Graphs with Lin-Lu-Yau curvature at least one and regular bone-idle graphs

Moritz Hehl

Institute of Mathematics, Universität Leipzig, 04109 Leipzig, Germany

November 21, 2024

Abstract

We study the Ollivier-Ricci curvature and its modification introduced by Lin, Lu, and Yau on graphs. We provide a complete characterization of all graphs with Lin-Lu-Yau curvature at least one. We then explore the relationship between the Lin-Lu-Yau curvature and the Ollivier-Ricci curvature with vanishing idleness on regular graphs. An exact formula for the difference between these two curvature notions is established, along with an equality condition. This condition allows us to characterize edges that are bone-idle in regular graphs. Furthermore, we demonstrate the non-existence of 3-regular bone-idle graphs and present a complete characterization of all 4-regular bone-idle graphs. We also show that there exist no 5-regular bone-idle graphs that are symmetric or a Cartesian product of a 3-regular and a 2-regular graph.

Keywords: Graph curvature, Ollivier-Ricci curvature, Ricci-flat, regular graphs, optimal transportation

Mathematics Subject Classification (2020): 05C75, 53C21, 05C81, 05C30, 68R10

1 Introduction and statement of results

Since the introduction of the geometric notion of curvature by Gauss and Riemann over 150 years ago, it has played a central role in differential geometry. Among the different types of curvature, Ricci curvature is of particular importance, serving as a fundamental tool in the study of Riemannian manifolds. Given its importance in differential geometry, it is natural to seek extensions of Ricci curvature to broader classes of metric spaces beyond Riemannian manifolds. This pursuit has led to the development of various generalized curvature notions for non-smooth or discrete structures, see, e.g., Bakry-Émery [1], Erbar-Maas [2], Mielke [3] and Forman [4].

In this work, we study a notion of Ricci-curvature introduced by Ollivier in 2009 [5]. Von Renesse and Sturm [6] established a connection between Ricci curvature and optimal

E-mail address: moritz.hehl@uni-leipzig.de

transport on smooth Riemannian manifolds. Building upon their results, Ollivier developed a discrete notion of Ricci curvature on metric spaces equipped with Markov chains or a measure, known as the *Ollivier-Ricci curvature*. This approach leverages optimal transport theory, as Ollivier's definition of curvature is based on the Wasserstein distance.

In the context of locally finite graphs, Ollivier's notion of Ricci curvature has recently received considerable attention. In this setting, the Ollivier-Ricci curvature κ_{α} is defined on the edges of the graph and depends on an *idleness parameter* $\alpha \in [0, 1]$. Ollivier considered idleness parameter $\alpha = 0$ and $\alpha = \frac{1}{2}$. In 2011, Lin, Lu, and Yau [7] introduced a modification of the Ollivier-Ricci curvature, by computing the derivative of the curvature with respect to the idleness parameter. We will refer to this modification as *Lin-Lu-Yau curvature* and denote it by κ .

Our first result is a complete characterization of the graphs with Lin-Lu-Yau curvature greater than or equal to one for every edge.

Theorem 1.1. Let G = (V, E) be a locally finite graph. Then $\kappa(x, y) \ge 1$ for every edge $x \sim y \in E$ if and only if the minimum degree $\delta(G) \ge |V| - 2$.

We then focus on the relationship between the 0-Ollivier-Ricci curvature κ_0 and the Lin-Lu-Yau curvature κ . For regular graphs, we derive an exact formula for the difference between these two curvature notions.

Theorem 1.2. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. If $|S_1(x) \cap S_1(y)| < d-1$, then

$$\kappa(x,y) - \kappa_0(x,y) = \frac{1}{d} \left(3 - \sup_{\phi \in \mathcal{O}_{xy}} \sup_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right)$$

where \mathcal{O}_{xy} denotes the set of optimal assignments between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$. If $|S_1(x) \cap S_1(y)| = d - 1$, then

$$\kappa(x,y) - \kappa_0(x,y) = \frac{2}{d}.$$

Finally, we study an analog of Ricci-flat manifolds. Ricci-flat Lorentzian manifolds play an important role in theoretical physics as solutions to Einstein's field equations in a vacuum with vanishing cosmological constant. As an analog, one might consider graphs where the Ollivier-Ricci curvature vanishes everywhere. In this work, we impose an even stronger condition. Namely, that the Ollivier-Ricci curvature κ_{α} vanishes everywhere for every idleness parameter α . We refer to such graphs as *bone-idle*. It turns out that bone-idleness is equivalent to the vanishing of both the 0-Ollivier-Ricci curvature and the Lin-Lu-Yau curvature κ everywhere. Therefore, we can apply our results on the relation between the Ollivier-Ricci curvature and the Lin-Lu-Yau curvature. Using this, we characterize edges that are bone-idle in regular graphs. Furthermore, we show that no 3-regular bone-idle graph exists.

Theorem 1.3. Let G = (V, E) be a locally finite graph. Suppose that G is bone-idle, then G is not 3-regular.

We also provide a complete characterization of 4-regular bone-idle graphs in Section 5.3 and discuss the existence of 5-regular bone-idle graphs.

We conclude this introduction by providing an outline of the remainder of the paper. In Section 2 we review the relevant concepts of Graph Theory and Optimal Transport Theory, and introduce the Ollivier-Ricci curvature, as well as its modification by Lin, Lu, and Yau. In Section 3 we provide a complete characterization of all graphs with Lin-Lu-Yau curvature at least one. In Section 4 we explore the relationship between the Lin-Lu-Yau curvature and the 0-Ollivier-Ricci curvature on regular graphs. Finally, in Section 5, we present our findings on bone-idle graphs.

2 Definitions and notations

We begin by reviewing some fundamental concepts of Graph Theory and Optimal Transport Theory. We then introduce Ollivier's discrete notion of Ricci curvature on graphs, as well as its modification by Lin, Lu, and Yau.

2.1 Graph Theory

A simple graph G = (V, E) is an unweighted, undirected graph that contains no multiple edges or self-loops. For two vertices $x, y \in V$ we denote the existence of an edge between x and y by $x \sim y$. For any two vertices $x, y \in V$, the shortest-path distance d(x, y) is the number of edges in a shortest path connecting x and y. If no such path exists, d(x, y) is defined to be infinity. The diameter of G is denoted by $diam(G) = \max_{x,y \in V} d(x, y)$. The girth of G is the length of a shortest cycle contained in G. If G does not contain any cycles, the girth is defined to be infinity.

For $x \in V$ and $r \in \mathbb{N}$ we define the *r*-sphere centered at x as $S_r(x) = \{y \in V : d(x, y) = r\}$ and the *r*-ball centered at x as $B_r(x) = \{y \in V : d(x, y) \leq r\}$. For an edge $x \sim y$, we denote by $N_{xy} = S_1(x) \cap S_1(y)$ the set of common neighbors of x and y.

The degree of a vertex $x \in V$ is denoted by $d_x = |S_1(x)|$. The minimum degree of a graph G is denoted by $\delta(G) = \min_{x \in V} d_x$. The graph G is called *locally finite* if every vertex has finite degree. The graph is said to be *d*-regular if every vertex has the same degree d.

An automorphism of a graph G = (V, E) is a permutation σ of the vertex set V such that for any pair of vertices $x, y \in V$, $x \sim y$ if and only if $\sigma(x) \sim \sigma(y)$. In other words, an automorphism is an isomorphism of G to itself. An *edge automorphism* of G = (V, E) is a permutation of the edge set E that sends edges with a common endpoint into edges with a common endpoint.

A graph is called *vertex-transitive* if, for any two vertices $x, y \in V$, there exists an automorphism σ of the graph such that $\sigma(x) = y$. A graph is called *edge-transitive* if, for any two edges $e_1, e_2 \in E$, there exists an edge automorphism γ such that $\gamma(e_1) = e_2$.

We call a graph *symmetric* if it is both vertex-transitive and edge-transitive.

We conclude this section by defining the *Cartesian product* of graphs. Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the Cartesian product $G \Box H$, is a graph with vertex

set $V_G \times V_H$. Two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \Box H$ if and only if either $x_1 = x_2$ and $y_1 \sim y_2 \in E_H$, or $x_1 \sim x_2 \in E_G$ and $y_1 = y_2$.

2.2 Ollivier-Ricci curvature and its modification

The Wasserstein distance, a metric defined on the space of probability measures, is a fundamental concept in optimal transport theory.

Definition 2.1 (Wasserstein distance). Let G = (V, E) be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. The Wasserstein distance between μ_1 and μ_2 is defined as

$$W_1(\mu_1, \mu_2) = \inf_{\pi \in \Pi(\mu_1, \mu_2)} \sum_{x \in V} \sum_{y \in V} d(x, y) \pi(x, y),$$
(2.1)

where

$$\Pi(\mu_1, \mu_2) = \left\{ \pi : V \times V \to [0, 1] : \sum_{y \in V} \pi(x, y) = \mu_1(x), \ \sum_{x \in V} \pi(x, y) = \mu_2(y) \right\}.$$

Intuitively, imagine two distributions given by μ_1 and μ_2 as piles of earth. The Wasserstein distance measures the minimal effort required to transform one pile of earth into another. We call $\pi \in \Pi(\mu_1, \mu_2)$ a transport plan and if the infimum in 2.1 is attained, we call π an optimal transport plan transporting μ_1 to μ_2 .

It is a well-known fact in optimal transport theory that no mass needs to be moved when it is shared between the two probability measures.

Lemma 2.2. Let G = (V, E) be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. Then there exists an optimal transport plan π transporting μ_1 to μ_2 satisfying

$$\pi(x, x) = \min\{\mu_1(x), \mu_2(x)\}\$$

for all $x \in V$.

To introduce Ollivier's notion of Ricci curvature on graphs, we define the probability measures μ_x^{α} for $x \in V$ and $\alpha \in [0, 1]$ by

$$\mu_x^{\alpha}(y) = \begin{cases} \alpha, & \text{if } y = x; \\ \frac{1-\alpha}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise.} \end{cases}$$

Then, the Ollivier-Ricci curvature is defined as follows.

Definition 2.3 (Ollivier-Ricci curvature). Let G = (V, E) be a locally finite graph. We define the α -Ollivier-Ricci curvature of an edge $x \sim y$ by

$$\kappa_{\alpha}(x,y) = 1 - W_1(\mu_x^{\alpha}, \mu_y^{\alpha}).$$

The parameter α is called the *idleness*.

We present a more intuitive formula for the Ollivier-Ricci curvature, as provided in [8]. Let ν_i^{α} be the mass transported with distance *i* under an optimal transport plan transporting μ_x^{α} to μ_y^{α} . Then

$$\sum_{i=0}^{3} \nu_{i}^{\alpha} = 1 \text{ and } W_{1}(\mu_{x}^{\alpha}, \mu_{y}^{\alpha}) = \sum_{i=1}^{3} i\nu_{i}^{\alpha}.$$

Therefore, we obtain

$$\kappa_{\alpha}(x,y) = \nu_0^{\alpha} - \nu_2^{\alpha} - 2\nu_3^{\alpha}.$$

Ollivier considered idleness parameters $\alpha = 0$ and $\alpha = \frac{1}{2}$. In [9], the authors study the Ollivier-Ricci curvature as a function of the idleness parameter. To this end, they introduced the *Ollivier-Ricci idleness function* $\alpha \to \kappa_{\alpha}(x, y)$. It was first shown by Lin, Lu, and Yau in [7], that the idleness function is concave. Using that $\kappa_1(x, y) = 0$, this implies that the function $h(\alpha) = \frac{\kappa_{\alpha}(x,y)}{1-\alpha}$ is increasing over the interval [0, 1). They further showed that $h(\alpha)$ is bounded and thus, the limit $\lim_{\alpha \to 1} h(\alpha)$ exists. Lin, Lu, and Yau used this result to introduce a modified version of the Ollivier-Ricci curvature that does not depend on the idleness.

Definition 2.4 (Lin-Lu-Yau curvature). Let G = (V, E) be a locally finite graph. The *Lin-Lu-Yau curvature* of an edge $x \sim y$ is defined as

$$\kappa(x,y) = \lim_{\alpha \to 1} \frac{\kappa_{\alpha}(x,y)}{1-\alpha}.$$

Remark 2.5. Observe that $\kappa_1(x, y) = 0$ for any edge $x \sim y$. Thus, $\kappa(x, y)$ is the negative of the derivative of $\kappa_{\alpha}(x, y)$ with respect to the idleness parameter in $\alpha = 1$.

In what follows, we write $Ric(G) \ge k$ (Ric(G) = k) if $\kappa(x, y) \ge k$ $(\kappa(x, y) = k)$ for all edges $x \sim y$ in G.

Bourne et al. [9] showed that the idleness function is piecewise linear with at most 3 linear parts. They also derived the length of the last linear part.

Theorem 2.6 ([9], Theorem 4.4). Let G = (V, E) be a locally finite graph and let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then $\alpha \to \kappa_{\alpha}(x, y)$ is linear over $\left[\frac{1}{d_x+1}, 1\right]$.

Thus, we obtain the following relation between the α -Ollivier-Ricci curvature and its modification by Lin, Lu, and Yau as an immediate consequence of the mean value theorem.

Theorem 2.7. Let G = (V, E) be a locally finite graph and let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then

$$\kappa_{\alpha}(x,y) = (1-\alpha)\kappa(x,y)$$

for $\alpha \in \left[\frac{1}{d_x+1}, 1\right]$.

Hence, the Lin-Lu-Yau curvature coincides up to a scaling factor with the α -Ollivier-Ricci curvature for large values of α .

For regular graphs, the optimal transport problem reduces to an optimal assignment problem between subsets of the 1-spheres. In [10], the author uses this observation to derive a simplified formula for the Lin-Lu-Yau curvature. To formalize this approach, we introduce the concept of an optimal assignment. **Definition 2.8** (Optimal assignment). Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. We call a bijection $\phi : S_1(x) \setminus B_1(y) \to S_1(y) \setminus B_1(x)$ an assignment between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$. Denote by \mathcal{A}_{xy} the set of all such assignments. We call $\phi \in \mathcal{A}_{xy}$ an optimal assignment if

$$\sum_{z \in S_1(x) \backslash B_1(y)} d(z, \phi(z)) = \inf_{\psi \in \mathcal{A}_{xy}} \sum_{z \in S_1(x) \backslash B_1(y)} d(z, \psi(z)).$$

The set of all optimal assignments between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ is denoted by \mathcal{O}_{xy} .

Remark 2.9. Observe that the condition that x and y have the same degree is necessary to ensure that $|S_1(x) \setminus B_1(y)| = |S_1(y) \setminus B_1(x)|$.

Theorem 2.10 ([10], Theorem 4.3). Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then the Lin-Lu-Yau curvature

$$\kappa(x,y) = \frac{1}{d} \left(d + 1 - \inf_{\phi \in \mathcal{A}_{xy}} \sum_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right),$$

A similar formula holds true for the Ollivier-Ricci curvature in the case of vanishing idleness, i.e., $\alpha = 0$.

Theorem 2.11 ([10], Theorem 4.8). Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then

$$\kappa_0(x,y) = \frac{1}{d} \left(d - \inf_{\phi} \sum_{z \in S_1(x) \setminus S_1(y)} d(z,\phi(z)) \right),$$

where the infimum is taken over all bijections ϕ between $S_1(x) \setminus S_1(y)$ and $S_1(y) \setminus S_1(x)$.

3 Lin-Lu-Yau curvature at least one

In this section, we characterize all graphs for which the Lin-Lu-Yau curvature is greater than or equal to one. To this end, we first establish the following upper bound on the Lin-Lu-Yau curvature.

Theorem 3.1. Let G = (V, E) be a locally finite graph and let $x, y \in V$ with $x \sim y$ and $d_x \geq d_y$. Then

$$\kappa(x,y) \le \frac{|N_{xy}| + 2}{d_x}.$$

Remark 3.2. In [11, Theorem 4], the authors establish a similar upper bound for the 0-Ollivier-Ricci curvature.

Proof. If $d_x = d_y$, observe that $d(i, j) \ge 1$ for every $i \in S_1(x) \setminus B_1(y)$ and $j \in S_1(y) \setminus B_1(x)$. Using Theorem 2.10, we obtain

$$\kappa(x,y) = \frac{1}{d_x} \left(d_x + 1 - \inf_{\phi \in \mathcal{A}_{xy}} \sum_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right)$$
$$\leq \frac{1}{d_x} \left(d_x + 1 - |S_1(x) \setminus B_1(y)| \right)$$
$$= \frac{|N_{xy}| + 2}{d_x}.$$

Now, assume $d_x > d_y$ and set $\alpha = \frac{1}{d_y+1}$. According to Lemma 2.2, there exists an optimal transport plan π transporting μ_x^{α} to μ_y^{α} , such that $\pi(j,j) = \min\{\mu_x^{\alpha}(j), \mu_y^{\alpha}(j)\}$ for all $j \in V$. Therefore,

$$\nu_0^{\alpha} = (|N_{xy}| + 1)\frac{1 - \alpha}{d_x} + \frac{1 - \alpha}{d_y}$$

where ν_i^{α} denotes the mass transported with distance *i* under π .

Observe that the set $I = \{z \in B_1(x) \setminus \{y\} : \pi(z, y) > 0\}$ is non-empty, as $\mu_y^{\alpha}(y) - \pi(y, y) > 0$. Furthermore, I does not contain x and common neighbors of x and y. Therefore, every vertex in I is at distance two of y. Hence,

$$\nu_2^{\alpha} \ge \mu_y^{\alpha}(y) - \pi(y, y) = \alpha - \frac{1 - \alpha}{d_x}.$$

Using $\alpha = \frac{1-\alpha}{d_y}$, we conclude that

$$\kappa_{\alpha}(x,y) = \nu_{0}^{\alpha} - \nu_{2}^{\alpha} - 2\nu_{3}^{\alpha}$$
$$\leq \frac{|N_{xy}| + 2}{d_{x}}(1-\alpha).$$

Finally, we apply Theorem 2.7, and obtain

$$\kappa(x,y) = \frac{1}{1-\alpha} \kappa_{\alpha}(x,y) \le \frac{|N_{xy}| + 2}{d_x}.$$

The following Bonnet-Myers type theorem on graphs will be of importance.

Theorem 3.3 (Discrete Bonnet-Myers Theorem [7]). Let G = (V, E) be a locally finite graph. If $Ric(G) \ge k > 0$, then the diameter of the graph G is bounded as follows:

$$diam(G) \le \frac{2}{k}.$$

We are now prepared to prove the main result of this section.

Theorem 1.1. Let G = (V, E) be a locally finite graph. Then $Ric(G) \ge 1$ if and only if $\delta(G) \ge |V| - 2$.

Proof. " \Leftarrow " Let $x \sim y$ be an arbitrary edge in G and assume, without loss of generality, that $d_x \geq d_y$. As $\delta(G) \geq |V| - 2$, we have $V = S_1(x) \cup S_1(y)$. Thus,

$$|V| = d_x + d_y - |N_{xy}|.$$

Using $d_y \ge |V| - 2$, we obtain $|N_{xy}| \ge d_x - 2$. If $|N_{xy}| = d_x - 1$, then $d_x = d_y$ must hold and we can apply Theorem 2.10, yielding $\kappa(x, y) = \frac{d+1}{d} > 1$.

If $|N_{xy}| = d_x - 2$, then x has exactly one neighbor z that is not adjacent to y. As $\delta(G) \ge |V| - 2$, z must be adjacent to every vertex in G besides y. Now, assume $\alpha = \frac{1}{d_y+1}$ and let π be an optimal transport plan transporting μ_x^{α} to μ_y^{α} , satisfying the property stated in Lemma 2.2. As z is adjacent to every vertex in G besides y, we obtain

$$\nu_0^{\alpha} = (|N_{xy}|+1)\frac{1-\alpha}{d_x} + \frac{1-\alpha}{d_y},$$
$$\nu_2^{\alpha} = \frac{1-\alpha}{d_y} - \frac{1-\alpha}{d_x},$$
$$\nu_3^{\alpha} = 0.$$

Therefore,

$$\kappa_{\alpha}(x,y) = \frac{|N_{xy}| + 2}{d_x}(1-\alpha) = (1-\alpha)$$

Using Theorem 2.7, we conclude $\kappa(x, y) = 1$.

" \implies "We show this by contradiction. Let G be a graph with $\delta(G) < |V| - 2$ and $Ric(G) \ge 1$. Let $x \in V$ such that $d_x = \delta(G)$ and let y be an arbitrary neighbor of x. Observe that $|N_{xy}| \le d_x - 1$. If $d_y > d_x + 1$, then

$$\kappa(x,y) \leq \frac{|N_{xy}|+2}{d_y} \leq \frac{d_x+1}{d_y} < 1,$$

contradicting our assumption.

Therefore, we now assume $d_x \leq d_y \leq d_x + 1$. As

$$1 \le \kappa(x, y) \le \frac{|N_{xy}| + 2}{d_y},$$

we conclude $|N_{xy}| \ge d_y - 2 \ge d_x - 2$. First, assume $|N_{xy}| = d_x - 1$. As $d_x < |V| - 2$, there exist two vertices $i, j \in V$ such that $x \not\sim i, j$. According to Theorem 3.3, d(i, x) = 2 and d(j, x) = 2 must hold true. If both i and j are adjacent to y, then $d_y \ge d_x + 2$, which contradicts our assumption. Thus, there exists a $z \in N_{xy}$ such that, without loss of generality, $i \sim z$ and $i \not\sim y, i \not\sim x$. Therefore, $|N_{iz}| \le d_z - 3$, which leads to

$$\kappa(i,z) \le \frac{|N_{iz}|+2}{\max\{d_i, d_z\}} \le \frac{d_z - 1}{\max\{d_i, d_z\}} < 1,$$

contradicting $Ric(G) \ge 1$.

Finally, assume $|N_{xy}| = d_x - 2$. Observe that, in this case, it is necessary for the equality $d_y = d_x$ to hold. Denote the vertex in $S_1(x) \setminus B_1(y)$ by z_1 and the vertex in $S_1(y) \setminus B_1(x)$ by z_2 . According to Theorem 2.10, we have

$$\kappa(x,y) = \frac{1}{d} \Big(d + 1 - d(z_1, z_2) \Big).$$

Using $\kappa(x, y) \geq 1$, we conclude $d(z_1, z_2) = 1$. As $d_x < |V| - 2$, there exist two vertices $i, j \in V$ such that $x \not\sim i, j$. It is not possible for y to be adjacent to both i and j, as this would contradict $d_y = d_x$. Therefore, without loss of generality, we can assume $y \not\sim i$. Again, according to Theorem 3.3, we must have d(x, i) = 2. Thus, it exists a $z \in S_1(x) \setminus \{y\}$ such that $z \sim i$. If $z = z_1$, then $|N_{xz}| \leq d_z - 3$, because $x \not\sim z_2$ and $x \not\sim i$. Therefore, we conclude

$$\kappa(x,z) \le \frac{|N_{xz}|+2}{d_z} < 1,$$

contradicting out assumption. Hence, $z \in N_{xy}$ must hold. As before, we obtain $|N_{iz}| \leq d_z - 3$, because $i \not\sim x$ and $i \not\sim y$, which leads to

$$\kappa(i,z) \le \frac{|N_{iz}|+2}{\max\{d_i, d_z\}} \le \frac{d_z - 1}{\max\{d_i, d_z\}} < 1.$$

This contradiction concludes the proof.

Corollary 3.4. Let G = (V, E) be a regular graph with $Ric(G) \ge 1$. Then G is isomorphic to one of the following graphs:

- (i) A complete graph, satisfying $Ric(G) = \frac{|V|}{|V|-1}$,
- (i) a cocktail party graph, satisfying Ric(G) = 1.

Remark 3.5. A cocktail party graph G = (V, E) is a regular graph of degree d, where d = |V| - 2.

Corollary 3.6. Let G = (V, E) be a locally finite graph with Ric(G) = 1. If |V| is even, then G is a cocktail party graph. If |V| is odd, then G is the graph with degree sequence (|V| - 1, |V| - 2, ..., |V| - 2).

Proof. Assume G is a locally finite graph, satisfying Ric(G) = 1. If G is regular, then G is a cocktail party graph, according to Corollary 3.4. Note that in this case, |V| must be even.

Next, assume G is not regular. According to Theorem 1.1, we have $\delta(G) \geq |V| - 2$. If there exist two vertices x, y with degree d > |V| - 2, then $|N_{xy}| = d - 1$, leading to $\kappa(x, y) = \frac{d+1}{d} > 1$, contradicting Ric(G) = 1. Therefore, G can only have degree sequence $(|V| - 1, |V| - 2, \dots, |V| - 2)$. It remains to show that the graph with this degree sequence indeed satisfies Ric(G) = 1. To this end, let $x \sim y$ be an arbitrary edge in G. If $d_x = |V| - 1 > d_y = |V| - 2$, then $|N_{xy}| = d_y - 1$. This leads to

$$\kappa(x,y) \le \frac{|N_{xy}| + 2}{d_x} = 1.$$

According to Theorem 1.1, we also have $\kappa(x, y) \ge 1$, and therefore $\kappa(x, y) = 1$.

Finally, assume $d_x = d_y = |V| - 2$. If $|N_{xy}| = d_x - 1$, then

$$|S_1(x) \cup S_1(y)| = d_x + 1 = |V| - 1.$$

Hence, there exists a vertex that is not adjacent to both x and y, resulting in a degree less than |V|-2, which leads to a contradiction. Therefore, $|N_{xy}| = d_x - 2$ must hold, implying

that $\kappa(x, y) \leq 1$. According to Theorem 1.1, we also have $\kappa(x, y) \geq 1$, and therefore we conclude $\kappa(x, y) = 1$. Finally, note that the graph with this degree sequence must contain an odd number of vertices. This can be seen by observing that each vertex of degree |V|-2 is non-adjacent to exactly one other vertex of degree |V|-2.

We also provide a complete characterization of all graphs with Lin-Lu-Yau curvature greater than one, a result previously established in [12].

Corollary 3.7. Let G = (V, E) be a locally finite graph with Ric(G) > 1. Then G is a complete.

Proof. According to Theorem 1.1, $\delta(G) \geq |V| - 2$ must hold true. If every vertex is of degree |V| - 2, then G is a cocktail party graph and therefore Ric(G) = 1. Thus, there exists at least one vertex x of degree |V| - 1. Assume there exists a vertex y of degree |V| - 2. Then

$$\kappa(x,y) \le \frac{|N_{xy}| + 2}{d_x} \le 1,$$

contradicting Ric(G) > 1. Therefore, G must be the complete graph.

4 Relation between the two curvature notions

In this section, we derive an exact expression for the difference between the Lin-Lu-Yau curvature and the 0-Ollivier-Ricci curvature on regular graphs. We begin with the following Lemma, which addresses certain assumptions that can be imposed on an optimal assignment.

Lemma 4.1. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then there exists an optimal assignment ϕ between $S_1(x)$ and $S_1(y)$, such that $\phi(i) = i$ for all $i \in N_{xy}$.

Furthermore, if $|N_{xy}| < d-1$, there exists an optimal assignment ϕ , satisfying the aforementioned property and $\phi(y) \neq x$.

Proof. The first part of the Lemma is an immediate consequence of the triangle inequality.

For the second part of the proof, assume that $|N_{xy}| < d-1$ and assume that $\phi(y) = x$. Choose an arbitrary $i \in S_1(x) \setminus B_1(y)$ and define a new assignment ϕ' between $S_1(x)$ and $S_1(y)$ by

$$\phi'(z) = \begin{cases} \phi(i), & \text{if } z = y; \\ x, & \text{if } z = i; \\ \phi(z), & \text{otherwise.} \end{cases}$$

Since $i \notin N_{xy}$, we have $d(i, \phi(i)) \ge 1$, leading to

$$d(y,\phi'(y)) + d(i,\phi'(i)) = 2 \le d(y,\phi(y)) + d(i,\phi(i)).$$

Hence, the new assignment ϕ' is still optimal. Finally, note that $\phi'(i) = \phi(i) = i$ for all $i \in N_{xy}$. This concludes the proof.

We are now ready to prove the main theorem of this section.

Theorem 1.2. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. If $|N_{xy}| < d - 1$, then

$$\kappa(x,y) - \kappa_0(x,y) = \frac{1}{d} \left(3 - \sup_{\phi \in \mathcal{O}_{xy}} \sup_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right).$$

If $|N_{xy}| = d - 1$, then

$$\kappa(x,y) - \kappa_0(x,y) = \frac{2}{d}.$$

Proof. Case 1: $|N_{xy}| = d - 1$. According to Theorem 2.10, we have $\kappa(x, y) = \frac{d+1}{d}$, while Theorem 2.11 shows that $\kappa_0(x, y) = \frac{d-1}{d}$. Hence,

$$\kappa_0(x,y) = \kappa(x,y) - \frac{2}{d}$$

Case 2: $|N_{xy}| < d-1$. Choose $\phi \in \mathcal{O}_{xy}$ and $j \in S_1(x) \setminus B_1(y)$ such that

$$d(j,\phi(j)) = \sup_{\psi \in \mathcal{O}_{xy}} \sup_{z \in S_1(x) \setminus B_1(y)} d(z,\psi(z)).$$

Next, we define an assignment ψ between $S_1(x) \setminus S_1(y)$ and $S_1(y) \setminus S_1(x)$ by

$$\psi(z) = \begin{cases} x, & \text{if } z = j; \\ \phi(j), & \text{if } z = y; \\ \phi(z), & \text{otherwise} \end{cases}$$

We claim that ψ is an optimal assignment between $S_1(x) \setminus S_1(y)$ and $S_1(y) \setminus S_1(x)$. Assume this is not the case. Let ψ' be an optimal assignment. According to Lemma 4.1, we can assume that $\psi'(y) \neq x$. As ψ is not optimal,

$$\sum_{z \in S_1(x) \setminus S_1(y)} d(z, \psi'(z)) < \sum_{z \in S_1(x) \setminus S_1(y)} d(z, \psi(z))$$
(4.1)

must hold. Using $d(j, \psi(j)) = d(y, \psi(y)) = 1$, we obtain

$$\sum_{\substack{z \in S_1(x) \setminus S_1(y) \\ z \neq j}} d(z, \psi(z)) = \sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq j}} d(z, \phi(z)) + 2.$$
(4.2)

On the other hand, by assumption, we have $\psi'(y) \neq x$. Denote by k the preimage of x under ψ' . Then

$$\sum_{\substack{z \in S_1(x) \setminus S_1(y) \\ z \neq k}} d(z, \psi'(z)) = \sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq k}} d(z, \psi'(z)) + 2.$$
(4.3)

Combining equations 4.1, 4.2 and 4.3 leads to

$$\sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq k}} d(z, \psi'(z)) < \sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq j}} d(z, \phi(z)).$$
(4.4)

Next, we define an assignment $\phi' \in \mathcal{A}_{xy}$ between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ by

$$\phi'(z) = \begin{cases} \psi'(y), & \text{if } z = k; \\ \psi'(z), & \text{if } z \in S_1(x) \setminus B_1(y) \text{ and } z \neq k \end{cases}$$

Due to the optimality of ϕ , we have

$$\sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi'(z)) \ge \sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi(z)).$$

We now distinguish the following two cases:

Case 1: $\sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi'(z)) = \sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi(z))$. In this case, $\phi' \in \mathcal{O}_{xy}$ and by our choice of ϕ and j, we have

$$d(k,\phi'(k)) \le d(j,\phi(j)),$$

leading to

$$\sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq k}} d(z, \phi'(z)) \ge \sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq j}} d(z, \phi(z)).$$

contradicting equation 4.4.

Case 2: $\sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi'(z)) > \sum_{z \in S_1(x) \setminus B_1(y)} d(z, \phi(z))$. Due to equation 4.4, we have

$$2 + d(j,\phi(j)) \le d(k,\phi'(k))).$$

As $d(k, \phi'(k)) \leq 3$ and $1 \leq d(j, \phi(j))$, we must have $d(j, \phi(j)) = 1$ and therefore

$$1 = d(j, \phi(j)) \ge d(z, \phi(z)) \ge 1$$

holds for any $z \in S_1(x) \setminus B_1(y)$, by the choice of j and ϕ . This contradicts equation 4.4, as

$$d(z, \psi'(z)) \ge 1, \ \forall z \in S_1(x) \setminus B_1(y).$$

Therefore our assumption was wrong and ψ is an optimal assignment between $S_1(x) \setminus S_1(y)$ and $S_1(y) \setminus S_1(x)$. Using Theorem 2.11, we obtain

$$\begin{split} \kappa_0(x,y) &= \frac{1}{d} \left(d - \sum_{\substack{z \in S_1(x) \setminus S_1(y)}} d(z,\psi(z)) \right) \\ &= \frac{1}{d} \left(d - \sum_{\substack{z \in S_1(x) \setminus B_1(y), \\ z \neq j}} d(z,\phi(z)) - 2 \right) \\ &= \frac{1}{d} \left(d + 1 - \sum_{\substack{z \in S_1(x) \setminus B_1(y)}} d(z,\phi(z)) \right) - \frac{1}{d} \left(3 - d(j,\phi(j)) \right), \end{split}$$

where we used equation 4.2 for the second equality. Using the optimality of ϕ and Theorem 2.10, we obtain

$$\kappa_0(x,y) = \kappa(x,y) - \frac{1}{d} \left(3 - d(j,\phi(j)) \right).$$

The choice of ϕ and j concludes the proof.

The following result was previously established in [9] and is an immediate consequence of Theorem 1.2.

Corollary 4.2. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then,

$$\kappa_0(x,y) = \kappa(x,y) - \frac{c}{d},$$

where $c \in \{0, 1, 2\}$.

The following result provides a necessary and sufficient condition for $\kappa(x, y) = \kappa_0(x, y)$ on edges with endpoints of equal degree.

Corollary 4.3. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then

$$\kappa_0(x,y) = \kappa(x,y)$$

if and only if there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ such that

$$\exists z \in S_1(x) \setminus B_1(y) : d(z, \phi(z)) = 3.$$

Remark 4.4. Therefore, in a regular graph with diameter at most two, every edge $x \sim y$ satisfies $\kappa(x, y) > \kappa_0(x, y)$. One class of such graphs are *d*-regular graphs with $d \geq \frac{|V|-1}{2}$.

Next, we use Corollary 4.3 to determine an interval where κ and κ_0 always coincide.

Corollary 4.5. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. If

$$\kappa(x,y) < -1 + \frac{2|N_{xy}| + 3}{d}.,$$

then $\kappa(x,y) = \kappa_0(x,y)$.

Proof. We argue by contradiction. Assume $\kappa(x, y) \neq \kappa_0(x, y)$ and

$$\kappa(x,y) < -1 + \frac{2|N_{xy}| + 3}{d}.$$

Let $\phi \in \mathcal{O}_{xy}$ be an optimal assignment between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$. According to Corollary 4.3, we have $d(z, \phi(z)) \leq 2$ for any $z \in S_1(x) \setminus B_1(y)$. Hence,

$$\begin{aligned} \kappa(x,y) &= \frac{1}{d} \left(d + 1 - \sum_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right) \\ &\geq \frac{1}{d} \left(d + 1 - 2|S_1(x) \setminus B_1(y)| \right) \\ &= \frac{1}{d} \left(-d + 3 + 2|N_{xy}| \right), \end{aligned}$$

contradicting our assumption.

5 Bone-idleness

In this section, we examine edges with Ollivier-Ricci curvature equal to zero for every idleness parameter α . This concept was originally introduced by Bourne et al. in [9] and referred to as *bone-idle*.

Definition 5.1 (Bone-idle). Let G = (V, E) be a locally finite graph. We say an edge $x \sim y$ is *bone-idle* if $\kappa_{\alpha}(x, y) = 0$ for every $\alpha \in [0, 1]$. We say that G is *bone-idle* if every edge of G is bone-idle.

The notion of Ricci-flatness is weaker but strongly related to bone-idleness.

Definition 5.2 (Ricci-flat). Let G = (V, E) be a locally finite graph. We call G Ricci-flat if $\kappa(x, y) = 0$ for every edge $x \sim y \in E$. We call $G \alpha$ -Ricci flat if $\kappa_{\alpha}(x, y) = 0$ for every edge $x \sim y \in E$.

Due to the following Lemma, bone-idleness and the various notions of Ricci-flatness are closely related.

Lemma 5.3. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ with $x \sim y$. Then the following are equivalent:

(i) $\kappa_{\alpha}(x,y) = 0$ for all $\alpha \in [0,1]$.

(*ii*)
$$\kappa_0(x, y) = \kappa(x, y) = 0.$$

Proof. (*ii*) \implies (*i*). Let $\alpha \in (0, 1)$ be arbitrary. Assume $\kappa_0(x, y) = \kappa(x, y) = 0$. Recall that the idleness function is concave, and also note that $\kappa_1(x, y) = 0$. Hence,

$$\kappa_{\alpha}(x,y) \ge \alpha \kappa_1(x,y) + (1-\alpha)\kappa_0(x,y) = 0.$$

The other inequality follows from the fact that the graph of a concave function lies below its tangent line at each point and that $\kappa'_1 = -\kappa$:

$$\kappa_{\alpha}(x,y) \leq \kappa_1(x,y) + \kappa_1'(x,y) \cdot (\alpha - 1) = \kappa(x,y) \cdot (1 - \alpha) = 0.$$

 $(i) \implies (ii)$. This is an immediate consequence of the definition of κ .

Therefore, a graph is bone-idle if and only if it is Ricci-flat and 0-Ricci-flat. Previous works have addressed the classification of Ricci-flat graphs. Cushing et al. [14] classified all Ricci-flat graphs with girth at least five.

Theorem 5.4 ([14], Theorem 1). Let G = (V, E) be a locally finite graph with girth at least five. Suppose that G is Ricci-flat. Then G is isomorphic to one of the following graphs:

- (i) The infinite path,
- (ii) the cycle graph C_n for $n \ge 6$,
- (*iii*) the dodecahedral graph,

- (iv) the Petersen graph,
- (v) the half-dodecahedral graph,
- (vi) the Triplex graph.

On the other hand, Bhattacharya et al. [13] classified all graphs that are 0-Ricci-flat and have girth at least five.

Theorem 5.5 ([13], Corollary 4.1). Let G = (V, E) be a locally finite graph with girth at least five. Suppose that G is 0-Ricci-flat. Then G is isomorphic to one of the following graphs:

- (i) The infinite path,
- (ii) the infinite ray,
- (iii) the path P_n for $n \ge 2$,
- (iv) the cycle graph C_n for $n \geq 5$,
- (v) the star graph T_n for $n \geq 3$.

Combining Theorem 5.5 and Theorem 5.4 yields the following result:

Corollary 5.6. Let G = (V, E) be a locally finite graph with girth at least five. Suppose that G is bone-idle. Then G is isomorphic to one of the following graphs:

- (i) The infinite path,
- (ii) the cycle graph C_n for $n \ge 6$.

Remark 5.7. Hence, there are no bone-idle graphs with girth equal to five.

The full classification of Ricci-flat and bone-idle graphs appears to be a difficult graph theory problem, which is still open. In the following, we leverage our previous findings to investigate the local structure of regular bone-idle graphs.

5.1 Local structures

For the subsequent discussion, we associate the following two quantities with an assignment $\phi \in \mathcal{A}_{xy}$:

- $N_1(\phi)$: The number of neighbors of x, forming a 4-cycle based at the edge $x \sim y$, with their image under ϕ . That is, $N_1(\phi) = |\{z \in S_1(x) \setminus B_1(y) : d(z, \phi(z)) = 1\}|$.
- $N_2(\phi)$: The number of neighbors of x, forming a 5-cycle based at the edge $x \sim y$, with their image under ϕ . That is, $N_2(\phi) = |\{z \in S_1(x) \setminus B_1(y) : d(z, \phi(z)) = 2\}|$.

Using this notation, we can examine the local structure of regular Ricci-flat graphs of girth four.

Theorem 5.8. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Furthermore, assume that $N_{xy} = \emptyset$. Then $\kappa(x, y) = 0$ if and only if one of the following holds:

(i) There exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ such that $N_1(\phi) = d-2$ and $N_2(\phi) = 0$.

(ii) There exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ such that $N_1(\phi) = d - 3$ and $N_2(\phi) = 2$.

Proof. " \implies "Assume $\kappa(x, y) = 0$. Let $\phi \in \mathcal{O}_{xy}$ be an optimal assignment between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$. According to Theorem 2.10, we have

$$\kappa(x,y) = \frac{1}{d} \left(d+1 - \sum_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right)$$
$$= \frac{1}{d} \left(-2d + 4 + 2N_1(\phi) + N_2(\phi) \right)$$
$$= 0.$$

This can only be the case if one of the cases stated in Theorem 5.8 holds true.

" \Leftarrow " This is an immediate consequence of Theorem 2.10.

Next, we present a necessary and sufficient condition for an edge $x \sim y$ to have $\kappa_0(x, y) = 0$. This condition was already established by Bhattacharya et al. [13].

Theorem 5.9. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Furthermore, assume that $N_{xy} = \emptyset$. Then $\kappa_0(x, y) = 0$ if and only if there exists a perfect matching between $S_1(x)$ and $S_1(y)$.

Therefore, a 0-Ricci-flat, regular graph of girth four must have a perfect matching between the neighborhoods $S_1(x)$ and $S_1(y)$ for every edge $x \sim y$. Examples are the *n*-dimensional hypercube Q_n , the *n*-dimensional integer lattice \mathbb{Z}^n and the complete bipartite graph $K_{n,n}$.

The subsequent theorem provides a necessary and sufficient condition for an edge in a graph of girth four to be bone-idle.

Theorem 5.10. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Furthermore, assume that $N_{xy} = \emptyset$. Then the edge $x \sim y$ is bone-idle if and only if there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ such that $N_1(\phi) = d-2$ and $N_2(\phi) = 0$.

Proof. Assume the edge $x \sim y$ is bone-idle, i.e., $\kappa(x,y) = \kappa_0(x,y) = 0$. According to Corollary 4.3 there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ and a $z \in S_1(x) \setminus B_1(y)$, such that $d(z, \phi(z)) = 3$. Therefore,

$$N_1(\phi) + N_2(\phi) < |S_1(x) \setminus B_1(y)| = d - 1.$$
(5.1)

As $\kappa(x, y) = 0$ we can apply Theorem 5.8. Using equation 5.1, we conclude that $N_1(\phi) = d - 2$ and $N_2(\phi) = 0$ must hold.

Conversely, assume that there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ that satisfies $N_1(\phi) = d - 2$ and $N_2(\phi) = 0$. Since

$$N_1(\phi) + N_2(\phi) = d - 2 < |S_1(x) \setminus B_1(y)|,$$

there exists a $z \in S_1(x) \setminus B_1(y)$ such that $d(z, \phi(z)) = 3$. Hence, according to Corollary 4.3, we have $\kappa(x, y) = \kappa_0(x, y)$. According to Theorem 5.8, we have $\kappa(x, y) = 0$, which concludes the proof.

Both the complete bipartite graph $K_{n,n}$ and the *n*-dimensional hypercube Q_n are 0-Ricciflat, regular graphs of girth four. Note that there exists a perfect matching between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ for any edge $x \sim y$ in the complete bipartite graph $K_{n,n}$. The same holds for the *n*-dimensional hypercube Q_n . Therefore, according to the previous theorem, neither of the graphs is bone-idle. Using Theorem 2.10 for the Lin-Lu-Yau curvature, we obtain that for both the complete bipartite graph $K_{n,n}$ and the *n*-dimensional hypercube Q_n

$$\kappa(x,y) = \frac{2}{n},$$

for every edge $x \sim y$. Thus, the graphs $K_{n,n}$ and Q_n satisfy $\kappa(x,y) > 0$ and $\kappa_0(x,y) = 0$ for all edges $x \sim y$.

A direct implication of Theorem 5.10 is that the *n*-dimensional integer lattice \mathbb{Z}^n is boneidle.

We conclude this section by extending Theorem 5.10 to arbitrary regular graphs as follows.

Theorem 5.11. Let G = (V, E) be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then the edge $x \sim y$ is bone-idle if and only if there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ such that $N_1(\phi) + N_2(\phi) < d - 1 - |N_{xy}|$ and

$$2d - 4 - 3|N_{xy}| = 2N_1(\phi) + N_2(\phi)$$

Proof. " \implies " Assume the edge $x \sim y$ is bone-idle, i.e., $\kappa(x, y) = \kappa_0(x, y) = 0$. According to Corollary 4.3 there exists an optimal assignment $\phi \in \mathcal{O}_{xy}$ between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ and a $z \in S_1(x) \setminus B_1(y)$, such that $d(z, \phi(z)) = 3$. Therefore,

$$N_1(\phi) + N_2(\phi) < |S_1(x) \setminus B_1(y)| = d - 1 - |N_{xy}|.$$
(5.2)

Using $\kappa(x, y) = 0$ and Theorem 2.10, we obtain

$$\kappa(x,y) = \frac{1}{d} \left(d + 1 - \sum_{z \in S_1(x) \setminus B_1(y)} d(z,\phi(z)) \right)$$
$$= \frac{1}{d} \left(-2d + 4 + 3|N_{xy}| + 2N_1(\phi) + N_2(\phi) \right)$$
$$= 0,$$

or equivalently, $2N_1(\phi) + N_2(\phi) = 2d - 4 - 3|N_{xy}|$.

" \Leftarrow " Assume ϕ is an optimal assignment such that

$$N_1(\phi) + N_2(\phi) < d - 1 - |N_{xy}| = |S_1(x) \setminus B_1(y)|.$$



Figure 1: Illustration of the two possible cases in Theorem 1.3

Therefore, there exists an $z \in S_1(x) \setminus B_1(y)$ such that $d(z, \phi(z)) = 3$. Hence, according to Corollary 4.3, we have $\kappa(x, y) = \kappa_0(x, y)$. Using Theorem 2.10 and

$$2d - 4 - 3|N_{xy}| = +2N_1(\phi) + N_2(\phi),$$

we obtain $\kappa(x, y) = 0$, and the edge $x \sim y$ is bone-idle.

5.2 3-regular bone-idle graphs

The objective of this section is to demonstrate that no 3-regular bone-idle graphs exist.

Theorem 1.3. Let G = (V, E) be a locally finite graph. Suppose that G is bone-idle, then G is not 3-regular.

Proof. We argue by contradiction. Assume G is 3-regular and bone-idle. According to Corollary 5.6, the girth of G must be less than five. Assume there exists an edge $x \sim y$ such that $|N_{xy}| > 0$. Then $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$ each contain only a single vertex, which we denote by z_1 and z_2 , respectively. Recall that $d(z_1, z_2) \leq 3$. Thus,

$$\kappa(x,y) = \frac{1}{3} \Big(4 - d(z_1, z_2) \Big) > 0.$$

This contradicts the bone-idleness of the graph. Hence, the girth of G must be equal to four. Let $x \sim y$ be an edge contained in a 4-cycle. Denote by x_1, x_2 and y_1, y_2 the other two neighbors of x and y, respectively. Without loss of generality, we may assume that $x_1 \sim y_1$, as $x \sim y$ is contained in a 4-cycle. Observe that $x_2 \sim y_1$ and $y_2 \sim x_1$ cannot hold true at the same time. Otherwise, there exists a perfect matching between $S_1(x) \setminus \{y\}$ and $S_1(y) \setminus \{x\}$, leading to $\kappa(x, y) = \frac{2}{3} > 0$. Therefore, we may assume, without loss of generality, that $x_2 \not\sim y_1$.

According to Theorem 5.10, there must be a 4-cycle based on the edge $x \sim x_2$. Using that $x_2 \not\sim y_1$, one of the following cases must be true:

Case 1: There is a $z \in S_1(x_2) \setminus \{x\}$ such that $z \sim x_1$. In this case we have $\kappa(x, x_1) > 0$, contradicting the bone-idleness of G.

Case 2: $x_2 \sim y_2$. In this case, we have $\kappa(x, y) > 0$, contradicting the bone-idleness of G. This concludes the proof.



Figure 3: A primitive 4-regular Ricci-flat graph

Hence, there are no 3-regular bone-idle graphs.

5.3 4-regular bone-idle graphs

In this section, we aim to provide a complete classification of all 4-regular bone-idle graphs. We begin by examining graphs with a girth of three. In [15], the authors classify all Ricci-flat 4-regular graphs of girth three.

Theorem 5.12 ([15], Theorem 5). Let G = (V, E) be a 4-regular graph of girth three. If G is Ricci-flat, then it is isomorphic to the icosidodecahedron graph.

Remark 5.13. See Figure 2 for an illustration of the icosidodecahedron graph, a polyhedron with 20 triangular faces, 12 pentagonal faces on 30 vertices connected by 60 identical edges, each of which separates a triangle from a pentagon.

Corollary 5.14. Let G = (V, E) be a 4-regular graph of girth three. If G is bone-idle, then it is isomorphic to the icosidodecahedron graph.

Proof. Assume G is a 4-regular bone-idle graph of girth three. Since it is bone-idle, G is Ricci-flat and therefore must be isomorphic to the icosidodecahedron graph. It remains to verify that the the icosidodecahedron graph G is indeed bone-idle. To this end, let $x \sim y$ be an arbitrary edge in G. Let $\phi \in \mathcal{O}_{xy}$ be an optimal assignment between $S_1(x) \setminus B_1(y)$ and $S_1(y) \setminus B_1(x)$. Then there exists an $z \in S_1(x) \setminus B_1(y)$ such that $d(z, \phi(z)) = 3$. According to Corollary 4.3, we have $\kappa(x, y) = \kappa_0(x, y)$. Using the Ricci-flatness we have $\kappa(x, y) = 0$. Using Lemma 5.3, we conclude that the edge $x \sim y$ is bone-idle.

Therefore, we have classified all 4-regular bone-idle graphs of girth three. We now proceed to classify all such graphs of girth four.

In [15], the authors classify all 4-regular Ricci-flat graphs that contain two four-cycles sharing a common edge. To this end, they introduce the concept of a *primitive graph*,



Figure 4: A primitive 4-regular Ricci-flat graph of "lattice type"

which can be understood as the 1-skeleton of the universal cover of the CW-complex formed by gluing 2-cells to all cycles of length at most five. They obtain the following result.

Theorem 5.15 ([15], Theorem 8). Let G = (V, E) be a 4-regular Ricci-flat graph that contains two four-cycles sharing one edge. Then G is isomorphic to graphs with the primitive graphs showing in Figure 3 and Figure 4.

According to Theorem 5.10, there must be two four-cycles supported on every edge in a 4-regular bone-idle graph of girth four. Thus, we can apply the previous theorem. Using Theorem 5.10, it is easy to verify that the graphs with the primitive graphs showing in Figure 3 and Figure 4 are bone-idle. Therefore, we obtain the following Corollary.

Corollary 5.16. Let G = (V, E) be a 4-regular bone-idle graph of girth four. Then G is isomorphic to graphs with the primitive graphs showing in Figure 3 and Figure 4.

The finite graphs with the primitive graph depicted in Figure 3 can be constructed as follows: Start with an *n*-cycle consisting of vertices x_0, \ldots, x_{n-1} placed inside another *n*-cycle with vertices y_0, \ldots, y_{n-1} . Connect each vertex y_k to $x_{(k-1) \mod n}$ and $x_{(k+1) \mod n}$. These graphs are denoted by BI_n . Figure 5 illustrates the graphs BI_6 , BI_7 , and BI_8 .

Examples of finite graphs with the primitive graph shown in Figure 4 include potentially twisted tori and the Klein bottle graphs. These graphs can be constructed as follows: Take an $n \times m$ grid graph and denote the vertices by $x_{i,j}$ for $i = 0, \ldots, n-1$ and $j = 0, \ldots, m-1$. First, add the edges $x_{0,j} \sim x_{n-1,j}$ for $j = 0, \ldots, m-1$. Then, to obtain a Klein bottle graph add the edges $x_{i,0} \sim x_{n-1-i,m-1}$ for $i = 0, \ldots, n-1$. Note that $n, m \ge 6$ must hold. On the other hand, to obtain a twisted torus graph add the edges $x_{i,0} \sim x_{(i+l) \mod n,m-1}$ for $i = 0, \ldots, n-1$ and for a fixed $l \le \frac{n}{2}$. Note that $n \ge 6$ and $m + l \ge 6$ must hold. A twisted torus graph and a Klein bottle graph are depicted in Figure 6.



Figure 5: Illustration of the graphs BI_6 , BI_7 and BI_8

By Corollary 5.6, no 4-regular bone-idle graphs exist with girth greater than or equal to five. Consequently, we have completed the classification of all 4-regular bone-idle graphs.

5.4 5-regular bone-idle graphs

In [7], Lin, Lu, and Yau established the following result concerning the curvature of Cartesian products of graphs.

Theorem 5.17 ([7]). Let $G = (V_G, E_G)$ be a d_G -regular graph and $H = (V_H, E_H)$ be a d_H -regular graph. Let $x_1, x_2 \in V_G$ with $x_1 \sim x_2$ and $y \in V_H$. Then

$$\kappa_0^{G\Box H}((x_1, y), (x_2, y)) = \frac{d_G}{d_G + d_H} \kappa_0^G(x_1, x_2),$$

$$\kappa^{G\Box H}((x_1, y), (x_2, y)) = \frac{d_G}{d_G + d_H} \kappa^G(x_1, x_2).$$

Corollary 5.18. Let $G = (V_G, E_G)$ be a d_G -regular graph and $H = (V_H, E_H)$ be a d_H -regular graph. If G and H are Ricci-flat graphs, then the Cartesian product $G \Box H$ is also Ricci-flat.

Therefore, one method for constructing 5-regular Ricci-flat graphs is to take the Cartesian product of a Ricci-flat 3-regular graph and a Ricci-flat 2-regular graph. By Theorem 1.3, this approach is not applicable for constructing 5-regular bone-idle graphs, as no 3-regular bone-idle graphs exist.

The authors in [16] study 5-regular Ricci-flat graphs. They are able to identify a 5-regular, symmetric graph of order 72, denoted RF_{72}^5 , that is not of Cartesian product type.

Theorem 5.19 ([16], Theorem 4.1). Let G = (V, E) be a 5-regular, symmetric graph. If G is Ricci-flat, then it is isomorphic to RF_{72}^5 .

It is easy to verify that for every edge $x \sim y$ in RF_{72}^5 , $\kappa_0(x, y) = -\frac{1}{5}$ holds. Therefore, we obtain the following result.



Figure 6: Illustration of a twisted torus graph on the left and a Klein bottle graph on the right

Corollary 5.20. There exists no 5-regular, symmetric graph that is bone-idle.

The authors also formulate the following conjecture.

Conjecture 5.21 ([16], Conjecture 1). If G = (V, E) is a 5-regular Ricci-flat graph, then G is either isomorphic to RF_{72}^5 or G is of Cartesian product type.

Should this conjecture prove true, no 5-regular bone-idle graphs exist. The existence of regular bone-idle graphs with an odd vertex degree remains an open question.

Acknowledgements: I would like to express my deepest gratitude to Prof. Dr. Renesse and Dr. Münch for their invaluable support and insightful feedback throughout the course of this work.

Declarations

Funding: Partial financial support was received from the BMBF (Federal Ministry of Education and Research) in DAAD project 57616814 (SECAI, School of Embedded Composite AI).

Conflict of interest: The author certifies that he has no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

References

- Dominique Bakry and Michel Émery. "Diffusions hypercontractives". In: Séminaire de Probabilités XIX 1983/84 (1985), pp. 177-206. https://doi.org/10.1007/BFb0075847
- [2] Matthias Erbar and Jan Maas. "Ricci Curvature of Finite Markov Chains via Convexity of the Entropy". In: Archive for Rational Mechanics and Analysis 206 (2012), pp. 997-1038. https://doi.org/10.1007/s00205-012-0554-z

- [3] Alexander Mielke. "Geodesic convexity of the relative entropy in reversible Markov chains". In: Calculus of Variations and Partial Differential Equations 48 (2013), pp. 1-31. https://doi.org/10.1007/s00526-012-0538-8
- [4] Robin Forman. "Bochner's Method for Cell Complexes and Combinatorial Ricci Curvature". In: Discrete and Computational Geometry 29 (2003), pp. 323-374. https://doi.org/10.1007/s00454-002-0743-x
- [5] Yann Ollivier. "Ricci curvature of Markov chains spaces". on metric In: Journal ofFunctional Analysis, 256.3(2009),810-864. pp. https://doi.org/10.1016/j.jfa.2008.11.001
- [6] Max-K. von Renesse and Karl-Theodor Sturm. "Transport inequalities, gradient estimates, entropy and Ricci curvature". In: *Communications on pure and applied mathematics*, 58.7 (2005), pp. 923-940. https://doi.org/10.1002/cpa.20060
- [7] Yong Lin, Linyuan Lu, and Shing-Tung Yau. "Ricci curvature of graphs". In: *Tohoku Math. J.* 63.4 (2011), pp. 605- 627. https://doi.org/10.2748/tmj/1325886283
- [8] Marzieh Eidi and Jürgen Jost. "Ollivier Ricci curvature of directed hypergraphs". In: Scientific Reports 10.1 (2020), pp. 12466.
- [9] David P Bourne et al. "Ollivier-Ricci Idleness Functions of Graphs". In: SIAM Journal on Discrete Mathematics 32.2 (2018), pp. 1408-1424.
- [10] Moritz Hehl. "Ollivier-Ricci curvature of regular graphs". arXiv preprint arXiv: 2407.08854 (2024).
- [11] Jürgen Jost and Shiping Liu. "Ollivier's Ricci Curvature, Local Clustering and Curvature-Dimension Inequalities on Graphs". In: Discrete Comput Geom 51 (2014), pp. 300-322. https://doi.org/10.1007/s00454-013-9558-1
- [12] Vincent Bonini et al. "Condensed Ricci curvature of complete and strongly regular graphs". In: *Involve, a Journal of Mathematics* 13.4 (2020), pp. 559-576. https://doi.org/10.2140/involve.2020.13.559
- [13] Bhaswar B. Bhattacharya and Sumit Mukherjee. "Exact and asymptotic results on coarse Ricci curvature of graphs". In: *Discrete Mathematics* 338.1 (2015), pp. 23-42. https://doi.org/10.1016/j.disc.2014.08.012
- [14] David Cushing et al. "Erratum for Ricci-flat graphs with girth at least five". arXiv preprint arXiv:1802.02979 (2018).
- [15] Shuliang Bai, Linyuan Lu, and Shing-Tung Yau. "Ricci-flat graphs with maximum degree at most 4". In: Asian Journal of Mathematics 25.6 (2021), pp. 757-814.
- [16] Heidi Lei and Shuliang Bai. "Ricci-flat 5-regular graphs". In: Pure and Applied Mathematics Quarterly 18.6 (2022), pp. 2511-2535.