text{if } x_i \in V \setminus L, \\ y_i, & \text{if } x_i \in L, \end{cases} \right)$$

the scheme is bounded and monotone. Namely,  $\min y_i \leq u^k(x_i) \leq \max y_i$  and  $u^{k+1}(x_i) \geq u^k(x_i)$  (or  $u^{k+1}(x_i) \leq u^k(x_i)$ ) for any  $k \geq 0$  and  $x_i \in V$ . The convergence of (17) follows.

## 3. Numerical experiments

In this section, we discuss the numerical performance of the proposed simplified hypergraph *p*-Laplacian equation (13) for data interpolation and semi-supervised learning. We focus on the case p = 2, which is commonly used in practice. All

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experiments are performed using MATLAB on a mini PC equipped with an Intel Core is 2.0 GHz CPU.

3.1. Data interpolation in 1D. Let n = 1280 and  $V = \{x_i\}_{i=1}^n$  be random numbers on the interval [0, 1] that follows the standard uniform distribution. We assume that 6 of the points are labeled (denoted by red circles in Figure 3). The goal is to interpolate the remaining points.

A k-nearest neighbor graph  $G_k$  with vertex set V can be constructed. For any vertices  $x_i, x_j \in V$ , we connect them by an edge  $e_{i,j}$  if  $x_j$  is among the k-nearest neighbors of  $x_i$ , denoted by  $x_j \stackrel{k}{\sim} x_i$ . We also connect them if  $x_i \stackrel{k}{\sim} x_j$  for the sake of symmetry. The constant weight  $w_{i,j} = 1$  is adopted for edge  $e_{i,j}$ . Alternatively, we can also construct a k-nearest neighbor hypergraph  $H_k$  with the vertex set V. For every vertex  $x_i \in V$ , we define a hyperedge

$$e_i = \{ x_j \in V : x_j \stackrel{k}{\sim} x_i \}.$$

Weight  $w_i = 1$  is assigned for every hyperedge  $e_i, i = 1, \dots, n$ .

The interpolation problem becomes the semi-supervised learning on  $G_k$  or  $H_k$ , which can be solved by equation (13) (i.e., iteration scheme (17)). We also consider the graph *p*-Laplacian  $F_G^{con}$ , the hypergraph *p*-Laplacian  $F_{CE}^{con}$  and  $F_H^{con}$  for comparison, see (1)–(3) for their definitions.  $F_G^{con}$  is solved by the algorithm of [18]. We utilize the gradient descent scheme for  $F_{CE}^{con}$ , which is as follows

$$u^{k+1}(x_i) = \sum_{k=1}^{m} w_k \chi_k(x_i) \sum_{x_j \in e_k} \left( u^k(x_j) - u^k(x_i) \right),$$
(18)

for  $x_i \in V \setminus L$ .  $F_H^{con}$  is solved by the stochastic PDHG [13].

To compare the interpolation result and the computation time of different algorithms, we run four algorithms for a sufficiently long time to obtain the "true solutions"  $u^*$  (shown in Figure 3). The running time of hypergraph models with respect to the relative  $\ell^2$  error

$$\frac{\|u-u^*\|_{\ell^2}}{\|u^*\|_{\ell^2}}$$

is then plotted in Figure 4.

As illustrated in Figure 3, all four algorithms effectively interpolate the data when k = 9. However, as k increases, notable differences emerge.  $F_G^{con}$  develops spikes at the labeled points.  $F_{CE}^{con}$  exhibits similar spiking behavior as it is essentially the graph Laplacian. In contrast,  $F_H^{con}$  effectively suppresses spiky solutions. As an approximation of  $F_H^{con}$ , equation (13) produces solutions with a similar structure. The difference between the two is that  $F_H^{con}$  gives better interpolation results near the labeled points (see the 3rd and 4th labeled points), while equation (13) provides smoother results (see the case k = 72).

Figure 4 shows the computational costs of different algorithms. Equation (13) outperforms  $F_H^{con}$  by a large margin and is even better than  $F_{CE}^{con}$ . Notably, its running time decreases as the parameter k increases. This is due to the fact that a larger k increases the cardinality of vertices, which in turn accelerates the convergence of equation (13). The running time for large k = 72 is comparable to that of the graph model  $F_G^{con}$  with a recent algorithm [18] ( $\approx 0.15s$ ).

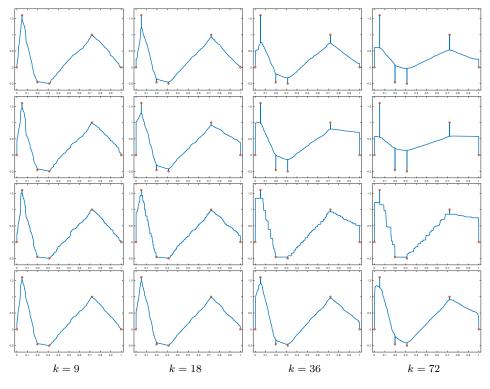


FIGURE 3. Results of graph Laplacian and hypergraph Laplacian for different k. From top to bottom:  $F_G^{con}$ ,  $F_{CE}^{con}$ ,  $F_H^{con}$ , equation (13).

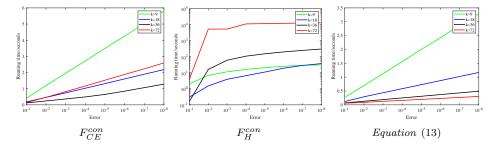


FIGURE 4. The running time of different methods with respect to the relative  $\ell_2$  error.

3.2. Semi-supervised learning. In the rest of this section, we consider the performance of  $F_H^{con}$  and equation (13) for semi-supervised learning on some real-world datasets summarized in Table 1. Both Mushroom and Covertype come from the UCI repository. Covertype(4,5) and Covertype(6,7) are derived from Covertype by selecting two classes, (4,5) and (6,7) respectively. They are provided by the first author of [10]. All three datasets contain only categorical features. We construct a hypergraph H = (V, E, W) from a given dataset as follows. For each feature and each category of the feature, we construct a hyperedge  $e \in E$  by joining all vertices

Dataset	Mushroom	Covertype (4,5)	Covertype (6,7)		Cora co-citation	Pubmed
V	8124	12240	37877	2708	2708	3312
E	110	104	123	1072	1579	1079
#labels	2	2	2	7	7	6

TABLE 1. Datasets for semi-supervised learning.

that belong to the same feature and category. duplicated hyperedges are removed. Cora and Pubmed [19] are hypergraph datasets, in which vertices are documents and hyperedges follow from co-authorship or co-citation. Weight  $w_k = 1$  is used for all hyperedges in all datasets.

Algorithm 1 provides the detail of semi-supervised learning on hypergraphs with  $F_H^{con}$  and equation (13).

## Algorithm 1 Semi-supervised learning with hypergraph *p*-Laplacian.

**Require:** A dataset represented by a hypergraph H = (V, E, W), a training set  $\{(x_i, y_i) : x_i \in L \subset V, y_i \in \{1, 2, \dots l\}\}.$ 

for k = 1: l do

Find the constraint: For any  $x_i \in L$ , let

$$u_k(x_i) = \begin{cases} 1, & \text{if } y_i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Interpolate  $u_k$  on  $V \setminus L$  by minimizing  $F_H^{con}$  or solving equation (13). end for

For any  $x_i \in V \setminus L$ , let

$$y_i = \arg \max_{1 \le k \le l} u_k(x_i), \quad u(x_i) = y_i.$$

#### return u.

The stochastic PDHG algorithm is not suitable for  $F_H^{con}$  when hyperedges have large cardinality (see Figure 4). We adopt the subgradient descent algorithm proposed in [16] for solving  $F_H^{con}$ . The step size is empirically chosen as

$$\tau(t) = \frac{1}{(t+1)^{\min(0.16t/10^5,1)}} \tag{19}$$

to speed up the algorithm, as suggested by the authors. In this setting, it is not easy to find a robust stopping criterion for the algorithm since it is not convergent in general. This is also the reason that we do not utilize it in the previous subsection. We manually select the smallest iteration number within [0, 5000] that reaches the minimum classification error for each dataset and labeling rate. While our simplified *p*-Laplacian equation is easy to implement. A simple stopping criterion like  $\max(u^{k-1}, u^k) \leq \varepsilon$  is stable and gives expected classification results. We choose  $\varepsilon = 10^{-4}$  for datasets Mushroom and Covertype and choose  $\varepsilon = 10^{-2}$  for datasets Cora and Pubmed.

TABLE 2. Classification error and standard deviation of  $F_H^{con}$  and equation (13).

L	20	40	60	80	100	120	140	160			
	Mushroom										
$\begin{array}{c} F_H^{con} \\ \text{E.}(13) \end{array}$	$15.8 \pm 4.2$ $8.4 \pm 2.8$	$7.6{\pm}2.1$ $5.9{\pm}2.8$		$5.1{\pm}2.6$ $4.3{\pm}1.8$			-				
Covertype(4,5)											
$\begin{array}{c} F_H^{con} \\ \text{E.}(13) \end{array}$	$18 \pm 3.1 \\ 6.5 \pm 5.7$	$14.1{\pm}2.8$ $3.6{\pm}3.1$						${}^{1.5\pm1.2}_{0.01\pm0.01}$			
Covertype(6,7)											
$\begin{array}{c} F_{H}^{con} \\ \mathrm{E.}(13) \end{array}$	$30.6 \pm 7.5 \\ 7.9 \pm 5.6$	$15.3 {\pm} 6.6$ $1.5 {\pm} 0.5$		$4.4{\pm}5.8$ $0.6{\pm}0.3$							
L	5%	)	10%	15	%	20%		25%			
		Cora co-authorship									
$F_H^{con}$ E.(13)		$8 \pm 1.4 \\ 8 \pm 1.5$	$41.6\pm0.$ $41.5\pm0.$		$0.0\pm 1.0$ $0.5\pm 0.8$	$32.8 \pm 32.1 \pm$		$30.0\pm0.8$ 29.2 $\pm0.9$			
		Cora co-citation									
$\begin{array}{c} F_{H}^{con} \\ \text{E.}(13) \end{array}$		$0\pm 1.4 \\ 7\pm 2.1$	31.0±1. 30.3±1.		$7.8 \pm 0.7$ $7.6 \pm 0.8$	$25.3 \pm 25.0 \pm$		$23.0\pm0.9$ $22.8\pm0.9$			
		Pubmed									
$\begin{array}{c} F_{H}^{con} \\ \text{E.}(13) \end{array}$		$6\pm 0.4 \\ 5\pm 0.6$	41.7±0. 41.4±0.		$0.0\pm0.2$ $0.8\pm0.2$	$36.6 \pm 36.3 \pm$	-	$34.2 \pm 0.2$ $34.0 \pm 0.1$			

To compare the classification accuracy of  $F_H^{con}$  and equation (13), we randomly select  $\{20, 40, \dots, 160\}$  points from the datasets Mushroom and Covertype and select  $\{5\%, 10\%, \dots, 25\%\}$  points from the datasets Cora and Pubmed as the training sets and run the algorithms for 10 times. The classification error and the standard deviation are summarized in Table 2. Surprisingly, equation (13) achieves better classification accuracy, even though it comes from an approximation of  $F_H^{con}$  and cannot suppress spiky solutions as well as  $F_H^{con}$ .

The main contribution of equation (13) lies in the computational efficiency and the stability. Figure 5 shows the average running time of two algorithms over 10 runs. For datasets Mushroom and Covertype, the computation time is greatly reduced by equation (13). While for datasets Cora and Pubmed, equation (13) is no longer favorable. Two facts should be noticed. Equation (13) requires more computational time for datasets Cora and Pubmed as they contain more hyperedges. The subgradient descent algorithm for solving  $F_H^{con}$  does not converge in general with the given step size (19), leading us to manually select the best step size. Whereas for Cora and Pubmed, it requires only a few hundred iterations to get the best classification error. This method is clearly inapplicable to practical problems where the real dataset is unknown. The numerical scheme (17) is convergent and does not involve any parameters, thus avoiding this problem.

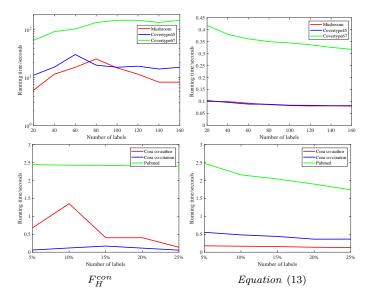


FIGURE 5. The average running time of  $F_H^{con}$  and equation (13).

# 4. Conclusion

In this paper, a new hypergraph *p*-Laplacian equation, deduced from an approximation of the hypergraph *p*-Laplacian regularization, has been proposed for semi-supervised learning. The unique solvability and the comparison principle of the equation have been established. Numerical experiments have confirmed the effectiveness of the new equation. It not only suppresses spiky solutions and improves classification accuracy but also significantly reduces computation time.

### Acknowledgements

KS is supported by China Scholarship Council. The authors acknowledge support from DESY (Hamburg, Germany), a member of the Helmholtz Association HGF.

#### DECLARATIONS

Data Availability Data will be made available on request.

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

#### References

- Songyang Zhang, Shuguang Cui, and Zhi Ding. Hypergraph-based image processing. In 2020 IEEE International Conference on Image Processing (ICIP), pages 216–220. IEEE, 2020.
- [2] Kehan Shi and Martin Burger. Continuum limit of p-biharmonic equations on graphs. arXiv preprint arXiv:2404.19689, 2024.
- [3] Steffen Klamt, Utz-Uwe Haus, and Fabian Theis. Hypergraphs and cellular networks. PLoS computational biology, 5(5):e1000385, 2009.

- [4] Rob Patro and Carl Kingsford. Predicting protein interactions via parsimonious network history inference. *Bioinformatics*, 29(13):i237–i246, 2013.
- [5] Alessia Antelmi, Gennaro Cordasco, Carmine Spagnuolo, and Przemysław Szufel. Social influence maximization in hypergraphs. *Entropy*, 23(7):796, 2021.
- [6] Ariane Fazeny, Daniel Tenbrinck, and Martin Burger. Hypergraph p-laplacians, scale spaces, and information flow in networks. In *International Conference on Scale Space and Variational Methods in Computer Vision*, pages 677–690. Springer, 2023.
- [7] Dengyong Zhou, Jiayuan Huang, and Bernhard Schölkopf. Learning with hypergraphs: Clustering, classification, and embedding. Advances in neural information processing systems, 19, 2006.
- [8] Sameer Agarwal, Kristin Branson, and Serge Belongie. Higher order learning with graphs. In Proceedings of the 23rd international conference on Machine learning, pages 17–24, 2006.
- [9] Ahmed El Alaoui, Xiang Cheng, Aaditya Ramdas, Martin J Wainwright, and Michael I Jordan. Asymptotic behavior of \ell\_p-based laplacian regularization in semi-supervised learning. In Conference on Learning Theory, pages 879–906. PMLR, 2016.
- [10] Matthias Hein, Simon Setzer, Leonardo Jost, and Syama Sundar Rangapuram. The total variation on hypergraphs-learning on hypergraphs revisited. Advances in Neural Information Processing Systems, 26, 2013.
- [11] Masahiro Ikeda and Shun Uchida. Nonlinear evolution equation associated with hypergraph laplacian. Mathematical Methods in the Applied Sciences, 46(8):9463–9476, 2023.
- [12] Takeshi Fukao, Masahiro Ikeda, and Shun Uchida. Heat equation on the hypergraph containing vertices with given data. arXiv preprint arXiv:2212.05446, 2022.
- [13] Kehan Shi and Martin Burger. Hypergraph p-laplacian regularization on point clouds for data interpolation. arXiv preprint arXiv:2405.01109, 2024.
- [14] Antonin Chambolle and Thomas Pock. A first-order primal-dual algorithm for convex problems with applications to imaging. *Journal of mathematical imaging and vision*, 40:120–145, 2011.
- [15] Antonin Chambolle, Matthias J Ehrhardt, Peter Richtárik, and Carola-Bibiane Schonlieb. Stochastic primal-dual hybrid gradient algorithm with arbitrary sampling and imaging applications. SIAM Journal on Optimization, 28(4):2783–2808, 2018.
- [16] Chenzi Zhang, Shuguang Hu, Zhihao Gavin Tang, and TH Hubert Chan. Re-revisiting learning on hypergraphs: confidence interval and subgradient method. In *International Conference* on Machine Learning, pages 4026–4034. PMLR, 2017.
- [17] Naum Zuselevich Shor. Minimization methods for non-differentiable functions, volume 3. Springer Science & Business Media, 2012.
- [18] Mauricio Flores, Jeff Calder, and Gilad Lerman. Analysis and algorithms for lp-based semisupervised learning on graphs. Applied and Computational Harmonic Analysis, 60:77–122, 2022.
- [19] Prithviraj Sen, Galileo Namata, Mustafa Bilgic, Lise Getoor, Brian Galligher, and Tina Eliassi-Rad. Collective classification in network data. AI magazine, 29(3):93–93, 2008.