# Weak pseudo-inverses and the associativity of two-place functions generated by left continuous monotone functions<sup>∗</sup>

Meng Chen<sup>†</sup>, Xue-ping Wang<sup>†</sup>

School of Mathematical Sciences, Sichuan Normal University, Chengdu 610066, Sichuan, People's Republic of China

Abstract This article introduces a weak pseudo-inverse of a monotone function, which is applied to prove that the associativity of a two-place function  $T : [0,1]^2 \rightarrow [0,1]$  defined by  $T(x,y) = t^{[-1]}(F(t(x), t(y)))$  where  $F : [0, \infty]^2 \to [0, \infty]$  is an associative function with neutral element in  $[0, \infty], t : [0, 1] \to [0, \infty]$  is a left continuous monotone function and  $t^{[-1]} : [0, \infty] \to [0, 1]$  is the weak pseudo-inverse of t depends only on properties of the range of t.

Keywords: Monotone function; Weak pseudo-inverse; Left continuous function; Associative function; Triangular norm

# 1 Introduction

In 1826, Abel [\[1\]](#page-15-0) obtained the easily checking result: Let  $t : Dom(t) \to R$  with  $Dom(t) \subseteq R$  be a continuous strictly monotone function whose range is closed under addition. Then the two-place function  $T: (Dom(t))^2 \to Dom(t)$  defined by

$$
T(x,y) = g(t(x) + t(y)),\tag{1}
$$

where  $Dom(t)$  is the domain of t,  $g: Dom(t) \to Dom(t)$  is the inverse function of t and R is the set of all real numbers, is associative. This result can be seen as the starting point of constructing a two-place real function that has nice algebraic properties through a monotone one-place real function. Following this idea, Schweizer and Sklar [\[5\]](#page-15-1) and Ling [\[4\]](#page-15-2) constructed triangular norms (t-norms for short) by continuous strictly decreasing functions, respectively. In particular, Klement, Mesiar and Pap [\[3\]](#page-15-3) defined an additive generator of a t-norm T as a strictly decreasing function  $t : [0, 1] \rightarrow [0, \infty]$  that is right continuous at 0 with  $t(1) = 0$  such that for all  $(x, y) \in [0, 1]^2$ ,

$$
t(x) + t(y) \in \text{Ran}(t) \cup [f(0), \infty],\tag{2}
$$

<span id="page-0-0"></span>and

$$
T(x,y) = g(t(x) + t(y))
$$
\n<sup>(3)</sup>

where g is a pseudo-inverse of t and Ran(t) is a range of t, and they further pointed out that we can generalize the additive generator of a t-norm  $T$  as it just satisfies [\(3\)](#page-0-0). This idea was identified by Viceník  $[8]$  when t is a strictly monotone function. The related work can refer to  $[6, 7, 9]$  $[6, 7, 9]$  $[6, 7, 9]$  also. Recently, Zhang and Wang [\[10\]](#page-15-8) also proved that a right continuous monotone function may be an additive generator of an associative two-place function (see Corollaries 5.2 and 5.3 of [\[10\]](#page-15-8)). One naturally wishes that a

<sup>∗</sup>Supported by the National Natural Science Foundation of China (No.12471440)

<sup>†</sup>E-mail address: mathchen2019@163.com

<sup>‡</sup>Corresponding author. xpwang1@hotmail.com; fax: +86-28-84761502

left continuous monotone function can also be a additive generator of an associative two-place function. However, Remark 4.1 of Zhang and Wang [\[10\]](#page-15-8) showed that generally, [\(3\)](#page-0-0) is not associative when t is a left continuous monotone function and  $q$  is a pseudo-inverse of  $t$ . In this article, we consider two problems: when can a left continuous monotone function  $t : [0, 1] \to [0, \infty]$  be an additive generator of an associative two-place function like the function  $T : [0,1]^2 \to [0,1]$  given by [\(3\)](#page-0-0) and what is the characterization of such left continuous monotone functions?

The rest of this article is organized as follows. In Section 2, we mainly recall some basic concepts. In Section 3, we define a weak pseudo-inverse of a monotone function, and develop its properties. In Section 4, we give a representation of the range  $\text{Ran}(t)$  of a left continuous non-decreasing function t. In Section 5, we first define an operation  $\otimes$  on the Ran(t), and then investigate some necessary and sufficient conditions for the operation  $\otimes$  being associative. In Section 6, we characterize what properties of Ran(t) are equivalent to the associativity of a function generated by a left continuous non-decreasing function  $t$ . A conclusion is drawn in Section 7.

## 2 Preliminaries

In this section, we recall some known basic concepts and results that will be used latter.

**Definition 2.1** ( [\[3\]](#page-15-3)) A t-norm is a binary operator  $T : [0,1]^2 \to [0,1]$  such that for all  $x, y, z \in [0,1]$ the following conditions are satisfied:

 $(T1) T(x, y) = T(y, x),$  $(T2) T(T(x,y),z) = T(x,T(y,z)),$  $(T3)$   $T(x,y) \leq T(x,z)$  whenever  $y \leq z$ ,  $(T4) T(x, 1) = x.$ 

A binary operator  $T : [0,1]^2 \rightarrow [0,1]$  is called a t-subnorm if it satisfies  $(T1), (T2), (T3),$  and  $T(x, y) \le \min\{x, y\}$  for all  $x, y \in [0, 1]$ .

<span id="page-1-1"></span>**Definition 2.2** ( [\[3\]](#page-15-3)) A t-conorm is a binary operator  $S : [0,1]^2 \to [0,1]$  such that for all  $x, y, z \in [0,1]$ the following conditions are satisfied:

- $(S1) S(x, y) = S(y, x),$  $(S2) S(S(x, y), z) = S(x, S(y, z)),$ (S3)  $S(x, y) \leq S(x, z)$  whenever  $y \leq z$ ,
- $(S4) S(x, 0) = x.$

A binary operator  $S : [0,1]^2 \rightarrow [0,1]$  is called a t-supconorm if it satisfies  $(S1), (S2), (S3),$  and  $S(x, y) > \max\{x, y\}$  for all  $x, y \in [0, 1]$ .

<span id="page-1-0"></span>**Definition 2.3** ( [\[3,](#page-15-3) [8\]](#page-15-4)) Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \to [m, n]$  be a monotone function. Then the function  $t^{(-1)} : [m, n] \to [a, b]$  defined by

 $t^{(-1)}(y) = \sup\{x \in [a, b] \mid (t(x) - y)(t(b) - t(a)) < 0\}$ 

is called a pseudo-inverse of the monotone function  $t$ .

Let  $[a, b] \subseteq [-\infty, \infty]$  with  $a \leq b$ . Then by convention,  $\sup \emptyset = a$  and  $\inf \emptyset = b$ .

**Definition 2.4 ( [\[2\]](#page-15-9))** Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \to [m, n]$  be a monotone non-decreasing function. Then each function  $t^* : [m, n] \to [a, b]$  satisfying

- (i)  $t \circ t^* \circ t = t$ ,
- (ii)  $t^{\wedge} \leq t^* \leq t^{\vee}$ ,

is called a quasi-inverse of t, where functions  $t^{\wedge} : [m, n] \to [a, b]$  and  $t^{\vee} : [m, n] \to [a, b]$  are defined by, respectively,  $t^{\wedge}(y) = \sup t^{-1}([m, y)), t^{\vee}(y) = \inf t^{-1}((y, n])$  in which  $t^{-1}$  is an inverse function of t.

<span id="page-2-1"></span>**Theorem 2.1** ( [\[3\]](#page-15-3)) Let  $t : [0, 1] \rightarrow [0, \infty]$  be a strictly decreasing function with  $t(1) = 0$  such that

$$
t(x) + t(y) \in \text{Ran}(t) \cup [t(0^+), \infty]
$$

*for all*  $(x, y) \in [0, 1]^2$ . Then the function  $T : [0, 1]^2 \rightarrow [0, 1]$  given by

$$
T(x, y) = t^{(-1)}(t(x) + t(y))
$$

*is a t-norm.*

# 3 Weak pseudo-inverses of monotone functions

This section first introduces a weak pseudo-inverse of a monotone function, and then discuss the properties of the weak pseudo-inverses. We also use the weak pseudo-inverse of a monotone function to construct a t-norm and a t-supconorm, respectively.

<span id="page-2-0"></span>**Definition 3.1** Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \to [m, n]$  be a monotone function. Then the function  $t^{[-1]} : [m, n] \to [a, b]$  defined by

$$
t^{[-1]}(y) = \sup\{x \in [a, b] \mid (t(x) - y)(t(b) - t(a)) \le 0\}
$$

is called a weak pseudo-inverse of the monotone function t.

As an immediate consequence of Definition [3.1,](#page-2-0) we get the following corollary.

Corollary 3.1 Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \to [m, n]$  be a monotone function

*(i)* If t is non-decreasing and non-constant, then for all  $y \in [m, n]$  we obtain the simpler formula

$$
t^{[-1]}(y) = \sup\{x \in [a, b] \mid t(x) \le y\}.
$$

*(ii)* If f is non-increasing and non-constant, then for all  $y \in [m, n]$  *we obtain the simpler formula* 

$$
t^{[-1]}(y) = \sup\{x \in [a, b] \mid t(x) \ge y\}.
$$

<span id="page-2-3"></span>*(iii)* If t is a constant function, then for all  $y \in [m, n]$  we have  $t^{[-1]}(y) = b$ .

**Remark 3.1** Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \to [m, n]$  be a monotone function, and let  $t^{[-1]}$  be its weak pseudo-inverse.

- (i) If t is non-decreasing, then the function  $t^{[-1]}$  is right continuous and non-decreasing, and for all  $y \in [m, t(a))$  we get  $t^{[-1]}(y) = a$ , and for all  $y \in (t(b), n]$  we have  $t^{[-1]}(y) = b$ .
- (ii) If t is non-increasing, then the function  $t^{[-1]}$  is left continuous and non-increasing, and for all  $y \in [m, t(b))$  we get  $t^{[-1]}(y) = b$ , and for all  $y \in (t(a), n]$  we have  $t^{[-1]}(y) = a$ .

<span id="page-2-2"></span>**Example 3.1** Let the function  $t : [0, 1] \rightarrow [0, \infty]$  be defined by

$$
t(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{4} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}], \\ x + \frac{1}{2} & \text{if } x \in (\frac{1}{2}, \frac{3}{4}), \\ 2 & \text{if } x \in [\frac{3}{4}, \frac{7}{8}], \\ x + \frac{5}{4} & \text{if } x \in [\frac{7}{8}, 1]. \end{cases}
$$

Then

$$
t^{(-1)}(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{4}, 1], \\ x - \frac{1}{2} & \text{if } x \in (1, \frac{5}{4}), \\ \frac{3}{4} & \text{if } x \in [\frac{5}{4}, 2], \\ \frac{7}{8} & \text{if } x \in (2, \frac{17}{8}), \\ x - \frac{5}{4} & \text{if } x \in [\frac{17}{8}, \frac{9}{4}), \\ 1 & \text{if } x \in [\frac{9}{4}, \infty]. \end{cases}
$$

$$
t^{[-1]}(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{4}, 1], \\ \frac{1}{2} & \text{if } x \in [\frac{1}{4}, 1], \\ \frac{1}{2} & \text{if } x \in [\frac{5}{4}, 2), \\ \frac{3}{4} & \text{if } x \in [\frac{5}{4}, 2), \\ \frac{7}{8} & \text{if } x \in [\frac{2}{3}, \frac{9}{4}), \\ 1 & \text{if } x \in [\frac{9}{8}, \frac{9}{4}]. \end{cases}
$$

Obviously,  $t^{(-1)} \le t^{[-1]}$  and  $t^* \le t^{[-1]}$ . If  $x \in [0, \frac{1}{4}) \cup (\frac{1}{2}, \frac{3}{4}) \cup [\frac{7}{8}, 1]$ , then  $t^{(-1)}(t(x)) = t^{[-1]}(t(x))$ . If  $x \in [0, \frac{3}{4}) \cup [\frac{7}{8}, 1]$ , then  $t^{[-1]}(t(x)) = t^*(t(x))$ .

<span id="page-3-0"></span>**Lemma 3.1** Let  $t : [a, b] \rightarrow [m, n]$  be a non-decreasing (resp. non-increasing) function and  $t^{[-1]}$  be its *weak pseudo-inverse. Then for all*  $x \in [a, b]$ ,

$$
t(t^{[-1]}(t(x))) \ge t(x) \ (resp. \ t(t^{[-1]}(t(x))) \le t(x)).
$$

**Proof.** If t is non-decreasing and  $x \in [a, b]$  then

$$
t^{[-1]}(t(x)) = \sup\{y \in [a, b] \mid t(y) \le t(x)\} \ge x,
$$

thus,

$$
t(t^{[-1]}(t(x))) = t(\sup\{y \in [a, b] \mid t(y) \le t(x)\}) \ge t(x).
$$

The case that t is non-increasing is completely analogous.

<span id="page-3-1"></span>**Lemma 3.2** *Let*  $t : [a, b] \rightarrow [m, n]$  *be a non-decreasing (resp. non-increasing) function, let*  $t^{[-1]}$  *be its*  $weak\ pseudo-inverse\ and\ x_0 \in [a, b].$  Then  $t(t^{[-1]}(t(x_0))) > t(x_0)$  (resp.  $t(t^{[-1]}(t(x_0))) < t(x_0)$ ) *if and only if there exists a*  $\delta_{x_0} > 0$  *such that*  $t(x) = t(x_0)$  *for all*  $x \in [x_0, x_0 + \delta_{x_0})$  *and*  $t(x_0) < t(x_0 + \delta_{x_0})$  $(resp. t(x_0) > t(x_0 + \delta_{x_0})).$ 

**Proof.** ( $\Leftarrow$ ). Let t be a non-decreasing function and  $x_0 \in [a, b)$ . If there is a  $\delta_{x_0} > 0$  such that  $t(x) = t(x_0)$  for all  $x \in [x_0, x_0 + \delta_{x_0})$  and  $t(x_0) < t(x_0 + \delta_{x_0})$ , then

$$
t^{[-1]}(t(x_0)) = \sup\{y \in [a, b] \mid t(y) \le t(x_0)\} = x_0 + \delta_{x_0},
$$

thus

$$
t(t^{[-1]}(t(x_0))) = t(\sup\{y \in [a,b] \mid t(y) \le t(x_0)\}) = t(x_0 + \delta_{x_0}) > t(x_0).
$$

(⇒). Suppose that  $t(t^{[-1]}(t(x_0))) > t(x_0)$ . Let  $x_0 \in [a, b)$  and  $\alpha = t^{[-1]}(t(x_0))$ . Then  $t(\alpha) > t(x_0)$ . Using the monotonicity of t we have  $\alpha > x_0$ . Take  $\delta_{x_0} = \alpha - x_0$ . Then from the monotonicity of t and  $\alpha = \sup\{y \in [a, b] \mid t(y) \le t(x_0)\}\$ , we immediately have  $t(x) = t(x_0)$  for all  $x \in [x_0, x_0 + \delta_{x_0})$  and  $t(x_0) < t(x_0 + \delta_{x_0}).$ 

The case that  $t$  is non-increasing is completely analogous.

 $\Box$ 

 $\Box$ 

From Lemmas [3.1](#page-3-0) and [3.2](#page-3-1) we easily get the following theorem.

<span id="page-3-2"></span>**Theorem 3.1** Let  $t : [a, b] \to [m, n]$  be a non-decreasing (resp. non-increasing) function and  $t^{[-1]}$  be its *weak pseudo-inverse. Then the following are equivalent:*

- $(i) \{x_0 \in [a, b) \mid \text{ there is a } \delta_{x_0} > 0 \text{ such that } t(x) = t(x_0) \text{ for all } x \in [x_0, x_0 + \delta_{x_0}) \text{ and } t(x_0) <$  $t(x_0 + \delta_{x_0})$  (resp.  $t(x_0) > t(x_0 + \delta_{x_0})$ ) =  $\emptyset$ ;
- *(ii)*  $t(t^{[-1]}(t(x_0))) = t(x_0)$  *for all*  $x_0 \in [a, b]$ *.*

From Definitions [2.3](#page-1-0) and [3.1,](#page-2-0) Lemma [3.1](#page-3-0) and Theorem [3.1,](#page-3-2) we easily deduce the following properties of a weak pseudo-inverse of a monotone function.

<span id="page-4-0"></span>**Proposition 3.1** Let  $a, b, m, n \in [-\infty, \infty]$  with  $a < b, m < n$  and  $t : [a, b] \rightarrow [m, n]$  be a monotone  $function$  and  $t^{[-1]}$  be its weak pseudo-inverse.

- (*i*)  $t^{[-1]}$  *coincides with*  $t^{(-1)}$  *if and only if t is strictly monotone. Moreover,*  $t^{[-1]}$  *coincides with*  $t^{-1}$  *if and only if* t *is a bijection.*
- (*ii*)  $t^{[-1]}$  *is continuous if and only if t is strictly monotone on the set*  $t^{[-1]}([m,n))$ *.*
- $(iii)$   $t^{[-1]} \circ t \ge id_{[a,b]}.$
- *(iv)* If t is either left continuous or strictly monotone then t  $\circ t^{[-1]} \circ t = t$ .
- (*v*) If t is strictly monotone then so is  $t^{[-1]}|_{\text{Ran}(t)}$ . Further, we have

$$
t \circ t^{[-1]} |_{\text{Ran}(t)} = id_{\text{Ran}(t)}, \quad t^{[-1]} \circ t = id_{[a,b]}.
$$

- (*vi*) If t is surjective then  $t \circ t^{[-1]} = id_{[m,n]}.$
- *(vii)* If both  $\mu : [a, b] \to [a, b]$  and  $\nu : [m, n] \to [m, n]$  are monotone bijections then

$$
(t \circ \mu)^{[-1]} = \mu^{-1} \circ t^{[-1]}, \ \ (\nu \circ t)^{[-1]} = t^{[-1]} \circ \nu^{-1}.
$$

From Proposition [3.1](#page-4-0) (i) and Theorem [2.1,](#page-2-1) we have the following corollary.

**Corollary 3.2** Let  $t : [0,1] \rightarrow [0,\infty]$  be a strictly decreasing function with  $t(1) = 0$  such that

 $t(x) + t(y) \in \text{Ran}(t) \cup [t(0^+), \infty]$ 

*for all*  $(x, y) \in [0, 1]^2$ . Then the function  $T : [0, 1]^2 \rightarrow [0, 1]$  given by

<span id="page-4-1"></span>
$$
T(x, y) = t^{[-1]}(t(x) + t(y))
$$

<span id="page-4-2"></span>*is a t-norm.*

**Proposition 3.2** Let  $t : [0,1] \rightarrow [0,\infty]$  be a left continuous non-decreasing function such that

$$
t(x) + t(y) \in \text{Ran}(t) \cup [t(1^-), \infty]
$$
\n
$$
(4)
$$

*for all*  $(x, y) \in [0, 1]^2$ . Then the function  $T : [0, 1]^2 \rightarrow [0, 1]$  given by

$$
T(x, y) = t^{[-1]}(t(x) + t(y))
$$

*is a t-supconorm.*

**Proof.** Replacing  $f^{(-1)}$  by  $t^{[-1]}$ , in completely analogous to the proof of Theorem 3.23 in [\[3\]](#page-15-3) we can show the monotonicity, the associativity and the commutativity of  $T$ , respectively. On the other hand,  $T(x,y) = t^{[-1]}(t(x) + t(y)) \ge t^{[-1]}(t(x)) \ge x$  for all  $x, y \in [0,1]$ , analogously,  $T(x,y) \ge y$ . Thus  $T(x, y) \ge \max\{x, y\}.$  Therefore, by Definition [2.2](#page-1-1) T is a t-supconorm.  $\Box$ 

Note that if  $t : [0, 1] \to [0, \infty]$  is a left continuous non-decreasing function but not strictly increasing and satisfies [\(4\)](#page-4-1) then one easily check that the function  $T : [0,1]^2 \to [0,1]$  given by  $T(x,y) = t^{(-1)}(t(x) +$  $t(y)$  isn't a t-supconorm.

Generally, we can prove the following result through a analogous way to the proof of Proposition [3.2.](#page-4-2)

<span id="page-5-1"></span>**Proposition 3.3** Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function and  $F : [0,\infty]^2 \to$  $[0, \infty]$  *be such that*  $([0, \infty], F, \leq)$  *is a fully ordered Abel semigroup with*  $F(x, 0) \geq x$  *for all*  $x \in [0, \infty]$ *. If* 

$$
F(t(x), t(y)) \in \text{Ran}(t) \cup [t(1^-), \infty]
$$
\n
$$
(5)
$$

*for all*  $(x, y) \in [0, 1]^2$ . Then the function  $T : [0, 1]^2 \rightarrow [0, 1]$  given by

$$
T(x, y) = t^{[-1]} F(t(x), t(y))
$$
\n(6)

*is a t-supconorm.*

In what follows, we consider what is a characterization of left continuous non-decreasing functions  $t : [0, 1] \rightarrow [0, \infty]$  such that the function  $T : [0, 1]^2 \rightarrow [0, 1]$  given by

<span id="page-5-0"></span>
$$
T(x,y) = t^{[-1]}(F(t(x),t(y)))
$$
\n(7)

is associative, where  $F: [0, \infty]^2 \to [0, \infty]$  is an associative function and  $t^{[-1]}: [0, \infty] \to [0, 1]$  is the weak pseudo-inverse of t.

## 4 The range of a left continuous non-decreasing function

In this section we give a representation of the range of a left continuous non-decreasing function. Let  $t : [0,1] \to [0,\infty]$  be a function. We write  $t(a^-) = \lim_{x \to a^-} t(x)$  for each  $a \in (0,1]$  and  $t(a^+) = \lim_{x \to a^+} t(x)$  for each  $a \in [0,1)$ . Define  $t(1^+) = \infty$  whenever t is non-decreasing. Further, let

 $\mathcal{A} = \{M \mid \text{there is a left continuous non-decreasing function } t : [0,1] \to [0,\infty] \text{ such that } \text{Ran}(t) = M \}$ 

and denoted by  $A \setminus B = \{x \in A \mid x \notin B\}$  for two sets A and B. Then the following lemma presents the range of a left continuous non-decreasing function.

**Lemma 4.1** Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function and  $M \in \mathcal{A}$  with  $M \neq$  $[t(0), \infty]$ . Then there exist a uniquely determined non-empty countable system  $\mathcal{U} = \{[b_k, d_k] \subseteq [0, \infty] \mid k \in \mathbb{Z}\}$  $K$ } of closed intervals of a positive length which satisfy that for all  $[b_k, d_k]$ ,  $[b_l, d_l] \in \mathcal{U}$ ,  $[b_k, d_k] \cap [b_l, d_l] = \emptyset$  $or [b_k, d_k] \cap [b_l, d_l] = \{d_k\}$  when  $d_k \leq b_l$ , and a uniquely determined non-empty countable set  $\mathcal{V} = \{c_k \in \mathcal{V} \mid d_l\}$  $[0, \infty] \mid k \in \overline{K}$  *such that*  $[b_k, d_k] \cap \mathcal{V} = \{b_k\}$  *or*  $[b_k, d_k] \cap \mathcal{V} = \{b_k, d_k\}$  *for all*  $k \in K$  *and* 

$$
M = \{c_k \in [0, \infty] \mid k \in \overline{K}\} \cup \left([t(0), \infty] \setminus \left(\bigcup_{k \in K} [b_k, d_k]\right)\right)
$$

*where*  $|K| \leq |\overline{K}|$ *.* 

**Proof.** We first prove the existence of both U and V. Take  $\mathcal{U} = \{ [t(x), t(x^+)] \mid x \in [0, 1], t(x) < t(x^+) \}$ and  $V = \{t(x) \mid x \in [0,1], t(x) < t(x^+) \} \cup \{t(x^+) \mid x \in [0,1], t(x) < t(x^+), t(x^+) \in M \}$ . It is easy to see that U is countable. Let K be a countable index set and  $|K| = |\mathcal{U}|$ . We can rewrite U as  $\mathcal{U} = \{ [b_k, d_k] \subseteq [0, \infty] \mid k \in K \}$  in which  $b_k = f(k)$ ,  $d_k = f(k^+)$  for each  $k \in K$ . Clearly,  $\mathcal{V}$  is also countable. Now, let K be a countable index set and  $|K| = |\mathcal{V}|$ . Obviously,  $|K| \leq |K|$ . Rearrange V from small to large and assign its every element an index  $k \in K$ , for example, the k-th element in the rearranged set V is denoted by  $c_k$  with  $k \in K$ . Then  $V = \{c_k \in [0,\infty] \mid k \in K\}$ . Notice that for every  $k \in K$ , there are two elements  $m, n \in \overline{K}$  with  $m + 1 = n$  such that  $d_k = t(k^+) = c_n$  and  $b_k = t(k) = c_m$ when  $t(k^+) \in M$ , and there exits a  $j \in \overline{K}$  such that  $b_k = c_j$  when  $t(k^+) \notin M$ . Obviously, U and V have all required properties, respectively.

Now, we prove the uniqueness of both U and V. Suppose that both the system  $\mathcal{U}_1 = \{[u_l, v_l] \subseteq$  $[0,\infty] \mid l \in L$  and the set  $\mathcal{V}_1 = \{c_l \in [0,\infty] \mid l \in \overline{L}\}\$ also have all required properties. Below, we prove that  $\mathcal{U} = \mathcal{U}_1$ . Fix an arbitrary interval  $[b_k, d_k] \in \mathcal{U}$ . Choose  $a \in [b_k, d_k]$  such that  $a \notin M$ . Then there exists a  $[u_l, v_l] \in \mathcal{U}_1$  such that  $a \in [u_l, v_l]$  where  $[u_l, v_l] \cap M = \{u_l\}$  or  $[u_l, v_l] \cap M = \{u_l, v_l\}$ . Next, we prove that  $[b_k, d_k] = [u_l, v_l]$  by distinguishing two steps.

Supposing that  $d_k \leq u_l < v_l$ , this contradicts the fact that  $a \in [b_k, d_k]$  and  $a \in [u_l, v_l]$  with  $a \notin M$ . Now, assume  $u_l < d_k < v_l$ . If  $d_k \in M$ , then  $d_k \in [u_l, v_l] \cap M$ , contrary to  $[u_l, v_l] \cap M \in \{\{u_l\}, \{u_l, v_l\}\}\.$  If  $d_k \notin M$ , then the set  $[u_l, v_l] \cap M$  is infinite, a contradiction since  $[u_l, v_l] \cap M \in \{\{u_l\}, \{u_l, v_l\}\}\.$  Therefore,  $u_l < v_l \leq d_k$ . In a completely analogous way, we have  $b_k \leq u_l < v_l$ . Thus  $b_k \leq u_l < v_l \leq d_k$ .

Suppose that  $b_k < u_l$ . Then  $u_l \in [b_k, d_k]$  since  $u_l \in M$ , contrary to  $[b_k, d_k] \cap M \in \{\{b_k\}, \{b_k, d_k\}\}.$ Consequently,  $b_k = u_l \langle v_l \leq d_k$ . In the following, we prove that  $v_l = d_k$ . Suppose that  $v_l \langle d_k$ . Then  $v_l \in [b_k, d_k]$ . If  $v_l \in M$ , then  $v_l \in [b_k, d_k] \cap M$ , contrary to  $[b_k, d_k] \cap M \in \{\{b_k\}, \{b_k, d_k\}\}\$ . If  $v_l \notin M$ , then the set  $[b_k, d_k] \cap M$  is infinite, a contradiction since  $[b_k, d_k] \cap M \in \{\{b_k\}, \{b_k, d_k\}\}\.$  Therefore,  $v_l = d_k$ .

We finally come to  $[b_k, d_k] = [u_l, v_l]$ . From the arbitrariness of  $[b_k, d_k]$ , we have  $\mathcal{U} \subseteq \mathcal{U}_1$ . The case  $U \supseteq U_1$  is completely analogous. Therefore,  $U = U_1$ . This follows that  $V = V_1$ . In particular, both U and  $V$  are independent of a choice of  $t$ .  $\Box$ 

**Definition 4.1** Let  $M \in \mathcal{A}$ . A pair  $(\mathcal{U}, \mathcal{V})$  is said to be associated with  $M \neq [t(0), \infty]$  if  $\mathcal{U} = \{[b_k, d_k] \subseteq$  $[0, \infty] \mid k \in K$  is a non-empty countable system of closed intervals of a positive length which satisfy that for all  $[b_k, d_k]$ ,  $[b_l, d_l] \in \mathcal{U}$ ,  $[b_k, d_k] \cap [b_l, d_l] = \emptyset$  or  $[b_k, d_k] \cap [b_l, d_l] = \{d_k\}$  when  $d_k \leq b_l$ , and  $\mathcal{V} = \{c_k \in \mathcal{V} \mid d_l\}$  $[0,\infty] \mid k \in \overline{K} \}$  is a non-empty countable set such that  $[b_k, d_k] \cap \mathcal{V} = \{b_k\}$  or  $[b_k, d_k] \cap \mathcal{V} = \{b_k, d_k\}$  for all  $k \in K$  and

$$
M = \{c_k \in [0, \infty] \mid k \in \overline{K}\} \cup \left([t(0), \infty] \setminus \left(\bigcup_{k \in K} [b_k, d_k]\right)\right).
$$

A pair  $(\mathcal{U}, \mathcal{V})$  is said to be associated with  $M = [t(0), \infty]$  if  $\mathcal{U} = \{\infty, \infty\}$  and  $\mathcal{V} = \{\infty\}$ .

We briefly write  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  instead of  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \subseteq [0, \infty] \mid k \in \overline{K}\})$  $k \in K$ ,  $\{c_k \in [0,\infty] \mid k \in \overline{K}\}$ ).

#### <span id="page-6-1"></span>Example 4.1

(i) Let the function  $t_1 : [0,1] \to [0,\infty]$  be defined by

$$
t_1(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, \frac{3}{4}], \\ x & \text{if } x \in (\frac{3}{4}, 1]. \end{cases}
$$

Then the pair  $(\{[\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [1, \infty]\}, \{\frac{1}{4}, \frac{1}{2}, 1\})$  is associated with  $[0, \frac{1}{4}] \cup \{\frac{1}{2}\} \cup (\frac{3}{4}, 1] \in \mathcal{A}$ .

(ii) Let the function  $t_2 : [0,1] \to [0,\infty]$  be defined by

$$
t_2(x) = \begin{cases} 1+x & \text{if } x \in [0, \frac{1}{5}], \\ \frac{6}{5} & \text{if } x \in (\frac{1}{5}, \frac{1}{4}], \\ \frac{3}{2} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}], \\ 2+x & \text{if } x \in (\frac{1}{2}, \frac{3}{4}], \\ \frac{1}{1-x} & \text{if } x \in (\frac{3}{4}, 1), \\ \infty & \text{otherwise.} \end{cases}
$$

Then the pair  $(\{[\frac{6}{5}, \frac{3}{2}], [\frac{3}{2}, \frac{5}{2}], [\frac{11}{4}, 4]\}, \{\frac{6}{5}, \frac{3}{2}, \frac{11}{4}\})$  is associated with  $[1, \frac{6}{5}] \cup \{\frac{3}{2}\} \cup [\frac{5}{2}, \frac{11}{4}] \cup [4, \infty] \in \mathcal{A}$ .

## 5 An operation on  $\text{Ran}(t)$  and its properties

In this section we first define an operation  $\otimes$  on Ran(t) with t a left continuous non-decreasing function, and then establish some necessary and sufficient conditions for the operation ⊗ being associative.

<span id="page-6-0"></span>**Definition 5.1** Let  $M \in \mathcal{A}$ . Define a function  $G_M : [0, \infty] \to M$  by

$$
G_M(x) = \min\{M \cap [\sup([0, x] \cap M), \inf([x, \infty] \cap M)]\}
$$

for all  $x \in [0, \infty]$ .

The next proposition describes the relationship between  $M$  and  $G_M$ .

<span id="page-7-1"></span>**Proposition 5.1** Let  $M \in \mathcal{A}$  and  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  be associated with M. *Then for all*  $x \in [0, \infty]$  *and*  $k \in K$ *,* 

- *(i)* If  $x \in [0, t(0)]$  then  $G_M(x) = t(0)$ .
- *(ii)*  $G_M(x) = x$  *if and only if*  $x \in M$ *.*
- *(iii)* If  $x \notin M$  and  $x > t(0)$  then  $G_M(x) = b_k$  *if and only if*  $x \in [b_k, d_k] \setminus \{c_k\}.$
- *(iv)*  $G_M$  *is a non-decreasing function.*

**Proof.** From Definition [5.1,](#page-6-0) (i), (ii) and (iv) are immediate.

(iii) Let  $x \notin M$  and  $x > t(0)$ . Then there is a  $k \in K$  such that  $x \in [b_k, d_k] \setminus \{c_k\}$ . Conversely, let  $x > t(0)$  and  $x \in [b_k, d_k] \setminus \{c_k\}$ . Then sup $([0, x] \cap M) = b_k$  and  $\inf([x, 1] \cap M) = d_k$ . Thus from Definition [5.1,](#page-6-0)  $G_M(x) = b_k$ .  $\Box$ 

<span id="page-7-0"></span>Example 5.1 In Example [3.1,](#page-2-2)

$$
(\mathrm{i})
$$

$$
G_M(x) = \begin{cases} \frac{1}{4} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, \frac{3}{4}], \\ x & \text{otherwise.} \end{cases}
$$

(ii)

$$
G_M(x) = \begin{cases} \frac{6}{5} & \text{if } x \in (\frac{6}{5}, \frac{3}{2}), \\ \frac{3}{2} & \text{if } x \in (\frac{3}{2}, \frac{5}{2}), \\ \frac{11}{4} & \text{if } x \in (\frac{11}{4}, 4], \\ x & \text{otherwise.} \end{cases}
$$

In what follows, we always suppose that  $F : [0, \infty]^2 \to [0, \infty]$  is an associative function. We need the following definition.

<span id="page-7-2"></span>**Definition 5.2** Let  $M \in \mathcal{A}$  and  $G_M$  be determined by M. Define an operation ⊗ :  $M^2 \to M$  by

$$
x \otimes y = G_M(F(x, y)).
$$

Example 5.2 In Example [5.1,](#page-7-0)

(i)

$$
x \otimes y = \begin{cases} \frac{1}{4} & \text{if } F(x, y) \in (\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{2} & \text{if } F(x, y) \in (\frac{1}{2}, \frac{3}{4}], \\ F(x, y) & \text{otherwise.} \end{cases}
$$

(ii)

$$
x \otimes y = \begin{cases} \frac{6}{5} & \text{if } F(x, y) \in (\frac{6}{5}, \frac{3}{2}), \\ \frac{3}{2} & \text{if } F(x, y) \in (\frac{3}{2}, \frac{5}{2}), \\ \frac{11}{4} & \text{if } F(x, y) \in (\frac{11}{4}, 4], \\ F(x, y) & \text{otherwise.} \end{cases}
$$

<span id="page-7-3"></span>**Proposition 5.2** Let  $M \in \mathcal{A}$  and  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  be associated with M. *Then for all*  $x, y \in M$  *and*  $k \in K$ *,* 

*(i)* If  $F(x, y) \in [0, t(0)]$ , then  $x \otimes y = t(0)$ .

*(ii)*  $x \otimes y = F(x, y)$  *if and only if*  $F(x, y) \in M$ .

*(iii)* If  $F(x, y) \notin M$  and  $F(x, y) > t(0)$  then  $x \otimes y = b_k$  if and only if  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}.$ 

*(iv)* ⊗ *is a non-decreasing function.*

<span id="page-8-2"></span>Proof. It is an immediate matter of Proposition [5.1](#page-7-1) and Definition [5.2.](#page-7-2)

**Proposition 5.3** Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function with  $\text{Ran}(t) = M$ *and*  $M \in \mathcal{A}$ ,  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  *be associated with* M. Then  $G_M(x) = t(t^{[-1]}(x))$ *for all*  $x \in [0, \infty]$ *.* 

 $\Box$ 

 $\Box$ 

**Proof.** If  $x \in \text{Ran}(t)$  then, from Proposition [3.1](#page-4-0) (iv), we have  $t(t^{[-1]}(x)) = x$ . Thus, in the case of  $x \notin M$  and  $x < t(0)$ , by Remark [3.1](#page-2-3) we have  $t^{[-1]}(x) = 0$ , hence  $t(t^{[-1]}(x)) = t(0)$ ; in the case of  $x \notin M$ and  $x > t(0)$ , there is  $k \in K$  such that  $x \in [b_k, d_k] \setminus \{c_k\}$  with  $b_k \in M$ . Consequently,

$$
t(t^{[-1]}(x)) = t(\sup\{y \in [0,\infty] \mid t(y) \le x\})
$$
  
=  $\sup\{t(y) \in [0,\infty] \mid t(y) \le x\}$   
=  $b_k$ .

Therefore, by Proposition [5.1,](#page-7-1) we get  $G_M(x) = t(t^{[-1]}(x))$ .

Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function with  $\text{Ran}(t) = M$  and  $(\mathcal{U}, \mathcal{V}) =$  $({\{[b_k, d_k] | k \in K\}, \{c_k | k \in \overline{K}\}})$  be associated with M. Denote

$$
\mathbb{H} = \{c \mid \text{ there are an } x_0 \in [0, 1] \text{ and } \varepsilon > 0 \text{ such that } t|_{[x_0, x_0 + \varepsilon]} = c\},\
$$

$$
\mathbb{N} = \{\sup\{x \in [0, 1] \mid t(x) = y\} \mid y \in \mathbb{H}\}, \mathbb{W} = \{x \in [0, 1] \mid t(x) \in M \setminus \mathbb{H}\},\
$$

$$
\mathbb{D} = \mathbb{N} \cup \mathbb{W}.
$$

In particular,  $t^{[-1]}(x) \in \mathbb{D}$  for all  $x \in [0, \infty]$ , and we have the following definition.

<span id="page-8-0"></span>**Definition 5.3** Let  $t : [0,1] \rightarrow [0,\infty]$  be a left continuous non-decreasing function. Define a function  $t^* : \mathbb{D} \to [0, \infty]$  by

$$
t^*(x) = t(x) \text{ for all } x \in \mathbb{D},
$$

and a two-place function  $F^* : \mathbb{D}^2 \to \mathbb{D}$  by

$$
F^*(x, y) = t^{[-1]}(F(t^*(x), t^*(y)))
$$

for all  $x, y \in \mathbb{D}$ , respectively.

Then we immediately have the following remark.

<span id="page-8-1"></span>**Remark 5.1** Let  $t : [0, 1] \rightarrow [0, \infty]$  be a left continuous non-decreasing function. Then

- (i)  $t^*$  is a strictly increasing function.
- (ii)  $t^{[-1]}(t^*(x)) = x$  for all  $x \in \mathbb{D}$ .
- <span id="page-8-3"></span>(iii)  $t^*(t^{[-1]}(x)) = x$  for all  $x \in [0, \infty]$ .

**Lemma 5.1** *Let*  $t : [0, 1] \rightarrow [0, \infty]$  *be a left continuous non-decreasing function. Then* 

$$
x \otimes y = t^*(F^*(t^{[-1]}(x), t^{[-1]}(y)))
$$

*for all*  $x, y \in M$  *and* 

$$
F^{\star}(x,y) = t^{[-1]}(t^{\star}(x) \otimes t^{\star}(y))
$$

*for all*  $x, y \in \mathbb{D}$ *.* 

**Proof.** From Definitions [5.2](#page-7-2) and [5.3,](#page-8-2) Remark [5.1](#page-8-1) and Proposition 5.3, for all  $x, y \in M$  we get

$$
x \otimes y = G_M(F(x, y))
$$
  
=  $t(t^{[-1]}(F(x, y)))$   
=  $t^*(t^{[-1]}(F(x, y)))$   
=  $t^*(t^{[-1]}(F(t^*(t^{[-1]}(x)), t^*(t^{[-1]}(y))))))$   
=  $t^*(F^*(t^{[-1]}(x), t^{[-1]}(y))).$ 

Because of  $x, y \in M$ , there exist two elements  $u, v \in \mathbb{D}$  such that  $t^*(u) = x$ ,  $t^*(v) = y$ . Thus, from Remark [5.1,](#page-8-1) we have

$$
t^{\star}(u) \otimes t^{\star}(v) = x \otimes y = t^{\star}(F^{\star}(t^{[-1]}(x), t^{[-1]}(y))) = t^{\star}(F^{\star}(u, v)).
$$

This follows that  $t^{[-1]}(t^*(u) \otimes t^*(v)) = t^{[-1]}(t^*(F^*(u, v)) = F^*(u, v).$ 

Furthermore, we have the following proposition.

<span id="page-9-1"></span>**Proposition 5.4** *Let*  $t : [0, 1] \rightarrow [0, \infty]$  *be a left continuous non-decreasing function. Then the following are equivalent:*

- *(i)* ⊗ *is associative.*
- (*ii*)  $F^*$  *is associative.*

**Proof.** Let  $\otimes$  be associative. Then, by Lemma [5.1,](#page-8-3) for all  $x, y, z \in \mathbb{D}$  we have

$$
F^*(F^*(x, y), z) = t^{[-1]}(t^*(F^*(x, y)) \otimes t^*(z))
$$
  
\n
$$
= t^{[-1]}(t^* \circ t^{[-1]}(t^*(x) \otimes t^*(y)) \otimes t^*(z))
$$
  
\n
$$
= t^{[-1]}(t^*(x) \otimes t^*(y) \otimes t^*(z))
$$
  
\n
$$
= t^{[-1]}(t^*(x) \otimes t^* \circ t^{[-1]}(t^*(y) \otimes t^*(z)))
$$
  
\n
$$
= t^{[-1]}(t^*(x) \otimes t^*(F^*(y, z)))
$$
  
\n
$$
= F^*(x, F^*(y, z)).
$$

Let  $F^*$  be associative. Then, by Lemma [5.1,](#page-8-3) for all  $x, y, z \in M$  we have

$$
(x \otimes y) \otimes z = t^*(F^*(t^{[-1]}(x \otimes y), t^{[-1]}(z)))
$$
  
\n
$$
= t^*(F^*(t^{[-1]} \circ t^*(F^*(t^{[-1]}(x), t^{[-1]}(y))), t^{[-1]}(z)))
$$
  
\n
$$
= t^*(F^*(F^*(t^{[-1]}(x), t^{[-1]}(y)), t^{[-1]}(z)))
$$
  
\n
$$
= t^*(F^*(t^{[-1]}(x), F^*(t^{[-1]}(y), t^{[-1]}(z)))
$$
  
\n
$$
= t^*(F^*(t^{[-1]}(x), t^{[-1]} \circ t^*(F^*(t^{[-1]}(y), t^{[-1]}(z))))
$$
  
\n
$$
= t^*(F^*(t^{[-1]}(x), t^{[-1]}(y \otimes z))
$$
  
\n
$$
= x \otimes (y \otimes z).
$$

<span id="page-9-0"></span>**Lemma 5.2** *Let*  $t : [0,1] \rightarrow [0,\infty]$  *be a left continuous non-decreasing function and*  $T : [0,1]^2 \rightarrow [0,1]$ *be a function defined by Eq.*[\(7\)](#page-5-0)*.* Then, for each  $x, y \in [0, 1]$ , there are two elements  $m, n \in \mathbb{D}$  such that  $t^{*}(m) = f(x), t^{*}(n) = f(y)$  and  $T(x, y) = F^{*}(m, n)$ . In particular,  $T(x, y) = F^{*}(x, y)$  for all  $x, y \in \mathbb{D}$ .

**Proof.** If  $x, y \in \mathbb{D}$  then from Definition [5.3](#page-8-0) we have  $t^*(x) = t(x)$  and  $t^*(y) = t(y)$ . If  $x \notin \mathbb{D}$ , then  $t(x) \in \mathbb{H}$ . Let  $m = \max\{s \in [0,1] \mid t(s) = t(x), t(x) \in \mathbb{H}\}$ . Obviously,  $m \in \mathbb{D}$  and  $t^*(m) = t(x)$ . Analogously, if  $y \notin \mathbb{D}$  then there is an  $n \in \mathbb{D}$  such that  $t^*(n) = t(y)$ .

Therefore, from Definition [5.3](#page-8-0) we have

$$
T(x,y) = t^{[-1]}(F(t(x), t(y)))
$$
  
=  $t^{[-1]}(F(t^*(m), t^*(n)))$   
=  $F^*(m, n)$ 

for arbitrarily  $x, y \in [0, 1]$ .

 $\Box$ 

 $\Box$ 

 $\Box$ 

The next proposition describes the relation between  $T$  and  $F^*$ .

<span id="page-10-0"></span>**Proposition 5.5** Let  $t : [0,1] \rightarrow [0,\infty]$  be a left continuous non-decreasing function and  $T : [0,1]^2 \rightarrow$ [0, 1] *be a function defined by Eq.[\(7\)](#page-5-0). Then the following are equivalent:*

- *(i)* T *is associative.*
- (*ii*)  $F^*$  *is associative.*

**Proof.** Suppose that  $F^*$  is associative. Let us prove that  $T(T(x, y), z) = T(x, T(y, z))$  for all  $x, y, z \in$ [0, 1]. Let  $x, y, z \in [0, 1]$ . Then by Lemma [5.2,](#page-9-0) there exist  $m, n, v \in \mathbb{D}$  such that  $t^*(m) = t(x), t^*(n) = t(y)$ and  $t^*(v) = t(z)$ , respectively, and  $T(x, y) = F^*(m, n)$ ,  $T(x, z) = F^*(m, v)$  and  $T(y, z) = F^*(n, v)$ , respectively. Therefore,

$$
T(T(x, y), z) = T(F^*(m, n), z)
$$
  
=  $F^*(F^*(m, n), v)$   
=  $F^*(m, F^*(n, v))$   
=  $F^*(m, T(y, z))$   
=  $T(x, T(y, z)).$ 

Conversely, if T is associative, i,e.,  $T(T(x, y), z) = T(x, T(y, z)$  for all  $x, y, z \in [0, 1]$ , then by Eq.[\(7\)](#page-5-0), we have

$$
t^{[-1]}(F(t \circ t^{[-1]}(F(t(x), t(y))), t(z))) = t^{[-1]}(F(t(x), t \circ t^{[-1]}(F(t(x), t(y))))).
$$

So that

$$
t^{[-1]}(F(t^{\star} \circ t^{[-1]}(F(t^{\star}(x), t^{\star}(y))), t^*(z))) = t^{[-1]}(F(t^{\star}(x), t^{\star} \circ t^{[-1]}(F(t^{\star}(x), t^{\star}(y))))
$$

for all  $x, y, z \in \mathbb{D}$ . Thus from Definition [5.3,](#page-8-0)  $F^*(F^*(x, y), z) = F^*(x, F^*(y, z))$ , therefore,  $F^*$  is associative.  $\Box$ 

The following is an immediate consequence of Propositions [5.4](#page-9-1) and [5.5.](#page-10-0)

<span id="page-10-1"></span>**Theorem 5.1** *Let*  $t : [0,1] \to [0,\infty]$  *be a left continuous non-decreasing function and*  $T : [0,1]^2 \to [0,1]$ *be a function defined by Eq.* $(7)$ *. Then* T *is associative if and only if*  $\otimes$  *is associative.* 

# 6 Associativity of the operation ⊗

This section devotes to exploring some necessary and sufficient conditions for the operation ⊗ being associative, which answer what properties of M are equivalent to the associativity of  $\otimes$ .

Let  $M \subseteq [0, \infty]$ . Define  $O(M) = \bigcup_{x,y \in M} (\min\{x, y\}, \max\{x, y\}]$  when  $M \neq \emptyset$  (where  $(x, x] = \emptyset$ ), and  $O(M) = \emptyset$  when  $M = \emptyset$ . Let  $\emptyset \neq A, B \subseteq [0, \infty]$  and  $c \in [0, \infty]$ . Denote  $F(A, B) = \{F(x, y) \mid x \in A, y \in A\}$ B} and  $F(\emptyset, A) = \emptyset = F(A, \emptyset)$ . It is clear that  $F(c, O(M)) = O(F(c, M))$  and  $F(O(M), c) = O(F(M, c))$ .

<span id="page-10-2"></span>**Definition 6.1** Let  $M \in \mathcal{A}$  and  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] | k \in K\}, \{c_k | k \in \overline{K}\})$  be associated with M. For all  $y \in M$  and  $k, l \in K$ , set  $K^* = K \cup {\tau}$  where  $\tau \notin K$ . Define

$$
M_k^y = \{x \in M \mid F(x, y) \in [b_k, d_k] \setminus \{c_k\}, M_y^k = \{x \in M \mid F(y, x) \in [b_k, d_k] \setminus \{c_k\},
$$
  
\n
$$
M_\tau^y = \{x \in M \mid F(x, y) < t(0)\}, M_y^\tau = \{x \in M \mid F(y, x) < t(0)\},
$$
  
\n
$$
M^y = \{x \in M \mid F(x, y) \in M\}, M_y = \{x \in M \mid F(y, x) \in M\},
$$
  
\n
$$
H_k^y = O(\{b_k\} \cup F(M_k^y, y)), H_\tau^y = O(\{t(0)\} \cup F(M_\tau^y, y)),
$$
  
\n
$$
H_y^k = O(\{b_k\} \cup F(y, M_y^k)), H_y^\tau = O(\{t(0)\} \cup F(y, M_y^\tau)),
$$
  
\n
$$
J_{k,l}^y = \begin{cases} O(F(M_k^y, b_l) \cup F(b_k, M_y^l)), & \text{if } M_k^y \neq \emptyset, M_y^l \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}
$$

$$
J_{\tau,l}^y = \begin{cases} O(F(M_\tau^y, b_l) \cup F(t(0), M_y^l)), & \text{if } M_\tau^y \neq \emptyset, M_y^l \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}
$$
  

$$
J_{k,\tau}^y = \begin{cases} O(F(M_k^y, t(0)) \cup F(b_k, M_y^{\tau})), & \text{if } M_k^y \neq \emptyset, M_y^{\tau} \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}
$$
  

$$
J_{\tau,\tau}^y = \begin{cases} O(F(M_\tau^y, t(0)) \cup F(t(0), M_y^{\tau})), & \text{if } M_\tau^y \neq \emptyset, M_y^{\tau} \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}
$$

 $\text{Put } \mathfrak{T}_{1}(M) = \bigcup_{y \in M} \bigcup_{k \in K^{*}} F(H_{k}^{y}, M_{y}), \mathfrak{T}_{2}(M) = \bigcup_{y \in M} \bigcup_{k \in K^{*}} F(M^{y}, H_{y}^{k}), \mathfrak{T}_{3}(M) = \bigcup_{y \in M} \bigcup_{k, l \in K^{*}} J_{k, l}^{y}$ <br>and  $\mathfrak{T}(M) = \mathfrak{T}_{1}(M) \cup \mathfrak{T}_{2}(M) \cup \mathfrak{T}_{3}(M).$ 

In the following, we further suppose that  $F : [0, \infty]^2 \to [0, \infty]$  is a monotone and associative function with neutral element in  $[0, \infty]$ . We have the following two lemmas.

<span id="page-11-0"></span>**Lemma 6.1** *Let*  $M \in \mathcal{A}$  *and*  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  *be associated with* M. Then for *any*  $x, y \in [0, 1]$ *,*  $G_M(x) = G_M(y)$  *if and only if*  $(\min\{x, y\}, \max\{x, y\}) \cap (M \setminus t(0)) = \emptyset$ *.* 

**Proof.** Let  $G_M(x) = G_M(y)$ . If  $(\min\{x, y\}, \max\{x, y\}) \cap (M \setminus t(0)) \neq \emptyset$ , then  $x \neq y$ , without loss of generality, say  $x < y$ , thus there is an  $a \in (x, y] \cap (M \setminus t(0))$ . By Proposition [5.1,](#page-7-1)  $G_M(a) = a > t(0)$ . If  $x \in M$ , then by Proposition [5.1](#page-7-1)  $G_M(x) = x$ , thus  $G_M(x) < G_M(a)$ . If  $x \notin M$ , then  $x \in [0, t(0))$ or there is a  $k \in K$  such that  $x \in [b_k, d_k] \setminus \{c_k\}$ . In the case  $x \in [0, t(0))$ , by Proposition [5.1](#page-7-1) we have  $G_M(x) = t(0) < G_M(a)$ . In the case  $x \in [b_k, d_k] \setminus \{c_k\}$ , by Proposition [5.1](#page-7-1) we have  $G_M(x) = b_k$ , thus  $G_M(x) < x < a = G_M(a)$ . In summary, we always have  $G_M(x) < G_M(a) \leq G_M(y)$ , a contradiction.

Conversely, if  $(\min\{x, y\}, \max\{x, y\}] \cap (M \setminus t(0)) = \emptyset$ , without loss of generality, say  $x < y$ , then  $(x, y] \subseteq (0, t(0)]$  or there is a  $k \in K$  such that  $(x, y] \subseteq [b_k, d_k] \setminus \{c_k\}$ . If  $(x, y] \subseteq (0, t(0)]$  then, from Proposition [5.1,](#page-7-1)  $G_M(x) = 0 = G_M(y)$ . If  $(x, y] \subseteq [b_k, d_k] \setminus \{c_k\}$  then, from Proposition 5.1,  $G_M(x) = b_k =$  $G_M(y).$  $\Box$ 

<span id="page-11-1"></span>**Lemma 6.2** *Let*  $M \in \mathcal{A}$  *and*  $(U, V) = (\{[b_k, d_k] | k \in K\}, \{c_k | k \in \overline{K}\})$  *be associated with*  $M$ *. Let*  $M_1, M_2 \subseteq [0, \infty]$  *be two non-empty sets and*  $c \in [0, \infty]$ *. Then* 

*(1)*  $F(O(M_1 \cup M_2), c) ∩ (M \setminus \{t(0)\}) \neq \emptyset$  *if and only if there exist*  $x \in M_1$  *and*  $y \in M_2$  *such that* 

 $(\min\{F(x, c), F(y, c)\}, \max\{F(x, c), F(y, c)\}\cap (M \setminus \{t(0)\}) \neq \emptyset.$ 

 $(2)$   $F(c, O(M_1 \cup M_2)) \cap (M \setminus \{t(0)\}) \neq \emptyset$  *if and only if there exist*  $x \in M_1$  *and*  $y \in M_2$  *such that* 

$$
(\min\{F(c,x), F(c,y)\}, \max\{F(c,x), F(c,y)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset.
$$

**Proof.** In a completely analogous to the proof of Lemma 5.3 in [\[10\]](#page-15-8).

 $\Box$ 

The following theorem characterizes what properties of M are equivalent to the associativity of  $\otimes$ .

<span id="page-11-2"></span>**Theorem 6.1** Let  $M \in \mathcal{A}$  and  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  be associated with M. Then *the operation*  $\otimes$  *on M is associative if and only if*  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$ *.* 

**Proof.** Let  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  be associated with  $M \in \mathcal{A}$ . We prove that the operation ⊗ on M is not associative if and only if  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

Suppose that the operation ⊗ is not associative, i.e., there exist three elements  $x, y, z \in M$  such that  $(x \otimes y) \otimes z \neq x \otimes (y \otimes z)$ . Then we claim that  $F(x, y) \notin M$  or  $F(y, z) \notin M$ . Otherwise, from Proposition [5.2,](#page-7-3)  $F(x, y) \in M$  and  $F(y, z) \in M$  would imply  $(x \otimes y) \otimes z = G_M(F(F(x, y), z)) = G_M(F(x, F(y, z))) =$  $x \otimes (y \otimes z)$ , a contradiction. We consider three cases as below.

(i) Let  $F(x, y) \notin M$  and  $F(y, z) \in M$ . Then  $y \otimes z = F(y, z)$  and either  $F(x, y) \in [0, t(0))$  or there exists a  $k \in K$  such that  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$ . If  $F(x, y) \in [0, t(0))$  then, by Proposition [5.2,](#page-7-3)  $x \otimes y = t(0)$ . It follows from Definition [5.2](#page-7-2) that  $G_M(F(t(0), z)) = (x \otimes y) \otimes z \neq x \otimes (y \otimes z) = G_M(F(x, F(y, z)))$ . On the other hand, by the associativity of F, we have  $G_M(F(x, F(y, z))) = G_M(F(F(x, y), z))$ . Thus  $G_M(F(t(0), z)) \neq G_M(F(F(x, y), z))$ . Therefore, by Lemma [6.1,](#page-11-0)

 $(\min\{F(t(0), z), F(F(x, y), z)\}, \max\{F(t(0), z), F(F(x, y), z)\} \cap (M \setminus \{t(0)\}) \neq \emptyset.$ 

Obviously,  $H^y_\tau = O({t(0)} \cup F(M^y_k, y)),$   $z \in M_y$  and  $x \in M^y_k$ , so that

$$
(\min\{F(t(0),z),F(F(x,y),z)\},\max\{F(t(0),z),F(F(x,y),z)\}]\subseteq F(H^y_\tau,M_y),
$$

which implies  $F(H^y_\tau, M_y) \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

If  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$ , then  $x \otimes y = b_k$ . It follows from Definition [5.1](#page-6-0) that  $G_M(F(b_k, z)) =$  $(x \otimes y) \otimes z \neq x \otimes (y \otimes z) = G_M(F(x, F(y, z)))$ . On the other hand, by the associativity of F, we have  $G_M(F(x, F(y, z))) = G_M(F(F(x, y), z))$ . Thus  $G_M(F(b_k, z)) \neq G_M(F(F(x, y), z))$ . Therefore, by Lemma [6.1,](#page-11-0)

 $(\min\{F(b_k, z), F(F(x, y), z)\}, \max\{F(b_k, z), F(F(x, y), z)\} \cap (M \setminus \{t(0)\}) \neq \emptyset.$ 

Obviously,  $H_k^y = O({b_k} \cup F(M_k^y, y)), z \in M_y$ , and  $x \in M_k^y$ . So that

$$
(\min\{F(b_k, z), F(F(x, y), z)\}, \max\{F(b_k, z), F(F(x, y), z)\}]\subseteq F(H_k^y, M_y),
$$

which implies  $F(H_k^y, M_y) \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

(ii) Let  $F(x, y) \in M$  and  $F(y, z) \notin M$ . In completely analogous to (i),  $F(M^y, H_y^k) \cap (M \setminus \{t(0)\}) \neq \emptyset$ where  $k \in K^*$ .

(iii) Let  $F(x, y) \notin M$  and  $F(y, z) \notin M$ . Then from  $F(x, y) \notin M$ , we have  $F(x, y) \in [0, t(0))$  or there exists a  $k \in K$  such that  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$ . In the case  $F(x, y) \in [0, t(0))$ , from Proposition [5.2](#page-7-3) we have  $x \otimes y = t(0)$ . In the case  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$ , we have  $x \otimes y = b_k$ . Hence  $(x \otimes y) \otimes z =$  $G_M(F(t(0), z))$  or  $(x \otimes y) \otimes z = G_M(F(b_k, z))$ . From  $F(y, z) \notin M$ , we have  $F(y, z) \in [0, t(0))$  or there exists an  $l \in K$  such that  $F(y, z) \in [b_l, d_l] \setminus \{c_l\}$ . If  $F(y, z) \in [0, t(0))$  then, from Proposition [5.2,](#page-7-3)  $y \otimes z = t(0)$ . If  $F(y, z) \in [b_l, d_l] \setminus \{c_l\}$  then  $y \otimes z = b_l$ . Thus  $x \otimes (y \otimes z) = G_M(F(x, t(0)))$  or  $x \otimes (y \otimes z) = G_M(F(x, b_l)).$ 

Since  $(x \otimes y) \otimes z \neq x \otimes (y \otimes z)$ , by Lemma [6.1](#page-11-0) there are four cases as follows.

Case (1).  $(\min\{F(t(0), z), F(x, t(0))\}, \max\{F(t(0), z), F(x, t(0))\}] \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Obviously,  $x \in M_k^y$  and  $F(x,t(0)) \in F(M_k^y,t(0))$ . Similarly,  $z \in M_y^l$  and  $F(t(0),z) \in F(t(0),M_y^l)$ . Therefore,

$$
(\min\{F(t(0), z), F(x, t(0))\}, \max\{F(t(0), z), F(x, t(0))\}] \subseteq J^y_{\tau, \tau}.
$$

This follows  $J^y_{\tau,\tau} \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

Case (2).  $(\min\{F(t(0), z), F(x, b_l)\}, \max\{F(t(0), z), F(x, b_l)\} \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Obviously,  $x \in M_k^y$ and  $F(x, b_i) \in F(M_k^y, b_i)$ . Similarly,  $z \in M_y^l$  and  $F(t(0), z) \in F(t(0), M_y^l)$ . Therefore,

$$
(\min\{F(t(0), z), F(x, b_l)\}, \max\{F(t(0), z), F(x, b_l)\}] \subseteq J_{\tau, l}^y.
$$

This follows  $J_{\tau,l}^y \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

Case (3).  $(\min\{F(b_k, z), F(x, t(0))\}, \max\{F(b_k, z), F(x, t(0))\} \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Obviously,  $x \in M_k^y$ and  $F(x,t(0)) \in F(M_k^y,t(0))$ . Similarly,  $z \in M_y^l$  and  $F(b_k, z) \in F(b_k, M_y^l)$ . Therefore,

$$
(\min\{F(b_k, z), F(x, t(0))\}, \max\{F(b_k, z), F(x, t(0))\}] \subseteq J_{k, \tau}^y.
$$

This follows  $J_{k,\tau}^y \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

Case (4).  $(\min\{F(b_k, z), F(x, b_l)\}, \max\{F(b_k, z), F(x, b_l)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Obviously,  $x \in M_k^y$ and  $F(x, b_i) \in F(M_k^{\tilde{y}}, b_i)$ . Similarly,  $z \in M_y^l$  and  $F(b_k, z) \in F(b_k, M_y^l)$ . Therefore,

$$
(\min\{F(b_k, z), F(x, b_l)\}, \max\{F(b_k, z), F(x, b_l)\}] \subseteq J_{k,l}^y
$$
.

This follows  $J_{k,l}^y \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

(i), (ii) and (iii) mean that we finally have  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) \neq \emptyset$ .

Conversely, suppose  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Then there exist a  $y \in M$  and two elements  $k, l \in K^*$ such that  $F(H_k^y, M_y) \cap (M \setminus \{t(0)\}) \neq \emptyset$  or  $H(M^y, H_y^k) \cap (M \setminus \{t(0)\}) \neq \emptyset$  or  $J_{k,l}^y \cap (M \setminus \{t(0)\}) \neq \emptyset$ . We distinguish three cases as follows.

(i) Let  $F(H_k^y, M_y) \cap (M \setminus \{t(0)\}) \neq \emptyset$ . If  $k \in K^*$ , then there exists a  $z \in M_y$  such that  $F(H_k^y, z) \cap (M \setminus \{t(0)\})$  $\{t(0)\}\neq \emptyset$ . Thus by the definition of  $H_k^y$ ,  $F(M_k^y, y) \neq \emptyset$ . Applying Lemma [6.2,](#page-11-1) there exist  $u \in \{t(0), b_k\}$ and  $v \in F(M_k^y, y)$  such that

$$
(\min\{F(u,z), F(v,z)\}, \max\{F(u,z), F(v,z)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset.
$$

Because of  $v \in F(M_k^y, y)$ , there exists an  $x \in M_k^y$  such that  $F(x, y) = v$ . Therefore, there exist two elements  $u \in \{t(0), b_k\}$  and  $x \in M_k^y$  such that

$$
(\min\{F(u,z), F(F(x,y),z)\}, \max\{F(u,z), F(F(x,y),z)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset.
$$

Consequently, from Lemma [6.1](#page-11-0) we have  $G_M(F(u, z)) \neq G_M(F(F(x, y), z))$ . On the other hand, from  $z \in M_y$ , we have  $F(y, z) \in M$ . This follows  $y \otimes z = F(y, z)$ . From  $x \in M_k^y$  we have  $F(x, y) \in [0, t(0))$  or  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}.$  If  $F(x, y) \in [0, t(0))$  then  $x \otimes y = t(0)$ . Therefore,  $(x \otimes y) \otimes z = G_M(F(t(0), z)) \neq$  $G_M(F(F(x,y), z)) = G_M(F(x, F(y, z))) = x \otimes (y \otimes z)$ . If  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$  then  $x \otimes y = b_k$ . Therefore,  $(x \otimes y) \otimes z = G_M(F(b_k, z)) \neq G_M(F(F(x, y), z)) = G_M(F(x, F(y, z))) = x \otimes (y \otimes z).$ 

(ii) Let  $F(M^y, H_y^k) \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Then in complete analogy to (i),  $(x \otimes y) \otimes z \neq x \otimes (y \otimes z)$ . (iii) Let  $J_{k,l}^y \cap (M \setminus \{t(0)\}) \neq \emptyset$ . Then  $J_{k,l}^y \neq \emptyset$ . Thus by the definition of  $J_{k,l}^y$ , we have  $F(O(F(M_k^y, a))$ 

 $F(b, M_y^l), e) \cap (M \setminus \{t(0)\}) \neq \emptyset$  where  $a \in \{t(0), b_l\}, b \in \{t(0), b_k\}$  and e is a neutral element of F, which means  $F(M_k^y, a) \neq \emptyset$  and  $F(b, M_y^l) \neq \emptyset$ . Applying Lemma [6.2,](#page-11-1) there exist two elements  $u \in F(b, M_y^l)$ and  $v \in F(M_k^y, a)$  such that

$$
(\min\{F(u,e), F(v,e)\}, \max\{F(u,e), F(v,e)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset.
$$

Because  $u \in F(b, M_y^l)$  and  $v \in F(M_k^y, a)$ , there exist a  $z \in M_y^l$  and an  $x \in M_k^y$  such that  $u = F(b, z)$ ,  $v = F(x, a)$ . Therefore, there exist an  $x \in M_k^y$  and a  $z \in M_y^l$  such that

$$
(\min\{F(b,z), F(x,a)\}, \max\{F(b,z), F(x,a)\}] \cap (M \setminus \{t(0)\}) \neq \emptyset
$$

since e is a neutral element of F. Further, by Lemma [6.1](#page-11-0) we have  $G_M(F(b, z)) \neq G_M(F(x, a))$ .

On the other hand, from  $x \in M_k^y$  we have  $F(x, y) \in [0, t(0))$  or  $F(x, y) \in [b_k, d_k] \setminus \{c_k\}$ . Thus  $x \otimes y = t(0)$  or  $x \otimes y = b_k$ . From  $z \in M_y^l$  we have  $F(y, z) \in [0, t(0))$  or  $F(y, z) \in [b_l, d_l] \setminus \{c_l\}$ . Thus  $y \otimes z = t(0)$  or  $y \otimes z = b_l$ .

Therefore, 
$$
(x \otimes y) \otimes z = G_M(F(b, z)) \neq G_M(F(x, a)) = x \otimes (y \otimes z)
$$
.  
Further, from Theorems 5.1 and 6.1 we have the following one.

<span id="page-13-0"></span>**Theorem 6.2** Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function and  $T : [0,1]^2 \to [0,1]$ *be a function defined by Eq.* $(7)$ *. Then the function* T *is associative if and only if*  $\mathfrak{I}(M) \cap (M \setminus \{t(0)\}) = \emptyset$ *.* 

From Definition [6.1,](#page-10-2) we have  $0 \notin \mathfrak{T}(M)$ . Thus if  $t(0) = 0$ , then  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$  if and only if  $\mathfrak{T}(M) \cap M = ∅$ . Therefore, we have the following corollary.

Corollary 6.1 Let  $t : [0,1] \rightarrow [0,\infty]$  be a left continuous non-decreasing function with  $t(0) = 0$  and  $T : [0,1]^2 \rightarrow [0,1]$  be a function defined by Eq.[\(7\)](#page-5-0). Then the function T is associative if and only if  $\mathfrak{T}(M) \cap M = \emptyset$ .

From Proposition [3.3,](#page-5-1) we have the following corollary.

Corollary 6.2 Let  $t : [0,1] \to [0,\infty]$  be a left continuous non-decreasing function and  $T : [0,1]^2 \to [0,1]$ *be a function defined by Eq.*[\(7\)](#page-5-0)*.* If  $F : [0, \infty]^2 \to [0, \infty]$  *is a function such that*  $([0, \infty], F, \leq)$  *is a fully ordered Abel semigroup with neutral element in*  $[0, \infty]$  *and*  $F(x, 0) \geq x$  *for all*  $x \in [0, \infty]$ *, then* T *is a t*-supconorm if and only if  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$ .

In particular, we have the following remark.

**Remark 6.1** Let  $t : [0,1] \to [0,1]$  be a left continuous non-decreasing function and  $T : [0,1]^2 \to [0,1]$  be a function defined by Eq.[\(7\)](#page-5-0).

(i) If F is a t-conorm, then  $T(x, 0) = t^{[-1]}(F(t(x), t(0))) \ge t^{[-1]}(t(x)) \ge x$  for all  $x \in [0, 1]$ . So, 0 isn't necessary a neutral element of T. Therefore, if  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$  then T isn't necessary a t-conorm. However, if t is further a strictly increasing function with  $t(0) = 0$  then T is a t-conorm. Another way is to slightly modify the function T as for all  $x, y \in [0, 1]$ ,

$$
T(x,y) = \begin{cases} \max\{x,y\} & \text{if } \min\{x,y\} = 0, \\ t^{(-1)}(F(t(x),t(y))) & \text{otherwise.} \end{cases}
$$

Then one can check that T is a t-conorm.

(*ii*) If F is a t-norm, then  $T(x, 1) = t^{[-1]}(F(t(x), t(1))) \le t^{[-1]}(t(x))$  and  $t^{[-1]}(t(x)) \ge x$  for all  $x \in [0, 1]$ . So, 1 isn't necessary a neutral element of T. Therefore, if  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$  then T isn't necessary a t-subnorm. However, if t is further a strictly increasing function with  $t(1) = 1$  then T is a t-norm.

**Example 6.1** Let  $M \in \mathcal{A}$  and  $(\mathcal{U}, \mathcal{V}) = (\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in \overline{K}\})$  be associated with M.

(1) Let  $F(x, y) = \max\{x, y\}$  for all  $x, y \in [0, 1]$ . In Example [4.1](#page-6-1) (i),  $M = [0, \frac{1}{4}] \cup {\frac{3}{2}} \cup {\frac{3}{4}}$ , 1], and we have  $\mathfrak{T}_{1}(M) = \mathfrak{T}_{2}(M) = \emptyset$  and  $\mathfrak{T}_{3}(M) = (\frac{1}{2}, \frac{3}{4}]$ . So,  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$  and by Theorem [6.2,](#page-13-0) the following function  $T : [0,1]^2 \to [0,1]$  given by Eq.[\(7\)](#page-5-0) is associative:

$$
T(x,y) = \begin{cases} \frac{3}{4} & \text{if } (x,y) \in [0, \frac{3}{4}] \times [\frac{1}{2}, \frac{3}{4}] \cup [\frac{1}{2}, \frac{3}{4}] \times [0, \frac{3}{4}],\\ \max\{x, y\} & \text{otherwise.} \end{cases}
$$

(2) Let  $F(x, y) = xy$  for all  $x, y \in [0, 1]$  and the function  $t : [0, 1] \rightarrow [0, \infty]$  be defined by

$$
t(x) = \begin{cases} 1 & \text{if } x = 0, \\ \infty & \text{otherwise.} \end{cases}
$$

Then  $M = \{1, \infty\}$ , and  $\mathfrak{T}(M) = \emptyset$ . So,  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$  and by Theorem [6.2,](#page-13-0) the following function  $T : [0, 1]^2 \to [0, 1]$  given by Eq.[\(7\)](#page-5-0) is associative:

$$
T(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1]^2, \\ 0 & \text{otherwise.} \end{cases}
$$

(3) Let  $F(x, y) = x + y$  and the function  $t : [0, 1] \rightarrow [0, \infty]$  be defined by

$$
t(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}], \\ \frac{1}{4} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}], \\ x & \text{if } x \in (\frac{1}{2}, 1] \end{cases}
$$

Then  $M = [0, \frac{1}{4}] \cup (\frac{1}{2}, 1]$ , and we have  $\mathfrak{T}_1(M) = \mathfrak{T}_2(M) = (\frac{1}{2}, 1]$ . So,  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) \neq \emptyset$  and by Theorem [6.2,](#page-13-0) the following function  $T : [0,1]^2 \to [0,1]$  given by Eq.[\(7\)](#page-5-0) isn't associative:

$$
T(x,y) = \begin{cases} x+y & \text{if } 0 \le x+y < \frac{1}{4} \text{ or } \frac{1}{2} < x+y \le 1, \\ \frac{1}{2} & \text{if } \frac{1}{4} \le x+y \le \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}
$$

Indeed, put  $x = \frac{1}{5}$ ,  $y = \frac{1}{4}$ ,  $z = \frac{1}{2}$ . Then  $T(T(x, y), z) = T(T(\frac{1}{5}, \frac{1}{4}), \frac{1}{2}) = T(\frac{1}{2}, \frac{1}{2}) = 1$ ,  $T(x, T(y, z)) =$  $T(\frac{1}{5}, T(\frac{1}{4}, \frac{1}{2})) = T(\frac{1}{5}, \frac{3}{4}) = \frac{19}{20}$ . Thus, T isn't an associative function.

## 7 Conclusions

One can easily check that our results are suitable for all left continuous non-increasing functions also. So that the main contributions of this article conclude that we gave the concept of a weak pseudoinverse of a monotone function for overcoming the difficulty that a function  $T : [0,1]^2 \rightarrow [0,1]$  defined by Eq.[\(3\)](#page-0-0) isn't associative when t is a left continuous monotone function, and we answered what is the characterization of a left continuous monotone function  $t : [0,1] \to [0,\infty]$  such that the function  $T : [0,1]^2 \to [0,1]$  given by Eq.[\(7\)](#page-5-0) is associative. It is regrettable that generally, our results aren't true when t is a right continuous monotone function. For instance, let  $F(x, y) = \max\{x, y\}$  and the function  $t:[0,1] \rightarrow [0,1]$  be defined by

$$
t(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in (\frac{1}{2}, \frac{3}{4}), \\ \frac{3}{4} & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}
$$

It is easy to see that t is a right continuous non-decreasing function. Then  $M = [0, \frac{1}{2}] \cup {\frac{3}{4}}$ . Thus from Definition 5.2 of [\[10\]](#page-15-8), we have  $I(M) = \emptyset$ , so that  $I(M) \cap (M \setminus \{t(1)\}) = \emptyset$ . Therefore, by Corollary 5.3 of [\[10\]](#page-15-8) we know that the following function  $T : [0,1]^2 \to [0,1]$  given by Eq.(6) of [10] is associative since  $t(1) < 1$ :

$$
T(x,y) = \begin{cases} \max\{x,y\} & \text{if } (x,y) \in [0,\frac{1}{2})^2, \\ \frac{3}{4} & \text{otherwise.} \end{cases}
$$

On the other hand, from Definition [6.1](#page-10-2) we have  $\mathfrak{T}(M) = \emptyset$ . This follows that  $\mathfrak{T}(M) \cap (M \setminus \{t(0)\}) = \emptyset$ . However, from Eq.[\(7\)](#page-5-0),

$$
T(x,y) = \begin{cases} \max\{x,y\} & \text{if } (x,y) \in [0,\frac{1}{2})^2, \\ \frac{3}{4} & \text{if } (x,y) \in [0,\frac{3}{4})^2 \setminus [0,\frac{1}{2})^2, \\ 1 & \text{otherwise.} \end{cases}
$$

Put  $x = \frac{1}{4}$ ,  $y = \frac{1}{4}$ ,  $z = \frac{1}{2}$ . Then  $T(T(x, y), z) = T(T(\frac{1}{4}, \frac{1}{4}), \frac{1}{2}) = T(\frac{1}{4}, \frac{1}{2}) = \frac{3}{4}$ ,  $T(x, T(y, z)) =$  $T(\frac{1}{4}, T(\frac{1}{4}, \frac{1}{2})) = T(\frac{1}{4}, \frac{3}{4}) = 1$ . Thus, T isn't an associative function.

# <span id="page-15-0"></span>References

- [1] N. H. Abel, Untersuchung der Function zweier unabhangig veranderlichen Grossen x und y wie  $f(x, y)$ , welche die Eigenschaft haben, dass  $f(z, f(x, y))$  eine symmetrische Function von x, y und z ist, J. Reine Angew. Math. 1 (1826) 11-15.
- <span id="page-15-9"></span>[2] E. P. Klement, R. Mesiar, E. Pap, Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms, Fuzzy Sets Syst. 104 (1999) 3-13.
- <span id="page-15-3"></span><span id="page-15-2"></span>[3] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
- <span id="page-15-1"></span>[4] C. H. Ling, Representation of associative functions, Publ, Math, Debrecen 12 (1965) 189-212.
- [5] B. Schweizer, A. Sklar, Associative functions and satistical triangle inequalities, Publ. Math. Debrecen 8 (1961) 169-186.
- <span id="page-15-6"></span><span id="page-15-5"></span>[6] P. Viceník, A note on generators of t-norms, BUSEFAL 75 (1998) 33-38.
- <span id="page-15-4"></span>[7] P. Viceník, Additive generators and discontinuity, BUSEFAL 76 (1998) 25-28.
- <span id="page-15-7"></span>[8] P. Viceník, Additive generators of associative functions, Fuzzy Sets Syst. 153 (2005) 137-160.
- <span id="page-15-8"></span>[9] P. Viceník, Additive generators of border-continuous triangular norms, Fuzzy Sets Syst. 159 (2008) 1631-1645.
- [10] Y. M. Zhang, X, P, Wang, Characterizations of monotone right continuous finctions which generate associative functions, Fuzzy Sets Syst. 477 (2024) 108799.