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Asymptotic behavior of the generalized Derrida-Retaux recursive model

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Abstract: We study the max-type recursive model introduced by Hu and Shi (J. Stat. Phys., 2018), which generalizes the model of Derrida and Retaux (J. Stat. Phys., 2014). We show that the class of geometric-type distributions are preserved by the model with geometric offspring distribution. The key result is a characterization for the long-time asymptotic behavior of the marginal distributions. From this result, we derive the conditional limit law and the asymptotics of the sustainability probability and the first moment in the critical case.

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1 Introduction

A discrete-time max-type recursive model was introduced by Derrida and Retaux [5] in their study of the depinning transition in the limit of strong disorder. Let X_0 be a random variable taking values in $\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$. By a *Derrida-Retaux process*, or *DR process* for short, we mean a sequence of random variables $(X_n : n \geq 0)$ defined recursively in the distribution sense by

$$X_{n+1} \stackrel{\mathrm{d}}{=} (X_n + \tilde{X}_n - 1)_+, \quad n \geq 0, \tag{1.1}$$

where $(z)_+ := \max(0, z)$ and $\tilde{X}_n$ is an independent copy of X_n. By (1.1) it is easy to see that

$$\mathbf{E}(X_{n+1}) \leq 2\mathbf{E}(X_n).$$

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Then the following limit exists:

$$F_\infty := \lim_{n\to\infty} 2^{-n}\mathbf{E}(X_n).$$

The limit F_∞ is known as the *free energy*. The DR process is referred to as *pinned* if $F_\infty > 0$, and as *unpinned* if $F_\infty = 0$. One main problem in this study is to determine for which distribution of X_0 the model is pinned or unpinned. Another basic question is the asymptotic behavior of the *sustainability probability* $\mathbf{P}(X_n \geq 1)$ as $n \to \infty$. The model is said to be *critical* if $\mathbf{E}(2^{X_0}) = \mathbf{E}(X_0 2^{X_0}) < \infty$. In this case, it is expected that

$$\mathbf{P}(X_n \geq 1) = \frac{4}{n^2} + o\Big(\frac{1}{n^2}\Big), \quad n \to \infty. \tag{1.2}$$

We refer the reader to [1, 4, 5] for the physical explanations of the above prediction.

A number of variations of the DR process have also been studied. Let η be a random variable taking values in $\{1, 2, \cdots\}$. Instead of (1.1), we can also define a discrete-time max-type recursive process $(Y_n : n \geq 0)$ by the formula

$$Y_{n+1} \stackrel{\mathrm{d}}{=} (Y_{n,1} + Y_{n,2} + \cdots + Y_{n,\eta} - 1)_+, \tag{1.3}$$

where $\{Y_{n,1}, Y_{n,2}, \ldots\}$ is a sequence of independent copies of Y_n independent of η. The model was first studied by Hu and Shi [7]. Following [7], we call $(Y_n : n \geq 0)$ a *generalized DR process*. A scaling limit theorem for the process was proved in [8], which leads to a generalization of the continuous-time Derrida–Retaux process introduced by Hu et al. [6]. For the generalized DR model, a weaker form of (1.2) was obtained by Chen et al. [3]. Their result is presented in the following theorem:

Theorem (Chen et al. [3]) Suppose that $\eta \stackrel{\text{a.s.}}{=} m \geq 2$ and $\mathbf{E}[(m+\delta)^{Y_0}] < \infty$ for some $\delta > 0$. If

$$\mathbf{E}(m^{Y_0}) = (m-1)\mathbf{E}(Y_0 m^{Y_0}), \tag{1.4}$$

then

$$\mathbf{P}(Y_n \geq 1) = \frac{1}{n^{2+o(1)}}, \quad n \to \infty. \tag{1.5}$$

In this work, we study the asymptotic behavior of the discrete-time generalized DR process with *geometric offspring distribution*. More precisely, we assume the random variable η in (1.3) satisfies

$$\mathbf{P}(\eta = k) = \frac{1}{m}\Big(1 - \frac{1}{m}\Big)^{k-1}, \quad k = 1, 2, \cdots, \tag{1.6}$$

where $m > 1$ is a constant. Then we have $\mathbf{E}(\eta) = m$. For this model, the *free energy* is defined as the limit:

$$F_\infty := \lim_{n\to\infty} m^{-n}\mathbf{E}(Y_n).$$

Given the parameters $(r,p) \in (0,1)^2$, we denote by $\nu = G(r,p)$ the *geometric-type distribution*:

$$\nu(\mathrm{d}x) = p\delta_0(\mathrm{d}x) + (1-p)r\sum_{k=1}^{\infty}(1-r)^{k-1}\delta_k(\mathrm{d}x),$$

where δ_0 is the unit mass at zero. For $n \geq 0$ let μ_n be the distribution of Y_n. Our main results are as follows.

Theorem 1.1 *Suppose that $\mu_0 = G(r_0,p_0)$ for some $(r_0,p_0) \in (0,1)^2$. Then we have $\mu_n = G(r_n,p_n)$ for $n \geq 1$, where $\{(r_n,p_n) : n \geq 1\} \subset (0,1)^2$ is defined recursively by*

$$r_{n+1} = \frac{r_n}{m-(m-1)p_n}, \quad p_{n+1} = 1-(1-r_{n+1})\Big(1-\frac{r_{n+1}p_n}{r_n}\Big). \tag{1.7}$$

Theorem 1.2 *Let $\{(r_n,p_n) : n \geq 1\} \subset (0,1)^2$ be defined by (1.7) from any initial value $(r_0,p_0) \in (0,1)^2$. Then the following limits exist:*

$$r_* := \lim_{n\to\infty} r_n, \quad p_* := \lim_{n\to\infty} p_n. \tag{1.8}$$

Moreover, one of the following holds:

$$\text{(1) } r_* = 0 \text{ and } p_* = 0; \quad \text{(2) } 1-m^{-1} \leq r_* < 1 \text{ and } p_* = 1.$$

Theorem 1.3 *Let (r_*,p_*) be given by (1.8). Then:*

(1) (Supercritical case) *When $r_* = 0$ and $p_* = 0$, we have*

$$F_\infty = \frac{1}{m-1}\Big(\frac{1}{r_1}-\frac{1}{r_0}\Big)\prod_{n=1}^{\infty}(1-r_n) \in (0,\infty) \tag{1.9}$$

and, as $n \to \infty$,

$$\begin{cases} r_n = F_\infty^{-1}m^{-n} + o(m^{-n}), \\ p_n = F_\infty^{-1}nm^{1-n} + o(nm^{-n}). \end{cases}$$

(2) (Subcritical case) *When $1-m^{-1} < r_* < 1$ and $p_* = 1$, we have $F_\infty = 0$ and, as $n \to \infty$,*

$$\begin{cases} r_n = r_* + \displaystyle\prod_{i=1}^{\infty}\frac{1-r_i}{1-r_*}\frac{(r_0-r_1)r_*^2}{(m(1-r_*)-1)r_0r_1}[m(1-r_*)]^n + o([m(1-r_*)]^n), \\ p_n = 1 - \displaystyle\prod_{i=1}^{\infty}\frac{1-r_i}{1-r_*}\frac{r_*(r_0-r_1)}{(m-1)r_0r_1}[m(1-r_*)]^n + o([m(1-r_*)]^n). \end{cases}$$

(3) (Critical case) *When* $r_* = 1 - m^{-1}$ *and* $p_* = 1$, *we have* $F_\infty = 0$ *and, as* $n \to \infty$,

$$\begin{cases} r_n = 1 - \dfrac{1}{m} + \dfrac{2}{m}\dfrac{1}{n} - \dfrac{4(m+1)}{3m(m-1)}\dfrac{\log n}{n^2} + o\Big(\dfrac{\log n}{n^2}\Big), \\ p_n = 1 - \dfrac{2}{(m-1)^2}\dfrac{1}{n^2} - \dfrac{8(m+1)}{3m(m-1)^3}\dfrac{\log n}{n^3} + o\Big(\dfrac{\log n}{n^3}\Big). \end{cases}$$

As consequences of Theorem 1.3, for the generalized DR model at criticality we have the following:

Corollary 1.4 *Under the assumptions of Theorem 1.3, in the critical case, conditionally on* $Y_n \geq 1$, *the random variable* Y_n *converges weakly to the limit* Y_∞ *with geometric distribution:*

$$\mathbf{P}(Y_\infty = k) = \frac{m-1}{m^k}, \quad k = 1, 2, \cdots. \tag{1.10}$$

Corollary 1.5 *Under the assumptions of Theorem 1.3, in the critical case we have, as* $n \to \infty$,

$$\mathbf{P}(Y_n \geq 1) = \frac{2}{(m-1)^2}\frac{1}{n^2} + o\Big(\frac{1}{n^2}\Big). \tag{1.11}$$

Corollary 1.6 *Under the assumptions of Theorem 1.3, in the critical case we have, as* $n \to \infty$,

$$\mathbf{E}(Y_n) = \frac{2m}{(m-1)^3}\frac{1}{n^2} + o\Big(\frac{1}{n^2}\Big). \tag{1.12}$$

We mention that (1.10) agrees with Conjecture 2 of Chen et al. [2], but the coefficients in (1.11) and (1.12) do not agree exactly with those in Conjectures 1 and 3 of the same paper.

Remark 1.7 The results of Theorem 1.3 are not quite satisfactory as the supercritical, subcritical and critical regimes are characterized only by the limit $(r_*, p_*) = \lim_{n\to\infty}(r_n, p_n)$. By (1.7) the sequence $\{r_n\} \subset (0,1)$ is strictly decreasing. Then we have $\lim_{n\to\infty} r_n = r_* = 0$ if $0 < r_0 \leq 1 - m^{-1}$. Consequently, the model belongs to the supercritical regime if $(r_0, p_0) \in D := (0, 1 - m^{-1}] \times (0,1)$. There seems a decreasing function $r \mapsto g(r)$ at the interval $[1 - m^{-1}, 1)$ with $g(1 - m^{-1}) = 1$ so that the supercritical, subcritical and critical zones $\mathscr{P}$, $\mathscr{U}$ and $\mathscr{C}$ of the model are given respectively by

$$\begin{aligned} \mathscr{P} &= D \cup \{(r,p) : 1 - m^{-1} < r < 1, 0 < p < g(r)\}, \\ \mathscr{U} &= \{(r,p) : 1 - m^{-1} < r < 1, g(r) < p < 1\}, \\ \mathscr{C} &= \{(r,p) : 1 - m^{-1} < r < 1, p = g(r)\}. \end{aligned}$$

We have not been able to find an accurate expression of the function g.

Remark 1.8 Given the parameters $(\lambda, p) \in (0, \infty) \times (0, 1)$, we denote by $\mu = E(\lambda, p)$ the *exponential-type distribution* given by:

$$\mu(\mathrm{d}x) = p\delta_0 + (1-p)\lambda \mathrm{e}^{-\lambda x}\mathrm{d}x, \quad x \geq 0,$$

where δ_0 is the unit mass at zero. As in the proof of Theorem 1.1, one can show that if $\mu_0 = E(\lambda_0, p_0)$ for $(\lambda_0, p_0) \in (0, \infty) \times (0, 1)$, then $\mu_n = E(\lambda_n, p_n)$ for $n \geq 1$, where the sequence $\{(\lambda_n, p_n) : n \geq 1\} \subset (0, \infty) \times (0, 1)$ is defined recursively by

$$\lambda_{n+1} = \frac{\mathrm{e}^{-\alpha}\lambda_n}{1 - (1 - \mathrm{e}^{-\alpha})p_n}, \quad p_{n+1} = 1 - \mathrm{e}^{-\lambda_{n+1}}\Big(1 - \frac{\lambda_{n+1}p_n}{\lambda_n}\Big).$$

All the results obtained this note can be extended to the exponential-type marginal distributions by similar arguments.

The rest of the note is organized as follows. The basic structures of the geometric-type marginal distributions are discussed in Section 2, where the proofs of Theorems 1.1 and 1.2 are also given. The proof of Theorem 1.3 is given in Section 3.

2 Geometric-type marginal distributions

Proof of Theorem 1.1. We show $\mu_n = G(r_n, p_n)$ by induction. Suppose this is true for some $n \geq 0$. Then

$$\mathbf{E}(s^{Y_n}) = p_n + \frac{(1-p_n)r_n s}{1-(1-r_n)s} = \frac{p_n + (r_n - p_n)s}{1-(1-r_n)s}.$$

Write $\xi_{n+1} = \sum_{k=1}^{\eta_n} Y_{n,k}$. By the independence of η_n and $\{Y_{n,k}\}$, we see that

$$\begin{aligned}
\mathbf{E}(s^{-\xi_{n+1}}) &= \sum_{k=1}^{\infty} m^{-1}(1-m^{-1})^{k-1}\Big[\frac{p_n + (r_n - p_n)s}{1-(1-r_n)s}\Big]^k \\
&= \frac{m^{-1}[p_n + (r_n - p_n)s]/[1-(1-r_n)s]}{1-(1-m^{-1})[p_n + (r_n - p_n)s]/[1-(1-r_n)s]} \\
&= \frac{p_n + (r_n - p_n)s}{m[1-(1-r_n)s] - (m-1)[p_n + (r_n - p_n)s]} \\
&= \frac{p_n + (r_n - p_n)s}{m - (m-1)p_n - [m(1-r_n) + (m-1)(r_n - p_n)]s} \\
&= \frac{p_n + (r_n - p_n)s}{m - (m-1)p_n - [m - (m-1)p_n - r_n]s} \\
&= \frac{q_{n+1}}{1-(1-r_{n+1})s} + \frac{(r_{n+1} - q_{n+1})s}{1-(1-r_{n+1})s}
\end{aligned}$$

$$= q_{n+1} + (1 - q_{n+1})\frac{r_{n+1}s}{1-(1-r_{n+1})s},$$

where

$$r_{n+1} = \frac{r_n}{m-(m-1)p_n}, \quad q_{n+1} = \frac{p_n}{m-(m-1)p_n} = \frac{r_{n+1}p_n}{r_n}.$$

Then ξ_{n+1} follows the geometric-type distribution $G(r_{n+1}, q_{n+1})$, that is,

$$\mathbf{P}(\xi_{n+1} \in \mathrm{d}x) = q_{n+1}\delta_0(\mathrm{d}x) + (1-q_{n+1})r_{n+1}\sum_{k=1}^{\infty}(1-r_{n+1})^{k-1}\delta_k(\mathrm{d}x).$$

By the total probability formula and the memoryless of the geometric distribution, we have

$$\begin{aligned}\mathbf{P}(Y_{n+1} \in \mathrm{d}x) &= \mathbf{P}(\xi_{n+1} \geq 2)\mathbf{P}(\xi_{n+1} - 1 \in \mathrm{d}x | \xi_{n+1} \geq 2) + \mathbf{P}(\xi_{n+1} \leq 1)\delta_0(\mathrm{d}x) \\ &= (1-q_{n+1})(1-r_{n+1})\sum_{k=1}^{\infty} r_{n+1}(1-r_{n+1})^{k-1}\delta_k(\mathrm{d}x) \\ &\quad + [1-(1-r_{n+1})(1-q_{n+1})]\delta_0(\mathrm{d}x),\end{aligned}$$

where

$$(1-q_{n+1})(1-r_{n+1}) = (1-r_{n+1})\Big(1-\frac{r_{n+1}p_n}{r_n}\Big) = 1-p_{n+1}.$$

Then we have $\mu_{n+1} = G(r_{n+1}, p_{n+1})$. □

Proposition 2.1 *Let* $\{(r_n, p_n) : n \geq 1\}$ *be the sequence defined by* (1.7)*. Then for any* $n \geq 0$ *we have*

$$\Big(\frac{1}{r_{n+2}} - \frac{1}{r_{n+1}}\Big) = m(1-r_{n+1})\Big(\frac{1}{r_{n+1}} - \frac{1}{r_n}\Big). \tag{2.1}$$

Proof. From the first equality in (1.7) it follows that

$$m-(m-1)p_n = \frac{r_n}{r_{n+1}}, \tag{2.2}$$

and hence

$$1-p_n = \frac{1}{m-1}\Big(\frac{r_n}{r_{n+1}} - 1\Big). \tag{2.3}$$

By (2.2) and the second equality in (1.7) we have

$$p_{n+1} = 1-(1-r_{n+1})\Big(1-\frac{r_{n+1}p_n}{r_n}\Big)$$

$$
\begin{aligned}
&= 1 - (1 - r_{n+1})\Big(1 - \frac{p_n}{m - (m-1)p_n}\Big) \\
&= 1 - \frac{m(1-p_n)(1-r_{n+1})}{m - (m-1)p_n}. \qquad (2.4)
\end{aligned}
$$

It follows that

$$
1 - p_{n+1} = m(1-p_n)\frac{r_{n+1}}{r_n}(1 - r_{n+1}).
$$

Using (2.3) we obtain

$$
\begin{aligned}
\frac{1}{m-1}\Big(\frac{r_{n+1}}{r_{n+2}} - 1\Big) &= \frac{m}{m-1}\Big(\frac{r_n}{r_{n+1}} - 1\Big)\frac{r_{n+1}}{r_n}(1 - r_{n+1}) \\
&= \frac{m}{m-1}\Big(1 - \frac{r_{n+1}}{r_n}\Big)(1 - r_{n+1}).
\end{aligned}
$$

This proves (2.1). □

Proposition 2.2 *Let* $\{(r_n, p_n) : n \geq 1\}$ *be the sequence defined by* (1.7). *Then for any* $n \geq 0$ *we have*

$$
\frac{p_{n+1}}{r_{n+1}} - \frac{p_n}{r_n} = m(1 - p_n). \qquad (2.5)
$$

Proof. By (2.2) and (2.4) it follows that

$$
\begin{aligned}
p_{n+1} &= \frac{m(1-p_n) + p_n - m(1-r_{n+1})(1-p_n)}{m - (m-1)p_n} \\
&= \frac{m(1-p_n)r_n + p_n}{m - (m-1)p_n} = \frac{r_{n+1}}{r_n}[m(1-p_n)r_n + p_n] \\
&= r_{n+1}\Big[m(1-p_n) + \frac{p_n}{r_n}\Big].
\end{aligned}
$$

Then the desired relation holds. □

Proposition 2.3 *Let* $\{(r_n, p_n) : n \geq 1\}$ *be the sequence defined by* (1.7). *Then for any* $n \geq 0$ *we have*

$$
m^{-n}\mathbf{E}(Y_n) = \frac{1}{m-1}\Big(\frac{1}{r_1} - \frac{1}{r_0}\Big)\prod_{i=1}^{n}(1 - r_i). \qquad (2.6)
$$

Proof. By Theorem 1.1 we know that Y_n has the geometric-type distribution $G(r_n, p_n)$. From (2.1) it follows that

$$
\frac{1}{r_{n+1}} - \frac{1}{r_n} = m^n\prod_{i=1}^{n}(1 - r_i)\Big(\frac{1}{r_1} - \frac{1}{r_0}\Big), \qquad (2.7)
$$

Then we can use (2.3) to see that

$$\mathbf{E}(Y_n) = \frac{1-p_n}{r_n} = \frac{1}{m-1}\Big(\frac{1}{r_{n+1}} - \frac{1}{r_n}\Big)$$
$$= \frac{m^n}{m-1}\prod_{i=1}^{n}(1-r_i)\Big(\frac{1}{r_1} - \frac{1}{r_0}\Big).$$

Then we have the expression (2.6). □

Proof of Theorem 1.2. By (1.7) it is easy to see the sequence $\{r_n\}$ is strictly decreasing. Set $r_* = \lim_{n\to\infty} r_n$. We claim that either $r_* = 0$ or $r_* \in [1-m^{-1}, \infty)$. Otherwise, we have $r_* \in (0, 1-m^{-1})$, and so there exists $N \geq 1$ such that $0 < r_* \leq r_n < 1-m^{-1}$ for all $n \geq N$. In this case, it follows from (2.1) that

$$\Big|\frac{1}{r_{n+2}} - \frac{1}{r_{n+1}}\Big| > \Big|\frac{1}{r_{n+1}} - \frac{1}{r_n}\Big|,$$

which is in contradiction to the existence of the limit $\lim_{n\to\infty} 1/r_n = 1/r_*$. Therefore, we must have $r_* = 0$ or $r_* \in [1-m^{-1}, \infty)$. We next examine the convergence of the sequence $\{p_n\}$ in the two cases.

(1) In the case $r_* = 0$, the sequence $1/r_n$ strictly increases to ∞ as $n \to \infty$. By applying the Stolz–Cesàro theorem in conjunction with (2.1) we get

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \lim_{n\to\infty} \frac{\frac{1}{r_n}}{\frac{1}{r_{n+1}}} = \lim_{n\to\infty} \frac{\frac{1}{r_n} - \frac{1}{r_{n-1}}}{\frac{1}{r_{n+1}} - \frac{1}{r_n}} = \lim_{n\to\infty} m^{-1}(1-r_n)^{-1} = m^{-1}. \tag{2.8}$$

Then $p_* = \lim_{n\to\infty} p_n = 0$ by (2.2).

(2) In the case $r_* \in [1-m^{-1}, \infty)$, we have $\lim_{n\to\infty}(r_n/r_{n+1}) = 1$ and hence $p_* = \lim_{n\to\infty} p_n = 1$ by (2.3). □

3 Asymptotic behavior of the dynamics

Lemma 3.1 *Suppose that $r_* = 1-m^{-1}$ and let $v_n = r_n - r_* = r_n - 1 + m^{-1}$. Then we have*

$$\lim_{n\to\infty} nv_n = \frac{2}{m}, \quad \lim_{n\to\infty} n^2(v_n - v_{n+1}) = \lim_{n\to\infty} \frac{m}{2} n^2 v_n v_{n+1} = \frac{2}{m}. \tag{3.1}$$

Proof. To simplify the presentation, we introduce the difference operator Δ in the following way: For any sequence $\{a_n\}$ write $\Delta a_n = a_{n+1} - a_n$ and

$$\Delta^2 a_n = \Delta(\Delta a_n) = a_{n+2} - 2a_{n+1} + a_n.$$

By (2.1) we have

$$\begin{aligned}(r_{n+2}-r_{n+1}) &= m(1-r_{n+1})\frac{r_{n+2}}{r_n}(r_{n+1}-r_n)\\ &= (1-mv_{n+1})\frac{r_{n+2}}{r_n}(r_{n+1}-r_n),\end{aligned}$$

which implies

$$\Delta v_{n+1} = (1-mv_{n+1})\frac{v_{n+2}+1-m^{-1}}{v_n+1-m^{-1}}\Delta v_n. \tag{3.2}$$

It follows that

$$\begin{aligned}\Delta^2 v_n &= (1-mv_{n+1})\frac{v_{n+2}+1-m^{-1}}{v_n+1-m^{-1}}\Delta v_n - \Delta v_n\\ &= \left[(1-mv_{n+1})\Big(1+\frac{v_{n+2}-v_n}{v_n+1-m^{-1}}\Big)-1\right]\Delta v_n\\ &= \left[-mv_{n+1}+(1-mv_{n+1})\frac{v_{n+2}-v_n}{v_n+1-m^{-1}}\right]\Delta v_n.\end{aligned} \tag{3.3}$$

Note that v_n strictly decreases to zero as $n\to\infty$. By applying the Stolz–Cesàro theorem,

$$\lim_{n\to\infty}\frac{v_{n+1}}{v_n} = \lim_{n\to\infty}\frac{\Delta v_{n+1}}{\Delta v_n} = \lim_{n\to\infty}(1-mv_{n+1})\frac{v_{n+2}+1-m^{-1}}{v_n+1-m^{-1}} = 1.$$

Then we deduce

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{v_n}\left[(1-mv_{n+1})\Big(1+\frac{v_{n+2}-v_n}{v_n+1-m^{-1}}\Big)-1\right]\\ &\quad=\lim_{n\to\infty}\left(-m\frac{v_{n+1}}{v_n}+\frac{v_{n+2}/v_n-1}{v_n+1-m^{-1}}-\frac{v_n-v_{n+2}}{v_n+1-m^{-1}}\frac{v_{n+1}}{v_n}\right) = -m.\end{aligned} \tag{3.4}$$

By (3.2) the sequence $-\Delta v_{n+1}$ strictly decreases to zero. Then we can use (3.4) and the Stolz–Cesàro theorem to obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{v_{n+1}v_n}{v_n-v_{n+1}} &= \lim_{n\to\infty}\frac{v_{n+1}(v_{n+2}-v_n)}{(v_{n+1}-v_{n+2})-(v_n-v_{n+1})}\\ &= \lim_{n\to\infty}\frac{v_{n+1}(\Delta v_n+\Delta v_{n+1})}{-\Delta^2 v_n}\\ &= \lim_{n\to\infty}\frac{-v_n(1+\Delta v_{n+1}/\Delta v_n)}{(1-mv_{n+1})\big(1+\frac{v_{n+2}-v_n}{v_n+1-m^{-1}}\big)-1} = \frac{2}{m}.\end{aligned}$$

By another application of the Stolz–Cesàro theorem we get

$$\lim_{n\to\infty} nv_n = \lim_{n\to\infty}\frac{1}{\frac{1}{v_{n+1}}-\frac{1}{v_n}} = \lim_{n\to\infty}\frac{v_nv_{n+1}}{v_n-v_{n+1}} = \frac{2}{m}.$$

This gives the limits in (3.1). □

Lemma 3.2 *Suppose that $r_* = 1 - m^{-1}$ and let $v_n = \lambda_n - r_* = r_n - 1 + m^{-1}$. Then we have*

$$\lim_{n\to\infty} n^3\Big(v_n - v_{n+1} - \frac{m}{2}v_n v_{n+1}\Big) = \frac{4(m+1)}{3m(m-1)}. \tag{3.5}$$

Proof. In view of (3.1), as $n \to \infty$ we have

$$v_n = \frac{2m^{-1}}{n} + o\Big(\frac{1}{n}\Big), \quad \Delta v_n = -\frac{2m^{-1}}{n^2} + o\Big(\frac{1}{n^2}\Big). \tag{3.6}$$

Then by Taylor's expansion for the function $1/(1+x)$ we see that

$$\begin{aligned}\frac{v_{n+2} - v_n}{v_n + 1 - m^{-1}} &= \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\frac{1}{1 + mv_n/(m-1)} \\ &= \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\Big[1 - \frac{m}{m-1}v_n + O\Big(\frac{1}{n^2}\Big)\Big] \\ &= \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1}) + O\Big(\frac{1}{n^3}\Big).\end{aligned}$$

Substituting the above expression into (3.3) we obtain

$$\Delta^2 v_n = -mv_{n+1}\Delta v_n + \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\Delta v_n + O\Big(\frac{1}{n^5}\Big).$$

It follows that

$$\begin{aligned}\Delta\Big(\Delta v_n + \frac{m}{2}v_{n+1}v_n\Big) &= \Delta^2 v_n + \frac{m}{2}v_{n+1}\Delta v_{n+1} + \frac{m}{2}v_{n+1}\Delta v_n \\ &= -\frac{m}{2}v_{n+1}\Delta v_n + \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\Delta v_n \\ &\quad + \frac{m}{2}v_{n+1}\Delta v_{n+1} + O\Big(\frac{1}{n^5}\Big) \\ &= \frac{m}{2}v_{n+1}\Delta^2 v_n + \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\Delta v_n + O\Big(\frac{1}{n^5}\Big) \\ &= -\frac{m^2}{2}v_{n+1}^2\Delta v_n + \frac{m}{m-1}(\Delta v_n + \Delta v_{n+1})\Delta v_n + O\Big(\frac{1}{n^5}\Big) \\ &= \frac{4}{m}\frac{1}{n^4} + \frac{8}{m(m-1)}\frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big) \\ &= \frac{4(m+1)}{m(m-1)}\frac{1}{n^4} + O\Big(\frac{1}{n^5}\Big).\end{aligned}$$

By applying Stolz–Cesàro theorem we deduce that

$$\begin{aligned}\lim_{n\to\infty} n^3\Big(\Delta v_n + \frac{m}{2}v_n v_{n+1}\Big) &= \lim_{n\to\infty}\frac{\Delta\big(\Delta v_n + \frac{m}{2}v_n v_{n+1}\big)}{\frac{1}{(n+1)^3} - \frac{1}{n^3}} = \lim_{n\to\infty}\frac{\frac{4(m+1)}{m(m-1)}\frac{1}{n^4}}{\frac{n^3-(n+1)^3}{(n+1)^3 n^3}} \\ &= \lim_{n\to\infty}\frac{\frac{4(m+1)}{m(m-1)}}{\frac{n[n^3-(n+1)^3]}{(n+1)^3}} = -\frac{4(m+1)}{3m(m-1)}.\end{aligned}$$

This proves the desired result. ☐

Proof of Theorem 1.3. (1) Consider the supercritical case, where $r_* = 0$ and $p_* = 0$. From (2.8) we see that

$$0 < \prod_{n=1}^{\infty}(1 - r_n) < \infty.$$

Then (1.9) follows from (2.6). Since $1/r_n$ strictly increases to ∞ as $n \to \infty$, in view of (2.7) and (2.8), we can use the Stolz–Cesàro theorem to see that

$$\begin{aligned}\lim_{n\to\infty}\frac{r_n}{m^{-n}} &= \lim_{n\to\infty}\frac{m^n}{\frac{1}{r_n}} = \lim_{n\to\infty}\frac{m^{n+1}-m^n}{\frac{1}{r_{n+1}}-\frac{1}{r_n}} = \lim_{n\to\infty}\frac{m-1}{m^{-n}\left(\frac{1}{r_{n+1}}-\frac{1}{r_n}\right)}\\ &= \frac{m-1}{\left(\frac{1}{r_1}-\frac{1}{r_0}\right)}\prod_{n=1}^{\infty}(1-r_n)^{-1} = F_\infty^{-1},\end{aligned}$$

which gives the asymptotics of r_n. By using (2.5), (2.8) and the Stolz–Cesàro theorem, we see that

$$\lim_{n\to\infty}\frac{p_n}{nr_n} = \lim_{n\to\infty}\frac{\frac{p_n}{r_n}}{n} = \lim_{n\to\infty}\left(\frac{p_{n+1}}{r_{n+1}} - \frac{p_n}{r_n}\right) = \lim_{n\to\infty} m(1-p_n) = m.$$

This proves the asymptotics of p_n.

(2) Consider the subcritical case, where $r_* > 1 - m^{-1}$ and $p_* = 1$. By (2.6) we have $F_\infty = \lim_{n\to\infty} m^{-n}\mathbf{E}(Y_n) = 0$. From the relation (2.1) it follows that

$$r_n - r_{n+1} = m(1-r_n)\frac{r_{n+1}}{r_{n-1}}(r_{n-1} - r_n). \tag{3.7}$$

Then an application of the Stolz–Cesàro theorem leads to

$$\begin{aligned}\lim_{n\to\infty}\frac{r_{n+1}-r_*}{r_n - r_*} &= \lim_{n\to\infty}\frac{r_{n+1}-r_n}{r_n - r_{n-1}}\\ &= \lim_{n\to\infty} m(1-r_n)\frac{r_{n+1}}{r_{n-1}} = m(1-r_*) < 1,\end{aligned}$$

which implies that

$$0 < \prod_{i=1}^{\infty}\frac{1-r_i}{1-r_*} < \infty.$$

By using (3.7) again we deduce that

$$r_n - r_{n+1} = m^n\frac{r_n r_{n+1}}{r_0 r_1}(r_0 - r_1)\prod_{i=1}^{n}(1-r_i)$$

$$= [m(1-r_*)]^n \frac{r_n r_{n+1}}{r_0 r_1}(r_0 - r_1) \prod_{i=1}^{n} \frac{1-r_i}{1-r_*}. \tag{3.8}$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_n - r_{n+1}}{[m(1-r_*)]^n} &= \lim_{n\to\infty} \frac{r_n r_{n+1}}{r_0 r_1}(r_0 - r_1) \prod_{i=1}^{n} \frac{1-r_i}{1-r_*} \\
&= \frac{(r_0 - r_1) r_*^2}{r_0 r_1} \prod_{i=1}^{\infty} \frac{1-r_i}{1-r_*}.
\end{aligned}$$

By using the Stolz–Cesàro theorem we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_n - r_*}{[m(1-r_*)]^n} &= \lim_{n\to\infty} \frac{r_n - r_{n+1}}{[1-m(1-r_*)][m(1-r_*)]^n} \\
&= \frac{(r_0 - r_1) r_*^2}{[1-m(1-r_*)] r_0 r_1} \prod_{i=1}^{\infty} \frac{1-r_i}{1-r_*}
\end{aligned}$$

and, also using (2.3) and (3.8),

$$\begin{aligned}
p_n &= 1 - \frac{1}{m-1}\Big(\frac{r_n}{r_{n+1}} - 1\Big) \\
&= 1 - \frac{1}{(m-1) r_{n+1}} [m(1-r_*)]^n \frac{r_n r_{n+1}}{r_0 r_1}(r_0 - r_1) \prod_{i=1}^{n} \frac{1-r_i}{1-r_*} \\
&= 1 - \frac{r_*(r_0 - r_1)}{(m-1) r_0 r_1} [m(1-r_*)]^n \prod_{i=1}^{\infty} \frac{1-r_i}{1-r_*} + o([m(1-r_*)]^n).
\end{aligned}$$

These give the asymptotics of (r_n, p_n).

(3) Finally, we deal with the critical case, where $r_* = 1 - m^{-1}$ and $p_* = 1$. By (2.6) we have $F_\infty = \lim_{n\to\infty} m^{-n}\mathbf{E}(Y_n) = 0$. Write $v_n = r_n - r_* = r_n - 1 + m^{-1}$ as in Lemma 3.1 and 3.2. By (3.1) and (3.5),

$$\Big(\frac{1}{v_{n+1}} - \frac{1}{v_n}\Big) - \frac{m}{2} = \frac{1}{v_n v_{n+1}}\Big(v_n - v_{n+1} - \frac{m}{2} v_n v_{n+1}\Big) = \frac{m(m+1)}{3(m-1)n} + o\Big(\frac{1}{n}\Big).$$

Applying Stolz–Cesàro theorem again we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\big(\frac{1}{v_n} - \frac{1}{v_0}\big) - \frac{m}{2} n}{\log n} &= \lim_{n\to\infty} \frac{\big(\frac{1}{v_{n+1}} - \frac{1}{v_n}\big) - \frac{m}{2}}{\log(n+1) - \log n} \\
&= \lim_{n\to\infty} \frac{\big(\frac{1}{v_{n+1}} - \frac{1}{v_n}\big) - \frac{m}{2}}{\log\big(1 + \frac{1}{n}\big)} \\
&= \lim_{n\to\infty} \frac{\big(\frac{1}{v_{n+1}} - \frac{1}{v_n}\big) - \frac{m}{2}}{\frac{1}{n}} = \frac{m(m+1)}{3(m-1)}.
\end{aligned}$$

It follows that

$$\frac{1}{v_n} = \frac{mn}{2} + \frac{m(m+1)}{3(m-1)}\log n + o(\log n),$$

and hence

$$\begin{aligned} v_n &= \frac{1}{\frac{mn}{2} + \frac{m(m+1)}{3(m-1)}\log n + o(\log n)} \\ &= \frac{\frac{2}{mn}}{1 + \frac{2(m+1)}{3(m-1)}\frac{\log n}{n} + o\big(\frac{\log n}{n}\big)}. \end{aligned}$$

By Taylor's expansion for the function $1/(1+x)$,

$$\begin{aligned} v_n &= \frac{2}{mn}\Big[1 - \frac{2(m+1)}{3(m-1)}\frac{\log n}{n} + o\Big(\frac{\log n}{n}\Big)\Big] \\ &= \frac{2}{m}\frac{1}{n} - \frac{4(m+1)}{3m(m-1)}\frac{\log n}{n^2} + o\Big(\frac{\log n}{n^2}\Big). \end{aligned} \tag{3.9}$$

This leads to the asymptotics of r_n. By (3.5) and (3.9) we have

$$\begin{aligned} v_n - v_{n+1} &= \frac{m}{2}v_n v_{n+1} + \frac{4(m+1)}{3m(m-1)}\frac{1}{n^3} + o\Big(\frac{1}{n^3}\Big) \\ &= \frac{2}{m}\frac{1}{n^2} - \frac{8(m+1)}{3m^2(m-1)}\frac{\log n}{n^3} + o\Big(\frac{\log n}{n^3}\Big). \end{aligned} \tag{3.10}$$

In view of (2.3) and (3.10), we have

$$\begin{aligned} p_n &= 1 - \frac{v_n - v_{n+1}}{(m-1)(1-m^{-1}+v_{n+1})} = 1 - \frac{m}{(m-1)^2}\frac{v_n - v_{n+1}}{1 + v_{n+1}/(1-m^{-1})} \\ &= 1 - \frac{m}{(m-1)^2}\Big[\frac{2}{m}\frac{1}{n^2} - \frac{8(m+1)}{3m^2(m-1)}\frac{\log n}{n^3} + o\Big(\frac{\log n}{n^3}\Big)\Big]\frac{1}{1 + v_{n+1}/(1-m^{-1})}. \end{aligned}$$

Then we get the asymptotics of p_n by (3.9) and Taylor's expansion for the function $1/(1+x)$. □

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