The Picard group of the Baily-Borel compactification of the moduli space of polarized K3 surfaces

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Abstract

In this paper, we study the Picard group of the Baily-Borel compactification of orthogonal Shimura varieties. As a result, we determine the Picard group of the Baily-Borel compactification of the moduli space of quasi-polarized K3 surfaces. Interestingly, in contrast to the situation observed in the moduli space of curves, we find that the Picard group of the Baily-Borel compactification is isomorphic to \mathbb{Z} .

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1. Introduction

Exploring the Picard group of moduli spaces stands as a vital and inspiring challenge in moduli theory, rooted in the groundbreaking contributions of Mumford [26] in 1965. Let \mathcal{M}_g be the moduli space of smooth curves of genus g and $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. Harer [18] and Arbarello-Cornalba [1] showed that

$$\operatorname{Pic}(\mathcal{M}_g) \cong \mathbb{Z} \text{ and } \operatorname{Pic}(\overline{\mathcal{M}}_g) \cong \mathbb{Z}^{2+\left\lfloor \frac{g}{2} \right\rfloor}$$
 (1.1)

are free abelian groups.

For K3 surfaces, let \mathscr{F}_g be the (coarse) moduli space of quasi-polarized K3 surfaces of genus g. By global Torelli theorem for K3 surfaces, \mathscr{F}_g is isomorphic to a Shimura variety of orthogonal type. It has been shown in [3, 14] that its Picard group $\operatorname{Pic}(\mathscr{F}_g) \cong \mathbb{Z}^{r_g}$ is a free abelian group generated by linear

combinations of Noether-Lefschetz divisors. Moreover, its rank is

$$r_{g} = \frac{31g + 24}{24} - \frac{1}{4} \left(\frac{2g - 2}{2g - 3}\right) \left(\frac{g}{2}\right) - \frac{1}{6} \left(\frac{g - 1}{4g - 5}\right) - \frac{1}{6} (-1)^{-\left(\frac{g - 1}{3}\right)} - \sum_{k=0}^{g-1} \left\{\frac{k^{2}}{4g - 4}\right\} - \sharp \left\{k \mid \frac{k^{2}}{4g - 4} \in \mathbb{Z}, 0 \le k \le g - 1\right\}$$

where $(\frac{\cdot}{2})$ is the Jacobi symbol. On \mathscr{F}_g , there exists a canonical Satake compactification, denoted as $\overline{\mathscr{F}}_g$, as established by the work of Baily and Borel. A natural problem is to figure out the Picard group of $\overline{\mathscr{F}}_g$. The main result of this paper is

Theorem 1.1 (Theorem 6.5). $\operatorname{Pic}(\overline{\mathcal{F}}_g) \cong \mathbb{Z}$, which is spanned by an integral multiple of the extended Hodge line bundle $\overline{\lambda}$.

Remark 1.2. Since $\overline{\lambda}$ is currently only known to be a \mathbb{Q} -line bundle, an integral multiple of $\overline{\lambda}$ is needed in the statement of Theorem 1.1. A generator of $\operatorname{Pic}(\overline{\mathcal{F}}_g)$ can be chosen to be $\overline{\lambda}$ once one can show that $\overline{\lambda}$ is a line bundle.

Our proof relies heavily on the study of the behavior of Heegner divisors on the boundary component of the Baily-Borel compactification established in [9] and a surjectivity result of theta liftings for vectorvalued modular forms. In fact, we prove that the Baily-Borel compactification of a Shimura variety of K3 type has Picard number one.

Theorem 1.3 (Corollary 6.4). Let M be an even lattice of signature (2, n) of K3 type, and let $\Gamma \subset O(M)(\mathbb{Q})$ be an arithmetic subgroup containing $\widetilde{O}(M) = \ker(O(M) \to O(M^{\vee}/M))$. If n > 10, then

$$\dim_{\mathbb{Q}} \operatorname{Pic}_{\mathbb{Q}} \left(\overline{\operatorname{Sh}}_{\Gamma}^{\operatorname{BB}}(M) \right) = 1.$$

Our result can be applied to the Baily-Borel compactification of moduli spaces of lattice-polarized K3 surfaces, polarized hyper-Kähler varieties, K-stable log Fano pairs and also moduli space of bounded polarized Calabi-Yau pairs (cf. [15, 21, 4]).

Remark 1.4. We believe that the conditions as stated in Theorem 1.3 can certainly be relaxed, though it might entail more sophisticated developments of our methods. More precisely, we believe that Theorem 1.3 holds for a general even lattice M of signature (2, n) and a possibly more general arithmetic subgroup Γ , whenever dim Sh_{Γ} $(M) \ge 3$.

Another motivation for studying this problem concerns the tautological ring of \mathcal{F}_g . Let

$$\mathbf{R}^*(\mathscr{F}_g) \subseteq \mathrm{CH}^*(\mathscr{F}_g)_{\mathbb{Q}}$$

be the subring generated by irreducible Noether-Lefschetz cycles. It is conjectured by Oprea and Pandharipande (see [12, Conjecture 2]) that

$$\mathbf{R}^{17}(\mathscr{F}_g)\cong\mathbb{Q}$$

is spanned by λ^{17} . This is clearly a very challenging problem, especially when g gets larger. As $CH^{17}(\mathscr{F}_g)_{\mathbb{Q}} \cong CH_2(\mathscr{F}_g)_{\mathbb{Q}} \cong CH_2(\mathscr{F}_g)_{\mathbb{Q}}$, one can regard $R^{17}(\mathscr{F}_g)$ as a subgroup of $CH_2(\mathscr{F}_g)_{\mathbb{Q}}$. Then a natural first attempt is to ask whether it is one-dimensional modulo *numerical equivalence*. Typically, one can consider intersecting the classes in $R^{17}(\mathscr{F}_g)$ with complete intersections in $CH^2(\mathscr{F}_g)_{\mathbb{Q}}$ (via cap product in the sense of [17, Chapter 17]), which is well-understood set-theoretically in Shimura geometry. According to the Noether-Lefschetz conjecture, complete intersections of divisors on \mathscr{F}_g are rationally equivalent to linear combinations of Shimura subvarieties on \mathscr{F}_g , which possess very rich structures.

However, Theorem 1.1 indicates that all complete intersections in $CH^2(\overline{\mathcal{F}}_g)_{\mathbb{Q}}$ are proportional to $\overline{\lambda}^2$. This first test is now almost trivially passed. Although it suggests supportive evidence for the original conjecture, one still need to find other interesting classes in $CH^2(\overline{\mathcal{F}}_g)_{\mathbb{Q}}$.

1.1. Organization of the paper

In Section 2, we review the basic theory of vector-valued Siegel modular forms including the theta series and Siegel-Eisenstein series. Section 3 and Section 4 contain the most technical part. We use the Bruinier-Stein and Shimura Hecke operators to prove a surjectivity result of theta liftings by adapting the proof of [25] to the odd rank case. Some computations are similar and we give the details in the appendix. Readers with a preference for geometry may choose to skip this part.

In Section 5 and Section 6, we study Heegner divisors on Shimura varieties of orthogonal type and their behavior on the boundary component of the Baily-Borel compactification. The main theorem is proved in Subsection 6.3.

Notations and conventions

• A discriminant form is a finite abelian group G with a quadratic form $q: G \to \mathbb{Q}/\mathbb{Z}$ such that

$$(x, y) := \mathfrak{q}(x + y) - \mathfrak{q}(x) - \mathfrak{q}(y)$$

is a non-degenerate symmetric bilinear form. More generally, for any two elements $\mathbf{u} = (u_1, ..., u_d)$, $\mathbf{v} = (v_1, ..., v_d) \in G^{(d)}$, we define $(\mathbf{u}, \mathbf{v}) := ((u_i, v_j)_{i,j}) \in \text{Sym}_d(\mathbb{Q}/\mathbb{Z})$. The level of G is

 $\ell(G) := \min\{m \in \mathbb{Z}_{>0} \mid m\mathfrak{q}(x) = 0 \text{ for all } x \in G\}.$

Note that since the bilinear form (-, -) is non-degenerate, for every $\gamma \in G$, we have $\ell(G) \cdot \gamma = 0$. Finally, the *p*-rank of *G* is rank_{*p*}(*G*) := dim_{**F**_{*p*}(*G* \otimes **F**_{*p*}).}

- We let G^n be the image of the map $G \xrightarrow{\times n} G$ and $G_n = \ker(G \xrightarrow{\times n} G)$. We also define $G^{n*} \subseteq G$ as the coset $\left\{ \gamma \mid n\mathfrak{q}(\mu) + (\mu, \gamma) \equiv 0 \mod \mathbb{Z}, \forall \mu \in G_n \right\}$. (1.2)
- Let $\mathbb{C}[G^{(d)}]$ be the group algebra of $G^{(d)} = G \times G \dots \times G$. There is a standard Hermitian inner product $\langle -, \rangle$ on $\mathbb{C}[G^{(d)}]$ given by

$$\langle \sum_{\gamma \in G^{(d)}} a_{\gamma} \mathbf{e}_{\gamma}, \sum_{\gamma \in G^{(d)}} b_{\gamma} \mathbf{e}_{\gamma} \rangle = \sum_{\gamma \in G^{(d)}} a_{\gamma} \overline{b}_{\gamma}.$$
(1.3)

By identifying $\mathbb{C}[G^{(2)}]$ with $\mathbb{C}[G] \otimes \mathbb{C}[G]$, we also use $\langle -, - \rangle$ to denote the antilinear map $\mathbb{C}[G] \times \mathbb{C}[G^{(2)}] \to \mathbb{C}[G]$ given by

$$\langle v, u \otimes w \rangle = \langle v, u \rangle \cdot \bar{w} \tag{1.4}$$

where $u, v, w \in \mathbb{C}[G]$.

• Let $(M, \langle -, - \rangle)$ be an even lattice. We can identify the dual lattice M^{\vee} as the sublattice

 $\{x \in M \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in M\}$

of $M \otimes \mathbb{Q}$. The discriminant group $G_M := M^{\vee}/M$ is a finite group of order $|\det(M)|$. There is a natural quadratic function $\mathfrak{q}_M : M^{\vee}/M \to \mathbb{Q}/\mathbb{Z}$ given by

$$\mathfrak{q}_M(v) = \frac{1}{2} \langle v, v \rangle \mod \mathbb{Z}, \tag{1.5}$$

for $v \in M^{\vee}$.

- Given a discriminant form G, we use $\operatorname{Gen}_{b^+,b^-}(G)$ to denote the genus of even lattices of signature (b^+, b^-) with discriminant form G. For a lattice $M \in \operatorname{Gen}_{b^+,b^-}(G)$, the signature of the discriminant form M^{\vee}/M is defined as $\operatorname{sign}(M) := b^+ b^- \mod 8 \in \mathbb{Z}/8\mathbb{Z}$. For all (b^+, b^-) such that $\operatorname{Gen}_{b^+,b^-}(G) \neq \emptyset$ and for all $M \in \operatorname{Gen}_{b^+,b^-}(G)$, this quantity is the same. Thus we also use $\operatorname{sign}(G)$ to denote it.
- We use $\mathbf{e}(x)$ to denote the function $e^{2\pi i x}$, and we use $\sqrt{z} := z^{\frac{1}{2}}$ to denote the principal branch of the square root function on \mathbb{C} .

2. Vector-valued Siegel modular forms

2.1. Weil representation

Let \mathbb{H}_d be the Siegel upper half-space. The *metaplectic double cover* $\operatorname{Mp}_{2d}(\mathbb{Z})$ of $\operatorname{Sp}_{2d}(\mathbb{Z})$ consists of the elements $\widetilde{g} = (g, \phi_g(\tau))$, where for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}_{2d}(\mathbb{Z}), \phi_g(\tau)$ is a square root of the function $\det(C\tau + D)$. It is well-known that $\operatorname{Mp}_{2d}(\mathbb{Z})$ is generated by

$$J_d = \left(\begin{pmatrix} 0 & -\mathrm{I}_\mathrm{d} \\ \mathrm{I}_\mathrm{d} & 0 \end{pmatrix}, \sqrt{\mathrm{det}(\tau)} \right), \quad n(B) = \left(\begin{pmatrix} \mathrm{I}_\mathrm{d} & B \\ 0 & \mathrm{I}_\mathrm{d} \end{pmatrix}, 1 \right),$$

where $B \in \text{Sym}_d(\mathbb{Z})$.

Let G be a discriminant form. For each $d \in \mathbb{Z}_{\geq 1}$, there is a unitary representation $\rho_G^{(d)}$ of $\operatorname{Mp}_{2d}(\mathbb{Z})$ on the group ring $\mathbb{C}[G^{(d)}]$, called the Weil representation associated to G. It is defined by

$$\rho_{G}^{(d)}(n(B))\mathbf{e}_{\gamma} = \mathbf{e}\left(\frac{1}{2}\mathrm{tr}((\gamma,\gamma)B)\right)\mathbf{e}_{\gamma},$$

$$\rho_{G}^{(d)}(J_{d})\mathbf{e}_{\gamma} = \frac{\mathbf{e}\left(-\frac{d}{8}\mathrm{sign}(G)\right)}{|G|^{\frac{d}{2}}}\sum_{\delta\in G^{(d)}}\mathbf{e}(-\mathrm{tr}(\gamma,\delta))\mathbf{e}_{\delta},$$
(2.1)

where e_{γ} is the standard basis of $\mathbb{C}[G^{(d)}]$ for $\gamma = (\gamma_1, \dots, \gamma_d) \in G^{(d)}$.

Remark 2.1. For
$$m(U) = \left(\begin{pmatrix} U & 0 \\ 0 & (U^{-1})^{\mathsf{T}} \end{pmatrix}, \sqrt{\det U^{-1}} \right) \in \operatorname{Mp}_{2n}(\mathbb{Z})$$
, we have
 $\rho_G^{(d)}(m(U)) \mathfrak{e}_{\gamma} = \sqrt{\det U^{-1}}^{\operatorname{sign}(G)} \mathfrak{e}_{\gamma U^{-1}}.$

Conventions:

• When $G \cong M^{\vee}/M$ for some lattice M, we may simply write it as $\rho_M^{(d)}$. And if furthermore d = 1, we may simplify it as ρ_M .

2.2. Siegel modular forms

For any $k \in \frac{1}{2}\mathbb{Z}$, define the Petersson slash operator $|_{k,G^{(d)}}$ on the space of vector-valued functions $f : \mathbb{H}_d \to \mathbb{C}[G^{(d)}]$ by

$$(f \mid_{k,G^{(d)}} [\tilde{g}])(\tau) := \phi_g(\tau)^{-2k} \cdot \det(g)^{k/2} \cdot \rho_G^{(d)}(\tilde{g})^{-1} f(g\tau),$$

with $\tilde{g} = (g, \phi_g(\tau)) \in \widetilde{\operatorname{GL}}_2^+(\mathbb{R})$ where

$$\widetilde{\operatorname{GL}}_2^+(\mathbb{R}) := \left\{ \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2^+(\mathbb{R}), \phi_g(\tau) \right) \mid \exists t \in \mathbb{C}^\times, \phi_g(\tau)^2 = t \cdot (c\tau + d) \right\}$$

is the metaplectic \mathbb{C}^{\times} -extension of $\operatorname{GL}_{2}^{+}(\mathbb{R})^{-1}$.

Definition 2.2. A Siegel modular form $f(\tau)$ of weight k and type $\rho_G^{(d)}$ is a vector-valued function

$$f: \mathbb{H}_d \to \mathbb{C}[G^{(d)}]$$

¹This is different from the definition given by Shimura in [31]. Here we adopt the convention of Bruinier-Stein in [11].

satisfying the following two conditions.

- For all $\tilde{g} = (g, \phi_g(\tau)) \in \operatorname{Mp}_{2d}(\mathbb{Z})$, we have $f \mid_{k, G^{(d)}} [\tilde{g}] = f$.
- *The function* f *is holomorphic on* \mathbb{H}_d *and at* ∞ *.*

When $G \cong G_M$ for an even lattice M, a Siegel modular form of type $\rho_G^{(d)}$ has a Fourier expansion

$$f(\tau) = \sum_{\gamma \in G^{(d)}} \sum_{\substack{T \in \operatorname{Sym}_d(\mathbb{Z})_{\operatorname{even}} + (\gamma, \gamma), \\ T \ge 0}} c(T, \gamma) \mathbf{e}\left(\frac{1}{2}\operatorname{tr}(T\tau)\right) \mathbf{e}_{\gamma}.$$

It is called a cusp form if $c(T, \gamma) \neq 0$ implies T > 0.

We set $\operatorname{Mod}_k(\rho_G^{(d)})$ (resp. $\operatorname{Cusp}_k(\rho_G^{(d)})$) to be the space of Siegel modular forms (resp. cusp forms) of weight k and type $\rho_G^{(d)}$. It is a standard fact that $\operatorname{Mod}_k(\rho_G^{(d)}) = 0$ if $2k + \operatorname{sign}(G) \neq 0 \mod 2$. To see this, note that the element $(I_{2d}, -1)$ acts on $\mathbb{C}[G^{(d)}]$ by a scalar $(-1)^{\operatorname{sign}(G)}$. From now on, we assume that the parity condition

$$2k + \operatorname{sign}(G) \equiv 0 \mod 2 \tag{2.2}$$

always holds.

When d = 1, let

$$\langle f, g \rangle_{\text{Pet}} := \int_{\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \langle f(z), g(z) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}$$

be the Petersson inner product on $Mod_k(\rho_M)$, which is well-defined if at least one of f, g is a cusp form.

2.3. Some functoriality results regarding Weil representations

This subsection gathers additional notions and results related to various Weil representations for future reference.

Isotropic lift/descent

Let $H \leq G$ be an isotropic subgroup.

• The isotropic lift map $\uparrow_H : \mathbb{C}[H^{\perp}/H] \to \mathbb{C}[G]$ is defined by

$$\uparrow_H (\mathfrak{e}_{\gamma+H}) = \sum_{\mu \in H} \mathfrak{e}_{\gamma+\mu}$$

for all $\gamma \in H^{\perp}$.

• The isotropic descent map $\downarrow_H : \mathbb{C}[G] \to \mathbb{C}[H^{\perp}/H]$ is defined by

$$\downarrow_H (\mathfrak{e}_{\gamma}) = \mathfrak{e}_{\gamma+H}$$

if $\gamma \in H^{\perp}$ and 0 otherwise.

It is well-known that they are adjoint with respect to the inner product on $\mathbb{C}[G]$ and $\mathbb{C}[H^{\perp}/H]$. Further, they commute with Weil representations, which indicates that they take modular forms to modular forms and preserve cuspidality.

Embeddings and tensor products

For 0 < r < d, there is an inclusion map

$$\iota: \operatorname{Mp}_{2r}(\mathbb{Z}) \times \operatorname{Mp}_{2d-2r}(\mathbb{Z}) \to \operatorname{Mp}_{2d}(\mathbb{Z})$$
(2.3)

given by

$$\Big(\begin{pmatrix}A & B\\ C & D\end{pmatrix}, \phi\Big), \Big(\begin{pmatrix}A' & B'\\ C' & D'\end{pmatrix}, \phi'\Big) \longmapsto \left(\begin{pmatrix}A & B\\ A' & B'\\ C & D\\ C' & D'\end{pmatrix}, \tilde{\phi}\Big),$$

where $\tilde{\phi}(\operatorname{diag}(z, z')) = \phi(z) \cdot \phi'(z')$. They satisfy

$$\rho_M^{(d)}(\iota(\tilde{g}_1, \tilde{g}_2))(\mathbf{e}_{\gamma} \otimes \mathbf{e}_{\gamma'}) = \rho_M^{(r)}(\tilde{g}_1)(\mathbf{e}_{\gamma}) \otimes \rho_M^{(d-r)}(\tilde{g}_2)(\mathbf{e}_{\gamma'}).$$

(cf. [19, Lemma 2.2])

2.4. Siegel-Eisenstein series and Theta series

Suppose *M* is a negative definite even lattice of rank *r* under the pairing (-, -). Embedd *M* into \mathbb{R}^r so that the pairing (-, -) is the standard scalar product on \mathbb{R}^r .

Siegel-Eisenstein series

Let $\Gamma_{\infty}^{(d)} \subseteq \operatorname{Sp}_{2d}(\mathbb{Z})$ be the subgroup generated by

$$\begin{pmatrix} \mathbf{I}_{\mathrm{d}} & B \\ \mathbf{0} & \mathbf{I}_{\mathrm{d}} \end{pmatrix} \text{ and } \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & (U^{-1})^{\mathrm{T}} \end{pmatrix}$$

for $B \in \text{Sym}_d(\mathbb{Z})$ and $U \in \text{GL}_d(\mathbb{Z})$. Let $\widetilde{\Gamma}_{\infty}^{(d)} \subseteq \text{Mp}_{2d}(\mathbb{Z})$ be the preimage of $\Gamma_{\infty}^{(d)}$. For $k \in \frac{1}{2}\mathbb{Z}$ and k > d + 1, the vector-valued Siegel-Eisenstein series is defined by

$$\mathbf{E}_{k,M}^{(d)}(\tau) = \sum_{\tilde{g}=(g,\phi_g(\tau))\in\tilde{\Gamma}_{\infty}^{(d)}\setminus\mathrm{Mp}_{2d}(\mathbb{Z})} \mathbf{e}_0 \mid_{k,G_M^{(d)}} [\tilde{g}]$$
$$= \sum_{\tilde{g}=(g,\phi_g(\tau))\in\tilde{\Gamma}_{\infty}^{(d)}\setminus\mathrm{Mp}_{2d}(\mathbb{Z})} \phi_g(\tau)^{-2k} \cdot \rho_M^{(d)}(\tilde{g})^{-1}(\mathbf{e}_0),$$

which is well-defined and converges normally. It is a vector-valued modular form of weight k and type $\rho_M^{(d)}$. The summand $\mathbf{e}_0 \mid_{k, G_M^{(d)}} [\tilde{g}]$ is $\tilde{\Gamma}_{\infty}^{(d)}$ -invariant since the parity condition (2.2) holds.

Theta series

Definition 2.3. Let H(X) be a harmonic polynomial in the matrix variable $X = (X_{ij})_{\substack{i=1,...,r \ j=1,...,d}}$ of degree *h*. We define the Theta series

$$\Theta_{M,H}^{(d)}(\tau) = \sum_{\mathbf{v}\in(M^{\vee})^{(d)}} H(\mathbf{v}) \mathbf{e}\left(-\frac{1}{2} \operatorname{tr}((\mathbf{v},\mathbf{v})\tau)\right) \mathbf{e}_{\mathbf{v}+M^d}$$
(2.4)

and the genus Theta series

$$\Theta_{\mathbf{Gen}(M)}^{(d)} = \frac{\sum\limits_{L \in \mathbf{Gen}(M)} |\operatorname{Aut}(L)|^{-1} \sum\limits_{\sigma \in \operatorname{Iso}(G_M, G_L)} |\operatorname{Aut}(G_L)|^{-1} \sigma^* \Theta_{L,1}^{(d)}}{|\operatorname{Iso}(G_M, G_M)| \sum\limits_{L \in \mathbf{Gen}(M)} |\operatorname{Aut}(L)|^{-1}},$$
(2.5)

where **Gen**(M) is short for **Gen**_{0,r}(G_M) and $\sigma^* \mathfrak{e}_{\gamma} = \mathfrak{e}_{\sigma^{-1}(\gamma)}$ for $\gamma \in G_L^{(d)}$.

By [5, Theorem 4.1], $\Theta_{M,H}^{(d)}$ is an element in $\operatorname{Mod}_{h+\frac{r}{2}}(\rho_M^{(d)})$. Moreover, $\Theta_{M,H}^{(d)}$ is a cusp form if h > 0. We define $\operatorname{Mod}_k^{\theta}(\rho_M^{(d)}) \subseteq \operatorname{Mod}_k(\rho_M^{(d)})$ to be the subspace

Span
$$\left\{ \sigma^* \Theta_{L,H}^{(d)} \mid L \in \mathbf{Gen}(M), H \text{ harmonic of degree } k - \frac{r}{2}, \sigma \in \mathrm{Iso}(G_M, G_N) \right\},\$$

and define $\operatorname{Cusp}_k^{\theta}(\rho_M^{(d)})$ as the subspace $\operatorname{Mod}_k^{\theta}(\rho_M^{(d)}) \cap \operatorname{Cusp}_k(\rho_M^{(d)})$. The main question we will consider in the following two sections is whether the equality $\operatorname{Cusp}_k^{\theta}(\rho_M^{(d)}) = \operatorname{Cusp}_k(\rho_M^{(d)})$ holds. In other words, we will consider whether any cusp form can be written as a linear combination of Theta series. For lattices of even rank this was done in [25]. We will adapt the proof to the odd rank case.

Remark 2.4. When d = 1, the space of harmonic polynomials in r variables of degree h has dimension $\binom{r+h-1}{r-1} - \binom{r+h-3}{r-1}$ and there is an explicit basis via the Kelvin transform (cf. [2, Theorem 5.25]). When h = 2, the harmonic polynomials are linear combinations of

$$H_u(v) = \langle u, v \rangle^2 - \frac{\langle u, u \rangle \langle v, v \rangle}{r}$$
(2.6)

for $u \in M$.

One can identify the genus Theta series with the Siegel-Eisenstein series using the Siegel-Weil formula.

Theorem 2.5. Suppose M is a negative definite even lattice of rank r with $\frac{r}{2} > d + 1$. Then

$$\Theta_{\operatorname{Gen}(M)}^{(d)} = \mathbf{E}_{\frac{r}{2},M}^{(d)}.$$

Proof. The proof will be given in Appendix A. See Theorem A.5.

3. Vector-valued Hecke operator

3.1. Bruinier-Stein's vector-valued Hecke operator

Let *G* be a finite abelian group equipped with a discriminant form whose level is $\ell(G) =: N$. Bruinier and Stein have introduced a Hecke operator acting on $Mod_k(\rho_G)$. For each integer $\alpha > 0$, set

$$\widetilde{\mathbf{g}}_{\alpha} = \left(\begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \widetilde{\operatorname{GL}}_2^+(\mathbb{R}).$$

The (inverse of the) Weil representation can be extended to the double coset

$$\widetilde{\mathbf{Y}}_{\alpha^2} := \mathrm{Mp}_2(\mathbb{Z}) \cdot \widetilde{\mathbf{g}}_{\alpha} \cdot \mathrm{Mp}_2(\mathbb{Z}).$$

For any element $\tilde{g} \cdot \tilde{\mathbf{g}}_{\alpha} \cdot \tilde{g}' \in \tilde{\mathbf{Y}}_{\alpha^2}$, define the action

$$\mathbf{e}_{\lambda} \mid [\tilde{g} \cdot \widetilde{\mathbf{g}}_{\alpha} \cdot \tilde{g}'] \coloneqq \mathbf{e}_{\lambda} \mid_{G} [\tilde{g}] \cdot \mid_{G} [\widetilde{\mathbf{g}}_{\alpha}] \cdot \mid_{G} [\tilde{g}'],$$

where $\mathbf{e}_{\gamma} \mid_G [\tilde{g}] = \rho_G(\tilde{g})^{-1}(\mathbf{e}_{\gamma})$ is the Weil representation and $\mathbf{e}_{\gamma} \mid_G [\tilde{\mathbf{g}}_{\alpha}] = \mathbf{e}_{\alpha\gamma}$. This action is proved to be well-defined in [11, Proposition 5.1].

Remark 3.1. It is easy to see that this extension can be made also in the case when $\alpha = 0$. This further extension will be used in Section 4.

Armed with this extended representation, we can define the vector-valued Hecke operator as follows.

$$\bigsqcup_{i} \operatorname{Mp}_{2}(\mathbb{Z}) \cdot \tilde{\delta}_{i}$$

for some $\tilde{\delta}_i = (\delta_i, \phi_{\delta_i}(\tau)) \in \tilde{\mathbf{Y}}_{\alpha^2}$. The Hecke operator $\mathbf{T}_{\alpha^2} : \operatorname{Mod}_k(\rho_G) \to \operatorname{Mod}_k(\rho_G)$ is defined by

$$\mathbf{T}_{\alpha^{2}}(f) \coloneqq \alpha^{k-2} \sum_{i} \sum_{\gamma \in G} (f_{\gamma} \mid_{k} [\tilde{\delta}_{i}]) \otimes (\mathfrak{e}_{\gamma} \mid [\tilde{\delta}_{i}]),$$

where $f = \sum_{\gamma \in G} f_{\gamma} \otimes \mathfrak{e}_{\gamma}$.

This definition is proved to be well-defined in [11, Section 5]. The main properties of these Hecke operators are summarized in the following proposition.

Proposition 3.3. [11, Theorem 4.12, Theorem 5.6] The Hecke operators defined in Definition 3.2 satisfy the following properties.

- These Hecke operators take cusp forms to cusp forms. They are self-adjoint with respect to the Petersson inner product, and the Hecke operators $\{\mathbf{T}_{\alpha^2} : \gcd(\alpha, N) = 1\}$ generate a commutative subalgebra of $\operatorname{End}_{\mathbb{C}}(\operatorname{Cusp}_k(\rho_G))$.
- For each pair α , β of coprime positive integers, $\mathbf{T}_{\alpha^2} \circ \mathbf{T}_{\beta^2} = \mathbf{T}_{(\alpha\beta)^2}$.

As a consequence, one has

Corollary 3.4. There exists a basis of $\operatorname{Cusp}_k(\rho_G)$ consisting of simultaneous eigenforms for all Hecke operators in $\{\mathbf{T}_{\alpha^2} : \gcd(\alpha, N) = 1\}$.

3.2. Non-vanishing of L-values

Let $f \in \text{Cusp}_k(\rho_G)$ be a non-zero simultaneous eigenform for all Hecke operators $\{\mathbf{T}_{\alpha^2} : \text{gcd}(\alpha, N) = 1\}$ with eigenvalues $\{\lambda(\alpha^2) : \text{gcd}(\alpha, N) = 1\}$. One can define the *L*-series

$$L(f,s) := \sum_{\substack{\alpha \ge 1 \\ \gcd(\alpha,N)=1}} \frac{\lambda(\alpha^2)}{\alpha^s}.$$
(3.1)

The main analytic properties of the *L*-series relevant to our goals are summarized in the following theorem.

Theorem 3.5. Suppose $k \ge 2$. The L-series (3.1) converges absolutely for Re(s) > k + 1. In particular, when Re(s) > k + 1,

$$L(f,s) \neq 0.$$

Proof. The case when $k \in \mathbb{Z}$ is [25, Proposition 3.3] and the result is slightly stronger. Now we consider the case when $k \in \mathbb{Z} + \frac{1}{2}$, then sign(G) is odd. We will establish the theorem by comparing Bruinier-Stein's Hecke operators with the scalar-valued Hecke operators defined by Shimura [31].

(i) Since sign(G) is odd, the oddity formula [13, p. 383 (30)] implies that $4 \mid N$. There is a map

$$\Gamma_0(4) \xrightarrow{\kappa} \widetilde{\operatorname{GL}}_2^+(\mathbb{R})$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon_d^{-1} \begin{pmatrix} c \\ d \end{pmatrix} \sqrt{c\tau + d} \right)$$

where $\epsilon_d = 1$ if $d \equiv 1 \mod 4$, $\epsilon_d = i$ if $d \equiv -1 \mod 4$, and $(\frac{1}{2})$ is the Jacobi symbol. This map lifts to $\operatorname{Mp}_2(\mathbb{Z})$ when restricted to $\Gamma_1(4)$. We denote by $\Delta(N)$ the image of any principal congruence subgroup $\Gamma(N) \subset \Gamma_1(4)$ under this lift. For a character χ of $(\mathbb{Z}/N\mathbb{Z})^{\times}$ and $\gamma \in G$, define the element

$$v_{\gamma,\chi} := \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \chi(d)^{-1} \mathbf{e}_{d\gamma}.$$

Since the Weil representation is trivial on $\Delta(N)$, the function $F_{\gamma,\chi} := \langle f, v_{\gamma,\chi} \rangle$ is then an element in $\operatorname{Cusp}_k(\Delta(N))$, where $\operatorname{Cusp}_k(\Delta(N))$ is the space of scalar-valued cusp forms of weight k and level $\Delta(N)$ (cf. [31]).

(*ii*) We extend κ to

$$\bigcup_{\alpha=1}^{\infty} \Gamma_0(4) \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4) \xrightarrow{\kappa} \widetilde{\operatorname{GL}}_2^+(\mathbb{R})$$

by $\begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \widetilde{\mathbf{g}}_{\alpha}$. For every prime *p* satisfying gcd(p, N) = 1, choose an element $R_{p^{2n-a}} \in \mathrm{SL}_2(\mathbb{Z})$ as a lift of the element $\begin{pmatrix} p^{-2n+a} & 0 \\ 0 & p^{2n-a} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and consider the set

$$\Sigma_{p^{2n}} := \left\{ \kappa \left(R_{p^{2n-a}} \begin{pmatrix} p^{2n-a} & bN \\ 0 & p^a \end{pmatrix} \right) \mid 0 \le a \le 2n, \ 0 \le b < p^a, \ \gcd(b, p^{\min\{a, 2n-a\}}) = 1 \right\}.$$

Since $R_{p^{2n-a}} \begin{pmatrix} p^{2n-a} & bN \\ 0 & p^a \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & p^{2n} \end{pmatrix} \mod N$, the element $\kappa \begin{pmatrix} R_{p^{2n-a}} \begin{pmatrix} p^{2n-a} & bN \\ 0 & p^a \end{pmatrix} \end{pmatrix}$ lies in the metaplectic *double* cover of $GL_2(\mathbb{R})$. As shown in [35, §4], this set can be served as a system of representatives for

$$\mathrm{Mp}_{2}(\mathbb{Z})\backslash \mathrm{Mp}_{2}(\mathbb{Z})\Big(\left(\begin{array}{cc}p^{2n} & 0\\ 0 & 1\end{array}\right), 1\Big)\mathrm{Mp}_{2}(\mathbb{Z}) \cong \Delta(N)\backslash \Delta(N)\Big(\left(\begin{array}{cc}1 & 0\\ 0 & p^{2n}\end{array}\right), p^{n}\Big)\Delta(N)$$

(*iii*) The advantage of choosing $\Sigma_{p^{2n}}$ as coset representatives is that the actions of these representatives via the Weil representation are particularly simple. Let $\tilde{g} = \begin{pmatrix} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi_g(\tau) \end{pmatrix} \in Mp_2(\mathbb{Z})$ with $N \mid b, N \mid c$. For $\gamma \in G$, we have $\rho_G(\tilde{g}) \mathfrak{e}_{\gamma} = \chi_G(\tilde{g}) \mathfrak{e}_{d\gamma}$ for an explicit character χ_G (cf. [7, Theorem 5.4]). It follows that for every $\tilde{g} \in \Sigma_{p^{2n}}$, we have

$$\mathbf{e}_{\gamma} \mid [\tilde{g}] = \mathbf{e}_{p^{-n}\gamma}.$$

(*iv*) In [31], Shimura introduced a Hecke operator

$$\mathcal{T}_{\alpha^2}^{\Delta(N)} : \operatorname{Cusp}_k(\Delta(N)) \to \operatorname{Cusp}_k(\Delta(N))$$

defined by

$$\mathcal{T}_{\alpha^2}^{\Delta(N)}(F) = \alpha^{k-2} \sum_{\tilde{g} \in \Delta(N) \setminus \Delta(N) \left(\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{2n} \end{pmatrix}, \alpha^n \right) \Delta(N)} F \mid_k [\tilde{g}].$$

By the discussions in (ii), it is easy to see that

$$\langle \mathbf{T}_{\alpha^2}(f), v_{\gamma,\chi} \rangle = \chi(p^n)^{-1} \mathcal{T}_{p^{2n}}^{\Delta(N)}(F_{\gamma,\chi}).$$

Now suppose *f* is a non-zero simultaneous eigenform for all Hecke operators in { \mathbf{T}_{α^2} : $gcd(\alpha, N) = 1$ } with eigenvalues $\lambda(\alpha^2)$. Since the elements $v_{\gamma,\chi}$ span $\mathbb{C}[G]$ and $f \neq 0$, one can find a pair (γ, χ) such that $F_{\gamma,\chi} \neq 0$. Then $F_{\gamma,\chi}$ is a non-zero simultaneous eigenform for all Hecke operators in { $\mathcal{T}_{\alpha^2}^{\Delta(N)}$: $gcd(\alpha, N) = 1$ } with eigenvalues $\chi(\alpha)\lambda(\alpha^2)$. We claim that the series

$$\sum_{\substack{\alpha \ge 1 \\ \operatorname{cd}(\alpha, N) = 1}} \frac{\chi(\alpha)\lambda(\alpha^2)}{\alpha^s}$$

is absolutely convergent for $\operatorname{Re}(s) > k + 1$. Hence L(f, s) is also absolutely convergent for $\operatorname{Re}(s) > k + 1$. Since we have a formal product expansion

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$$L(f,s) = \prod_{\gcd(p,N)=1} \sum_{n=0}^{\infty} \frac{\lambda(p^{2n})}{p^{ns}},$$

general theory regarding Euler products now shows that the absolute convergence for Re(s) > k+1implies $L(f, s) \neq 0$.

Now we prove the claim. First observe that $F_{\gamma,\chi}(N \cdot) \in \operatorname{Cusp}_k(\Delta_0(N^2), \psi)$ for some character ψ of $(\mathbb{Z}/N^2\mathbb{Z})^{\times}$. Furthermore, we have $(\mathcal{T}_{\alpha^2}^{\Delta(N)}F_{\gamma,\chi})(N \cdot) = \mathcal{T}_{\alpha^2}^{\Delta_0(N^2),\psi}(F_{\gamma,\chi}(N \cdot))$. This is an analog of the classical way of treating a modular form of level $\Gamma(N)$ as a modular form of level $\Gamma_0(N^2)$ with a Nebentypus. Now let p be a prime such that $\gcd(p, N) = 1$. By the theory of Shimura lifts, $\chi(p)\lambda(p^2)$ is the p-th Fourier coefficient of a normalized cuspidal Hecke eigenform of weight 2k - 1. Hence the Ramanujan conjecture proved by Deligne implies that

$$\left|\chi(p)\lambda(p^2)\right| \le 2p^{k-1}$$

Since $p \ge 3$, by the explicit algebraic relations between the Hecke operators $\{\mathcal{T}_{p^{2n}}^{\Delta_0(N^2),\psi} \mid n \ge 0\}$ in [29, Theorem 1.1], one can inductively prove that

$$\left|\chi(p^n)\lambda(p^{2n})\right| \le 3^n p^{n(k-1)} \le p^{nk}.$$

This immediately yields the claim.

3.3. The case when $p \mid N$

The Hecke operators $\{\mathbf{T}_{\alpha^2} : \gcd(\alpha, N) \neq 1\}$ are trickier since the Hecke algebra generated by $\{\mathbf{T}_{p^{2n}} : n \geq 1\}$ is in general non-commutative if $p \mid N$ (cf. [11, Remark 5.7]). However, regarding our goals, a comparison result of these Hecke operators with certain scalar-valued Hecke operators is sufficient.

Lemma 3.6. Fix a prime p. Let $f \in Mod_k(\rho_G)$ and let $\gamma \in G$ satisfy the following conditions $(*_p)$ for p:

- γ is not divisible by p in G;
- when p = 2, there exists $\mu \in G$, $2\mu = 0$, such that $2\mathfrak{q}(\mu) + (\mu, \gamma) \neq 0 \mod \mathbb{Z}$.

Then

$$\langle \mathbf{T}_{p^{2n}}(f), \mathbf{e}_{\gamma} \rangle = \mathrm{T}_{p^{2n}}^{\mathfrak{q}(\gamma)} \langle f, \mathbf{e}_{p^{2n}\gamma} \rangle,$$

where

$$\mathbf{T}_{p^{2n}}^{\mathfrak{q}(\gamma)}(F) := p^{(k-2)n} \sum_{b=0}^{p^{2n}-1} \mathbf{e}(-b\mathfrak{q}(\gamma))F \mid_{k} \left[\left(\begin{pmatrix} 1 & b \\ 0 & p^{2n} \end{pmatrix}, p^{n} \right) \right]$$

for any scalar-valued function F on \mathbb{H} .

Proof. When $\alpha = p^n$, there is an explicit coset decomposition

$$\widetilde{\mathbf{Y}}_{p^{2n}} = \mathrm{Mp}_{2}(\mathbb{Z}) \cdot \left(\begin{pmatrix} p^{2n-a} & b \\ 0 & p^{a} \end{pmatrix}, \sqrt{p^{a}} \right)$$

where $0 \le a \le 2n$, $0 \le b < p^a$ and $gcd(b, p^{\min\{a, 2n-a\}}) = 1$ (cf. [32, §5.1]). Set $\tilde{\delta}_{a,b} = \left(\begin{pmatrix} p^{2n-a} & b \\ 0 & p^a \end{pmatrix}, \sqrt{p^a} \right)$. Then we have

$$\frac{1}{p^{(k-2)n}} \langle \mathbf{T}_{p^{2n}}(f), \mathbf{e}_{\gamma} \rangle = \sum_{a=0}^{2n} \sum_{\substack{0 \le b < p^a \\ \gcd(b, p^{\min\{a, 2n-a\}})=1}} \sum_{\mu \in G} \langle f_{\mu} \mid_{k} [\tilde{\delta}_{a,b}] \otimes (\mathbf{e}_{\mu} \mid [\tilde{\delta}_{a,b}]), \mathbf{e}_{\gamma} \rangle$$

From the definition of $T_{p^{2n}}^{\mathfrak{q}(\gamma)}$, it suffices to show

$$\langle \mathbf{e}_{\mu} \mid [\widetilde{\delta}_{a,b}], \mathbf{e}_{\gamma} \rangle = 0$$

for every a < 2n. The action of $\tilde{\delta}_{a,b}$ is well-understood. For instance, if p is odd, there is an explicit formula of $\mathfrak{e}_{\mu} \mid [\tilde{\delta}_{a,b}]$ proved in [32, Theorem 5.2] and [8, Proposition 5.3]. In general, it is of the form

$$\mathbf{e}_{\mu} \mid [\tilde{\delta}_{a,b}] = \sum_{\lambda} c_{\lambda} \mathbf{e}_{\lambda}$$
(3.2)

with $\lambda \in G^p$ if p is odd, and $\lambda = 2m\mu + \beta$ for some $\beta \in G^{2*}$ and $m \in \mathbb{Z}$ if p = 2 (cf. [30, Theorem 4.7] and [33, Theorem 1]). As γ satisfies the condition $(*_p)$, then $\langle \mathbf{e}_{\mu} \mid [\widetilde{\delta}_{a,b}], \mathbf{e}_{\gamma} \rangle = 0$ for every a < 2n. \Box

By using this lemma inductively, one formally obtains a useful result for later detecting when a modular form is annihilated by those Hecke operators not coprime to N.

Proposition 3.7. [25, Corollary 3.5] Fix a finite set S of primes. Let $f \in Mod_k(\rho_G)$ and let $\gamma, \mu \in G$ satisfy the following conditions:

- $\left(\prod_{p \in S} p\right) \gamma = \left(\prod_{p \in S} p\right) \mu \text{ and } \mathfrak{q}(\gamma) = \mathfrak{q}(\mu);$
- for any $p \in S$, γ, μ satisfy conditions $(*_p)$.

Set

$$v^{\mu}_{\gamma,S} := \sum_{I \subset S} (-1)^{|I|} \mathfrak{e}_{\gamma^{\mu}_{I}},$$

where $\gamma_I^{\mu} \in G$ is the element whose *p*-adic component is equal to that of μ when $p \in I$ and other *p*-adic components remain the same as those of γ . Then if $\prod_{p \in S} \sum_{n=1}^{\infty} \frac{\mathbf{T}_{p^{2n}}}{p^{ns}}(f) = 0, \langle f, v_{\gamma,S}^{\mu} \rangle = 0.$

Proof. By using Lemma 3.6, one can verify the following formula

$$\langle \mathbf{T}_{p^{2n}}(f), v_{\gamma, S \setminus \{p\}}^{\mu_{\{p\}}^{\gamma}} \rangle = \mathbf{T}_{p^{2n}}^{\mathfrak{q}(\gamma)} \langle f, v_{p^n \gamma, S \setminus \{p\}}^{p^n \mu_{\{p\}}^{\gamma}} \rangle,$$
(3.3)

and

$$\langle \mathbf{T}_{p^{2n}}(f), v^{\mu}_{\gamma^{\mu}_{\{p\}}, S \setminus \{p\}} \rangle = \mathbf{T}_{p^{2n}}^{\mathfrak{q}(\gamma)} \langle f, v^{p^{n}\mu}_{p^{n}\gamma^{\mu}_{\{p\}}, S \setminus \{p\}} \rangle.$$
(3.4)

By induction on the cardinality of S, one can obtain the assertion by (3.3), (3.4) and the fact

$$v_{\gamma,S}^{\mu} = v_{\gamma,S\setminus\{p\}}^{\mu_{\{p\}}^{\gamma}} - v_{\gamma_{\{p\}}^{\mu},S\setminus\{p\}}^{\mu}.$$
(3.5)

4. Modular forms as Theta series

This section studies the space of cusp forms $\operatorname{Cusp}_k(\rho_M)$ with respect to an even negative definite lattice M. The goal is to show that under certain conditions of ρ_M , the space $\operatorname{Cusp}_k(\rho_M)$ is spanned by the Theta series. Throughout this section, M is an even negative definite lattice of rank r.

4.1. Projection to $\operatorname{Cusp}_{k}^{\theta}(\rho_{M})$

In [16], Eichler and Zagier introduced a differential operator transforming a C^{∞} function on \mathbb{H}_2 into a C^{∞} function on $\mathbb{H} \times \mathbb{H}$. To define this, we need Gegenbauer polynomials $G_r^h(x, y)$, which are defined by

$$\frac{1}{(1-2xT+yT^2)^{\frac{r}{2}-1}} = \sum_{h=0}^{\infty} G_r^h(x,y)T^h,$$

and $G_r^h(1,1) = \binom{r-3+h}{h}$. Then we can define the differential operator

$$\partial_h : C^{\infty}(\mathbb{H}_2) \to C^{\infty}(\mathbb{H} \times \mathbb{H})$$
$$f \mapsto G_r^h \left(\frac{1}{2} \frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1 \partial z_4} \right) f(Z) \mid_{z_2 = 0},$$

where the coordinate $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \mathbb{H}_2$. For the Theta series, we have the following facts.

Lemma 4.1. Suppose h > 0. Let $\{H_i\}$ be an orthonormal basis of the space of harmonic forms of degree h. For $(z, z') \in \mathbb{H} \times \mathbb{H} \subseteq \mathbb{H}_2$, we have

$$\Theta_{M,1}^{(2)}(z,z') = \Theta_{M,1}^{(1)}(z) \otimes \Theta_{M,1}^{(1)}(z')$$
(4.1)

and

$$(\partial_h \Theta_{M,1}^{(2)})(z,z') = C \sum_i \Theta_{M,H_i}^{(1)}(z) \otimes \Theta_{M,\overline{H}_i}^{(1)}(z')$$
(4.2)

for some non-zero constant C.

Proof. For the first assertion, this follows from the definition of the Theta series. The second assertion then follows from [25, Proposition 2.7].

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Let $k = \frac{r}{2} + h$ for some non-negative integer h. When h = 0, assume further r > 4. As in [25], setting $\vartheta_{M,k} := \partial_h \Theta_{\text{Gen}(M)}^{(2)}$, we can define a Theta lifting map

$$\Psi: \operatorname{Cusp}_k(\rho_M) \to \operatorname{Mod}_k(\rho_M) \tag{4.3}$$

by sending f to

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(z), \vartheta_{M,k}(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

as a function in z'. Then one has

Proposition 4.2. Let k, M be as above. The linear map Ψ is diagonalizable and surjective onto $\operatorname{Cusp}_k^{\theta}(\rho_M)$. In particular, there is a decomposition

$$\operatorname{Cusp}_k(\rho_M) = \operatorname{Cusp}_k^{\theta}(\rho_M) \oplus \ker \Psi.$$

Proof. The proof is essentially [25, Proposition 6.2]. According to Lemma 4.1, we have

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$$\Psi(f)(z') = \int_{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \left(\sum_{L \in \mathbf{Gen}(M)} \sum_{\sigma \in \mathrm{Iso}(G_M, G_L)} C_{L,\sigma} \langle f(z), \sigma^* \partial_h \Theta_{L,1}^{(2)}(z, -\overline{z'}) \rangle \right) y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}$$
$$= C \sum_i \sum_{L \in \mathbf{Gen}(M)} \sum_{\sigma \in \mathrm{Iso}(G_M, G_L)} C_{L,\sigma} \langle f, \sigma^* \Theta_{L,H_i}^{(1)} \rangle_{\mathrm{Pet}} \cdot \sigma^* \Theta_{L,H_i}^{(1)}(z')$$

where the constant C and the harmonic polynomials $\{P_i\}$ are those in Lemma 4.1, and

$$\Theta_{\operatorname{Gen}(M)}^{(2)} = \sum_{L \in \operatorname{Gen}(M)} \sum_{\sigma \in \operatorname{Iso}(G_M, G_L)} C_{L,\sigma} \sigma^* \Theta_{L,1}^{(2)}$$

is given in (2.5). When h > 0, as $\Theta_{L,H_i}^{(1)}$ is cuspidal, we have $\Psi(f) \in \text{Cusp}_k^{\theta}(\rho_M)$. When h = 0, we have all $H_i = 1$. Note that $\Theta_{\text{Gen}(M)}^{(1)} = \mathbf{E}_{k,M}^{(1)}$ (by Theorem 2.5) is orthogonal to f under the Petersson inner product, one can rewrite $\Psi(f)$ as

$$\Psi(f)(z') = \sum_{L \in \mathbf{Gen}(M)} \sum_{\sigma \in \mathrm{Iso}(G_M, G_L)} C_{L,\sigma} \langle f, \sigma^* \Theta_{L,1} - \Theta_{\mathbf{Gen}(M)}^{(1)} \rangle_{\mathrm{Pet}} \cdot (\sigma^* \Theta_{L,1} - \Theta_{\mathbf{Gen}(M)}^{(1)}).$$

Since $\sigma^* \Theta_{L,1} - \Theta_{\text{Gen}(M)}^{(1)}$ is cuspidal, the above formula implies $\Psi(f) \in \text{Cusp}_k^{\theta}(\rho_M)$ as well. The assertion now follows from the Lemma 4.3 below.

Lemma 4.3. [25, Lemma 6.1] Let V be a \mathbb{C} -vector space with an inner product $\langle \cdot, \cdot \rangle$, and let $\{v_i\}_{i \in I} \subset V$ be an arbitrary finite family. Then $f : V \to V$ defined by $v \to \sum_{i \in I} \langle v, v_i \rangle \cdot v_i$ is diagonalizable and surjective onto Span $\{v_i : i \in I\}$.

Thus a statement like "Cusp = $Cusp^{\theta}$ " is equivalent to "ker $\Phi = 0$ ".

4.2. Ψ as Hecke operators

Let $\alpha \in \mathbb{Z}^{>0}$. Let $K_{\alpha}(z, z') : \mathbb{H} \times \mathbb{H} \to \mathbb{C}[G_M] \otimes \mathbb{C}[G_M]$ be given by

$$\sum_{\gamma \in G_M} \sum_{\tilde{g} = (g, \phi_g(z)) \in \tilde{\mathbf{Y}}_{a^2}} \frac{\phi_g(z)^{-2k}}{(z' + g \cdot z)^k} \rho_M(\tilde{g})^{-1}(\mathfrak{e}_{\gamma}) \otimes \mathfrak{e}_{\gamma}.$$
(4.4)

More explicitly, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\phi_g(z) = \pm \sqrt{cz + d}$, then

$$\frac{\phi_g(z)^{-2k}}{(z'+g(z))^k} = \frac{(\pm 1)^{2k}}{(az+b+czz'+dz')^k}.$$
(4.5)

As a function of z, one can check $\mathbf{T}_{\alpha^2}(K_{\alpha}(z,z')) = \alpha^{2k-2}K_{\alpha}(z,z')$. As shown in [25, Proposition 3.7] (see also Appendix C for the odd signature case), it can be viewed as the kernel function of \mathbf{T}_{α^2} up to a scalar, i.e. it satisfies

$$\frac{\mathbf{T}_{\alpha^2}(f)(z')}{\alpha^{2k-2}} = C \int_{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \langle f, K_\alpha(z, -\overline{z}') \rangle y^{k-2} \mathrm{d}x \mathrm{d}y,$$
(4.6)

for some non-zero constant C.

Theorem 4.4. Assume that $r = \operatorname{rank}(M) > 6$, and let $k = \frac{r}{2} + h$ for some non-negative integer h. For any $f \in \operatorname{Cusp}_k(\rho_M)$, we have

$$\Psi(f) = C' \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{2k-2-h}} \mathbf{T}_{\alpha^2}(f).$$

for some non-zero constant C'.

Proof. Let $\widetilde{C} := h! \binom{r-3+h}{h} \binom{k-1}{h}$. For $\tau = (z, z') \in \mathbb{H} \times \mathbb{H} \subseteq \mathbb{H}_2$, we claim that

$$\vartheta_{M,k}(\tau) = \partial_h \Theta_{\mathbf{Gen}(M)}^{(2)} = \begin{cases} \mathbf{E}_{k,M}^{(1)}(z) \otimes \mathbf{E}_{k,M}^{(1)}(z') + \frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} \sum_{\alpha=1}^{\infty} K_{\alpha}(z,z') & \text{if } h = 0\\ \widetilde{C} \frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} \sum_{\alpha=1}^{\infty} \alpha^h K_{\alpha}(z,z') & \text{if } h > 0 \end{cases}$$
(4.7)

If the claim holds, note that

$$\int_{\mathrm{SL}_{2}(\mathbb{Z})\backslash\mathbb{H}} \langle f(z), \mathbf{E}_{k,M}^{(1)}(z) \otimes \mathbf{E}_{k,M}^{(1)}(-\overline{z'}) \rangle y^{k} \frac{\mathrm{d}x\mathrm{d}y}{y^{2}} = \langle f(z), \mathbf{E}_{k,M}^{(1)}(z) \rangle_{\mathrm{Pet}} \cdot \mathbf{E}_{k,M}^{(1)}(z') = 0$$
(4.8)

for any $f \in \text{Cusp}_k(\rho_M)$, then we have

$$\Psi(f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(z), \vartheta_{M,k}(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

= $\widetilde{C} \frac{\mathbf{e}(-\mathrm{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} \sum_{\alpha=1}^{\infty} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(z), \alpha^h K_\alpha(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$
= $\widetilde{C} \frac{\mathbf{e}(-\mathrm{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} C^{-1} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{2k-2-h}} \mathbf{T}_{\alpha^2}(f),$

by using (4.7) and (4.6).

To prove the claim, let us first collect some facts.

(i) By Theorem 2.5, we have

$$\Theta_{\mathbf{Gen}(M)}^{(2)} = \mathbf{E}_{\frac{r}{2},M}^{(2)} = \sum_{\tilde{g}^{(2)} \in \widetilde{\Gamma}_{\infty}^{(2)} \setminus \mathrm{Mp}_{4}(\mathbb{Z})} \left(1 \mid_{\frac{r}{2}} [\tilde{g}^{(2)}] \right) \rho_{M}^{(2)} (\tilde{g}^{(2)})^{-1} (\mathfrak{e}_{0} \otimes \mathfrak{e}_{0}).$$

(*ii*) It's shown in Appendix B that for any coset $[\tilde{g}^{(2)}] \in \tilde{\Gamma}_{\infty}^{(2)} \setminus Mp_4(\mathbb{Z})$, it is uniquely determined by a pair

$$(\varphi(\tilde{g}^{(2)}), \alpha) \in \widetilde{\operatorname{Mat}}_2(\mathbb{Z}) \times \mathbb{Z}^{\leq 0},$$

(up to ± 1 when $\alpha = 0$), where $\varphi(\tilde{g}^{(2)}) = (g, \phi_g)$ satisfies $g \in \mathbf{Y}_{\alpha^2}$ and

$$\frac{\phi_g(z)^{-2k}}{(z'+g(z))^k} = \phi_{g^{(2)}}(z,z')^{-2k}.$$
(4.9)

Moreover, we can find a representative $\tilde{g}_{0}^{(2)} \in [\tilde{g}^{(2)}]$ such that

$$\rho_M^{(2)}(\tilde{g}_0^{(2)})^{-1}(\mathfrak{e}_0 \otimes \mathfrak{e}_0) = \frac{\mathfrak{e}(\operatorname{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} \sum_{\gamma \in G_M} (\mathfrak{e}_\gamma \mid [\varphi(\tilde{g}^{(2)})]) \otimes \mathfrak{e}_\gamma.$$
(4.10)

(*iii*) As shown in the proof of [25, Theorem 4.7], recall $k = \frac{r}{2} + h$, we have

$$\partial_h \left(1 \mid_{\frac{r}{2}} [\tilde{g}^{(2)}] \right) (\tau) = G_r^h(1,1) |\alpha|^h \prod_{i=0}^{h-1} \left(\frac{r}{2} + i \right) \left(1 \mid_k [\tilde{g}^{(2)}] \right) (\tau)$$

$$= \widetilde{C} |\alpha|^h \left(1 \mid_k [\tilde{g}^{(2)}] \right) (\tau).$$
(4.11)

for $\tau = (z, z') \in \mathbb{H} \times \mathbb{H}$, where the non-zero constant $\widetilde{C} = h! \binom{r-3+h}{h} \binom{k-1}{h}$.

Now we are ready to derive the formula. For simplicity, let us first assume h > 0. Then by (4.11), (4.10) and (B.2), we have

$$\begin{split} \partial_{h} \mathbf{E}_{\frac{t}{2},M}^{(2)}(z,z') &= \sum_{\alpha=0}^{\infty} \sum_{\substack{\tilde{g}^{(2)} \in \widetilde{\Gamma}_{\alpha}^{(2)} \setminus Mp_{4}(\mathbb{Z}) \\ \varphi(\tilde{g}^{(2)}) \in \widetilde{\mathbf{Y}}_{\alpha^{2}}}} \partial_{h} \left(1 \mid_{\underline{r}} [\tilde{g}^{(2)}] \right) (z,z') \rho_{M}^{(2)} (\tilde{g}^{(2)})^{-1} (\mathfrak{e}_{0} \otimes \mathfrak{e}_{0}) \\ &= \widetilde{C} \sum_{\alpha=1}^{\infty} \sum_{\varphi(\tilde{g}^{(2)}) \in \widetilde{\mathbf{Y}}_{\alpha^{2}}} \alpha^{h} \left(1 \mid_{k} [\tilde{g}^{(2)}] \right) (z,z') \rho_{M}^{(2)} (\tilde{g}^{(2)})^{-1} (\mathfrak{e}_{0} \otimes \mathfrak{e}_{0}) \\ &= \widetilde{C} \frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_{M}|^{\frac{1}{2}}} \sum_{\alpha=1}^{\infty} \sum_{\varphi(\tilde{g}^{(2)}) \in \widetilde{\mathbf{Y}}_{\alpha^{2}}} \alpha^{h} \left(1 \mid_{k} [\tilde{g}^{(2)}] \right) (z,z') \left(\sum_{\gamma \in G_{M}} (\mathfrak{e}_{\gamma} \mid [\varphi(\tilde{g}^{(2)})]) \otimes \mathfrak{e}_{\gamma} \right) \\ &= \widetilde{C} \frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_{M}|^{\frac{1}{2}}} \sum_{\alpha=1}^{\infty} \frac{1}{2} \sum_{\tilde{g} \in \widetilde{\mathbf{Y}}_{\alpha^{2}}} \alpha^{h} \frac{\phi_{g}(z)^{-2k}}{(z'+g\cdot z)^{k}} \left(\sum_{\gamma \in G_{M}} (\mathfrak{e}_{\gamma} \mid [\tilde{g}]) \otimes \mathfrak{e}_{\gamma} \right) \\ &= \widetilde{C} \frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_{M}|^{\frac{1}{2}}} \frac{1}{2} \sum_{\alpha=1}^{\infty} \alpha^{h} K_{\alpha}(z,z'). \end{split}$$

When h = 0, one has to deal with the determinant zero part and show that

$$\frac{\mathbf{e}(\operatorname{sign}(M)/8)}{|G_M|^{\frac{1}{2}}} \frac{1}{2} \sum_{\tilde{g} \in \widetilde{\mathbf{Y}}_0/\pm 1} \frac{\phi_g(z)^{-2k}}{(z'+g \cdot z)^k} \sum_{\gamma \in G_M} (\mathbf{e}_\gamma \mid [\tilde{g}]) \otimes \mathbf{e}_\gamma = \mathbf{E}_{k,M}(z) \otimes \mathbf{E}_{k,M}(z').$$

The argument is essentially the same (see [25, Theorem 4.6]) and we omit the details.

Theorem 4.5. Assume that M is a negative definite even lattice of level N and rank r > 6. Furthermore, assume that for all primes p, $M \otimes \mathbb{Z}_p$ splits a hyperbolic plane. For $k \ge r/2$, we have

$$\operatorname{Cusp}_k^{\theta}(\rho_M) = \operatorname{Cusp}_k(\rho_M).$$

Proof. As mentioned before, it suffices to show that the linear map Ψ is injective. When sign(*M*) is even, this is exactly [25, Theorem 6.7]. When sign(*M*) is odd, the argument is very similar and we may sketch the proof for the ease of readers.

(*i*) Suppose $f \in \text{Cusp}_k(\rho_M)$ satisfies $\Psi(f) = 0$. By Theorem 4.4,

$$\Psi(f) = C' \sum_{\alpha=1}^{\infty} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-h}}(f) = C' \left(\sum_{\substack{\alpha \ge 1 \\ \gcd(\alpha,N)=1}} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-h}} \right) \left(\prod_{p \mid N} \sum_{n=0}^{\infty} \frac{\mathbf{T}_{p^{2n}}}{p^{n(2k-2-h)}} \right) (f)$$

for some non-zero constant C'. By Corollary 3.4, choose a basis of $\operatorname{Cusp}_k(\rho_M)$ consisting of simultaneous eigenforms for all Hecke operators in $\{\mathbf{T}_{\alpha^2} : \operatorname{gcd}(\alpha, N) = 1\}$. Now by Theorem 3.5, for each element g in this basis,

$$\left(\sum_{\substack{\alpha \ge 1\\ \gcd(\alpha,N)=1}} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-h}}\right)(g) = L(g, 2k-2-h)g \neq 0.$$

Therefore the operator $\sum_{\substack{\alpha \ge 1 \\ \gcd(\alpha, N)=1}} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-h}}$ is invertible, which implies $\prod_{p|N} \sum_{n=1}^{\infty} \frac{\mathbf{T}_{p^{2n}}}{p^{n(2k-2-h)}}(f) = 0.$

(*ii*) Let $n := \prod_{p \mid N} p$ be the radical of *N*. By [25, Lemma 6.6], according to our assumption on *M*, there

exists a sublattice $L \subseteq M$ such that $G_L \cong G_M \oplus G'$, where $G' = \begin{pmatrix} 0 & \frac{1}{n} \\ \frac{1}{n} & 0 \end{pmatrix} \mod \mathbb{Z}$ under a basis $\{x_1, x_2\}$ and $nx_i = 0$. Then the cyclic subgroup $H = \langle x_1 \rangle$ is an isotropic subgroup of G_L and $H^{\perp}/H \cong G_M$.

- (*iii*) For any $\gamma \in G_M$, we have
 - $x_1 + \gamma, x_2 + \gamma$ are not divisible by p in G_L for any $p \mid N$.
 - when *l* is even, then

$$2\mathfrak{q}\left(\frac{nx_1}{2}\right) + \left(\frac{n}{2}x_1, x_2 + \gamma\right) = 2\mathfrak{q}\left(\frac{nx_2}{2}\right) + \left(\frac{n}{2}x_2, x_1 + \gamma\right) = \frac{1}{2} \neq 0 \mod \mathbb{Z}.$$

(*iv*) According to [25, Lemma 6.3], also $\prod_{p|N} \sum_{n=1}^{\infty} \frac{\mathbf{T}_{p^{2n}}}{p^{n(2k-2-h)}} \uparrow_H (f) = 0. \text{ Let } v = v_{x_1+\gamma, x_2+\gamma, S} \text{ and then}$

$$\downarrow_{H} (v) = \sum_{I \subset \{p|N\}} (-1)^{|I|} \downarrow_{H} \mathfrak{e}_{(x_{1}+\gamma)_{I}^{x_{2}+\gamma}}$$
$$= \mathfrak{e}_{\gamma}$$

as $(x_1 + \gamma)_I^{x_2 + \gamma}$ is not orthogonal to *H* unless $I = \emptyset$. Then

$$\langle f, \mathbf{e}_{\gamma} \rangle = \langle \uparrow_H (f), v \rangle = 0,$$

where we used Proposition 3.7 for the last equality. Since γ is arbitrary, this implies that f = 0.

Remark 4.6. There is a well-established criterion for determining whether $M \otimes \mathbb{Z}_p$ splits a hyperbolic plane, as mentioned in [27]. For example, this occurs if $\operatorname{rank}_p(G_M) < r - 2$. In practical terms, this condition is not overly restrictive when r is sufficiently large, as $\operatorname{rank}_p(G_M) \leq r$ always holds.

5. Heegner divisors on Shimura varieties of orthogonal type

Throughout this section, we let M be an even lattice of signature (2, n) under a pairing $\langle -, - \rangle$ and $G_M = M^{\vee}/M$. For a normal variety X, $Cl_{\mathbb{Q}}(X)$ will denote the class group of Weil divisors on X with rational coefficients.

5.1. Shimura variety attached to a lattice

Let $\mathbf{D} = \mathbf{D}(M)$ be the type IV Hermitian symmetric domain associated to M defined by

$$\mathbf{D}(M) = \left\{ w \in \mathbb{P}(M \otimes \mathbb{C}) \mid \langle w, w \rangle = 0, \langle w, \overline{w} \rangle > 0 \right\}$$
(5.1)

Let $\widetilde{O}(M) = \ker(O(M) \to O(G_M))$ be the stable orthogonal group. It acts naturally on $\mathbf{D}(M)$ and the arithmetic quotient $\widetilde{O}(M) \setminus \mathbf{D}$ is a normal quasi-projective variety, denoted by $\operatorname{Sh}(M)$. The tautological line bundle O(-1) on $\mathbf{D}(M) \subseteq \mathbb{P}(M \otimes \mathbb{C})$ is $\widetilde{O}(M)$ -invariant and descends to a line bundle on $\operatorname{Sh}(M)$, denoted by λ_M . It is called the Hodge line bundle on $\operatorname{Sh}(M)$.

Example 5.1. Set

$$\Lambda_g = \langle 2 - 2g \rangle \oplus E_8(-1)^{\oplus 2} \oplus U^{\oplus 2}.$$

Its discriminant group $\Lambda_g^{\vee}/\Lambda_g$ is a cyclic group of order 2g - 2. By global Torelli theorem for K3 surfaces, the period map induces an isomorphism

$$\mathcal{F}_g \cong \mathrm{Sh}(\Lambda_g).$$

which identifies the Noether-Lefschetz divisors on \mathcal{F}_g as Heegner divisors on $\mathrm{Sh}(\Lambda_g)$.

5.2. Heegner divisors on Sh(M)

Let us consider the divisor class group of Sh(M).

Definition 5.2. For $m \in \mathbb{Q}$ and $\gamma \in G_M$, if $(m, \gamma) \neq (0, 0)$, we define

$$\mathbf{H}_{m,\gamma} = \widetilde{\mathbf{O}}(M) \setminus \sum_{\substack{v \in M + \gamma \\ \mathfrak{q}_M(v) = m}} v^{\perp},$$

to be the Heegner divisor on Sh(M). If $(m, \gamma) = (0, 0)$, then we may set $H_{0,0} = -\lambda_M$.

In general, the Heegner divisor $H_{m,\gamma} \in Cl_{\mathbb{Q}}(Sh(M))$ is not irreducible. When $\gamma = -\gamma$ in M^{\vee}/M , the Heegner divisor $H_{m,\gamma}$ has multiplicity two. The irreducible component of $H_{m,\gamma}$ is the image of

$$\Gamma_{\nu} \setminus \mathbf{D}(\nu^{\perp}) \to \widetilde{\mathbf{O}}(M) \setminus \mathbf{D}$$
 (5.2)

for some $v \in M^{\vee} \subseteq M \otimes \mathbb{Q}$ satisfying $\mathfrak{q}_M(v) = m$ and $v \equiv \gamma \mod M$, where $\Gamma_v \subseteq \widetilde{O}(M)$ is the stabilizer of v.

When *M* contains two hyperbolic planes, by using Eichler's lemma, there is a triangular relation between the Heegner divisors $H_{m,\gamma}$ and its irreducible components. In this case, the span of all Heegner divisors $H_{m,\gamma}$ equals the span of irreducible components of $H_{m,\gamma}$.

5.3. Picard group of Sh(M)

Let $\operatorname{Pic}_{\mathbb{Q}}(\operatorname{Sh}(M))$ be the Picard group of $\operatorname{Sh}(M)$ with rational coefficients. As $\operatorname{Sh}(M)$ has only quotient singularity, $\operatorname{Sh}(M)$ is \mathbb{Q} -factorial and we have an isomorphism

$$\operatorname{Pic}_{\mathbb{O}}(\operatorname{Sh}(M)) \cong \operatorname{Cl}_{\mathbb{O}}(\operatorname{Sh}(M)),$$

which identifies $H_{m,\gamma}$ as an element in $Pic_{\mathbb{Q}}(Sh(M))$. Let us define

$$\operatorname{ACusp}_k(\rho_M) \subseteq \operatorname{Mod}_k(\rho_M)$$

as the subspace generated by elements in $\operatorname{Cusp}_k(\rho_M)$ and the Siegel-Eisenstein series $E_{k,M}$, called the space of almost cusp forms of weight k and type ρ_M . Then we have

Theorem 5.3 ([3]). Assume that $n \ge 3$ and M splits two hyperbolic planes. Then $Pic_{\mathbb{Q}}(Sh(M))$ is spanned by the Heegner divisors and there is an isomorphism

$$\Phi$$
: Pic_C(Sh(M)) \cong ACusp $_{\frac{2+n}{2}}(\rho_M)^{\vee}$

by sending $H_{m,\gamma}$ to the coefficient function

$$f = \sum_{\gamma \in G_M} \sum_{i \in \mathbb{Q}_{\ge 0}} c_{i,\gamma} q^i \mathfrak{e}_{\gamma} \mapsto c_{-m,\gamma}, \ \forall f \in \operatorname{ACusp}_{\frac{2+n}{2}}(\rho_M).$$
(5.3)

As a consequence, we have

Theorem 5.4 (cf. [5, 6]). The assumption on M is the same as above. A finite linear combination $H = \sum_{\gamma \in G_M} \sum_{m \in Q} a_{m,\gamma} H_{m,\gamma}$ is proportional to the Hodge line bundle $\lambda_M = H_{0,0}$ if and only if

$$\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} c_{-m,\gamma} = 0$$
(5.4)

for all cusp forms $\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} c_{m,\gamma} q^m \mathfrak{e}_{\gamma} \in \operatorname{Cusp}_{\frac{2+n}{2}}(\rho_M).$

Proof. If *H* satisfies (5.4), then the map $\Phi(H) \in ACusp_{\frac{2+n}{2}}(\rho_M)^{\vee}$ satisfies

$$\Phi(\mathbf{H})(f) = 0$$

for any $f \in \text{Cusp}_{\frac{2+n}{2}}(\rho_M)$. This means $\Phi(H)$ is proportional to the constant coefficients function $\Phi(H_{0,0})$. It follows that H is a multiple of $H_{0,0}$. The converse is obvious.

6. Heegner divisors on $\overline{Sh}(M)$

6.1. Baily-Borel compactification

There is a Baily-Borel compactification $\overline{Sh}(M)$ of Sh(M) by adding modular curves and cusps, which is a normal projective variety. Abstractly, it is given by the Proj of the graded ring of automorphic forms

on D. In the modern language, we have the following isomorphism

$$\overline{\mathrm{Sh}}(M) \cong \operatorname{Proj} \bigoplus_{m} \mathrm{H}^{0}\left(\mathrm{Sh}(M), \lambda_{M}^{\otimes m}\right).$$

In other words, the Hodge line bundle λ_M can be extended to an ample \mathbb{Q} -line bundle on $\overline{Sh}(M)$, denoted by $\overline{\lambda}_M$.

A more explicit description of $\overline{Sh}(M)$ as an arithmetic quotient is given below. If $V \subseteq M$ is a sublattice, we denote by

$$\pi_V: \mathbb{P}(M \otimes \mathbb{C}) - \mathbb{P}(V \otimes \mathbb{C}) \longrightarrow \mathbb{P}(M/V \otimes \mathbb{C})$$

the natural projection. Then we have

$$\overline{\mathrm{Sh}}(M) = \widetilde{\mathrm{O}}(M) \setminus \big(\mathbf{D} \bigsqcup_{I: \text{isotropic line}} \pi_{I^{\perp}} \mathbf{D} \cup \bigsqcup_{J: \text{isotropic plane}} \pi_{J^{\perp}} \mathbf{D} \big).$$

The 1-dimensional boundary components of $\overline{Sh}(M)$ correspond one-to-one to the orbits of rank 2 primitive isotropic sublattices of M under the action of $\widetilde{O}(M)$. We denote by $\partial_J(\overline{Sh}(M)) \subseteq \partial(\overline{Sh}(M))$ the boundary component associated to some isotropic plane J.

For each isotropic plane $J \subseteq M$, the lattice J^{\perp}/J is a negative definite lattice of rank n - 2. A natural question is whether, for any isotropic subgroup $H \subseteq G_M$, there is an inclusion

$$\operatorname{Gen}_{0,n-2}(H^{\perp}/H) \subseteq \{J^{\perp}/J \mid J \subseteq M \text{ is an isotropic plane}\}/\cong .$$
(6.1)

We provide a sufficient condition for this to be held.

Proposition 6.1. If M splits two hyperbolic planes, then there is an inclusion

$$\operatorname{Gen}_{0,n-2}(G_M) \subseteq \{J^{\perp}/J \mid J \subseteq M \text{ is an isotropic plane}\} \cong .$$
(6.2)

Proof. For any negative definite lattice $L \in \text{Gen}_{0,n-2}(G_M)$,

$$L \oplus U^{\oplus 2} \in \operatorname{Gen}_{2,n}(G_M).$$

Since $\sharp |\mathbf{Gen}_{2,n}(G_M)| = 1$ by [27, Theorem 1.13.1], we have $L \oplus U^{\oplus 2} \cong M$. Now it is sufficient to take J to be the image of the standard isotropic plane of $U^{\oplus 2}$ under this isomorphism.

6.2. Heegner divisors on the boundary

Let $x \in \partial(\overline{Sh}(M))$ be a boundary point. The (analytic) local Picard group of $\overline{Sh}(M)$ at x is defined by

$$\operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M), x) := \lim_{x \in U} \operatorname{Pic}_{\mathbb{Q}}(U \cap \operatorname{Sh}(M))$$

where U runs through all (analytic) open neighbourhoods of x. There is a restriction map

$$\operatorname{Pic}_{\mathbb{Q}}(\operatorname{Sh}(M)) \to \operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M), x)$$
 (6.3)

We say x is a generic point if it is not a cusp. For any isotropic plane $J \subseteq M$, we say a divisor $H \in \text{Pic}_{\mathbb{Q}}(\text{Sh}(M))$ is trivial at generic points of $\partial_J(\overline{\text{Sh}}(M))$ if there exists a generic point $x \in \partial_J(\overline{\text{Sh}}(M))$ such that the image of H in $\text{Pic}_{\mathbb{Q}}(\overline{\text{Sh}}(M), x)$ is trivial. The natural projection $O(M) \to O(G_M)$ is surjective if M splits two hyperbolic planes (cf. [27, Theorem 1.14.2]). Now the result of Bruinier and Fretaig about local Borcherds products immediately implies

Theorem 6.2. (cf. [9, Theorem 5.1]) Assume M splits two hyperbolic planes. Let x be a generic point on the component $\partial_J(\operatorname{Sh}(M))$ for some isotropic plane $J \subseteq M$ such that $J^{\perp}/J \in \operatorname{Gen}_{0,n-2}(G_M)$. The image of a finite linear combinations of Heegner divisors $\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} \operatorname{H}_{m,\gamma}$ is trivial in $\operatorname{Pic}_{\mathbb{Q}}(\operatorname{Sh}(M), g \cdot x)$

for all $g \in O(M)$ if and only if

$$\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} c_{-m,\gamma} = 0$$

for all theta series $\sigma^* \Theta_{J^{\perp}/J,H} = \sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} c_{m,\gamma} q^m \mathfrak{e}_{\gamma}$, where H runs over all harmonic polynomials of degree 2 and σ runs over $\operatorname{Iso}(G_M, G_{J^{\perp}/J})$.

For simplicity, we say that an even lattice *M* is of K3 type if *M* splits two hyperbolic planes and $M \otimes \mathbb{Z}_p$ can split three hyperbolic planes for all primes *p*. Then we have

Theorem 6.3. Let *M* be an even lattice of signature (2, n) of K3 type. Then if n > 10,

$$\dim_{\mathbb{Q}} \operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M)) = 1.$$

Proof. As the boundary of Sh(M) in $\overline{Sh}(M)$ are 1-dimensional and dim $Sh(M) \ge 3$, the pushforward map is an isomorphism

$$i_* : \operatorname{Cl}_{\mathbb{Q}}(\operatorname{Sh}(M)) \cong \operatorname{Cl}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M)).$$

By Theorem 5.3, $\operatorname{Cl}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M))$ is spanned by the Zariski closure of linear combinations of Heegner divisors. One can view $\operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M))$ as a subspace of $\operatorname{Pic}_{\mathbb{Q}}(\operatorname{Sh}(M))$. For any element $D \in \operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M))$, it can be written as a linear combination $\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} H_{m,\gamma}$. Since *D* is \mathbb{Q} -Cartier, its image in each local Picard group is a torsion element. By Proposition 6.1 and Theorem 6.2, we have

$$\sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} c_{-m,\gamma} = 0$$

for all cusp forms $f = \sum_{\gamma \in G_M} \sum_{m \in \mathbb{Q}} c_{m,\gamma} q^m \mathbf{e}_{\gamma} \in \operatorname{Cusp}_{\frac{2+n}{2}}^{\theta}(\rho_M)$. Applying Theorem 4.5 to J^{\perp}/J for an isotropic plane $J \subset M$ such that $J^{\perp}/J \in \operatorname{Gen}_{0,n-2}(G_M)$, the space $\operatorname{Cusp}_{\frac{2+n}{2}}^{\theta}(\rho_M)$ above is actually $\operatorname{Cusp}_{\frac{2+n}{2}}(\rho_M)$. Now Theorem 5.4 implies that D is proportional to the Hodge line bundle. This proves the theorem.

More generally, if $\Gamma \subset O(M)(\mathbb{Q})$ is an arithmetic subgroup containing O(M), denote by $Sh_{\Gamma}(M) := \Gamma \setminus \mathbf{D}$ the Shimura variety with respect to Γ . Let $\overline{Sh}_{\Gamma}(M)$ be the Baily-Borel compactification. Then we have

Corollary 6.4. With notations and assumptions as above, if n > 10, dim_Q Pic_Q($\overline{Sh}_{\Gamma}(M)$) = 1.

Proof. Note that the natural projection

$$\overline{\mathrm{Sh}}(M) \to \overline{\mathrm{Sh}}_{\Gamma}(M)$$

is a finite surjective morphism. By [3, Corollary 3.8], the first Chern class map induces an isomorphism

$$\operatorname{Pic}_{\mathbb{Q}}(\operatorname{Sh}_{(\Gamma)}(M)) \cong \operatorname{H}^{2}(\operatorname{Sh}_{(\Gamma)}(M), \mathbb{Q}).$$

Thus by the projection formula, the pullback homomorphism $\operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}_{\Gamma}(M)) \to \operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}(M))$ is injective. Since trivially $\dim_{\mathbb{Q}}\operatorname{Pic}_{\mathbb{Q}}(\overline{\operatorname{Sh}}_{\Gamma}(M)) \ge 1$, the result now follows from Theorem 6.3.

6.3. Proof of Theorem 1.1

Theorem 6.5 (Theorem 1.1). $\operatorname{Pic}(\overline{\mathcal{F}}_g) \cong \mathbb{Z}$, which is spanned by an integral multiple of the extended Hodge line bundle $\overline{\lambda}$.

Proof. This is the case when $M = \Lambda_g$. As $\operatorname{rank}_p(G_{\Lambda_g}) \leq 1$, by Remark 4.6, the conditions in Theorem 6.3 is automatically satisfied. Hence we have $\dim_{\mathbb{Q}} \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}}_g) = 1$. As shown in [14], $\operatorname{Pic}(\mathscr{F}_g)$ is torsion-free, then $\operatorname{Pic}(\overline{\mathscr{F}}_g)$ is also torsion-free as $\overline{\mathscr{F}}_g \setminus \mathscr{F}_g$ has codimension > 2.

It suffices to show that $\operatorname{Cl}(\overline{\mathscr{F}}_g) \cap \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}}_g) \subset \operatorname{Cl}_{\mathbb{Q}}(\overline{\mathscr{F}}_g)$ is generated by the extended Hodge line bundle $\overline{\lambda}$. To see this, let $\mathcal{S} \to \mathbb{P}^1$ be a family of unigonal K3 surfaces of genus g constructed in [22, §4.1.2]:

$$Y = \mathbb{P}(O_{\mathbb{P}^2}(4) \oplus O_{\mathbb{P}^2}) \xrightarrow{\varphi} \mathbb{P}^2$$

$$F = \varphi^{-1}(\text{line}), \quad A = \mathbb{P}(O_{\mathbb{P}^2}(4))$$
(6.4)

Let $B \in |3A + 12F|$ be generic. In particular, it is smooth and it does not intersect A. Let $\pi : \tilde{Y} \to Y$ be the double cover branched over A + B. We take a generic member

$$\mathcal{S} \in |\mathcal{O}_{\widetilde{Y}}(\pi^*F) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)|.$$

The resulting family $S \to \mathbb{P}^1$ is a family of double cover of the Hirzebruch surface $\Sigma := \mathbb{P}(O_{\mathbb{P}^1}(4) \oplus O_{\mathbb{P}^1})$ branched over $-2K_{\Sigma}$. The fiber of $S \to \mathbb{P}^1$ is a unigonal K3 surface with at worst A-D-E singularity and admits a polarization of genus g, induced by the restriction of $\frac{1}{2}\pi^*A + g\pi^*F$.

One can view \mathcal{F}_g as the coarse moduli space of primitive polarized K3 surfaces of genus g with at worst A-D-E singularities, then we obtain a morphism

$$\psi: \mathbb{P}^1 \to \mathcal{F}_g \subseteq \overline{\mathcal{F}}_g \tag{6.5}$$

By using the Grothendieck-Riemann-Roch theorem, Laza and O'Grady have computed that the degree of $\psi^*(\overline{\lambda}|_{\mathscr{F}_g}) = \psi^*(\lambda)$ is 1. This implies that $\overline{\lambda}$ is primitive in $\operatorname{Cl}(\overline{\mathscr{F}}_g) \cap \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}}_g)$ and hence

$$\operatorname{Cl}(\overline{\mathscr{F}}_g) \cap \operatorname{Pic}_{\mathbb{Q}}(\overline{\mathscr{F}}_g) = \langle \overline{\lambda} \rangle.$$

Appendix

A. A vector-valued Siegel-Weil formula

Following [25] and [34], we introduce the Siegel-Weil formula for metaplectic groups. For simplicity, let us assume M is a negative definite even lattice and $V = M \otimes \mathbb{Q}$. Let Sp_{2d} be the standard symplectic \mathbb{Z} -group scheme and $\operatorname{Mp}_{2d}(\mathbb{A}) \xrightarrow{\pi} \operatorname{Sp}_{2d}(\mathbb{A})$ be the metaplectic double cover. Then the inclusion $\operatorname{Sp}_{2d}(\mathbb{Q}) \hookrightarrow \operatorname{Sp}_{2d}(\mathbb{A})$ can be uniquely extended to an inclusion

$$\operatorname{Sp}_{2d}(\mathbb{Q}) \hookrightarrow \operatorname{Mp}_{2d}(\mathbb{A}).$$

Through this lifting, we will consider $\text{Sp}_{2d}(\mathbb{Q})$ as a subgroup of $\text{Mp}_{2d}(\mathbb{A})$.

Fix the standard additive character ψ : $\mathbb{Q}\setminus\mathbb{A}\to\mathbb{C}^{\times}$ whose archimedean component is given by $\psi_{\infty}:\mathbb{R}\to\mathbb{C}^{\times}, x_{\infty}\mapsto \mathbf{e}(x_{\infty})$ and the *p*-adic component is given by $\psi_p:\mathbb{Q}_p\to\mathbb{C}^{\times}, x_p\mapsto \mathbf{e}(-x'_p)$, where $x'_p\in\mathbb{Q}/\mathbb{Z}$ is the principal part of x_p . Let ω_{ψ} be the (automorphic) Weil representation of $O(V)(\mathbb{A})\times Mp_{2d}(\mathbb{A})$ realized in the Schrödinger model $S(V(\mathbb{A})^d)$, where $S(V(\mathbb{A})^d)$ is the space of Schwartz-Bruhat functions on $V(\mathbb{A})^d$. Similarly, one can define $\omega_{\psi,f}$ as the Weil representation acting on $S(V(\mathbb{A}_f)^d)$.

Remark A.1. There is a natural relation between this automorphic Weil representation and the representation $\rho_M^{(d)}$ defined in Subsection 2.1. Each element $\gamma \in G_M^{(d)}$ corresponds to a Schwartz function

$$\varphi_{\gamma} = \otimes_{p < \infty} \varphi_p \in \mathcal{S}(V(\mathbb{A}_f)^d),$$

where $\varphi_p \in \mathcal{S}(V(\mathbb{Q}_p)^d)$ is the characteristic function of $\gamma + (M \otimes \mathbb{Z}_p)^d$. Under the map

$$\iota: \mathbb{C}[G_M^{(d)}] \to \mathcal{S}(V(\mathbb{A}_f)^d)$$
$$\mathfrak{e}_{\gamma} \mapsto \varphi_{\gamma},$$

we have $\omega_{\psi,f}(g_f) \circ \iota = \iota \circ \overline{\rho_M^{(d)}}(g)$ (cf. [36]) for $g \in \operatorname{Mp}_{2d}(\mathbb{Z})$ and $g_f \in \operatorname{Mp}_{2d}(\hat{\mathbb{Z}})$ the unique element such that $gg_f \in \operatorname{Sp}_{2d}(\mathbb{Q}) \subset \operatorname{Mp}_{2d}(\mathbb{A})$.

Definition A.2. For $\varphi \in \mathcal{S}(V(\mathbb{A})^d)$, we can define the Theta series

$$\theta(g,h,\varphi) := \sum_{\mathbf{x} \in V(\mathbb{Q})^d} \omega_{\psi}(g) \varphi(h^{-1}\mathbf{x}), \quad g \in \mathrm{Mp}_{2d}(\mathbb{A}), \quad h \in \mathrm{O}(V)(\mathbb{A}).$$

which is automorphic on both $Mp_{2d}(\mathbb{A})$ and $O(V)(\mathbb{A})$.

Let $P = NA \subset \text{Sp}_{2d}$ be the standard Siegel parabolic subgroup. Here for any commutative ring *R*,

$$A(R) := \left\{ m(U) := \begin{pmatrix} U & 0 \\ 0 & (U^{-1})^{\mathsf{T}} \end{pmatrix} | U \in \mathrm{GL}_d(R) \right\},$$

and

$$N(R) := \left\{ n(B) := \begin{pmatrix} I_d & B \\ 0 & I_d \end{pmatrix} | B \in \operatorname{Sym}_d(R) \right\}.$$

We have the global Iwasawa decomposition $\operatorname{Sp}_{2d}(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K$ and $\operatorname{Mp}_{2d}(\mathbb{A}) = N(\mathbb{A})\widetilde{A}(\mathbb{A})\widetilde{K}$ for the standard maximal open compact subgroup.

Definition A.3. With the notations as above, the Eisenstein series associated with $\varphi \in \mathcal{S}(V^d(\mathbb{A}))$ is defined by

$$E(g, s, \varphi) := \sum_{\gamma \in P(\mathbb{Q}) \setminus Sp_{2d}(\mathbb{Q})} \Phi(\gamma g, s, \varphi) \quad g \in \mathrm{Mp}_{2d}(\mathbb{A}), \quad s \in \mathbb{C}$$

where $\Phi(g, s, \varphi) = |\det a(g)|^{s-s_0}(\omega_{\psi}(g)\varphi)(0)$ with $s_0 = \frac{r}{2} - \frac{d+1}{2}$ and $\pi(g) = n \cdot m(a(g)) \cdot k \in \operatorname{Sp}_{2d}(\mathbb{A})$ under the Iwasawa decomposition. It converges absolutely for $\operatorname{Re}(s) > \frac{d+1}{2}$.

The famous Siegel-Weil formula identifies the Theta lifting of the constant function 1 and the special value of the Eisenstein series.

Theorem A.4 (Siegel-Weil formula in the metaplectic case (cf. [34])). Let $\varphi \in S(V(\mathbb{A})^d)$ be a \widetilde{K} -finite function. Suppose r > d + 3, then the Eisenstein series $E(g, s, \varphi)$ is holomorphic at $s = s_0$ and

$$E(g, s_0, \varphi) = \int_{\mathcal{O}(V)(\mathbb{Q}) \setminus \mathcal{O}(V)(\mathbb{A})} \theta(g, h, \varphi) dh.$$

Here the Haar measure d*h is normalized so that* $vol(O(V)(\mathbb{Q})\setminus O(V)(\mathbb{A})) = 1$.

As an application, we can deduce Theorem 2.5.

$$\Theta_{\operatorname{Gen}(M)}^{(d)} = \mathbf{E}_{\frac{r}{2},M}^{(d)}.$$

Proof. The idea is to identify the component functions of both sides via the Siegel-Weil formula. When r is even, this is [25, Theorem 5.5]. Now suppose r is odd and we sketch the proof as below.

According to Kudla (cf. [20, p. 37 Proposition 4.3]), for the archimedean component of the Weil representation ω_{ψ} , there is a character $\chi_{V,\infty}^{\psi}$ of $\widetilde{A(\mathbb{R})}$ such that

$$(\omega_{\psi,\infty}(\tilde{m})\varphi_{\infty})(\mathbf{x}) = \chi_{V,\infty}^{\psi}(\tilde{m}) |\det(m(a))|^{\frac{r}{2}} \varphi_{\infty}(\mathbf{x} \cdot a), \quad \tilde{m} = (m(a), \phi_{m(a)}) \in \widetilde{A(\mathbb{R})},$$

where $\varphi_{\infty} \in \mathcal{S}(V(\mathbb{R})^d)$ and $a \in \mathrm{GL}_d(\mathbb{R})$. Let $\varphi_{\infty}(\mathbf{x}) = e^{-\pi \mathrm{tr}(\mathbf{x},\mathbf{x})}$ be the standard Gaussian function and for $\gamma \in (M^{\vee}/M)^d$, denote $\varphi_{\infty} \otimes \varphi_{\gamma}$ by $\varphi_{\infty,\gamma}$. For $\tau = x + iy \in \mathbb{H}_d$ and $a \in \mathrm{GL}_d(\mathbb{R})$ satisfying $aa^\top = y$, we consider the element

$$g_{\tau} = n(x)m(a) \in \operatorname{Sp}_{2d}(\mathbb{R}).$$

The component function of our previously defined vector-valued genus Theta series (resp. vector-valued Siegel-Eisenstein series) is given as follows.

• Genus Theta series:

$$\langle \Theta_{\mathbf{Gen}(M)}^{(d)}(\tau), \mathbf{e}_{\gamma} \rangle = \chi_{V,\infty}^{\psi}(\tilde{m})^{-1} |\det(m(a))|^{-\frac{r}{2}} \int_{\mathcal{O}(V)(\mathbb{Q}) \setminus \mathcal{O}(V)(\mathbb{A})} \theta(\tilde{g}_{\tau}, h, \varphi_{\infty, \gamma}) \mathrm{d}h,$$

where $\tilde{g}_{\tau} = n(x)\tilde{m} = (g_{\tau}, \phi_{g_{\tau}})$ and the Haar measure dh is normalized so that $vol(O(V)(\mathbb{Q})\setminus O(V)(\mathbb{A})) = 1$.

• Siegel-Eisenstein series:

$$\langle \mathbf{E}_{\frac{r}{2},M}^{(d)}(\tau), \mathbf{e}_{\gamma} \rangle = \chi_{V,\infty}^{\psi}(\tilde{m})^{-1} |\det(m(a))|^{-\frac{r}{2}} E(\tilde{g}_{\tau}, s_0, \varphi_{\infty,\gamma})$$

For the comparison of Theta series, the proof is exactly the same as in the classical case (cf. [23, Example 2.2.6]). For more details, one can see [25, Theorem 5.5]. For the comparison of Siegel-Eisenstein series. Since [25, Theorem 5.5] uses a computation by Kudla which might be difficult to locate a reference in the metaplectic case, we briefly present an alternative proof here.

Consider $q_{\tau}(\mathbf{x}) := e^{\pi i \operatorname{tr}((\mathbf{x}, \mathbf{x})\tau)}$. One has

$$\omega_{\psi,\infty}(n(B))(q_{\tau}(\mathbf{x})) = \psi_{\infty}\left(\frac{1}{2}\operatorname{tr}((\mathbf{x},\mathbf{x})B)\right)q_{\tau}(\mathbf{x}) = q_{n(B)\cdot\tau}(\mathbf{x})$$

for $n(B) = \left(\begin{pmatrix} I_d & B \\ 0 & I_d \end{pmatrix}, 1 \right)$, where $B \in \text{Sym}_d(\mathbb{Z})$ and

$$\omega_{\psi,\infty}(J_d)(q_\tau(\mathbf{x})) = \sqrt{\det(\tau)}^{-r} q_{J_d \cdot \tau}(\mathbf{x})$$

for $J_d = \left(\begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}, \sqrt{\det(\tau)} \right)$. Since $Mp_{2d}(\mathbb{Z})$ is generated by J_d and n(B)'s, combining these equations one gets

$$\omega_{\psi,\infty}(\tilde{g})(q_{\tau}(\mathbf{x})) = \phi_g(\tau)^{-r} q_{g\cdot\tau}(\mathbf{x})$$

for $\tilde{g} = (g, \phi_g(\tau)) \in \operatorname{Mp}_{2d}(\mathbb{Z})$. Hence

$$\chi^{\psi}_{V,\infty}(\tilde{m})^{-1} |\det(m(a))|^{-\frac{r}{2}} \Phi(\tilde{g} \cdot \tilde{g}_{\tau}, s_0, \varphi_{\infty,\gamma}) = \phi_g(\tau)^{-r} \omega_{\psi,f}(\tilde{g}_f)(\varphi_{\gamma})(0)$$

and the rest are exactly the same as the last several lines in [25, Theorem 5.5].

The result now follows from Theorem A.4. Note that in our case $r > 2d + 2 \ge d + 3$ for all $d \in \mathbb{Z}_{>0}$. So the condition r > d + 3 in Theorem A.4 always holds.

B. Coset decomposition and Weil representations

Recall there is a map

$$\iota: \mathrm{Mp}_2(\mathbb{Z}) \times \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{Mp}_4(\mathbb{Z}).$$

For $\tilde{A} = (A, \phi_A), \tilde{B} = (B, \phi_B) \in Mp_2(\mathbb{Z})$, we define $u(\tilde{A})$ to be the image of $\iota(\tilde{A}, (I_2, 1))$ and $d(\tilde{B})$ to be the image of $\iota((I_2, 1), \tilde{B})$. If we set

$$u(\hat{A}) = (u(A), \phi_{u(A)}) \text{ and } d(\hat{B}) = (d(B), \phi_{d(B)}),$$

with $u(A), d(B) \in SL_2(\mathbb{Z})$, then for $(z, z') \in \mathbb{H} \times \mathbb{H} \subseteq \mathbb{H}_2$, one has

$$\phi_{u(A)}(z, z') = \phi_A(z)$$
 and $\phi_{d(B)}(z, z') = \phi_B(z')$

from the definition. Next, for any $\alpha \in \mathbb{Z}$, we define

$$\widetilde{C}_{\alpha} = (C_{\alpha}, \phi_{C_{\alpha}}(\tau)) = \left(\begin{pmatrix} \alpha^{2} + \alpha & -\alpha - 1 & -1 & -\alpha - 1 \\ -\alpha - 1 & 1 & 0 & 0 \\ -\alpha & 1 & 0 & 0 \\ 0 & 0 & -1 & -\alpha \end{pmatrix}, \sqrt{\alpha^{2}z - 2\alpha w + z'} \right),$$

with $\tau = \begin{pmatrix} z & w \\ w & z' \end{pmatrix} \in \mathbb{H}_2$.

It has been shown in [25, Proposition 4.2] that for any $g^{(2)} \in \text{Sp}_4(\mathbb{Z})$ there exist some $\alpha \in \mathbb{Z}^{\leq 0}$, $A, B \in \text{SL}_2(\mathbb{Z})$ such that

$$\Gamma_{\infty}^{(2)}g^{(2)} = \Gamma_{\infty}^{(2)}C_{\alpha}u(A)d(B)$$

and

$$\varphi(DC_{\alpha}u(A)d(B)) = \pm B' \begin{pmatrix} \alpha^2 & 0\\ 0 & 1 \end{pmatrix} A,$$

is well-defined, where $D = n(S)m(U) \in \Gamma_{\infty}^{(2)}$ with $\det(U) = \pm 1$ and $B' = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $(I, -1) \in \widetilde{\Gamma}_{\infty}^{(2)}$, it follows that we also get a decomposition

$$\widetilde{\Gamma}_{\infty}^{(2)}\widetilde{g}^{(2)} = \widetilde{\Gamma}_{\infty}^{(2)}\widetilde{C}_{\alpha}u(\widetilde{A})d(\widetilde{B})$$
(B.1)

for any $\tilde{g}^{(2)} \in Mp_4(\mathbb{Z})$. For our purpose, we need to extend the map φ to metaplectic covers and study their actions under the Weil representation.

Definition B.1. • For $\tilde{B} = (B, \phi_B) \in Mp_2(\mathbb{Z})$, we define $\tilde{B}' = (B', \phi_{B'})$ where $\phi_{B'}$ is given by

$$\phi_{B'}(z)\sqrt{z'+B'z}=\phi_B(z')\sqrt{z+Bz'}.$$

for $z, z' \in \mathbb{H}$.

• For $\tilde{g}^{(2)} = (g^{(2)}, \phi_{g^{(2)}}) \in Mp_4(\mathbb{Z})$, we define $\tilde{\varphi}(\tilde{g}^{(2)}) = (\varphi(g^{(2)}), \phi_{\varphi(g^{(2)})})$, where $\phi_{\varphi(g^{(2)})}$ is chosen to satisfy

$$\phi_{g^{(2)}}(z,z') = \phi_{\varphi(g^{(2)})}(z)\sqrt{z' + \varphi(g^{(2)})z}.$$
(B.2)

Under this map, we have $\widetilde{\varphi}(\widetilde{C}_{\alpha}) = \widetilde{\mathbf{g}}_{\alpha} = \left(\begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$ and one verifies that $\phi_{B'}(z)$ and $\phi_{\varphi(g^{(2)})}(z)$ do not depend on z'.

Lemma B.2. For any $\tilde{A}, \tilde{B} \in Mp_2(\mathbb{Z})$, we have

- (i) $(\tilde{A}\tilde{B})' = \tilde{B}'\tilde{A}'$
- (*ii*) $\widetilde{\varphi}(\widetilde{C}_{\alpha}u(\widetilde{A})d(\widetilde{B})) = \widetilde{B}' \cdot \widetilde{\mathbf{g}}_{\alpha} \cdot \widetilde{A}$

Proof. For (i), from the definition, one can directly verify that

$$\phi_{(A\cdot B)'}(z)\sqrt{z'+(A\cdot B)'z} = \phi_{A'}(z)\phi_B(z')\sqrt{A'z+Bz'} = \phi_{B'\cdot A'}(z)\sqrt{z'+(A\cdot B)'z}.$$

For (*ii*), set $\widetilde{C}_{\alpha} \cdot u(\widetilde{A}) \cdot d(\widetilde{B}) = (g^{(2)}, \phi_{g^{(2)}})$. As $\varphi(g^{(2)}) = B' \cdot \mathbf{g}_{\alpha} \cdot A$, it suffices to show that

$$\frac{\phi_{g^{(2)}}(z,z')}{\sqrt{z'+\varphi(g^{(2)})z}}=\phi_{B'}(\mathbf{g}_{\alpha}\cdot Az)\phi_A(z).$$

This is clear because

LHS =
$$\frac{\left(\sqrt{\alpha^2 A z + B z'}\right) \phi_{u(A)}(z, B z') \phi_{d(B)}(z, z')}{\sqrt{z' + B' \alpha^2 A z}}$$
$$= \frac{\left(\sqrt{\alpha^2 A z + B z'}\right) \phi_A(z) \phi_B(z')}{\sqrt{z' + B' \alpha^2 A z}}$$
$$= \phi_{B'}(\alpha^2 A z) \phi_A(z)$$
$$= RHS.$$
(B.3)

Now it follows from [25, Proposition 4.2] that there is a well-defined injection

$$\widetilde{\Gamma}_{\infty}^{(2)} \setminus \operatorname{Mp}_{4}(\mathbb{Z}) \to (\widetilde{\operatorname{Mat}}_{2}(\mathbb{Z}) \times \mathbb{Z}^{\leq 0}) / (\langle (-I, i) \rangle \times 0)$$

$$[\widetilde{g}^{(2)}] \mapsto (\varphi(\widetilde{g}^{(2)}), \alpha).$$
(B.4)

We get

Lemma B.3. [25, Lemma 4.3] Let $\alpha \in \mathbb{Z}$, $\tilde{A} \in \widetilde{\mathbf{Y}}_{\alpha^2}$ and $\tilde{B} \in Mp_2(\mathbb{Z})$. Then

$$\sum_{\gamma \in G_M} (\mathfrak{e}_{\gamma} \mid [\tilde{A}]) \otimes \rho_M(\tilde{B})^{-1} \mathfrak{e}_{\gamma} = \sum_{\gamma \in G_M} (\mathfrak{e}_{\gamma} \mid [\tilde{B}'\tilde{A}]) \otimes \mathfrak{e}_{\gamma}.$$

Then we have

Proposition B.4. For $\tilde{g}^{(2)} = \tilde{C}_{\alpha}u(\tilde{A})d(\tilde{B})$ with $\alpha \leq 0$, we have

$$\rho_M^{(2)}(\tilde{g}^{(2)})^{-1}(\mathfrak{e}_0 \otimes \mathfrak{e}_0) = \frac{\mathfrak{e}(\operatorname{sign}(G_M)/8)}{\sqrt{|G_M|}} \sum_{\gamma \in G_M} (\mathfrak{e}_{\gamma} \mid [\widetilde{\varphi}(\tilde{g}^{(2)})]) \otimes \mathfrak{e}_{\gamma}.$$
(B.5)

Proof. It is straightforward to check that \widetilde{C}_{α} admits the following decomposition

$$\widetilde{C}_{\alpha} = J_2^{-1} \cdot n\left(\begin{pmatrix} 0 & -1 \\ -1 & -\alpha \end{pmatrix}\right) \cdot J_2^{-1} \cdot n\left(\begin{pmatrix} \alpha^2 + \alpha & -\alpha - 1 \\ -\alpha & -1 & 1 \end{pmatrix}\right) \cdot J_2^{-1} \cdot n\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

in Mp₄(\mathbb{Z}).

Given this decomposition, when $\tilde{g}^{(2)} = \tilde{C}_{\alpha}$, the equation B.5 holds by a direct computation (cf. [25, Proposition 4.4]).² Then from the decomposition (B.1), we have

$$\begin{split} \rho_{M}^{(2)}(\tilde{g}^{(2)})^{-1}(\mathfrak{e}_{0}\otimes\mathfrak{e}_{0}) &= \rho_{M}^{(2)}(\widetilde{C}_{\alpha}u(\tilde{A})d(\tilde{B}))^{-1}(\mathfrak{e}_{0}\otimes\mathfrak{e}_{0}) \\ &= \rho_{M}^{(2)}(u(\tilde{A})d(\tilde{B}))^{-1}\frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\varphi(\widetilde{C}_{\alpha})])\otimes\mathfrak{e}_{\gamma} \\ &= \frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\varphi(\widetilde{C}_{\alpha})] \mid [\tilde{A}])\otimes\rho_{M}(\tilde{B})^{-1}(\mathfrak{e}_{\gamma}) \\ &= \frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\tilde{B}'\varphi(\widetilde{C}_{\alpha})\tilde{A}])\otimes\mathfrak{e}_{\gamma} \\ &= \frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\varphi(\widetilde{C}_{\alpha}u(\tilde{A})d(\tilde{B}))])\otimes\mathfrak{e}_{\gamma} \\ &= \frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\varphi(\widetilde{C}_{\alpha}u(\tilde{A})d(\tilde{B}))])\otimes\mathfrak{e}_{\gamma} \\ &= \frac{\mathfrak{e}(\operatorname{sign}(G_{M})/8)}{\sqrt{|G_{M}|}}\sum_{\gamma\in G_{M}}(\mathfrak{e}_{\gamma} \mid [\varphi(\widetilde{C}_{\alpha}u(\tilde{A})d(\tilde{B}))])\otimes\mathfrak{e}_{\gamma}. \end{split}$$

Here we use Lemma B.3 in the fifth equation.

C. Hecke kernels

Proposition C.1. *For* $\alpha > 0$ *, we have*

$$\frac{\mathbf{T}_{\alpha^2}(f)(z')}{\alpha^{2k-2}} = C \int_{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \langle f, K_\alpha(z, -\overline{z'}) \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}, \tag{C.1}$$

where the constant $C = \frac{2^{k-4}(k-1)}{\pi i^k}$.

Proof. We only deal with the odd signature case. As in the even signature case, we have $\mathbf{T}_{\alpha^2}(K_{\alpha}(z, z')) = \alpha^{2k-2}K_{\alpha}(z, z')$ and it suffices to show that

$$\mathbf{T}_1(f)(z') = f(z') = C \int_{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \langle f, K_1(z, -\bar{z}') \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}.$$

Denote the element $\left(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \in Mp_2(\mathbb{Z})$ by \widetilde{T} . We have

²A careful reader may note that since our definition of "J" is different from that in [25], the action of "J₂" in [25] is actually the action of " $J_2^{-1"}$ in our notation. Therefore, the reason why we use J_2^{-1} in the above decomposition is to keep the consistency with the computations in [25, Proposition 4.4].

$$K_{1}(z, z') = \sum_{\gamma \in G_{M}} \sum_{\tilde{g} \in Mp_{2}(\mathbb{Z})} \frac{\phi_{g}(z)^{-2k}}{(z'+g \cdot z)^{k}} \rho_{M}(\tilde{g})^{-1}(\mathfrak{e}_{\gamma}) \otimes \mathfrak{e}_{\gamma}$$

$$= \sum_{\gamma \in G_{M}} \sum_{\tilde{g} \in \langle \widetilde{T} \rangle \setminus Mp_{2}(\mathbb{Z})} \sum_{n \in \mathbb{Z}} \frac{\phi_{T^{n}g}(z)^{-2k}}{(z'+T^{n}g \cdot z)^{k}} \rho_{M}(\widetilde{T}^{n}\tilde{g})^{-1}(\mathfrak{e}_{\gamma}) \otimes \mathfrak{e}_{\gamma}$$

$$= \sum_{\gamma \in G_{M}} \sum_{\tilde{g} \in \langle \widetilde{T} \rangle \setminus Mp_{2}(\mathbb{Z})} \sum_{n \in \mathbb{Z}} \frac{\phi_{g}(z)^{-2k}}{(z'+g \cdot z+n)^{k}} \mathfrak{e}(-n\mathfrak{q}(\gamma))\rho_{M}(\tilde{g})^{-1}(\mathfrak{e}_{\gamma}) \otimes \mathfrak{e}_{\gamma}$$

$$= 2 \frac{(-2\pi i)^{k}}{\Gamma(k)} \sum_{\gamma \in G_{M}} \sum_{\substack{m \in \mathbb{Z} + \mathfrak{q}(\gamma) \\ m > 0}} m^{k-1} \mathbf{P}_{\gamma,m}(z) \otimes \mathfrak{e}(mz') \mathfrak{e}_{\gamma}$$
(C.2)

where $\mathbf{P}_{\gamma,m}(z) = \frac{1}{2} \sum_{\tilde{g} \in \tilde{\Gamma}^+_{\infty} \setminus Mp_2(\mathbb{Z})} (\mathbf{e}(mz)\mathbf{e}_{\gamma}) |_{k,G_M} [\tilde{g}]$ is the Poincaré series defined in [10, §1.2]. Here the last identity comes from the Lipschitz Summation Formula in [24] (see also [28, Theorem 1]), which

Write the cusp form f as a Fourier expansion

we recall in Lemma C.2 below.

$$f(z) = \sum_{\substack{\gamma \in G_M}} \sum_{\substack{m \in \mathbb{Z} + \mathfrak{q}(\gamma) \\ m > 0}} c(m, \gamma) \mathbf{e}(mz) \mathbf{e}_{\gamma}.$$

By [10, Proposition 1.5], one has $\langle f, \mathbf{P}_{\gamma,m}(z) \rangle_{\text{Pet}} = \frac{2 \cdot \Gamma(k-1)}{(4\pi m)^{k-1}} c(m, \gamma)$. Combine this with (C.2), we finally get

$$\begin{split} &\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f, K_1(z, -\bar{z}') \rangle y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2} \\ =& 2 \frac{(2\pi i)^k}{\Gamma(k)} \frac{2 \cdot \Gamma(k-1)}{(4\pi)^{k-1}} \sum_{\gamma \in G_M} \sum_{\substack{m \in \mathbb{Z} + \mathfrak{q}(\gamma) \\ m > 0}} c(m, \gamma) \mathbf{e}(mz') \mathbf{e}_{\gamma} \\ =& \frac{\pi i^k}{2^{k-4}(k-1)} f(z'). \end{split}$$

Lemma C.2 (Lipschitz Summation Formula). Let $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 1$ and $z \in \mathbb{H}$. Then for $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} \mathbf{e}(nx)(z+n)^{-k} = \frac{(-2\pi i)^k}{\Gamma(k)} \sum_{\substack{r \in \mathbb{Z}-x \\ r>0}} r^{k-1} \mathbf{e}(rz)$$

where we chose a branch of the logarithm compatible with our choice of $\sqrt{\cdot}$.

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